QUANTUM GROUPS AND YANG-BAXTER EQUATIONS.

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Abstract.

The principles of the theory of quantum groups are reviewed from the point of view of the possibility of using them for deformations of symmetries in physics models. The R -matrix approach to the theory of quantum groups is discussed in detail and taken as the basis of quantization of classical Lie groups and also some Lie supergroups. Trigonometric solutions of the Yang-Baxter equation associated with the quantum groups $GL_q(N)$, $SO_q(N)$, $Sp_q(2n)$ and supergroups $GL_q(N|M)$, $Osp_q(N|2m)$, as well as their rational (Yangian) limits, are presented. Elliptic solutions of the Yang-Baxter equation are also considered. The notions of a group algebra of braid group and its finite dimensional qoutients (Hecke and Birman-Murakami-Wenzl algebras) are introduced. Applications of the theory of quantum groups and Yang-Baxter equations in different branches of theoretical physics are briefly discussed.

Contents

1 INTRODUCTION

In modern theoretical physics, the ideas of symmetry and invariance play a very important role. As a rule, symmetry transformations form groups, and therefore the most natural language for describing symmetries is the group theory language. About 25 years ago, in the study of quantum integrable systems [2, 3, 4] in particular in the framework of the quantum inverse scattering method [1] new algebraic structures arose, the generalizations of which were later called quantum groups [22][1]. Yang-Baxter equations became a unifying basis of all these investigations. The most important nontrivial examples of quantum groups are deformations or quantizations of ordinary classical Lie groups and algebras (more precisely, one considers the quantization of the algebra of functions on a Lie group and the universal enveloping of a Lie algebra). The quantization is accompanied by the introduction of an additional parameter q (the deformation parameter), which plays a role analogous to the role of Planck's constant in quantum mechanics. In the limit $q \to 1$, the quantum Lie groups and algebras go over into the classical ones. Although quantum groups are deformations of the usual groups, they nevertheless possess several properties that make it possible to speak of them as "symmetry groups". Moreover, one can claim that the quantum groups serve as the symmetries of the quantum integrable models (see, for example, Refs. [5, 6, 7])[2]. In this connection, the idea naturally arises of looking for and constructing other physical models possessing such quantum symmetries. Some of these approaches use the similarity of the representation theories of quantum and classical Lie groups and algebras (for q not equal to the roots of unity). As a result, we have, for example, identity of the dimensions of the irreducible representations (multiplets) for the group $SU(N)$ and dimensions for co-representations of the quantum group $SU_q(N)$. Thus, we can use quantum Lie groups and algebras both in the phenomenology of elementary particles and in nuclear spectroscopy investigations. Further, it is natural to wish to investigate the already existing field-theoretical models (for example, the Salam-Weinberg model or the standard model) with a view to establishing their connection (see e.g. Ref. [9]) with noncommutative geometry [8][3] and, in particular, the possibility of their being invariant with respect to quantum-group transformations. A very attractive idea is that of relating the deformation parameters of quantum groups to the mixing angles that occur in the standard model as free parameters. One of the possible realizations of this idea was proposed in Ref. [12] (see also [13]). We also mention here the numerous attempts to deform the Lorentz and Poincare groups and the construction of a covariant quantum space-time corresponding to these deformations [15, 16] [4]. It is clear that the approaches listed above (associated with quantization of symmetries in physics) represent only a small fraction of all the applications of the theory of quantum groups. Quantum groups and Yang-Baxter equations arise naturally in many problems of theoretical physics, and this makes it possible to speak of them and the theories of

[1] In pure mathematics the analogous structures have been appeared as nontrivial examples of "ring-groups" introduced by G.I. Kac; see [31] and references therein.

[2] Yangian symmetries are the symmetries of the same type.

[3] After the quantization of any Poisson manifold [10] and appearing of papers [11] the subject of field theories on noncommutative spaces has started to be very popular from the point of view of string theories.

[4] Discussions of different covariant quantum spaces can be found in [14] (see also refs. therein)

them as an important paradigm in mathematical physics. Unfortunately, the strict limits of the review make it impossible to discuss in detail all applications of quantum groups and Yang-Baxter equations. I have therefore restricted myself to a brief listing of certain areas in theoretical and mathematical physics in which quantum groups and Yang-Baxter equations play an important role. The list is given in the Conclusions. In Sec. 2, the mathematical foundations of the theory of quantum groups are presented. A significant part of Sec. 3, is a detailed exposition of the famous work of Faddeev, Reshetikhin and Takhtajan [17] who have formulated the R matrix approach to the theory of quantum groups. In this section, we also consider questions of invariant Baxterization of R matrices, many-parameter deformations of Lie groups, and the quantization of some Lie supergroups. At the end of Sec. 3, we outline the basic notions of a theory of quantum Knizhnik - Zamolodchikov equations and discuss elliptic solutions of the Yang-Baxter equation for which the algebraic basis (the type of quantum universal enveloping Lie algebras $U_q(g)$ in the case of trigonometric solutions) has not yet completely clarified (see, however [18], [19]). In Sec. 4, we briefly discuss a group algebra of braid group and its finite quotients such as Hecke and Birman-Murakami-Wenzl algebras. The content of this Section can be considered as a different presentation of some facts from Sec. 3.

As we have already mentioned, that some applications of quantum groups and the Yang-Baxter equations are briefly considered in Sec. 5.

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2 HOPF ALGEBRAS

This section of the review is based on the Refs. [20]-[30].

2.1 Coalgebras

We consider an associative unital algebra $\mathcal{A}$ (over the field of complex numbers $\mathbf{C}$; in what follows, all algebras that are introduced will also be understood to be over the field of complex numbers). Each element of $\mathcal{A}$ can be expressed as a linear combination of basis elements $\{e_i\}$, where $i = 1, 2, 3, \ldots$ and $E^i e_i = I$ ($E^i \in \mathbf{C}$) is the identity element (we imply the summation over repeated indices). This means that for any two elements e_i and e_j we can define their multiplication in the form

$$\mathcal{A} \otimes \mathcal{A} \xrightarrow{m} \mathcal{A} \Rightarrow e_i \cdot e_j = m_{ij}^k e_k, \tag{2.1.1}$$

where m_{ij}^k is certain set of complex numbers that satisfy the condition

$$E^i m_{ij}^k = m_{ji}^k E^i = \delta_j^k \tag{2.1.2}$$

for the identity element, and also the condition

$$m_{ij}^l m_{lk}^n = m_{il}^n m_{jk}^l \equiv m_{ijk}^n \;, \tag{2.1.3}$$

which is equivalent to the condition of associativity for the algebra $\mathcal{A}$:

$$(e_i e_j)e_k = e_i(e_j e_k). \tag{2.1.4}$$

The condition of associativity (2.1.4) for the multiplication (2.1.1) can obviously be represented in the form of the commutativity of the diagram:

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "m"] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "m"] & \mathcal{A}
\end{tikzcd}$$

Fig. 1. Associativity axiom.

In Fig. 1, m represents multiplication: $\mathcal{A}\otimes\mathcal{A} \xrightarrow{m} \mathcal{A}$, and id denotes the identity mapping. The existence of the unit element I means that one can define a mapping i: $C \to \mathcal{A}$ (embedding of $\mathbf{C}$ in $\mathcal{A}$)

$$k \xrightarrow{\mathbf{i}} k\cdot I \ , \quad k \in \mathbf{C} \tag{2.1.5}$$

For I we have the condition (2.1.2), which is equivalent to the diagram of Fig.2,

$$\begin{tikzcd}
& \mathcal{A}\otimes\mathcal{A} \arrow[dd, "m"] & \\
\mathbf{C}\otimes\mathcal{A} \arrow[ur, "\mathbf{i}\otimes id"] \arrow[dr, leftrightarrow] & & \mathcal{A}\otimes\mathbf{C} \arrow[ul, "id\otimes\mathbf{i}"'] \arrow[dl, leftrightarrow] \\
& \mathcal{A} &
\end{tikzcd}$$

Fig. 2. Axioms for the identity.

in which the mappings $C\otimes\mathcal{A} \leftrightarrow \mathcal{A}$ and $\mathcal{A}\otimes\mathbf{C} \leftrightarrow \mathcal{A}$ are natural isomorphisms. One of the advantages of the diagrammatic language used here is that it leads directly to the definition of a new fundamental object – the coalgebra – if we reverse all the arrows in the diagrams of Fig.1 and Fig.2.

Definition 1. *A coalgebra $\mathcal{C}$ is a vector space (with the basis $\{e_i\}$) equipped with a mapping $\Delta : \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$*

$$\Delta(e_i) = \Delta_i^{kj} e_k \otimes e_j \ , \tag{2.1.6}$$

which is called comultiplication, and also equipped with a mapping $\epsilon : \mathcal{C} \to \mathbf{C}$, which is called the coidentity. The coalgebra $\mathcal{C}$ is called coassociative if the mapping Δ satisfies the condition of coassociativity (cf. the first diagram with arrows reversed)

$$(id\otimes\Delta)\Delta = (\Delta\otimes id)\Delta \Rightarrow \Delta_i^{nl}\Delta_l^{kj} = \Delta_i^{lj}\Delta_l^{nk} \equiv \Delta_i^{nkj} \ . \tag{2.1.7}$$

The coidentity ϵ must satisfy the following conditions (cf. the second diagram)

$$m(\epsilon\otimes id)\Delta(\mathcal{C}) = m(id\otimes\epsilon)\Delta(\mathcal{C}) = \mathcal{C} \Rightarrow \epsilon_i\Delta_k^{ij} = \Delta_k^{ji}\epsilon_i = \delta_k^j \ . \tag{2.1.8}$$

Here m is multiplication map: $m(c \otimes e_i) = m(e_i \otimes c) = c \cdot e_i$, and the complex numbers ϵ_i are determined from the relations $\epsilon(e_i) = \epsilon_i$.

For algebras and coalgebras, the concepts of modules and comodules can be introduced. Thus, if $\mathcal{A}$ is an algebra, the left $\mathcal{A}$-module can be defined as a vector space N and a mapping $\psi: \ \mathcal{A} \otimes N \to N$ (action of $\mathcal{A}$ on N) such that the diagrams on Fig. 3 are commutative.

$$
\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \otimes N \arrow[r, "id \otimes \psi"] \arrow[d, "m \otimes id"'] & \mathcal{A} \otimes N \arrow[d, "\psi"] \\
\mathcal{A} \otimes N \arrow[r, "\psi"] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{A} \otimes N \arrow[dd, "\psi"] \\
\mathbf{C} \otimes N \arrow[ur, "\mathbf{i} \otimes id"] \arrow[dr, leftrightarrow] & \\
 & N
\end{tikzcd}
$$

Fig. 3 Axioms for left $\mathcal{A}$-module.

In other words, the space N is the space of a representation for the algebra $\mathcal{A}$.

If N is a (co)algebra and the mapping ψ preserves the (co)algebraic structure of N (see below), then N is called a left $\mathcal{A}$-module (co)algebra. The concepts of right module and module (co)algebra are introduced similarly. If N is simultaneously a left and a right $\mathcal{A}$-module, then N is called a two-sided $\mathcal{A}$-module. It is obvious that the algebra $\mathcal{A}$ itself is a two-sided $\mathcal{A}$-module for which the left and right actions are given by the left and right multiplications in the algebra.

Now suppose that $\mathcal{C}$ is a coalgebra; then a left $\mathcal{C}$-comodule can be defined as a space M together with a mapping $\Delta_L : \quad M \to \mathcal{C} \otimes M$ (coaction of $\mathcal{C}$ on M) satisfying the axioms of Fig. 4 (in the diagrams in Fig. 3 defining modules it is necessary to reverse all the arrows).

$$
\begin{tikzcd}
\mathcal{C} \otimes \mathcal{C} \otimes M & \mathcal{C} \otimes M \arrow[l, "id \otimes \Delta_L"'] \\
\mathcal{C} \otimes M \arrow[u, "\Delta \otimes id"] & M \arrow[l, "\Delta_L"'] \arrow[u, "\Delta_L"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{C} \otimes M \arrow[dl, "\epsilon \otimes id"'] \\
\mathbf{C} \otimes M \arrow[dr, leftrightarrow] & \\
 & M \arrow[uu, "\Delta_L"']
\end{tikzcd}
$$

Fig. 4 Axioms for left $\mathcal{A}$-comodule.

If M is a (co)algebra and the mapping Δ_L preserves the (co)algebraic structure (for example, is a homomorphism; see below), then M is called a left $\mathcal{C}$-comodule (co)algebra. Right comodules are introduced similarly, after which two-sided comodules are defined in the natural manner. It is obvious that the coalgebra $\mathcal{C}$ is a two-sided $\mathcal{C}$-comodule.

Let $\mathcal{V}$, $\tilde{\mathcal{V}}$ be two vector spaces with bases $\{e_i\}$, $\{\tilde{e}_i\}$. We denote by $\mathcal{V}^*$, $\tilde{\mathcal{V}}^*$ the corresponding dual linear spaces whose basis elements are linear functionals $\{e^i\}: \ \mathcal{V} \to \mathbf{C}$, $\{\tilde{e}^i\}: \tilde{\mathcal{V}} \to \mathbf{C}$. For the values of these functionals, we shall use the expressions $\langle e^i|e_j\rangle$ and $\langle \tilde{e}^i|\tilde{e}_j\rangle$. For every mapping $L: \ \mathcal{V} \to \tilde{\mathcal{V}}$ it is possible to define a unique mapping $L^*: \ \tilde{\mathcal{V}}^* \to \mathcal{V}^*$ induced by the equations

$$\langle \tilde{e}^i|L(e_j)\rangle = \langle L^*(\tilde{e}^i)|e_j\rangle, \tag{2.1.9}$$

if the matrix $\langle e^i|e_j\rangle$ is invertible. In addition, for the dual objects there exists the linear injection

$$\rho: \ \mathcal{V}^* \otimes \tilde{\mathcal{V}}^* \to (\mathcal{V} \otimes \tilde{\mathcal{V}})^* ,$$

which is given by the equations

$$\langle \rho(e^i \otimes \tilde{e}^j)|e_k \otimes \tilde{e}_l\rangle = \langle e^i|e_k\rangle\langle \tilde{e}^j|\tilde{e}_l\rangle .$$

A consequence of these facts is that for every coalgebra $(\mathcal{C}, \ \Delta, \ \epsilon)$ it is possible to define an algebra $\mathcal{C}^* = \mathcal{A}$ (as dual object to $\mathcal{C}$) with multiplication $m = \Delta^* \cdot \rho$ and the unit element I that satisfy the relations

$$\langle a|c_{(1)}\rangle\langle a'|c_{(2)}\rangle = \langle \rho(a \otimes a')|\Delta(c)\rangle = \langle \Delta^* \cdot \rho(a \otimes a')|c\rangle = \langle a\, a' \mid c\rangle ,$$
$$\langle I|c\rangle = \epsilon(c) \ , \ \ \forall a, a' \in \mathcal{A} \ , \ \forall c \in \mathcal{C} \ .$$

Here we have used the convenient Sweedler notation of Ref. [11] for comultiplication in $\mathcal{C}$: $\Delta(c) = \sum_c c_{(1)} \otimes c_{(2)}$ [cf. (2.1.6)] (we also use the Sweedler notations for left and right coactions $\Delta_L(v) = \sum_v \bar{v}^{(-1)} \otimes v^{(0)}$ and $\Delta_R(v) = \sum_v v^{(0)} \otimes \bar{v}^{(1)}$, where index (0) is reserved for the comodule elements). The summation symbol $\sum_c$ will usually be omitted in the equations.

Thus, duality in the diagrammatic definitions of the algebras and coalgebras (reversal of the arrows) has in particular the consequence that the algebras and coalgebras are indeed duals of each other.

It is natural to expect that an analogous duality can also be traced for modules and comodules. Let $\mathcal{V}$ be a left comodule for $\mathcal{C}$. Then the left coaction of $\mathcal{C}$ on $\mathcal{V}$: $v \to \sum_v \ \bar{v}^{(-1)} \otimes v^{(0)}$ $(\bar{v} \in \mathcal{C}, \ \ v \in \mathcal{V})$ induces a right action of $\mathcal{A} = \mathcal{C}^*$ on $\mathcal{V}$:

$$(v, a) \to v \triangleleft a = \langle a|\bar{v}^{(-1)}\rangle v^{(0)} \ , \quad a \in \mathcal{A}$$

(here and in what follows, we omit the summation sign $\sum_v$) and therefore $\mathcal{V}$ is a right module for $\mathcal{A}$. Conversely, the right coaction of $\mathcal{C}$ on $\mathcal{V}$: $v \to v^{(0)} \otimes \bar{v}^{(1)}$ induces the left action of $\mathcal{A} = \mathcal{C}^*$ on $\mathcal{V}$:

$$(a, v) \to a \triangleright v = v^{(0)}\langle a|\bar{v}^{(1)}\rangle .$$

From this we immediately conclude that the coassociative coalgebra $\mathcal{C}$ (which coacts on itself by the coproduct) is a natural module for its dual algebra $\mathcal{A}$. Indeed, the right action $\mathcal{C} \otimes \mathcal{A} \to \mathcal{C}$ is determined by the equations

$$(c, a) \to c \triangleleft a = \langle a|c_{(1)}\rangle c_{(2)} \tag{2.1.10}$$

whereas for the left action $\mathcal{A} \otimes \mathcal{C} \to \mathcal{C}$ we have

$$(a, c) \to a \triangleright c = c_{(1)}\langle a|c_{(2)}\rangle . \tag{2.1.11}$$

Here $a \in \mathcal{A}$ $c \in \mathcal{C}$. The module axioms (shown as the diagrams in Fig. 3) hold by virtue of the coassociativity of $\mathcal{C}$.

Finally, we note that the action of a certain algebra H on $\mathcal{C}$ from the left (from the right) induces an action of H on $\mathcal{A} = \mathcal{C}^*$ from the right (from the left). This obviously follows from relations of the type (2.1.9).

2.2 Bialgebras

So-called bialgebras are the next important objects that are used in the theory of quantum groups.

Definition 2. *An associative algebra $\mathcal{A}$ with identity that is simultaneously a coassociative coalgebra with coidentity is called a bialgebra if the algebraic and coalgebraic structures are self-consistent. Namely, the comultiplication and coidentity must be homomorphisms of the algebras:*

$$\Delta(e_i)\Delta(e_j) = m_{ij}^k \Delta(e_k) \Rightarrow \Delta_i^{i'i''}\Delta_j^{j'j''} m_{i'j'}^{k'} m_{i''j''}^{k''} = m_{ij}^k \Delta_k^{k'k''} \ , \tag{2.2.1}$$

$$\Delta(I) = I \otimes I \ , \ \ \epsilon(e_i e_j) = \epsilon(e_i)\epsilon(e_j) \ , \ \ \epsilon(I) = E^i \, \epsilon_i = 1 \ .$$

Note that for every bialgebra we have a certain freedom in the definition of the multiplication (2.1.1) and the comultiplication (2.1.6). Indeed, all the axioms (2.1.3), (2.1.7), and (2.2.1) are satisfied if instead of (2.1.1) we take

$$e_i \cdot e_j = m_{ji}^k e_k,$$

or instead of (2.1.6) choose

$$\Delta'(e_i) = \Delta_i^{jk} e_k \otimes e_j \ , \tag{2.2.2}$$

(such algebras are denoted as $\mathcal{A}^{op}$ and $\mathcal{A}^{cop}$, respectively). Then the algebra $\mathcal{A}$ is called noncommutative if $m_{ij}^k \neq m_{ji}^k$, and noncocommutative if $\Delta_k^{ij} \neq \Delta_k^{ji}$.

In quantum physics, it is usually assumed that all algebras of observables are bialgebras. Indeed, a coalgebraic structure is needed to define the action of the algebra $\mathcal{A}$ of observables on the state $|\psi_1\rangle \otimes |\psi_2\rangle$ of the system that is the composite system formed from two independent systems with wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$

$$a \triangleright (|\psi_1\rangle \otimes |\psi_2\rangle) = \Delta(a)\,(|\psi_1\rangle \otimes |\psi_2\rangle) = a_{(1)}\,|\psi_1\rangle \otimes a_{(2)}\,|\psi_2\rangle \ \ (\forall a \in \mathcal{A}) \ . \tag{2.2.3}$$

In other words, it is only for bialgebras that it is possible to construct a theory of representations in which new representations can be obtained by multiplying old ones.

A classical example of a bialgebra is the universal enveloping algebra of a Lie algebra, in particular, the spin algebra in three-dimensional space. To demonstrate this, we consider the Lie algebra g with generators J_α $(\alpha = 1, 2, 3, \ldots)$, that satisfy the antisymmetric multiplication rule (defining relations)

$$[J_\alpha, \, J_\beta] = t_{\alpha\beta}^\gamma J_\gamma. \tag{2.2.4}$$

Here $t_{\alpha\beta}^\gamma = -t_{\beta\alpha}^\gamma$ are structure constants which satisfy Jacoby identity. The enveloping algebra of this algebra is the algebra U_g with basis elements consisting of the identity I and the elements $e_i = J_{\alpha_1} \cdots J_{\alpha_n}$ $\forall n \geq 1$, where the products of the generators J are ordered lexicographically, i.e., $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$. The coalgebraic structure for the algebra U_g is specified by means of the mappings

$$\Delta(J_\alpha) = J_\alpha \otimes I + I \otimes J_\alpha \ , \ \ \epsilon(J_\alpha) = 0 \ , \ \ \epsilon(I) = 1 \ , \tag{2.2.5}$$

which satisfy all the axioms of a bialgebra. The mapping Δ in (2.2.5) is none other than the rule for addition of spins. In fact one can quantize the coalgebraic structure (2.2.5) for universal enveloping algebra U_g and consider the noncocommutative comultiplications Δ. Such quantizations will be considered below in Section 3.3 and leads to the definition of Lie bialgebras.

Considering exponentials of elements of a Lie algebra, one can arrive at the definition of a group bialgebra of the group G with structure mappings

$$\Delta(h) = h \otimes h \ , \ \ \epsilon(h) = 1 \ (\forall h \in G), \tag{2.2.6}$$

which obviously follow from (2.2.5). The next important example of a bialgebra is the algebra $\mathcal{A}(G)$ of functions on a group ($f : G \to \mathbf{C}$). This algebra is dual to the group algebra of the group G, and its structure mappings have the form ($f, f' \in \mathcal{A}(G); \ \ h, h' \in G$):

$$(f \cdot f')(h) = f(h) f'(h) \ , \ \ (\Delta(f))(h, h') = f(h \cdot h') \ , \ \ \epsilon(f) = f(I) \ . \tag{2.2.7}$$

where I_G is the identity element in the group G. In particular, if the functions T^i_j realize a matrix representation of the group G, then we have

$$T^i_j(hh') = T^i_k(h) T^k_j(h') \Rightarrow \Delta(T^i_j) = T^i_k \otimes T^k_j \ , \tag{2.2.8}$$

(the functions T^i_j can be regarded as generators of a subalgebra in the algebra $\mathcal{A}(G)$). Note that if g is non-Abelian, then U_g and G are noncommutative but cocommutative bialgebras, whereas $\mathcal{A}(G)$ is a commutative but noncocommutative bialgebra. Anticipating, we mention that the most interesting quantum groups are associated with noncommutative and noncocommutative bialgebras.

It is obvious that for a bialgebra $\mathcal{H}$ it is also possible to introduce the concepts of left (co)modules and (co)module (co)algebras [right (co)modules and (co)module (co)algebras are introduced in exactly the same way]. Moreover, for the bialgebra $\mathcal{H}$ it is possible to introduce the concept of a left (right) bimodule B, i.e., a left (right) $\mathcal{H}$-module that is simultaneously a left (right) $\mathcal{H}$-comodule; at the same time, the module and comodule structures must be self-consistent:

$$\Delta_L(\mathcal{H} \triangleright B) = \Delta(\mathcal{H}) \triangleright \Delta_L(B) \ ,$$

$$(\epsilon \otimes id)\Delta_L(b) = b \ , \ \ b \in B \ .$$

On the other hand, in the case of bialgebras the conditions of conserving of the (co)algebraic structure of (co)modules can be represented in a more explicit form. For example, for the left $\mathcal{H}$-module algebra $\mathcal{A}$ we have ($a, b \in \mathcal{A}; \ \ h \in \mathcal{H}$):

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \ , \ \ h \triangleright I_A = \epsilon(h) I_A \ .$$

In addition, for the left $\mathcal{H}$-module coalgebra $\mathcal{A}$ we must have

$$\Delta(h \triangleright a) = \Delta(h) \triangleright \Delta(a) = h_{(1)} \triangleright a_{(1)} \otimes h_{(2)} \triangleright a_{(2)} \ , \ \ \epsilon(h \triangleright a) = \epsilon(h)\epsilon(a) \ .$$

Similarly, the algebra $\mathcal{A}$ is a left $\mathcal{H}$-comodule algebra if

$$\Delta_L(ab) = \Delta_L(a)\Delta_L(b) \ , \ \ \Delta_L(I_A) = I_{\mathcal{H}} \otimes I_A \ ,$$

and, finally, the coalgebra $\mathcal{A}$ is a left $\mathcal{H}$-comodule coalgebra if

$$(id \otimes \Delta)\Delta_L(a) = m_{\mathcal{H}}(\Delta_L \otimes \Delta_L)\Delta(a) \ , \quad (id \otimes \epsilon_A)\Delta_L(a) = I_{\mathcal{H}}\epsilon_A(a) \ , \tag{2.2.9}$$

where

$$m_{\mathcal{H}}(\Delta_L \otimes \Delta_L)(a \otimes b) = \bar{a}^{(-1)}\bar{b}^{(-1)} \otimes a^{(0)} \otimes b^{(0)} \ .$$

We now consider the bialgebra $\mathcal{H}$, which acts on a certain module algebra $\mathcal{A}$. One further important property of bialgebras is that we can define a new associative algebra $\mathcal{A}\sharp\mathcal{H}$ as the cross product (smash product) of $\mathcal{A}$ and $\mathcal{H}$. Namely:

Definition 3. *The left smash product $\mathcal{A}\sharp\mathcal{H}$ of the bialgebra $\mathcal{H}$ and its left module algebra $\mathcal{A}$ is an associative algebra such that:*
1) As a vector space, $\mathcal{A}\sharp\mathcal{H}$ is identical to $\mathcal{A} \otimes \mathcal{H}$
2) The product is defined in the sense ($h, g \in \mathcal{H}$; $a, b \in \mathcal{A}$)

$$(a\sharp g)(b\sharp h) = \sum_g a(g_{(1)} \triangleright b)\sharp(g_{(2)}h) \equiv (a\sharp I)(\Delta(g) \triangleright (b\sharp h)) \ ; \tag{2.2.10}$$

3) The identity element is $I\sharp I$.

If the algebra $\mathcal{A}$ is the bialgebra dual to the bialgebra $\mathcal{H}$, then the relations (2.2.10) and (2.1.11) define the rules for interchanging the elements $(I\sharp g)$ and $(a\sharp I)$:

$$(I\sharp g)(a\sharp I) = (a_{(1)}\sharp I)\langle g_{(1)}|a_{(2)}\rangle(I\sharp g_{(2)}) \ . \tag{2.2.11}$$

Thus, the subalgebras $\mathcal{A}$ and $\mathcal{H}$ in $\mathcal{A}\sharp\mathcal{H}$ do not commute with each other. The smash product depends on which action (left or right) of the algebra $\mathcal{H}$ on $\mathcal{A}$ we choose. In addition, the smash product generalizes the concept of the semidirect product. In particular, if we take as bialgebra $\mathcal{H}$ the Lorentz group algebra [see (2.2.6)], and as module $\mathcal{A}$ the group of translations in Minkowski space, then the smash product $\mathcal{A}\sharp\mathcal{H}$ defines the structure of the Poincare group.

The coanalog of the smash product, the smash coproduct $\mathcal{A}\underline{\sharp}\mathcal{H}$, can also be defined. For this, we consider the bialgebra $\mathcal{H}$ and its comodule coalgebra $\mathcal{A}$. Then on the space $\mathcal{A}\otimes\mathcal{H}$ it is possible to define the structure of a coassociative coalgebra:

$$\Delta(a\underline{\sharp}h) = (a_{(1)}\underline{\sharp}\bar{a}_{(2)}^{(-1)}h_{(1)}) \otimes (a_{(2)}{}^{(0)}\underline{\sharp}h_{(2)}) \ , \quad \epsilon(a\underline{\sharp}h) = \epsilon(a)\epsilon(h) \ . \tag{2.2.12}$$

The proof of the coassociativity reduces to verification of the identity

$$(m_{\mathcal{H}}(\Delta_L \otimes \Delta_{\mathcal{H}}) \otimes id)(id \otimes \Delta_L)\Delta_{\mathcal{A}}(a) = (id \otimes id \otimes \Delta_L)(id \otimes \Delta_{\mathcal{A}})\Delta_L(a) \ ,$$

which is satisfied if we take into account the axiom (2.2.9) and the comodule axiom

$$(id \otimes \Delta_L)\Delta_L(a) = (\Delta_{\mathcal{H}} \otimes id)\Delta_L(a) \ . \tag{2.2.13}$$

Note that from the two bialgebras $\mathcal{A}$ and $\mathcal{H}$, which act and coact on each other in a special manner, it is possible to organize a new bialgebra that is simultaneously the smash product and smash coproduct of $\mathcal{A}$ and $\mathcal{H}$ (bicross product; see Ref. [24]).

2.3 Hopf algebras

We can now introduce the main concept in the theory of quantum groups, namely, the concept of the Hopf algebra.

Definition 4. *A bialgebra $\mathcal{A}$ equipped with an additional mapping $S: \ \mathcal{A} \to \mathcal{A}$ such that*

$$\begin{array}{c} m(S \otimes id)\Delta = m(id \otimes S)\Delta = \mathbf{i} \cdot \epsilon \Rightarrow \\ S(a_{(1)})\, a_{(2)} = a_{(1)}\, S(a_{(2)}) = \epsilon(a) \cdot I \ \ (\forall a \in \mathcal{A}) \end{array} \tag{2.3.1}$$

is called a Hopf algebra. The mapping S is called the antipode and is an antihomomorphism with respect to both multiplication and comultiplication:

$$S(ab) = S(b)S(a) \ , \ \ (S \otimes S)\Delta(a) = \sigma \cdot \Delta(S(a)) \ , \tag{2.3.2}$$

where $a, b \in \mathcal{A}$ and σ denotes the operator of transposition, $\sigma(a \otimes b) = (b \otimes a)$.
If we set

$$S(e_i) = S_i^j e_j \ , \tag{2.3.3}$$

then the axiom (2.3.1) can be rewritten in the form

$$\Delta_k^{ij} S_i^n m_{nj}^l = \Delta_k^{ij} S_j^n m_{in}^l = \epsilon_k E^l \ . \tag{2.3.4}$$

From the axioms for the structure mappings of a Hopf algebra, it is possible to obtain the useful equations

$$\begin{array}{c} S_j^i \epsilon_i = \epsilon_j \ , \ \ S_j^i E^j = E^i \ , \\ \\ \Delta_k^{ji} (S^{-1})_i^n m_{nj}^l = \Delta_k^{ji} (S^{-1})_j^n m_{in}^l = \epsilon_k E^l \ , \end{array} \tag{2.3.5}$$

which we shall use in what follows. Note that, in general, the antipode S is not necessarily invertible. An invertible antipode is called bijective.

In quantum physics the existence of the antipode S is needed to define a space of contragredient states $\langle\psi|$ (contragredient module of $\mathcal{A}$) with pairing $\langle\psi|\phi\rangle$: $\langle\psi| \otimes |\phi\rangle \to \mathbf{C}$. Left actions of the Hopf algebra $\mathcal{A}$ of observables to the contragredient states are (cf. the actions (2.2.3) of $\mathcal{A}$ to the gredient states):

$$a \triangleright \langle\psi| := \langle\psi|\, S(a) \ \ (a \in \mathcal{A}) \ ,$$

$$a \triangleright (\langle\psi_1| \otimes \langle\psi_2|) := (\langle\psi_1| \otimes \langle\psi_2|)\Delta(S(a)) = \langle\psi_1|\, S(a_{(2)}) \otimes \langle\psi_2|\, S(a_{(1)}) \ .$$

The states $\langle\psi|$ are called left dual to the states $|\phi\rangle$; the right dual ones are introduced with the help of the inverse antipode S^{-1} (see e.g. [30]). Then, the covariance of the pairing $\langle\psi|\phi\rangle$ under the left action of $\mathcal{A}$ can be established:

$$a \triangleright \langle\psi|\phi\rangle = \langle\psi|S(a_{(1)})\, a_{(2)}|\phi\rangle = \epsilon(a)\langle\psi|\phi\rangle \ .$$

The universal enveloping algebra U_q and the group bialgebra of the group G that we considered above can again serve as examples of cocommutative Hopf algebras. An example of a commutative Hopf algebra is the bialgebra $\mathcal{A}(G)$, which we also considered above. The antipodes for these algebras have the form

$$\begin{array}{c} U_g: \ \ S(J_\alpha) = -J_\alpha \ , \ \ S(I) = I \ , \\ \\ G: \ \ S(h) = h^{-1} \ , \\ \\ \mathcal{A}(G): \ \ S(f)(h) = f(h^{-1}) \ , \end{array} \tag{2.3.6}$$

and satisfy the relation $S^2 = id$, which holds for all commutative or cocommutative Hopf algebras.

From the point of view of the axiom (2.3.1), $S(a)$ looks like the inverse of the element a, although in the general case $S^2 \neq id$. We recall that if a set of elements of $\mathcal{G}$ with associative multiplication $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ and with identity (semigroup) also contains all the inverse elements, then such a set $\mathcal{G}$ becomes a group. Thus, from the point of view of the presence of the mapping S, a Hopf algebra is a generalization of a group algebra (for which $S(h) = h^{-1}$), although by itself it obviously need not be a group algebra. In accordance with Drinfeld's definition [13] the concepts of a Hopf algebra and a quantum group are more or less equivalent. Of course, the most interesting examples of quantum groups arise when one considers noncommutative and noncocommutative Hopf algebras.

We consider a noncommutative Hopf algebra $\mathcal{A}$ which is also noncommutative $\Delta \neq \Delta'$.

Definition 5. *A Hopf algebra $\mathcal{A}$ for which there exists an invertible element $R \in \mathcal{A} \otimes \mathcal{A}$ such that $\forall a \in \mathcal{A}$*

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \ , \tag{2.3.7}$$

$$(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \ , \quad (id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12} \tag{2.3.8}$$

is called quasitriangular. Here the element

$$\mathcal{R} = \sum_{ij} R^{(ij)} e_i \otimes e_j \tag{2.3.9}$$

is called the universal $\mathcal{R}$ matrix, $R^{(ij)}$ are the constants and the symbols $\mathcal{R}_{12}, \ldots$ have the meaning

$$\mathcal{R}_{12} = \sum_{ij} R^{(ij)} e_i \otimes e_j \otimes I \ , \ \ \mathcal{R}_{13} = \sum_{ij} R^{(ij)} e_i \otimes I \otimes e_j \ , \ \ \mathcal{R}_{23} = \sum_{ij} R^{(ij)} I \otimes e_i \otimes e_j \ . \tag{2.3.10}$$

The relation (2.3.7) shows that the noncocommutativity in a quasitriangular Hopf algebra is kept "under control." It can be shown [23] that for such a Hopf algebra the universal $\mathcal{R}$ matrix (2.3.9) satisfies the Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ , \tag{2.3.11}$$

(to which a considerable part of the review will be devoted) and the relations

$$(id \otimes \epsilon)\mathcal{R} = (\epsilon \otimes id)\mathcal{R} = I \ , \tag{2.3.12}$$

$$(S \otimes id)\mathcal{R} = \mathcal{R}^{-1} \ , \ \ (id \otimes S)\mathcal{R}^{-1} = \mathcal{R} \ . \tag{2.3.13}$$

The proof of Eq. (2.3.11) reduces to writing out the expression $(id \otimes \Delta')(\mathcal{R})$ in two different ways:

$$\begin{gathered} (id \otimes \Delta')(\mathcal{R}) = R^{ij} e_i \otimes \mathcal{R}\Delta(e_j)\mathcal{R}^{-1} = \\ \mathcal{R}_{23}(id \otimes \Delta)(\mathcal{R})\mathcal{R}_{23}^{-1} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\mathcal{R}_{23}^{-1} \ . \end{gathered} \tag{2.3.14}$$

On the other hand, we have

$$(id \otimes \Delta')(\mathcal{R}) = (id \otimes \sigma)(id \otimes \Delta)(\mathcal{R}) = (id \otimes \sigma)\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}_{13} \ , \tag{2.3.15}$$

where σ is the transposition operator. Comparing (2.3.14) and (2.3.15), we readily obtain (2.3.11). It is easy to derive the relations (2.3.12) by applying $(\epsilon \otimes id \otimes id)$ and $(id \otimes id \otimes \epsilon)$ respectively to (2.3.8). Next, we prove the first relation in (2.3.13) (the second one can be proved in a similar manner). We consider the expression $\mathcal{R} \cdot (S \otimes id)\, \mathcal{R}$ and make use of the Hopf algebra axioms (2.3.1) and the first equation in (2.3.8). Then we obtain the desired equation

$$\mathcal{R}_{23} \cdot (id \otimes S \otimes id)\, \mathcal{R}_{23} = (m_{12} \otimes id)\mathcal{R}_{13}(id \otimes S \otimes id)\mathcal{R}_{23} =$$

$$= (m_{12} \otimes id)\,(id \otimes S \otimes id)\mathcal{R}_{13}\, \mathcal{R}_{23} = (\mathbf{i} \cdot \epsilon \otimes id)\, \mathcal{R} = I \; .$$

The next important concept that we shall need in what follows is the concept of the Hopf algebra $\mathcal{A}^*$ that is the dual of the Hopf algebra $\mathcal{A}$. We choose in $\mathcal{A}^*$ basis elements $\{e^i\}$ and define multiplication, the identity, comultiplication, the coidentity, and the antipode for $\mathcal{A}^*$ in the form

$$e^i e^j = m^{ij}_k e^k \; , \;\; I = \bar{E}_i e^i \; , \;\; \Delta(e^i) = \Delta^i_{jk} e^j \otimes e^k \; , \;\; \epsilon(e^i) = \bar{\epsilon}^i \; , \;\; S(e^i) = \bar{S}^i_j e^j \; . \tag{2.3.16}$$

Definition 6. *Two Hopf algebras $\mathcal{A}$ and $\mathcal{A}^*$ with corresponding bases $\{e_i\}$ and $\{e^i\}$ are said to be dual to each other if there exists a nondegenerate pairing $\langle . | . \rangle$: $\mathcal{A}^* \otimes \mathcal{A} \rightarrow \mathbf{C}$ such that*

$$\begin{gathered} \langle e^i e^j | e_k \rangle \equiv \langle e^i \otimes e^j | \Delta(e_k) \rangle = \langle e^i | e_{k'} \rangle \Delta^{k' k''}_k \langle e^j | e_{k''} \rangle \\ \langle e^i | e_j e_k \rangle \equiv \langle \Delta(e^i) | e_j \otimes e_k \rangle = \langle e^{i'} | e_j \rangle \Delta^i_{i' i''} \langle e^{i''} | e_k \rangle \\ \langle S(e^i) | e_j \rangle = \langle e^i | S(e_j) \rangle \; , \;\; \langle e^i | I \rangle = \epsilon(e^i) \; , \;\; \langle I | e_i \rangle = \epsilon(e_i) \; . \end{gathered} \tag{2.3.17}$$

Since the pairing $\langle . | . \rangle$ (2.3.17) is nondegenerate, we can always choose basis elements $\{e^i\}$ such that

$$\langle e^i | e_j \rangle = \delta^i_j \; . \tag{2.3.18}$$

Then from the axioms for the pairing (2.3.17) and from the definitions of the structure maps (2.1.1), (2.3.3), and (2.3.16) in $\mathcal{A}$ and $\mathcal{A}^*$ we readily deduce

$$m^{ij}_k = \Delta^{ij}_k \; , \;\; m^k_{ij} = \Delta^k_{ij} \; , \;\; \bar{S}^i_j = S^i_j \; , \;\; \bar{\epsilon}^i = E^i \; , \;\; \bar{E}_i = \epsilon_i \; . \tag{2.3.19}$$

Thus, the multiplication, identity, comultiplication, coidentity, and antipode in a Hopf algebra define, respectively, comultiplication, coidentity, multiplication, identity, and antipode in the dual Hopf algebra.

2.4 Example: the group algebra of finite group and its algebra of functions

In [32] L.S. Pontrjagin showed that the set of characters of an abelian locally compact group G is an abelian group, called the dual group G^* of G. The group G^* is also locally compact. Moreover, the dual group of G^* is isomorphic to G. This beautiful theory becomes wrong if G is a noncommutative group, even if it is finite. To restore the duality principle one can replace the set of characters for a finite noncommutative group G by the category of its irreducible representations (irreducible representations for the commutative groups are exactly characters). Indeed, T. Tannaka and M. Krein showed that the compact group G can be recovered from the set of its unitary irreducible representations. They proved a duality theorem for compact groups, involving irreducible representations of G (although no group-like structure is to be put on that class, since the tensor product of two irreducible representation may no longer be irreducible). However, the tensor product of two irreducible representations can be expanded as a sum of irreducible representations and, thus, the dual object has the structure of an algebra.

In this Subsection we consider the group algebra $\mathcal{G}$ of finite group G and the algebra $\mathcal{A}(G) \equiv \mathcal{G}^\star$ of functions on the group G as simplest examples of the Hopf algebras. The structure mappings for these algebras have been defined in (2.2.6), (2.2.7) and (2.3.6). Note that the algebras $\mathcal{G}$ and $\mathcal{G}^\star$ are Hopf dual to each other. Below we concentrate on the discussion of the detailed structure of $\mathcal{G}^\star$ from the point of view of the representation theory of finite groups (see e.g. [26]).

Let G be a finite group of order N with generic element g including unit 1: $(g_1 = 1,\ g_2,\ ,\dots, g_N)$. Consider the element e of $\mathcal{G}$ (an invariant average over G or "symmetrizer"):

$$e = \frac{1}{N}\sum_{i=1}^{N} g_i \equiv \frac{1}{N}\sum_{g\in G} g \ . \tag{2.4.1}$$

This element satisfies properties

$$1.)\ e^2 = e\ , \quad 2.)\ h\, e = e\, h = e \ \ \forall h \in G\ . \tag{2.4.2}$$

which can be immediately proved if we note that the sets of elements $\{h\, g_i\}$ and $\{g_i\, h\}$ $(i = 1, 2, \dots, N)$, for fixed $h \in G$, coincide with G, or in other words coincide with the set $\{g_1,\ g_2,\ ,\dots, g_N\}$ but written in a different order.

Let the functions $T^i_j \in \mathcal{G}^\star$ $(i, j = 1, 2, \dots, n)$ (see (2.2.8)) define a representation of finite group G in n-dimensional vector space V. Two representations $T^{(1)}$ and $T^{(2)}$ are called equivalent (or isomorphic $T^{(1)} = T^{(2)}$) if an isomorphic map $\tau : V_1 \to V_2$ exists such that $T^{(1)}(h) = \tau^{-1} T^{(2)}(h)\tau$ $(\forall h \in G)$. Any representation of finite group G is equivalent to the unitary representation. Indeed, let $\langle x, y\rangle_1 = \overline{\langle y, x\rangle_1}$ be a Hermitian scalar product in V $(x, y \in V)$. Then, one can prove that the pairing

$$\langle x, y\rangle := \frac{1}{N}\sum_{g\in G}\langle T(g)x, T(g)y\rangle_1 \tag{2.4.3}$$

is also a Hermitian scalar product in V. It is clear that $\langle x, y\rangle = \langle T(h)x, T(h)y\rangle$ for all $h \in G$ and, therefore, T is an unitary representation of G with respect to the scalar product (2.4.3).

Let the functions $(T^{(\nu)})^i_j \in \mathcal{G}^\star$ $(i, j = 1, 2, \dots, N_\nu)$ define all irreducible representations of G in N_ν- dimensional vector spaces V_ν (index ν enumerates inequivalent irreducible representations). Consider the set of matrices $A^{k_\mu}_{j_\nu}$, $B^{i_\nu}_{m_\mu}$:

$$(A^{k_\mu}_{j_\nu})^{i_\nu}_{m_\mu} = \langle\, S\left((T^{(\nu)})^{i_\nu}_{j_\nu}\right)\, (T^{(\mu)})^{k_\mu}_{m_\mu} \,|\, e\,\rangle = (B^{i_\nu}_{m_\mu})^{k_\mu}_{j_\nu}\ , \tag{2.4.4}$$

where $\langle .|.\rangle$ denotes the pairing: $\mathcal{G}^* \otimes \mathcal{G} \to \mathbf{C}$ (see Definition 6 in Subsect. 2.3) and the element e is given in (2.4.1). Below we use concise matrix notations such that eq. (2.4.4) is written in the form $A_{\nu\mu} = \langle\, S\,(T_\nu)\; T_\mu \,|\, e \,\rangle = B_{\mu\nu}$ where low indices ν and μ denote the numbers of vector spaces related to corresponding irreducible representations. From the property 2.) (2.4.2) and relations (2.2.8), (2.3.17) we deduce ($\forall h \in G$)

$$\begin{gathered} A_{\nu\mu} = \langle\, S\,(T_\nu)\; T_\mu |\, e\, h \,\rangle = \langle\, S\,(T_\nu)\; |\; e_{(1)}\, h \,\rangle \,\langle\, T_\mu \,|\; e_{(2)}\, h \,\rangle = \\ = \langle\, S\,(T_\nu)\; |\; h \,\rangle \,\langle\, S\,(T_\nu)\; |\, e_{(1)} \,\rangle \,\langle\, T_\mu \,|\, e_{(2)} \,\rangle \,\langle\, T_\mu \,|\, h \,\rangle = T_\nu^{-1}(h)\; A_{\nu\mu}\; T_\mu(h)\;, \end{gathered} \tag{2.4.5}$$

where $\Delta(h) = h \otimes h$ and

$$\Delta(e) = e_{(1)} \otimes e_{(2)} \equiv \frac{1}{N} \sum_{g \in G} g \otimes g\;. \tag{2.4.6}$$

Analogously we have

$$B_{\mu\nu} = \langle\, S\,(T_\nu)\; T_\mu \,|\; h\, e \,\rangle = T_\mu(h)\; B_{\mu\nu}\; T_\nu^{-1}(h) \quad \forall h \in G\;. \tag{2.4.7}$$

Thus, the matrices $A_{\nu\mu}$, $B_{\mu\nu}$ intertwine the representations $T^{(\nu)}$ and $T^{(\mu)}$. Since $T^{(\nu)}$ and $T^{(\mu)}$ are irreducible representations of G which are inequivalent for $\mu \neq \nu$, we conclude that

$$A_{\nu\mu} = B_{\mu\nu} = \lambda \delta^{\mu\nu}\, P_{\nu\mu} \;\Rightarrow\; (A^{k_\mu}_{j_\nu})^{i_\nu}_{m_\mu} = (B^{i_\nu}_{m_\mu})^{k_\mu}_{j_\nu} = \lambda\, \delta^{\mu\nu}\, \delta^{i_\nu}_{m_\mu}\, \delta^{k_\mu}_{j_\nu}\;, \tag{2.4.8}$$

where λ is a constant and $P_{\nu\mu}$ is a permutation matrix: $P^{i_\nu i_\mu}_{j_\nu j_\mu} = \delta^{i_\mu}_{j_\nu} \delta^{i_\nu}_{j_\mu}$. It follows from Schur's Lemma:

Lemma. *Let $T^{(1)} : G \to GL(V_1)$ and $T^{(2)} : G \to GL(V_2)$ are irreducible linear representations of G in vector spaces V_1, V_2 and A is a linear map: $V_1 \to V_2$ such that*

$$T^{(2)}(h) \cdot A = A \cdot T^{(1)}(h) \quad \forall h \in G\;. \tag{2.4.9}$$

Then,
(1) $A = 0$, *if representations $T^{(1)}$ and $T^{(2)}$ are not equivalent (isomorphic);*
(2) $A = \lambda \cdot I$ *(A is proportional to the unit map I), if $V_1 = V_2$ and $T^{(1)}$ is equivalent to $T^{(2)}$.*

Proof. Let the representations $T^{(1)}$ and $T^{(2)}$ are not equivalent. We assume that $A \neq 0$ and denote $W_1 = \mathrm{Ker(A)}$ (W_1 is a set of elements $x \in V_1$ such that $A(x) = 0$). If $x \in W_1$, then, in view of (2.4.9), we have $A \cdot T^{(1)}(h)\,(x) = T^{(2)}(h) \cdot A(x) = 0$ and $T^{(1)}(h)\,(x) \in W_1$ $\forall h$. It means that W_1 is invariant subspace in V_1. Since $T^{(1)}$ is irreducible we conclude that either $W_1 = V_1$, or $W_1 = \mathrm{Ker(A)} = 0$. The first case is excluded (it corresponds to $A = 0$) and, therefore, $W_1 = 0$. In the same way one can prove that $\mathrm{Img}(A)$ is invariant subspace of V_2 and, thus, $\mathrm{Img}(A) = V_2$. It means that the map $A : V_1 \to V_2$ is the isomorphism (i.e. $T^{(1)}$ and $T^{(2)}$ are equivalent representations) and it proves the statement (1).

Now we suppose that $V_1 = V_2$, $T^{(1)}$ is equivalent to $T^{(2)}$ and one can choose a basis in V_2 in such a way that $T^{(1)}(h) = T^{(2)}(h)$ $\forall h$. Let λ be an eigenvalue of A: $Av = \lambda v$, ($v \neq 0$). Consider the map $A' = A - \lambda I$ for which $v \in \mathrm{Ker}(A')$ and, thus, $\mathrm{Ker}(A') \neq 0$. For the map A' we also obtain $T^{(2)}(h) \cdot A' = A' \cdot T^{(1)}(h) \quad \forall h \in G$. But according to the first part of this proof we have $A' = 0$, i.e. $A = \lambda I$. $\square$

Using (2.4.1), (2.4.4), (2.4.6) we write (2.4.8) in the form

$$\frac{1}{N} \sum_{g \in G} T^{i_\nu}_{j_\nu}(g^{-1}) \, T^{k_\mu}_{m_\mu}(g) = \lambda \, \delta^{\mu\nu} \, \delta^{i_\nu}_{m_\mu} \, \delta^{k_\mu}_{j_\nu} \; . \tag{2.4.10}$$

The coefficient λ in (2.4.10) can be found if we put $\mu = \nu$, $k_\mu = j_\nu$ and take the sum over $j_\nu = 1, \dots, N_\nu$. As a result we obtain $\lambda = 1/N_\nu$, where N_ν is the dimension of the irreducible representation T_ν. According to the definition of the scalar product (2.4.3) one can choose the basis in the space V_ν in such a way that $T^{i_\nu}_{j_\nu}(g^{-1}) = (T^{j_\nu}_{i_\nu}(g))^*$ (i.e. T_ν is a unitary representation) and represent (2.4.10) as an orthogonality condition

$$\frac{1}{N} \sum_{g \in G} (T^{j_\nu}_{i_\nu}(g))^* \, T^{k_\mu}_{m_\mu}(g) = \frac{1}{N_\nu} \, \delta^{\mu\nu} \, \delta^{i_\nu}_{m_\mu} \, \delta^{k_\mu}_{j_\nu} \;\; \Leftrightarrow \tag{2.4.11}$$

$$\Leftrightarrow \;\; \langle T^\dagger_\nu T_\mu \, | \, e \rangle = \frac{1}{N_\nu} \, \delta^{\mu\nu} \, P_{\mu\nu} \; .$$

Consider a representation T of G in n-dimensional vector space V and define a function

$$\chi_T(h) := \sum_{i=1}^{n} T^i_i(h) = \mathrm{Tr}(T(h)) \; , \tag{2.4.12}$$

which is called the character of the group G in the representation T. It is clear that:
1.) $\chi_T(1) = n$;
2.) $\chi_T(g^{-1}) = \chi^*_T(g)$ (it is evident for the unitary representation);
3.) for a direct sum of two representations $T = T^{(1)} \oplus T^{(2)}$ we have $\chi_T = \chi_{(1)} + \chi_{(2)}$;
4.) for a direct product of two representations $T = T^{(1)} \otimes T^{(2)}$ we have $\chi_T = \chi_{(1)} \cdot \chi_{(2)}$;
5.) $\chi_T(h) = \chi_T(g \, h \, g^{-1})$ and for equivalent representations $T^{(1)}$ and $T^{(2)}$ we have $\chi_{(1)} = \chi_{(2)}$.
Property 2.) is correct for any representations of the finite group G in view of their equivalence to the unitary representations.

For two functions $\phi, \psi \in \mathcal{G}^\star$ on G we define the pairing (cf. (2.4.3), (2.4.4))

$$\langle \phi, \psi \rangle = \frac{1}{N} \sum_{g \in G} \phi(g)^* \, \psi(g) = \langle \phi^* \, \psi \, | \, e \rangle \; . \tag{2.4.13}$$

This pairing satisfies all axioms for scalar products. Using (2.4.13) we write (2.4.11) in the form $\langle T_\nu \, , \, T_\mu \rangle = \frac{1}{N_\nu} \, \delta^{\mu\nu} \, P_{\mu\nu}$ and then obtain by means of (2.4.12) the orthogonality conditions for characters of the irreducible representations T_ν and T_μ

$$\langle \chi_\nu, \chi_\mu \rangle = \frac{1}{N} \sum_{g \in G} \chi^*_\nu(g) \, \chi_\mu(g) = \delta^{\mu\nu} \; . \tag{2.4.14}$$

Let $\chi_1, \dots, \chi_h$ be different characters of all inequivalent irreducible representations $T_1, \dots, T_h$ of the group G. Then any representation T is isomorphic to $T = m_1 T_1 \oplus \dots \oplus m_h T_h$ where integers $m_i \geq 0$. In this case character χ of the representation T is $\chi = m_1 \chi_1 + \dots + m_h \chi_h$, and from the orthogonality condition (2.4.14) we deduce

$$\langle \chi, \chi_\nu \rangle = m_\nu \; , \quad \langle \chi, \chi \rangle = \sum_{\nu=1}^{h} m^2_\nu \; .$$

If we apply this consideration to a representation which is the tensor product $T_\nu \otimes T_\mu$ of two irreducible representations, then for the product of two characters we obtain the expansion

$$\chi_\nu \chi_\mu = \sum_\xi m^\xi_{\nu\mu} \chi_\xi \ ,$$

where $m^\xi_{\nu\mu}$ are positive integers.

Let T be a regular representation of the finite group G: $g_i g_k = \sum_{m=1}^N T_k^m(g_i) g_m$, where $T_k^m(g_i) = \delta^m_{k_i}$ if $g_i g_k = g_{k_i}$. It is clear that the character χ_R of the regular representation has the properties: $\chi_R(1) = N$, where N is the order of the group G and $\chi_R(g) = 0 \ \forall g \neq 1$, since $T^m_m(g) = 0$ for $g \neq 1$. Using this fact one can calculate the number of times each irreducible representation T_ν is contained in the regular representation T_R:

$$\langle \chi_R, \chi_\nu \rangle = \frac{1}{N} \sum_{g \in G} \chi^*_R(g)\, \chi_\nu(g) = \frac{1}{N}\, N\, \chi_\nu(1) = N_\nu \ ,$$

and an expansion of χ_R over the basis χ_ν is $\chi_R = \sum_{\nu=1}^h N_\nu \chi_\nu$. The direct consequence of these propositions is that the degree (dimension) N_ν of irreducible representations T_ν and the order N of the group G are related by the condition

$$N = \chi_R(1) = \sum_{\nu=1}^h N_\nu \chi_\nu(1) = \sum_{\nu=1}^h N_\nu^2 \ .$$

Let $f \in \mathcal{G}^\star$ be a *class function* on G, i.e. $f(hgh^{-1}) = f(g) \ \forall h, g \in G$. For some representation T we consider the matrix

$$A^m_k(f) = \frac{1}{N} \sum_{g \in G} f(g) T^m_k(g) = \langle f^* \, , \, T^m_k \rangle = \langle f\, T^m_k \,|\, e \rangle \ . \qquad (2.4.15)$$

If representation $T = T_\nu$ (of degree N_ν) is irreducible and has character χ_ν , then $A^m_k = \lambda \delta^m_k$ and λ is a constant:

$$\lambda = \frac{1}{N_\nu} \sum_{g \in G} f(g) \chi_\nu(g) \equiv \frac{N}{N_\nu} \langle f^*, \, \chi_\nu \rangle = \frac{N}{N_\nu} \langle f\, \chi_\nu \,|\, e \rangle \ . \qquad (2.4.16)$$

Indeed, it is readily to show that $T^r_m(h^{-1}) A^m_k T^k_j(h) = A^r_j \ (\forall h)$ and according to Schur's Lemma we have $A^m_k = \lambda \delta^m_k$. Taking the trace of both sides of this relation we deduce (2.4.16) and

$$A^m_k(f) = \frac{1}{N} \sum_{g \in G} f(g) T^m_k(g) = \frac{N}{N_\nu} \langle f^*, \, \chi_\nu \rangle \delta^m_k \ . \qquad (2.4.17)$$

Consider the space H of all class functions on G. The characters $\chi_1, \dots, \chi_h$ belong to this space. Now we prove that the characters $\chi_1, \dots, \chi_h$ form an orthonormal basis in the space H. Indeed, according to (2.4.14) $\chi_1, \dots, \chi_h$ forms the orthonormal system in H. We need to show that this system is complete. In other words we should prove that every element f of H orthogonal to any χ_ν is zero. Indeed, for this class function f consider the matrix $A^m_k(f^*)$ (2.4.15) which is zero for any irreducible representation T in view of (2.4.17). Since any representation can be decomposed into a dirrect sum of irreducible ones we conclude that $A_k^m(f^*) = 0$ for the regular representation, but it means that $(\forall g_k \in G)$

$$\left(\sum_{i=1}^{N} f^*(g_i)\, g_i \right) g_k = \sum_{m,i=1}^{N} f^*(g_i)\, T_k^m(g_i)\, g_m = 0 \;\Rightarrow\; \left(\sum_{i=1}^{N} f^*(g_i)\, g_i \right) = 0 \;,$$

and, therefore, $f^*(g_i) = 0 \Rightarrow f(g_i) = 0$ $(\forall g_i \in G)$.

Recall that the elements $g, g' \in G$ are said to be conjugated if there is an element $h \in G$ such that $g' = hgh^{-1}$. One can split all elements of G on conjugacy classes $C_1, \dots, C_k$. The number k of all conjugacy classes of the group G is equal to the number h of inequivalent irreducible representations of G. Indeed, consider a class function $f \in H$ which is a constant on each class C_m. Thus, f can be defined by k arbitrary constants. It means that the dimension of H is k. On the other hand the dimension of H is equal to the number of independent characters $\chi_1, \dots, \chi_h$ but this number coincides with the number of inequivalent irreducible representations. It proves the identity $k = h$.

Let f_h be the function equal to 1 on conjugacy class C_h of h and equal to 0 elsewhere. Since it is a class function, it can be expanded as

$$f_h = \sum_{\nu=1}^{h} \lambda_\nu\, \chi_\nu \;, \quad \lambda_\nu = \langle \chi_\nu \,,\, f_h \rangle = \frac{1}{N} \sum_{g \in C_h} \chi_\nu^*(g)\, f_h(g) = \frac{c(h)}{N} \chi_\nu^*(h) \;,$$

where $c(h) := \dim(C_h)$. Thus, we have for each $g \in G$

$$\frac{c(h)}{N} \sum_{\nu=1}^{h} \chi_\nu^*(h)\, \chi_\nu(g) = f_h(g) = \left\{ \begin{array}{l} 1 \;\text{ if }\; g \in h \;, \\ 0 \;\text{ if }\; g \notin C_h \;. \end{array} \right. \tag{2.4.18}$$

Remark. The above results about representations of finite groups are carried over to the representation theory of compact Lie groups G_L. It follows from the fact that the averaging over finite groups (see eqs. (2.4.1), (2.4.3), (2.4.10), (2.4.13) and (2.4.15)) has an analog in the case of compact Lie groups which is an invariant integration over G_L.

2.5 Heisenberg and Quantum doubles.

In Subsection 2.2 we have defined (see Definition 3) the notion of the smash (cross) product of the bialgebra and its module algebra. Since the Hopf dual algebra $\mathcal{A}^*$ is the natural right and left module algebra for the Hopf algebra $\mathcal{A}$ (2.1.10), (2.1.11), one can immediately define the right $\mathcal{A}^* \sharp \mathcal{A}$ and the left $\mathcal{A} \sharp \mathcal{A}^*$ cross products of the algebra $\mathcal{A}$ on $\mathcal{A}^*$. These cross-product algebras are called Heisenberg doubles of $\mathcal{A}$ and they are the associative algebras with nontrivial cross-multiplication rules (cf. eq. (2.2.11)):

$$a\, \bar{a} = (a_{(1)} \triangleright \bar{a})\, a_{(2)} = \bar{a}_{(1)}\, \langle a_{(1)} \,|\, \bar{a}_{(2)} \rangle\, a_{(2)} \;, \tag{2.5.1}$$

$$\bar{a}\, a = a_{(1)} (\bar{a} \triangleleft a_{(2)}) \; = a_{(1)}\, \langle \bar{a}_{(1)} \,|\, a_{(2)} \rangle\, \bar{a}_{(2)} \;, \tag{2.5.2}$$

where $a \in \mathcal{A}$ and $\bar{a} \in \mathcal{A}^*$. Here we discuss only the left cross product algebra $\mathcal{A} \sharp \mathcal{A}^*$ (2.5.1) (the other one (2.5.2) is considered analogously).

As in the previous subsection, we denote $\{e^i\}$ and $\{e_i\}$ the dual basis elements of $\mathcal{A}^*$ and $\mathcal{A}$, respectively. In terms of this basis we rewrite (2.5.1) in the form

$$e_r\, e^n = e^i\, \Delta^n_{if} \langle e_j | e^f \rangle\, \Delta^{jk}_r\, e_k = m^n_{ij}\, e^i\, e_k\, \Delta^{jk}_r \; . \qquad (2.5.3)$$

Let us define a right $\mathcal{A}^*$ - coaction and a left $\mathcal{A}$ - coaction on the algebra $\mathcal{A}\sharp\mathcal{A}^*$, such that these coactions respect the algebra structure of $\mathcal{A}\sharp\mathcal{A}^*$:

$$\Delta_R(z) = C\,(z \otimes 1)\, C^{-1} \; , \quad \Delta_L(z) = C^{-1}\,(1 \otimes z)\, C \; , \quad C \equiv e_i \otimes e^i \; . \qquad (2.5.4)$$

The inverse of the canonical element C is

$$C^{-1} = S(e_i) \otimes e^i = e_i \otimes S(e^i) \; ,$$

and Δ_R, Δ_L (2.5.4) are represented in the form

$$\Delta_R(z) = (e_{k(1)}\, z\, S(e_{k(2)})) \otimes e^k \; , \quad \Delta_L(z) = e_k \otimes S(e^k_{(1)})\, z\, e^k_{(2)} \; . \qquad (2.5.5)$$

Note that $\Delta_R(\bar{z}) = \Delta(\bar{z})$ $\forall \bar{z} \in \mathcal{A}^*$ and $\Delta_L(z) = \Delta(z)$ $\forall z \in \mathcal{A}$ (here $\mathcal{A}$ and $\mathcal{A}^*$ are understood as the Hopf subalgebras in $\mathcal{A}\sharp\mathcal{A}^*$ and Δ are corresponding comultiplications). Indeed, for $z \in \mathcal{A}$ we have

$$\Delta_L(z) = e_k \otimes S(e^k_{(1)})\, z\, e^k_{(2)} = e_k \otimes S(e^k_{(1)})\, e^k_{(2)}\, \langle z_{(1)} \,|\, e^k_{(3)} \rangle z_{(2)} =$$

$$= e_k\, \langle z_{(1)} \,|\, e^k \rangle \otimes z_{(2)} = z_{(1)} \otimes z_{(2)} \; ,$$

(the proof of $\Delta_R(\bar{z}) = \Delta(\bar{z})$ is similar). The axioms

$$(id \otimes \Delta)\Delta_R = (\Delta_R \otimes id)\Delta_R \;\; , \quad (id \otimes \Delta_L)\Delta_L = (\Delta \otimes id)\Delta_L \; ,$$

$$(id \otimes \Delta_R)\Delta_L(z) = C^{-1}_{13}\, (\Delta_L \otimes id)\Delta_R(z)\, C_{13} \; ,$$

can be verified directly by using relations (cf. (2.3.8))

$$(id \otimes \Delta)C_{12} = C_{13}\, C_{23} \; , \quad (\Delta \otimes id)C_{12} = C_{13}\, C_{23} \; ,$$

and the pentagon identity [27] for C

$$C_{12}\, C_{13}\, C_{23} = C_{23}\, C_{12} \; . \qquad (2.5.6)$$

The proof of (2.5.6) is straightforward (see (2.5.3)):

$$C_{12}\, C_{13}\, C_{23} = e_i\, e_j \otimes e^i\, e_k \otimes e^j\, e^k = e_n \otimes m^n_{ij}\, e^i\, e_k\, \Delta^{jk}_r \otimes e^r =$$

$$= e_n \otimes e_r\, e^n \otimes e^r = C_{23}\, C_{12} \; .$$

The pentagon identity (2.5.6) is used for the construction of the explicit solutions of the tetrahedron equations (3D generalizations of the Yang-Baxter equations).

Although $\mathcal{A}$ and $\mathcal{A}^*$ are Hopf algebras, their Heisenberg doubles $\mathcal{A}\sharp\mathcal{A}^*$, $\mathcal{A}^*\sharp\mathcal{A}$ are not Hopf algebras. But as we have seen just before the algebra $\mathcal{A}\sharp\mathcal{A}^*$ (as well as $\mathcal{A}^*\sharp\mathcal{A}$) still possesses some covariance properties, since the coactions (2.5.4) are covariant transformations (homomorphisms) of the algebra $\mathcal{A}\sharp\mathcal{A}^*$.

The natural question is the following: is it possible to invent such a cross-product of the Hopf algebra and its dual Hopf algebra to obtain a new Hopf algebra? Drinfeld [22] showed that there exists a quasitriangular Hopf algebra $\mathcal{D}(\mathcal{A})$ that is a special smash product of the Hopf algebras $\mathcal{A}$ and $\mathcal{A}^o$: $\mathcal{D}(\mathcal{A}) = \mathcal{A} \bowtie \mathcal{A}^o$, which is called the quantum double. Here we denote by $\mathcal{A}^o$ the algebra $\mathcal{A}^*$ with opposite comultiplication: $\Delta(e^i) = m^i_{kj} e^j \otimes e^k$, $\mathcal{A}^o = (\mathcal{A}^*)^{cop}$. It follows from (2.3.5) that the antipode for $\mathcal{A}^o$ will be not S but the skew antipode S^{-1}. Thus, the structure mappings for $\mathcal{A}^o$ have the form

$$e^i e^j = \Delta^{ij}_k e^k \ , \ \ \Delta(e^i) = m^i_{kj} e^j \otimes e^k \ , \ \ S(e^i) = (S^{-1})^i_j e^j \ . \tag{2.5.7}$$

The algebras $\mathcal{A}$ and $\mathcal{A}^o$ are said to be antidual, and for them we can introduce the antidual pairing $\langle\langle .|.\rangle\rangle$: $\mathcal{A}^o \otimes \mathcal{A} \to \mathbf{C}$, which satisfies the conditions

$$\begin{gathered}
\langle\langle e^i e^j | e_k \rangle\rangle \equiv \langle\langle e^i \otimes e^j | \Delta(e_k) \rangle\rangle = \Delta^{ij}_k \ , \\
\langle\langle e^i | e_k e_j \rangle\rangle \equiv \langle\langle \Delta(e^i) | e_j \otimes e_k \rangle\rangle = m^i_{kj} \ , \\
\langle\langle S(e^i) | e_j \rangle\rangle = \langle\langle e^i | S^{-1}(e_j) \rangle\rangle = (S^{-1})^i_j \ , \\
\langle\langle e^i | S(e_j) \rangle\rangle = \langle\langle S^{-1}(e^i) | e_j \rangle\rangle = S^i_j \ , \\
\langle\langle e^i | I \rangle\rangle = E^i \ , \ \ \langle\langle I | e_i \rangle\rangle = \epsilon_i \ .
\end{gathered} \tag{2.5.8}$$

The universal R matrix can be expressed in the form of the canonical element

$$\mathcal{R} = (e_i \bowtie I) \otimes (I \bowtie e^i) \ , \tag{2.5.9}$$

and the multiplication in $\mathcal{D}(\mathcal{A})$ is defined in accordance with (the summation signs are omitted)

$$(a \bowtie \alpha)(b \bowtie \beta) = a \left((\alpha_{(3)} \triangleright b) \triangleleft S(\alpha_{(1)}) \right) \bowtie \alpha_{(2)} \beta \ , \tag{2.5.10}$$

where $\alpha, \beta \in \mathcal{A}^o$, $a, b \in \mathcal{A}$, $\Delta^2(\alpha) = \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}$ and

$$\alpha \triangleright b = b_{(1)} \langle\langle \alpha | b_{(2)} \rangle\rangle \ , \ \ b \triangleleft \alpha = \langle\langle \alpha | b_{(1)} \rangle\rangle b_{(2)} \ . \tag{2.5.11}$$

The coalgebraic structure on the quantum double is defined by the direct product of the coalgebraic structures on the Hopf algebras $\mathcal{A}$ and $\mathcal{A}^o$:

$$\Delta(e_i \bowtie e^j) = \Delta(e_i \bowtie I)\Delta(I \bowtie e^j) = \Delta^{nk}_i m^j_{lp} (e_n \bowtie e^p) \otimes (e_k \bowtie e^l) \ . \tag{2.5.12}$$

Finally, the antipode and coidentity for $\mathcal{D}(\mathcal{A})$ have the form

$$S(a \bowtie \alpha) = S(a) \bowtie S(\alpha) \ , \ \ \epsilon(a \bowtie \alpha) = \epsilon(a)\epsilon(\alpha) \ . \tag{2.5.13}$$

All the axioms of a Hopf algebra can be verified for $\mathcal{D}(\mathcal{A})$ by direct calculation. A simple proof of the associativity of the multiplication (2.5.10) and the coassociativity of the comultiplication (2.5.12) can be found in Ref. [25].

Taking into account (2.5.11), we can rewrite (2.5.10) as the commutator for the elements $(I \bowtie \alpha)$ and $(b \bowtie I)$:

$$(I \bowtie \alpha)(b \bowtie I) = \langle\langle S(\alpha_{(1)}) | b_{(1)} \rangle\rangle (b_{(2)} \bowtie I)(I \bowtie \alpha_{(2)}) \langle\langle \alpha_{(3)} | b_{(3)} \rangle\rangle \tag{2.5.14}$$

or, in terms of the basis elements $\alpha = e^t$ and $b = e_s$ we have [22]

$$
\begin{gathered}
(I \bowtie e^t)(e_s \bowtie I) = m^t_{klp} \Delta^{njk}_s (S^{-1})^p_n (e_j \bowtie I)(I \bowtie e^l) \equiv \\
\left(m^t_{ip} (S^{-1})^p_n \Delta^{nr}_s \right) \left(m^i_{kl} \Delta^{jk}_r \right) (e_j \bowtie I)(I \bowtie e^l) \; ,
\end{gathered}
\tag{2.5.15}
$$

where m^t_{klp} and Δ^{njk}_s are defined in (2.1.3) and (2.1.7), and $(S^{-1})^p_n$ is the matrix of the skew antipode.

The consistence of definitions of left and right bimodules over the quantum double $\mathcal{D}(\mathcal{A})$ should be clarified in view of the nontrivial structure of the cross-multiplication rule (2.5.14), (2.5.15) for subalgebras $\mathcal{A}$ and $\mathcal{A}^o$. It can be done (see, e.g. [112]) if one considers left or right coinvariant bimodules (Hopf modules): $M^L = \{m : \; \Delta_L(m) = 1 \otimes m\}$ or $M^R = \{m : \; \Delta_R(m) = m \otimes 1\}$. E.g., for M^R one can define the left $\mathcal{A}$ and left $\mathcal{A}^o$-module actions as

$$
a \triangleright m = a_{(1)} \, m \, S(a_{(2)}) \; , \tag{2.5.16}
$$

$$
\alpha \triangleright m = \langle\langle S(\alpha), m_{(-1)} \rangle\rangle \, m_{(0)} \; , \tag{2.5.17}
$$

where $\Delta_L(m) = m_{(-1)} \otimes m_{(0)}$ is the left $\mathcal{A}$-coaction on M^R and $a \in \mathcal{A}$, $\alpha \in \mathcal{A}^o$. Note that left $\mathcal{A}$-module action (2.5.16) respects the right coinvariance of M^R. The compatibility condition for the left $\mathcal{A}$-action (2.5.16) and left $\mathcal{A}$-coaction Δ_L is written in the form (we represent $\Delta_L(a \triangleright m)$ in two different ways):

$$
(a \triangleright m)_{(-1)} \otimes (a \triangleright m)_{(0)} = a_{(1)} \, m_{(-1)} \, S(a_{(3)}) \otimes a_{(2)} \triangleright m_{(0)} \; . \tag{2.5.18}
$$

A module with the property (2.5.18) is called Yetter-Drinfeld module. Then, using (2.5.16), (2.5.18) and opposite coproduct for $\mathcal{A}^o$, we obtain

$$
\begin{gathered}
\alpha \triangleright (a \triangleright m) = \alpha \triangleright (a_{(1)} \, m \, S(a_{(2)})) = \langle\langle S(\alpha), a_{(1)} \, m_{(-1)} \, S(a_{(3)}) \rangle\rangle \, a_{(2)} \triangleright m_{(0)} = \\
= \langle\langle S(\alpha_{(1)}), \, a_{(1)} \rangle\rangle \, \langle\langle \alpha_{(3)}, \, a_{(3)} \rangle\rangle \, a_{(2)} \triangleright (\alpha_{(2)} \triangleright m) \; ,
\end{gathered}
\tag{2.5.19}
$$

and one can recognize in eq. (2.5.19) the quantum double multiplication formula (2.5.14).

It follows from Eqs. (2.1.3) and (2.1.7) and from the identities for the skew antipode (2.3.5) that

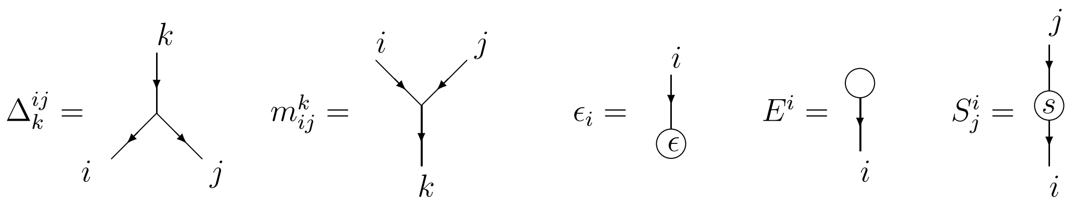
$$
\left(m^q_{tk} \Delta^{ks}_m \right) \left(m^t_{ip} (S^{-1})^p_n \Delta^{nr}_s \right) = \delta^q_i \delta^r_m \; , \tag{2.5.20}
$$

and this enables us to rewrite (2.5.15) in the form

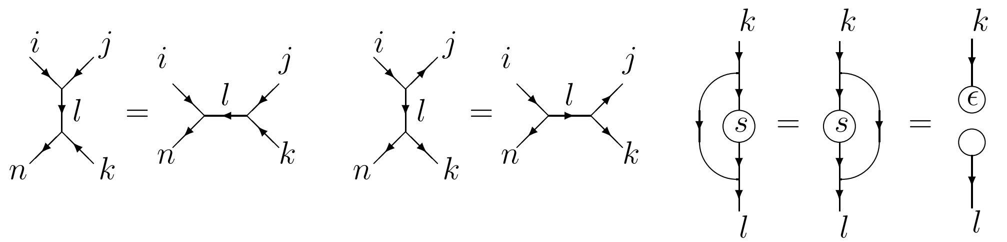
$$
\left(m^q_{tk} \Delta^{ks}_m \right) (I \bowtie e^t)(e_s \bowtie I) = (m^q_{kl} \Delta^{jk}_m)(e_j \bowtie I)(I \bowtie e^l) \; .
$$

This equation is equivalent to the axiom (2.3.7) for the universal matrix $\mathcal{R}$ (2.5.9). The relations (2.3.8) for $\mathcal{R}$ (2.5.9) are readily verified. Thus, $\mathcal{D}(\mathcal{A})$ is indeed a quasitriangular Hopf algebra with universal $\mathcal{R}$ matrix represented by (2.5.9).

In conclusion, we note that many relations for the structure constants of Hopf algebras [for example, the relation (2.45)] can be obtained and represented in a transparent form by means of the following diagrammatic technique:



For example, the axioms of associativity (2.3) and coassociativity (2.1.7) and the axioms for the antipode (2.3.4) can be represented in the form



Now we make three important remarks relating to the further development of the theory of Hopf algebras.

2.6 Twisted, ribbon and quasi- Hopf algebras

Remark 1. *Twisted Hopf algebras.*
Consider a Hopf algebra $\mathcal{A}$ $(\Delta,\, \epsilon,\, S)$. Let $\mathcal{F}$ be an invertible element of $\mathcal{A} \otimes \mathcal{A}$ such that:

$$(\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F} \ , \tag{2.6.1}$$

and we denote $\mathcal{F} = \sum_i \alpha_i \otimes \beta_i$, $\mathcal{F}^{-1} = \sum_i \gamma_i \otimes \delta_i$. Following the twisting procedure [30] one can define a new Hopf algebra $\mathcal{A}^{(F)}$ $(\Delta^{(F)},\, \epsilon^{(F)},\, S^{(F)})$ (twisted Hopf algebra) with the new structure mappings

$$\Delta^{(F)}(a) = \mathcal{F}\, \Delta(a)\, \mathcal{F}^{-1} \ , \tag{2.6.2}$$

$$\epsilon^{(F)}(a) = \epsilon(a) \ , \quad S^{(F)}(a) = U\, S(a)\, U^{-1} \quad (\forall a \in \mathcal{A}) \ , \tag{2.6.3}$$

where the twisting element $\mathcal{F}$ satisfies the cocycle equation

$$\mathcal{F}_{12}\, (\Delta \otimes id)\mathcal{F} = \mathcal{F}_{23}\, (id \otimes \Delta)\mathcal{F} \ , \tag{2.6.4}$$

and the element $U = \alpha_i\, S(\beta_i)$ is invertible and obeys

$$U^{-1} = S(\gamma_i)\, \delta_i \ , \quad S(\alpha_i)\, U^{-1}\, \beta_i = 1 \ , \tag{2.6.5}$$

(the summation over i is assumed). First of all we show that the algebra $\mathcal{A}^{(F)}$ $(\Delta^{(F)},\, \epsilon)$ is a bialgebra. Indeed, the cocycle equation (2.6.4) guarantees the coassociativity condition (2.1.7) for the new coproduct $\Delta^{(F)}$ (2.6.2). Then the axioms for counit ϵ (2.1.8) are easily deduced from (2.6.1). Considering the identity

$$m(id \otimes S \otimes id) \left(\mathcal{F}_{23}^{-1}\, \mathcal{F}_{12}\, (\Delta \otimes id)\mathcal{F} \right) = m(id \otimes S \otimes id)(id \otimes \Delta)\mathcal{F}$$

we obtain the form for U^{-1} (2.6.5). The second relation in (2.6.5) is obtained from the identity: $m(S \otimes id)\mathcal{F}^{-1}\mathcal{F} = 1$.

Now the new antipode $S^{(F)}$ (2.6.3) follows from equation

$$m(id \otimes S)\,(\Delta^{(F)}(a)\,\mathcal{F}) = m(id \otimes S)\,(\mathcal{F}\,\Delta(a))\;,$$

which is rewritten in the form $\tilde{a}_{(1)} U S(\tilde{a}_{(2)}) = \epsilon(a)\,U$, where $\Delta^{(F)}(a) = \tilde{a}_{(1)} \otimes \tilde{a}_{(2)}$.

If the algebra $\mathcal{A}$ is a quasitriangular Hopf algebra with the universal $\mathcal{R}$ matrix (2.3.7), then the new Hopf algebra $\mathcal{A}^{(F)}$ is also quasitriangular and a new universal $\mathcal{R}$-matrix is

$$\mathcal{R}^{(F)} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}\;, \qquad (2.6.6)$$

since we have

$$\Delta^{(F)\prime} = \mathcal{F}_{21}\,\Delta'\,\mathcal{F}_{21}^{-1} = \mathcal{F}_{21}\,\mathcal{R}\,\Delta\,\mathcal{R}^{-1}\,\mathcal{F}_{21}^{-1} = \left(\mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}\right)\Delta^{(F)}\left(\mathcal{F}\,\mathcal{R}^{-1}\,\mathcal{F}_{21}^{-1}\right)\;.$$

The Yang - Baxter eq. (2.3.11) for R matrix (2.6.6) can be directly checked with the help of (2.3.7) and (2.6.4).

Impose additional relations on $\mathcal{F}$

$$(\Delta \otimes id)\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{23}\;\;,\quad (id \otimes \Delta)\mathcal{F} = \mathcal{F}_{13}\,\mathcal{F}_{12}\;, \qquad (2.6.7)$$

which, together with (2.6.4), implies the Yang-Baxter equation for $\mathcal{F}$. Using (2.3.7) one deduces from (2.6.7) equations

$$\mathcal{R}_{12}\,\mathcal{F}_{13}\,\mathcal{F}_{23} = \mathcal{F}_{23}\,\mathcal{F}_{13}\,\mathcal{R}_{12}\;\;,\quad \mathcal{F}_{12}\,\mathcal{F}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{F}_{13}\,\mathcal{F}_{12}\;. \qquad (2.6.8)$$

Eqs. (2.6.8) and the Yang-Baxter relations for universal elements $\mathcal{R}$, $\mathcal{F}$ define the twist which is proposed in [28] (the additional condition $\mathcal{F}^{21}\mathcal{F} = 1 \otimes 1$ is assumed in [28]).

Note that if $\mathcal{A}$ is the Hopf algebra of functions on the group algebra of group G (2.2.7), then eq. (2.6.4) can be written in the form of 2-cocycle equation:

$$\mathcal{F}(a,b)\,\mathcal{F}(ab,c) = \mathcal{F}(b,c)\mathcal{F}(a,bc)\;,\quad (\forall a,b,c \in G)\;,$$

for the projective representation ρ of G: $\rho(a)\rho(b) = \mathcal{F}(a,b)\,\rho(a\,b)$. That is why eq. (2.6.4) is called cocycle equation.

Many explicit solutions of the cocycle equation (2.6.4) are known (see e.g. [33]–[35] and references therein).

Remark 2. *Ribbon Hopf algebras.*

Here we explain the notion of the ribbon Hopf algebras [29]. Consider quasitriangular Hopf algebra $\mathcal{A}$ and represent the universal $\mathcal{R}$-matrix in the form $\mathcal{R} = \sum_\mu \alpha_\mu \otimes \beta_\mu$, $\mathcal{R}^{-1} = \sum_\mu \gamma_\mu \otimes \delta_\mu$, where α_μ, β_μ, γ_μ, $\delta_\mu \in \mathcal{A}$. Using (2.3.13) we rewrite the identities $(id \otimes S)\mathcal{R}\mathcal{R}^{-1} = 1 = (id \otimes S)\mathcal{R}^{-1}\mathcal{R}$ as

$$\alpha_\mu\,\alpha_\nu \otimes \beta_\nu\,S(\beta_\mu) = 1 = \alpha_\mu\,\alpha_\nu \otimes S(\beta_\nu)\,\beta_\mu\;. \qquad (2.6.9)$$

(here and below the sum over μ and ν is assumed) while for $(S \otimes id)\mathcal{R}\mathcal{R}^{-1} = 1 = (S \otimes id)\mathcal{R}^{-1}\mathcal{R}$ we have

$$S(\gamma_\mu)\,\gamma_\nu \otimes \delta_\nu\,\delta_\mu = 1 = \gamma_\mu\,S(\gamma_\nu) \otimes \delta_\nu\,\delta_\mu\;. \qquad (2.6.10)$$

Consider the element $u = \sum_\mu S(\beta_\mu)\,\alpha_\mu$ for which the following proposition holds
Proposition 1. *(see [23])*
1.) For any $a \in \mathcal{A}$ we have

$$S^2(a)\,u = u\,a \; . \tag{2.6.11}$$

2.) the element u is invertible, with $u^{-1} = S^{-1}(\delta_\mu)\,\gamma_\mu$.
Proof.
1.) From the relation (2.3.7) it follows that $\forall a \in \mathcal{A}$ (the summation signs are omitted):

$$\alpha_\mu\,a_{(1)} \otimes \beta_\mu\,a_{(2)} \otimes a_{(3)} = a_{(2)}\,\alpha_\mu \otimes a_{(1)}\,\beta_\mu \otimes a_{(3)} \; .$$

From this we obtain

$$S^2(a_{(3)})\,S(\beta_\mu\,a_{(2)})\,\alpha_\mu a_{(1)} = S^2(a_{(3)})\,S(a_{(1)}\,\beta_\mu)\,a_{(2)}\alpha_\mu \; ,$$

or

$$S^2(a_{(3)})\,S(a_{(2)})\,u\,a_{(1)} = S^2(a_{(3)})\,S(\beta_i)\,S(a_{(1)})\,a_{(2)}\alpha_i \; .$$

Applying to this equation the axioms (2.3.1) and taking into account the fact that $\epsilon(S(a)) = \epsilon(a)$, we obtain (2.6.11).
2.) Putting $w = S^{-1}(\delta_\mu)\,\gamma_\mu$, we have

$$u\,w = u\,S^{-1}(\delta_\mu)\,\gamma_\mu = S(\delta_\mu)\,u\,\gamma_\mu = S(\beta_\nu\,\delta_\mu)\,\alpha_\nu\,\gamma_\mu \; .$$

Since $\mathcal{R}\cdot\mathcal{R}^{-1} = \alpha_\nu\gamma_\mu \otimes \beta_\nu\delta_\mu = 1$, we have $u\,w = 1$. It follows from last equation and from (2.6.11) that $S^2(w)\,u = 1$, and therefore the element u has both a right and left inverse. Thus, the element u is invertible and we can rewrite (2.6.11) in the form

$$S^2(a) = u\,a\,u^{-1} \; . \tag{2.6.12}$$

This relation shows, in particular, that the operation of taking the antipode is not involutive. In Ref. [23] it was noted that

$$\Delta(u) = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(u\otimes u) = (u\otimes u)(\mathcal{R}_{21}\mathcal{R}_{12})^{-1} \; .$$

In addition, it was shown that the relations (2.6.12) are satisfied if, for the element $u \equiv u_1$, we take any of the following three elements:

$$u_2 = S(\gamma_\mu)\,\delta_\mu \; , \quad u_3 = \beta_\mu\,S^{-1}(\alpha_\mu) \; , \quad u_4 = \gamma_\mu\,S^{-1}(\delta_\mu) \; .$$

Using the results of this Proposition, it can be seen that $S(u)^{-1} = u_2$, $S(u_3)^{-1} = u_4$. In addition, it turns out that all the u_i commute with each other, and the elements $u_i u_j^{-1} = u_j^{-1} u_i$ are central in $\mathcal{A}$ (see [23]). Consequently, the element $u\,S(u) = u_1\,u_2^{-1}$ is also central. On the basis of these remarks, we introduce the important concept of a ribbon Hopf algebra (see [29]):
Definition 7. *Consider a quasitriangular Hopf algebra $(\mathcal{A},\ \mathcal{R})$. Then the triplet $(\mathcal{A},\ \mathcal{R},\ v)$ is called a ribbon Hopf algebra if v is a central element in $\mathcal{A}$ and*

$$v^2 = u\,S(u) \; , \quad S(v) = v \; , \quad \epsilon(v) = 1 \; ,$$

$$\Delta(v) = (\mathcal{R}_{21}\,\mathcal{R}_{12})^{-1}\,(v\otimes v) \; .$$

For each quasitriangular Hopf algebra $\mathcal{A}$, we can define $\mathcal{A}$-colored ribbon graphs [29]. If, moreover, $\mathcal{A}$ is a ribbon Hopf algebra, then for each $\mathcal{A}$-colored ribbon graph we can associate the central element of $\mathcal{A}$ that generalizes the Jones polynomial being an invariant of a knot in $\mathbf{R}^3$ (see [29], [36]).

Remark 3. *Quasi-Hopf algebras.*

One can introduce a generalization of a Hopf algebra, called a quasi-Hopf algebra, [30] which is defined as an associative unital algebra $\mathcal{A}$ with homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, homomorphism $\epsilon : \ \mathcal{A} \to \mathbf{C}$, antiautomorphism $S : \ \mathcal{A} \to \mathcal{A}$ and invertible element $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$. At the same time Δ, ϵ, Φ and S satisfy the axioms

$$(id \otimes \Delta)\Delta(a) = \Phi \cdot (\Delta \otimes id)\Delta(a) \cdot \Phi^{-1}, \ \ a \in \mathcal{A}, \tag{2.6.13}$$

$$(id \otimes id \otimes \Delta)(\Phi) \cdot (\Delta \otimes id \otimes id)(\Phi) = (I \otimes \Phi) \cdot (id \otimes \Delta \otimes id)(\Phi) \cdot (\Phi \otimes I), \tag{2.6.14}$$

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta \ , \ \ (id \otimes \epsilon \otimes id)\Phi = I \otimes I \tag{2.6.15}$$

$$S(a_{(1)})\, \alpha\, a_{(2)} = \epsilon(a)\, \alpha \ , \ \ a_{(1)}\, \beta\, S(a_{(2)}) = \epsilon(a)\, \beta \ , \tag{2.6.16}$$

$$\phi_i\, \beta\, S(\phi'_i)\, \alpha\, \phi''_i = I \ , \ \ S(\bar{\phi}_i)\, \alpha\, \bar{\phi}'_i\, \beta\, S(\bar{\phi}''_i) = I \ ,$$

where α and β are certain fixed elements of $\mathcal{A}$, $\Delta(a) = a_{(1)} \otimes a_{(2)}$, and

$$\Phi := \phi_i \otimes \phi'_i \otimes \phi''_i \ , \ \ \Phi^{-1} := \bar{\phi}_i \otimes \bar{\phi}'_i \otimes \bar{\phi}''_i \ ,$$

(summation over i is assumed). Thus, a quasi-Hopf algebra differs from an ordinary Hopf algebra in that the axiom of coassociativity is replaced by the weaker condition (2.6.13). In other words, a quasi-Hopf algebra is noncoassociative, but this noncoassociativity is kept under control by means of the element Φ. The axioms (2.6.16) (which looks like different definitions of the left and right antipodes) generalize the axioms (2.3.1) for usual Hopf algebras and consequently the elements α and β involve into the play with the contragredient representations of the quasi-Hopf algebras.

To make the pentagonal condition (2.6.14) more transparent, let us consider (following Ref. [30]) the algebra $\mathcal{A}$ as the algebra of functions on a "noncommutative" space X equipped with a $*$ product: $X \times X \to X$. Then, elements $a \in \mathcal{A}$, $b \in \mathcal{A} \otimes \mathcal{A}, \dots$ are written in the form $a(x)$, $b(x,y)$ ... and $\Delta(a)$ is represented as $a(x * y)$. The homomorphism ϵ defines the point in X, which we denote 1 and instead of $\epsilon(a)$ we write $a(1)$. Then, equations (2.6.13) – (2.6.15) are represented in the form [30]:

$$a(x*(y*z)) = \Phi(x,y,z)\, a((x*y)*z)\, \Phi(x,y,z)^{-1} \ ,$$

$$\Phi(x,y,z*u)\, \Phi(x*y,z,u) = \Phi(y,z,u)\, \Phi(x,y*z,u)\, \Phi(x,y,z) \ , \tag{2.6.17}$$

$$a(1*x) = a(x) = a(x*1) \ , \quad \Phi(x,1,z) = 1 \ .$$

Now it is clear that (2.6.17) (and respectively (2.6.14)) is the sufficient condition for the commutativity of the diagram:

$$\begin{array}{ccccc}
a\,(((x*y)*z)*u) & \longrightarrow & a((x*y)*(z*u)) & \longrightarrow & a(x*(y*(z*u))) \\
\downarrow & & & & \downarrow \\
a((x*(y*z))*u) & & \xrightarrow{\qquad\qquad} & & a(x*((y*z)*u))
\end{array}$$

Applications of the theory of quasi-Hopf algebras to the solutions of the Knizhnik-Zamolodchikov equations are discussed in Ref. [30]. On the other hand, one can suppose that, by virtue of the occurrence of the pentagonal relation (2.6.14) for the element Φ, quasi-Hopf algebras will be associated with multidimensional generalizations of Yang-Baxter equations.

3 THE YANG-BAXTER EQUATION AND QUANTIZATION OF LIE GROUPS

In this section, we discuss the R -matrix approach to the theory of quantum groups [17], on the basis of which we perform a quantization of classical Lie groups and also some Lie supergroups. We present trigonometric solutions of the Yang-Baxter equation invariant with respect to the adjoint action of the quantum groups $GL_q(N)$, $SO_q(N)$, $Sp_q(2n)$ and supergroups $GL_q(N|M)$, $Osp_q(N|2m)$. We briefly discuss the corresponding Yangian (rational) solutions, and also quantum Knizhnik-Zamolodchikov equations and $Z_N \otimes Z_N$ symmetric elliptic solutions of the Yang-Baxter equation.

3.1 Numerical R-matrices

Some results of this Subsection have been presented in [36], [37].

Let $\mathcal{A}$ be a quasitriangular Hopf algebra. Consider a representation T of the algebra $\mathcal{A}$: $a \to T^i_j(a)$, $\forall a \in \mathcal{A}$ $(i, j = 1, \dots, N)$ in N-dimensional vector space V. In view of (2.3.11) and (2.2.8), the matrix representation $R^{ij}_{kl} = (T^i_k \otimes T^j_l)\mathcal{R}$ of the universal element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ satisfies the Yang-Baxter equation

$$R^{i_1 i_2}_{j_1 j_2}\, R^{j_1 i_3}_{k_1 j_3}\, R^{j_2 j_3}_{k_2 k_3} = R^{i_2 i_3}_{j_2 j_3}\, R^{i_1 j_3}_{j_1 k_3}\, R^{j_1 j_2}_{k_1 k_2} \tag{3.1.1}$$

(the summation over repeated indices is assumed). We stress that not all solutions R of the eqs. (3.1.1) are images $(T \otimes T)\mathcal{R}$ of some universal $\mathcal{R}$ matrix. We assume that $R \in End(V \otimes V)$ is an invertible matrix

$$R^{ij}_{kl}\,(R^{-1})^{kl}_{mn} = (R^{-1})^{ij}_{kl}\,R^{kl}_{mn} = \delta^i_m\,\delta^l_n\ . \tag{3.1.2}$$

In terms of concise matrix notation [17], the relation (3.1.1) is written in the following equivalent forms

$$R_{12}\,R_{13}\,R_{23} = R_{23}\,R_{13}\,R_{12} \Longrightarrow \tag{3.1.3}$$

$$\hat{R}_{12}\,\hat{R}_{23}\,\hat{R}_{12} = \hat{R}_{23}\,\hat{R}_{12}\,\hat{R}_{23} \Longrightarrow \tag{3.1.4}$$

$$\hat{R}_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}^{-1} = \hat{R}_{12}^{-1}\,\hat{R}_{23}^{-1}\,\hat{R}_{12}\;, \quad \hat{R}_{12}\,\hat{R}_{23}^{-1}\,\hat{R}_{12}^{-1} = \hat{R}_{23}^{-1}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}\;, \tag{3.1.5}$$

where $\hat{R} := P\,R$, P is the permutation matrix:

$$P^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\;, \quad \hat{R}^{i_1 i_2}_{j_1 j_2} = (P\,R)^{i_1 i_2}_{j_1 j_2} = R^{i_2 i_1}_{j_1 j_2}\;.$$

and the indices $1,2,3$ label the vector spaces V in which the corresponding R matrices act non-trivially, e.g. $R_{12} = R \otimes I$, $R_{23} = I \otimes R$ (I is a unit matrix in $Mat(N)$). We also use the following form of the Yang-Baxter equations (3.1.1), (3.1.3), (3.1.4)

$$\hat{R}_a\,\hat{R}_{a+1}\,\hat{R}_a = \hat{R}_{a+1}\,\hat{R}_a\,\hat{R}_{a+1}\;, \tag{3.1.6}$$

where $\hat{R}_a = I^{\otimes(a-1)} \otimes \hat{R} \otimes I^{\otimes(M-a)}$, $(a = 1,\dots,M)$. Note that the matrices $\hat{R}_a$ define the representation of the braid group $\mathcal{B}_{M+1}$ in view of relations (3.1.6) and

$$[\hat{R}_a,\,\hat{R}_b] = 0 \;\; (\text{for } \;|a-b| > 1)\;. \tag{3.1.7}$$

The direct consequences of (3.1.6) are the equations

$$X(\hat{R}_a)\,\hat{R}_{a+1}\,\hat{R}_a = \hat{R}_{a+1}\,\hat{R}_a\,X(\hat{R}_{a+1})\;, \;\; \hat{R}_a\,\hat{R}_{a+1}\,X(\hat{R}_a) = X(\hat{R}_{a+1})\,\hat{R}_a\,\hat{R}_{a+1}\;, \tag{3.1.8}$$

which make it possible to carry an arbitrary function $X(\hat{R}_a)$ through the operators $\hat{R}_a\hat{R}_{a+1}$ and $\hat{R}_{a+1}\hat{R}_a$.

Definition 8. *The matrix R is called skew-invertible if there exists matrix Ψ such that (cf. (3.1.2))*

$$R^{mk}_{jn}\,\Psi^{in}_{ml} = \delta^i_j\delta^k_l = \Psi^{ni}_{lm}\,R^{km}_{nj}\;.$$

The matrix forms of these relations are[5]

$$R^{t_1}_{12}\,\Psi^{t_1}_{12} = I_{12} = \Psi^{t_1}_{12}\,R^{t_1}_{12}\;, \tag{3.1.9}$$

$$Tr_2(\hat{R}_{12}\,\hat{\Psi}_{23}) = P_{13} = Tr_2\left(\hat{\Psi}_{12}\,\hat{R}_{23}\right)\;, \tag{3.1.10}$$

where $\hat{\Psi} = P\,\Psi$. We say that the invertible R- matrix is completely invertible if R_{12} is skew-invertible and the matrix R_{12}^{-1} is also skew-invertible, i.e. there exists the matrix Φ_{12} such that

$$Tr_2\left(\hat{\Phi}_{12}\,\hat{R}_{23}^{-1}\right) = P_{13} = Tr_2(\hat{R}_{12}^{-1}\,\hat{\Phi}_{23})\;, \tag{3.1.11}$$

where $\hat{R}^{-1} = R^{-1}\,P$.

Note that the skew-invertible R- matrices have been considered in [36] where the matrix Ψ_{12} is denoted as $((R^{t_1}_{12})^{-1})^{t_1}$ (see (3.1.9)).

Now we define four matrices

$$D_1 = Tr_2(\hat{\Psi}_{12})\;, \quad Q_2 = Tr_1(\hat{\Psi}_{12})\;,\;, \tag{3.1.12}$$

$$\overline{D}_1 = Tr_2(\hat{\Phi}_{12})\;, \quad \overline{Q}_2 = Tr_1(\hat{\Phi}_{12})\;, \tag{3.1.13}$$

which we use below. In view of the relations (3.1.10) and (3.1.11) we deduce

$$Tr_2\left(\hat{R}_{12}D_2\right) = I_1\;, \quad Tr_1(Q_1\,\hat{R}_{12}) = I_2\;. \tag{3.1.14}$$

[5] The form (3.1.10) is very convenient in calculations and has been proposed in [50].

$$Tr_2(\hat{R}^{-1}_{12}\,\overline{D}_2) = I_1\;, \quad Tr_1\left(\overline{Q}_1\,\hat{R}^{-1}_{12}\right) = I_2\;. \tag{3.1.15}$$

Proposition 2. *Let the Yang-Baxter matrix $\hat{R}$ be invertible and skew-invertible, then the following identities hold*

$$D_0\,P_{02} = Tr_3 D_3 \hat{R}^{-1}_{23}\,\hat{R}_{03}\;, \quad D_0\,P_{02} = Tr_3 D_3 \hat{R}_{23}\,\hat{R}^{-1}_{03}\;, \tag{3.1.16}$$

$$Q_0\,P_{02} = Tr_1 Q_1 \hat{R}^{-1}_{12}\,\hat{R}_{10}\;, \quad Q_0\,P_{02} = Tr_1 Q_1 \hat{R}_{12}\,\hat{R}^{-1}_{10}\;, \tag{3.1.17}$$

$$\hat{R}^{-1}_1\,D_2 = D_1\,\hat{\Psi}_{21}\;\;, \quad D_2\,\hat{R}^{-1}_1 = \hat{\Psi}_{21}\,D_1\;, \tag{3.1.18}$$

$$\hat{R}^{-1}_1\,Q_1 = Q_2\,\hat{\Psi}_{21}\;\;, \quad Q_1\,\hat{R}^{-1}_1 = \hat{\Psi}_{21}\,Q_2\;, \tag{3.1.19}$$

where the matrices D and Q satisfy

$$D_2\,Q_2 = Q_2\,D_2 = Tr_3(D_3\,\hat{R}^{-1}_2) = Tr_1(Q_1\,\hat{R}^{-1}_1)\;. \tag{3.1.20}$$

If the matrix R is completely invertible, then

$$\hat{R}_1\,D_2 = D_1\,\hat{\Phi}_{21}\;\;, \quad D_2\,\hat{R}_1 = \hat{\Phi}_{21}\,D_1\;. \tag{3.1.21}$$

$$\hat{R}_1\,Q_1 = Q_2\,\hat{\Phi}_{21}\;\;, \quad Q_1\,\hat{R}_1 = \hat{\Phi}_{21}\,Q_2\;. \tag{3.1.22}$$

and the matrices D and Q are invertible according to the relations

$$D^{-1} = \overline{Q}\;, \quad Q^{-1} = \overline{D}\;. \tag{3.1.23}$$

Conversely, if the matrix D (or Q) is invertible, then the matrix R is completely invertible. For invertible matrices D and Q one has the relations

$$Tr_1(D^{-1}_1\,\hat{R}^{-1}_1) = I_2 = Tr_3(Q^{-1}_3\,\hat{R}^{-1}_2)\;. \tag{3.1.24}$$

Proof. We multiply both equations (3.1.8) (for $a=1$) from the left by $\hat{\Psi}_{01}$ and from the right by $\hat{\Psi}_{34}$ and take the traces Tr_{13} ($\equiv Tr_1\,Tr_3$). Using (3.1.10) we obtain

$$Tr_1 \hat{\Psi}_{01} X(\hat{R}_1) P_{24} \hat{R}_1 = Tr_3 \hat{R}_2\,P_{02}\,X(\hat{R}_2)\,\hat{\Psi}_{34}\;, \tag{3.1.25}$$

$$Tr_1 \hat{\Psi}_{01} \hat{R}_1\,P_{24}\,X(\hat{R}_1) = Tr_3 X(\hat{R}_2)\,P_{02}\,\hat{R}_2\,\hat{\Psi}_{34}\;. \tag{3.1.26}$$

We put in (3.1.25), (3.1.26) $X(\hat{R}) = \hat{R}^{-1}$ and act on these relations by $Tr_4(\dots)$ or $Tr_0(\dots)$. Using (3.1.10) and (3.1.12) we obtain the following identities

$$D_0\,I_2 = Tr_3 D_3 \hat{R}^{\mp 1}_2\,P_{02}\,\hat{R}^{\pm 1}_2\;, \quad Q_0\,I_2 = Tr_1 Q_1 \hat{R}^{\pm 1}_1\,P_{02}\,\hat{R}^{\mp 1}_1\;, \tag{3.1.27}$$

which can be easily rewritten as (3.1.16) and (3.1.17)

$$D_0\,P_{02} = Tr_3 D_3 \hat{R}^{\mp 1}_{23}\,\hat{R}^{\pm 1}_{03}\;, \quad Q_0\,P_{02} = Tr_1 Q_1 \hat{R}^{\pm 1}_{12}\,\hat{R}^{\mp 1}_{10}\;.$$

Acting on the first relation (3.1.16) by $Tr_0(\hat{\Psi}_{10}...)$ and on the second relation (3.1.16) by $Tr_2(\hat{\Psi}_{12}...)$ we obtain (3.1.18). Similarly acting on the first relation (3.1.17) by $Tr_0(\dots\hat{\Psi}_{03})$ and on the second relation (3.1.17) by $Tr_2(\dots\hat{\Psi}_{23})$ we obtain (3.1.19). Taking the traces $Tr_2(\dots)$ and $Tr_1(\dots)$ of (3.1.18) and (3.1.19), respectively, gives (3.1.20).

If the matrix R is completely invertible, then acting on the first relation (3.1.16) by $Tr_2(\hat{\Phi}_{12}...)$ and on the second relation (3.1.16) by $Tr_0(\hat{\Phi}_{10}...)$ we obtain (3.1.21). Analogously, acting on the first relation (3.1.17) by $Tr_2(\ldots\hat{\Phi}_{23})$ and on the second relation (3.1.17) by $Tr_0(\ldots\hat{\Phi}_{03})$ we deduce (3.1.22). Eqs. (3.1.23) are obtained by taking traces $Tr_2(\ldots)$ and $Tr_1(\ldots)$ of (3.1.21) and (3.1.22), respectively.

Conversely, if the matrix D is invertible, then $D_1\hat{R}_{21}D_2^{-1}$ (cf. (3.1.21)) is the skew-inverse matrix for $\hat{R}^{-1}$. Indeed,

$$Tr_2\left(\hat{R}_{12}^{-1}\, D_2\, \hat{R}_{32}\, D_3^{-1}\right) = Tr_2\left(\hat{R}_{12}^{-1}\, D_2\, \hat{R}_{32}\right)\, D_3^{-1} =$$

$$= D_1\, Tr_2\left(\hat{\Psi}_{21}\, \hat{R}_{32}\right)\, D_3^{-1} = D_1\, P_{13}\, D_3^{-1} = P_{13}\ .$$

Thus, the R-matrix is completely invertible. The proof of the fact that the R- matrix is completely invertible for the case of the invertible matrix Q is analogous.

For invertible matrices D and Q one can rewrite the first relations of (3.1.18) and (3.1.19) as

$$D_1^{-1}\, \hat{R}_1^{-1} = \hat{\Psi}_{21}\, D_2^{-1}\ , \quad Q_2^{-1}\, \hat{R}_1^{-1} = \hat{\Psi}_{21}\, Q_1^{-1}\ .$$

Then acting on these relations by the traces $Tr_1(\ldots)$ and $Tr_2(\ldots)$ we obtain (3.1.24). $\square$

Corollary 1. Let $\hat{R}$ be skew-invertible and the matrix A_{12} be one of the matrices $\{\hat{R}_{12},\, \hat{\Psi}_{12}\}$. Then, from (3.1.18), (3.1.19) we obtain

$$[A_{12},\, D_1\, D_2] = 0 = [A_{12},\, Q_1\, Q_2]\ , \tag{3.1.28}$$

$$A_{12}^{\pm 1}\, D_1\, Q_1 = D_2\, Q_2\, A_{12}^{\pm 1}\ . \tag{3.1.29}$$

If $\hat{R}$ is completely invertible, then, using (3.1.21), (3.1.22) we prove eqs. (3.1.28), (3.1.29) for $A_{12} = \hat{\Phi}_{12}$ and deduce the relation on the matrices $\hat{\Phi}$ and $\hat{\Psi}$:

$$\hat{\Phi}_{12}^{-1} = D_2^2\, \hat{\Psi}_{12}\, D_1^{-2} = Q_1^2\, \hat{\Psi}_{12}\, Q_2^{-2}\ .$$

Corollary 2. For any quantum $(N \times N)$ matrix E (with noncommutative entries E^i_j) one can find the following identities

$$Tr(D\,E)\, I_1 = Tr_2\left(D_2 \hat{R}_1^{\mp 1}\, E_1\, \hat{R}_1^{\pm 1}\right)\ , \quad Tr(Q\,E)\, I_2 = Tr_1\left(Q_1 \hat{R}_1^{\pm 1}\, E_2\, \hat{R}_1^{\mp 1}\right), \tag{3.1.30}$$

which demonstrate the invariance properties of the quantum traces $Tr(D\,E) =: Tr_D(E)$ and $Tr(Q\,E) =: Tr_Q(E)$. To prove these identities we multiply eqs. (3.1.27) by the matrix E_0 and take the trace $Tr_0(\ldots)$. Note that in view of (3.1.28) the multiple quantum traces satisfy cyclic property:

$$\begin{array}{c} Tr_{\mathcal{D}(1\ldots m)}\left(X(\hat{R})\cdot Y\right) = Tr_{\mathcal{D}(1\ldots m)}\left(Y\cdot X(\hat{R})\right)\ , \\ Tr_{\mathcal{Q}(1\ldots m)}\left(X(\hat{R})\cdot Y\right) = Tr_{\mathcal{Q}(1\ldots m)}\left(Y\cdot X(\hat{R})\right)\ , \end{array} \tag{3.1.31}$$

where $X(\hat{R}) \in End(V^{\otimes m})$ denotes arbitrary element of the group algebra of the braid group $\mathcal{B}_m$ in R-matrix representation (see (3.1.6)) and $Y \in End(V^{\otimes m})$ are arbitrary operators.

Corollary 3. Let R be skew-invertible matrix. We multiply Yang-Baxter eqs. (3.1.5) by the matrices D_2, D_3 from the left and right:

$$D_2\,\hat{R}_2\,\hat{R}_1^{-1}\,\hat{R}_2^{-1}\,D_3 = D_2\,\hat{R}_1^{-1}\,\hat{R}_2^{-1}\,\hat{R}_1\,D_3\;, \quad D_3\,\hat{R}_1\,\hat{R}_2^{-1}\,\hat{R}_1^{-1}\,D_2 = D_3\,\hat{R}_2^{-1}\,\hat{R}_1^{-1}\,\hat{R}_2\,D_2\;,$$

and use the relations (3.1.18) and (3.1.28). As a result we deduce

$$\hat{\Psi}_{21}\,\hat{\Psi}_{32}\,\hat{R}_{12} = \hat{R}_{23}\,\hat{\Psi}_{21}\,\hat{\Psi}_{32}\;, \quad \hat{\Psi}_{32}\,\hat{\Psi}_{21}\,\hat{R}_{23} = \hat{R}_{12}\,\hat{\Psi}_{32}\,\hat{\Psi}_{21}\;. \tag{3.1.32}$$

Corollary 4. For any function X of $\hat{R}$ the matrix $Y_2 := Tr_1 Q_1 X(\hat{R}_1)$ (3.1.35) satisfies

$$Y_2 \hat{R}_1^{\pm 1} = \hat{R}_1^{\pm 1} Y_1\;, \tag{3.1.33}$$

and $[D,\, Y] = 0 = [Y,\, Q]$. The matrices

$$Y_2^{(n)} := Tr_3\left(D_3\,\hat{R}_2^n\right) = Tr_1\left(Q_1\,\hat{R}_1^n\right) \quad \forall n \in \mathbf{Z}\;, \tag{3.1.34}$$

generate the commutative set. Indeed,

$$Y_2 \hat{R}_1^{\pm 1} = Tr_3\left(D_3 X(\hat{R}_2) \hat{R}_1^{\pm 1} \hat{R}_2^{\pm 1} \hat{R}_2^{\mp 1}\right) = \hat{R}_1^{\pm 1} Tr_3\left(D_3 \hat{R}_2^{\pm 1} X(\hat{R}_1) \hat{R}_2^{\mp 1}\right) =$$

$$= \hat{R}_1^{\pm 1} Tr_2\left(D_2\,X(\hat{R}_1)\right) = \hat{R}_1^{\pm 1} Y_1\;.$$

Then, we have:

$$D_2\,Y_2 = Tr_3 D_2\,D_3\,X(\hat{R}_2) = Tr_3 X(\hat{R}_2)\,D_2\,D_3 = Y_2\,D_2\;.$$

The proof of the identity $[Y,\, Q] = 0$ is the same. The commutativity of the matrices $Y_2^{(n)}$ follows from (3.1.33), since for n is even and odd we have, respectively

$$Y_2\,Y_2^{(2k)} = Tr_3\left(D_3\,Y_2\,\hat{R}_2^{2k}\right) = Tr_3\left(D_3\,\hat{R}_2^{2k}\,Y_2\right) = Y_2^{(2k)}\,Y_2\;,$$

$$Y_2\,Y_2^{(2k+1)} = Tr_1\left(Q_1\,\hat{R}_2^{2k+1}\,Y_1\right) = Tr_1\left(Y_1\,Q_1\,\hat{R}_2^{2k+1}\right) =$$

$$= Tr_1\left(Q_1\,Y_1\,\hat{R}_2^{2k+1}\right) = Tr_1\left(Q_1\,\hat{R}_2^{2k+1}\,Y_2\right) = Y_2^{(2k+1)}\,Y_2\;.$$

Corollary 5. The trace $Tr_{04}(\ldots)$ of eq. (3.1.25) (or (3.1.26)) gives

$$Tr_1 Q_1 X(\hat{R}_1) = Tr_3 D_3 X(\hat{R}_2)\;, \tag{3.1.35}$$

where we have redefined the arbitrary function X: $X(\hat{R})\hat{R} \to X(\hat{R})$. In particular, for $X = 1$, we obtain $Tr(D) = Tr(Q)$. Eq. (3.1.35) leads to the following identity

$$Tr_{12}\left(Q_1 Q_2 X(\hat{R}_1)\right) = Tr_{23}\left(D_3 Q_2 X(\hat{R}_2)\right) = Tr_{34}\left(D_3 D_4 X(\hat{R}_3)\right)\;. \tag{3.1.36}$$

Proposition 3. *The identity (3.1.36) is generalized as*

$$Tr_{1\ldots n}\left(Q_1 \cdots Q_k D_{k+1} \cdots D_n X_{1\to n}\right) = Tr_{1\ldots n}\left(D_1 \cdots D_n X_{1\to n}\right) \quad (\forall k \leq n)\;, \tag{3.1.37}$$

where $X_{1\to n} := X(\hat{R}_1, \ldots, \hat{R}_{n-1})$ *is a function of matrices* $\hat{R}_i$ $(i = 1, \ldots, n-1)$.

Proof. Indeed, from (3.1.6), (3.1.7) we have $\hat{R}_1\hat{R}_2\cdots\hat{R}_n\, X_{1\to n} = X_{2\to n+1}\,\hat{R}_1\hat{R}_2\cdots\hat{R}_n$. Multiplying this eq. by the matrices Q_1 and D_{n+1} from the left and right and taking the trace $Tr_1 Tr_{n+1}(\ldots)$ we deduce

$$Tr_1\left(Q_1\,\hat{R}_1\cdots\hat{R}_{n-1}\,X_{1\to n}\right) = Tr_{n+1}\left(D_{n+1}\,X_{2\to n+1}\,\hat{R}_2\cdots\hat{R}_n\right)\ ,$$

which is written (for arbitrary function X) in the form

$$Tr_1\left(Q_1\,X_{1\to n}\right) = Tr_{n+1}\left(D_{n+1}\,(\hat{R}_2\cdots\hat{R}_n)^{-1}\,X_{2\to n+1}\,\hat{R}_2\cdots\hat{R}_n\right)\ . \tag{3.1.38}$$

Then, applying the trace $Tr_2(Q_2\ldots)$ to (3.1.38) (and again using (3.1.38)) we obtain

$$\begin{aligned} Tr_{12}\left(Q_1Q_2\,X_{1\to n}\right) &= Tr_{n+1}D_{n+1}Tr_2\left(Q_2\,(\hat{R}_2\cdots\hat{R}_n)^{-1}\,X_{2\to n+1}\,\hat{R}_2\cdots\hat{R}_n\right) = \\ &= Tr_{n+1,n+2}\left(D_{n+1}D_{n+2}\,(\hat{R}_3\cdots\hat{R}_{n+1})^{-2}\,X_{3\to n+2}\,(\hat{R}_3\cdots\hat{R}_{n+1})^2\right)\ . \end{aligned} \tag{3.1.39}$$

Applying the trace $Tr_3(Q_3\ldots)$ to (3.1.39) etc. we obtain

$$\begin{gathered} Tr_{1\ldots k}\left(Q_1\cdots Q_k\,X_{1\to n}\right) = \\ = Tr_{n+1\ldots n+k}\left(D_{n+1}\cdots D_{n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{-k}\,X_{k+1\to n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^k\right)\ , \end{gathered}$$

and finally applying $Tr_{k+1\ldots n}(D_{k+1}\cdots D_n\ \ldots)$ deduce (3.1.37)

$$\begin{gathered} Tr_{1\ldots n}\left(Q_1\cdots Q_k D_{k+1}\cdots D_n\,X_{1\to n}\right) = \\ = Tr_{k+1\ldots n+k}\left(D_{k+1}\cdots D_{n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^{-k}\,X_{k+1\to n+k}\,(\hat{R}_{k+1}\cdots\hat{R}_{n+k-1})^k\right) = \\ = Tr_{k+1\ldots n+k}\left(D_{k+1}\cdots D_{n+k}\,X_{k+1\to n+k}\right)\ , \end{gathered} \tag{3.1.40}$$

where we have used many times the cyclic property (3.1.31). □

The Proposition 3 is important from the point of veiw of the investigations of link invariants, since it demonstrates the equivalence (for the case of the skew-invertible R-matrices) of the left and right closuring of braids (functions $X_{1\to n}$) by means of the traces with matrices Q and D, respectively.

We shall now assume that the invertible $\hat{R}$ matrix obeys the characteristic equation

$$(\hat{R}-\lambda_1)(\hat{R}-\lambda_2)\cdots(\hat{R}-\lambda_M) = 0,\ \ (\lambda_i \neq \lambda_j\ \text{ if }\ i\neq j)\ , \tag{3.1.41}$$

where $\lambda_i \neq 0\ \forall i$. This equation can be represented in the form

$$\hat{R}^M - \sigma_1(\lambda)\,\hat{R}^{M-1} + \ldots + (-1)^{M-1}\,\sigma_{M-1}(\lambda)\,\hat{R} + (-1)^M\,\sigma_M(\lambda) = 0\ , \tag{3.1.42}$$

where

$$\sigma_k(\lambda) = \sum_{i_1<i_2<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k}\ ,$$

are elementary symmetric functions of λ_i $(i=1,\ldots,M)$. For R matrices satisfying (3.1.41), one can introduce a set of M projectors:

$$\mathbf{P}_k = \prod_{j\neq k}\frac{(\hat{R}-\lambda_j)}{(\lambda_k-\lambda_j)}\ ,\quad \sum_k \mathbf{P}_k = \mathbf{1}\ , \tag{3.1.43}$$

which project the $\hat{R}$ matrix to its eigenvalues $\mathbf{P}_k\,\hat{R} = \hat{R}\,\mathbf{P}_k = \lambda_k\,\mathbf{P}_k$ and can be used for the spectral decomposition

$$X(\hat{R}) = \sum_{k=1}^{M} X(\lambda_k)\mathbf{P}_k, \tag{3.1.44}$$

of an arbitrary function X of the R matrix. In particular, for $X = \mathbf{1}$ we obtain the completeness condition. For the R- matrix satisfying (3.1.41) the space of matrices Y (3.1.35) is M- dimensional and the basis is generated by $\{Y^{(n)}\}$ (3.1.34) or $\overline{Y}_2^{(n)} := Tr_1\,(Q_1\,\mathbf{P}_n)$ for $(n = 1,\ldots,M)$.

For known explicit examples of the completely invertible $\hat{R}$ matrices, being solutions of (3.1.6), all matrices (3.1.35) which satisfy (3.1.33) are proportional to the identity matrix. In fact one can prove the following statement.

Proposition 4. *Assume that Y satisfies eq. (3.1.33), where the invertible matrix $\hat{R}$ obeys the characteristic eqs. (3.1.41) and (3.1.42). If the eigenvalues λ_i of $\hat{R}$ are such that*

$$\sum_{k=0}^{[\frac{M}{2}]} (-1)^{M-2k}\,\sigma_{M-2k}(\lambda)\,\lambda_j^{2k} \neq 0 \quad \forall j = 1,\ldots,M\ , \tag{3.1.45}$$

($\sigma_0 = 1$) *then, the matrix Y is proportional to the identity matrix.*
Proof. Consider a matrix A which is the sum over the even powers of $\hat{R}$:

$$A := \sum_{k=0}^{[\frac{M}{2}]} (-1)^{M-2k}\,\sigma_{M-2k}(\lambda)\,\hat{R}^{2k}\ .$$

The matrix A is invertible iff all its eigenvalues are not equal to zero and this is equivalent to the conditions (3.1.45). In this case eq. (3.1.42) can be represented in the form

$$\left[\sum_{k=0}^{[\frac{M}{2}]} (-1)^{M-2k-1}\,\sigma_{M-2k-1}(\lambda)\,\hat{R}^{2k+1}\right]\,A^{-1} = I\ .$$

Multiplying the last identity by Y_1 and Y_2 from the left and right, respectively, and using (3.1.33) we deduce the equation $Y_1 - Y_2 = 0$ which gives $Y \sim I$ if condition (3.1.45) holds. $\square$

Here we consider two special cases $M = 2, 3$ for the characteristic equation (3.1.41). By renormalizing the matrix $\hat{R}$, it is always possible to fix first two eigenvalues in (3.1.41) so that $\lambda_1 = q \neq 0$ and $\lambda_2 = -q^{-1} \neq 0$. For $M = 2$ the Eqs. (3.1.41), (3.1.42) are represented in the form of the Hecke condition

$$\begin{gathered}(\hat{R} - q)(\hat{R} + q^{-1}) = 0 \quad \Rightarrow \quad \hat{R}^2 = \lambda\,\hat{R} + I \quad \Rightarrow \\ \hat{R} - \lambda\,I - \hat{R}^{-1} = 0\ , \quad \lambda := (q - q^{-1})\ .\end{gathered} \tag{3.1.46}$$

In this case, according to Proposition 4, all matrices Y are proportional to the identity matrix if $\lambda \neq 0$. Here we have only two basis matrices (3.1.34) $(n = 0, 1)$: $Y^{(0)} = Tr(D)\,I$ and $Y^{(1)} = I$ and any other matrices (3.1.35) are represented as a linear combination of $Y^{(0)}$ and $Y^{(1)}$. E.g. the matrix $Y^{(-1)}$ can be immediately found as (see (3.1.20))

$$Y_1^{(-1)} = Q_1\, D_1 = Tr_2 D_2 \hat{R}_1^{-1} = Tr_2 D_2 \left(\hat{R}_1 - \lambda\, I\right) = (1 - \lambda\, Tr(D))\; I_1 \; .$$

It means that for the Hecke case the matrix $QD \sim I$ everywhere and the matrices D and Q are always invertible if $\lambda Tr(D) \neq 1$.

For $M = 3$ the equations (3.1.41), (3.1.42) are the Birman-Wenzl cubic relation

$$(\hat{R} - q)(\hat{R} + q^{-1})(\hat{R} - \nu) = 0 \;\Rightarrow\; \hat{R}^3 - (\lambda + \nu)\, \hat{R}^2 + (\lambda\nu - 1)\hat{R} + \nu = 0 \; .$$

According to Proposition 4 one can deduce the conditions

$$\lambda \neq 0 \; , \quad (\nu^2 + \lambda\nu - 1) \neq 0 \; .$$

Thus, for the cubic characteristic equation, in the case $\{q^2 \neq 1 \; , \;\; \nu \neq \pm q^{\mp 1}\}$ all the matrices Y (3.1.35) are proportional to the identity matrix.

Numerical R-matrix which is the image $(T \otimes T)\mathcal{R}$ of the universal $\mathcal{R}$ matrix for quasitriangular Hopf algebra is obliged to be skew-invertible. Indeed, relations (2.6.9) are the algebraic counterparts of (3.1.10) where the matrix Ψ is given by the equation

$$\Psi^{in}_{ml} = T^i_m(\alpha_\mu)\, T^n_l(S(\beta_\mu)) = \hat{\Psi}^{ni}_{ml} \; . \tag{3.1.47}$$

Moreover, the image $(T \otimes T)\mathcal{R}^{-1} = R^{-1}$ is also skew-invertible. The matrix Φ_{12} in (3.1.11) is given by

$$\Phi^{in}_{ml} = T^i_m(\delta_\mu)\, T^n_l(S(\gamma_\mu)) = \hat{\Phi}^{ni}_{ml} \; . \tag{3.1.48}$$

and the algebraic counterpart of (3.1.11) is (2.6.10).

From eqs. (3.1.47), (3.1.48) we have

$$D = T(S(\beta_\mu)\alpha_\mu) = T(u) \; , \quad \overline{D} = T(S(\gamma_\mu)\delta_\mu) = T(u_2) \; ,$$

$$Q = T(\alpha_\mu S(\beta_\mu)) = T(S(u_3)) \; , \quad \overline{Q} = T(\delta_\mu S(\gamma_\mu)) = T(S(u_4)) \; ,$$

where elements u, u_1, u_2, u_3 were discussed in Remark 2 of Subsection 2.5.

3.2 Quantum matrix algebras (RTT, reflection equation algebras, etc)

We consider an algebra $\mathcal{A}^*$ of functions on a quasitriangular Hopf algebra $\mathcal{A}$ and generators of $\mathcal{A}^*$: the identity element 1 and elements of $N \times N$ matrix $T = ||T^i_j||$ $(i, j = 1, \dots, N)$, which define N- dimensional matrix representation of $\mathcal{A}$. We will use the following notation: $f(a) = \langle a,\, f\rangle$ for the functions $f \in \mathcal{A}^*$ of elements $a \in \mathcal{A}$. For the image $R_{12} = \langle \mathcal{R},\, T_1 \otimes T_2\rangle$ of the universal matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ we deduce $\forall a \in \mathcal{A}$ the identity (by using (2.2.8))

$$R_{12}\, \langle a,\, T_1\, T_2\rangle = R_{12}\, \langle a_{(1)},\, T_1\rangle\, \langle a_{(2)},\, T_2\rangle = \langle\, \mathcal{R}_{12}\Delta(a)\, ,\, T_1 \otimes T_2\, \rangle =$$

$$= \langle\, \Delta'(a)\mathcal{R}_{12}\, ,\, T_1 \otimes T_2\, \rangle = \langle\, \Delta'(a)\, ,\, T_1 \otimes T_2\, \rangle\, R_{12} = \langle a,\, T_2\, T_1\rangle\, R_{12} \; .$$

Since the element $a \in \mathcal{A}$ is not fixed here, one can conclude (for the nondegenerate pairing) that the elements T^i_j satisfy the following quadratic relations (RTT relations):

$$R^{i_1 i_2}_{j_1 j_2} \, T^{j_1}_{k_1} \, T^{j_2}_{k_2} = T^{i_2}_{j_2} \, T^{i_1}_{j_1} \, R^{j_1 j_2}_{k_1 k_2} \Leftrightarrow R_{12} T_1 T_2 = T_2 T_1 R_{12} \Leftrightarrow \hat{R} \, T_1 \, T_2 = T_1 \, T_2 \, \hat{R} \; , \quad (3.2.1)$$

where the indices 1 and 2 label the matrix spaces and the matrix R_{12} satisfies Yang-Baxter equation (3.1.1), (3.1.3).

In the case of nontrivial R matrices satisfying (3.1.1) the relations (3.2.1) define a noncommutative quadratic algebra (as the algebra of functions with the generators $\{1, \, T^i_j\}$) which is called RTT algebra. We stress that one can consider RTT algebra (3.2.1) determined by the Yang-Baxter R matrix which is not in general the image of any universal $\mathcal{R}$ matrix. The Yang-Baxter equation for R is necessary to ensure that on monomials of third degree in T no relations additional to (3.2.1) arise. We shall assume that R_{12} is a skew-invertible matrix. In this case matrices D and Q (3.1.12) define 1-dimensional representations $\rho_D(T^i_j) = D^i_j$ and $\rho_Q(T^i_j) = Q^i_j$ for RTT algebra (3.2.1) (see (3.1.28)). In some cases below we also assume that R_{12} is a lower triangular block matrix and its elements depend on the numerical parameter $q = exp(h)$, which is called the deformation parameter.

Suppose that the RTT algebra can be extended in such a way that it also contains all elements $(T^{-1})^i_j$:

$$(T^{-1})^i_k \, T^k_j = T^i_k \, (T^{-1})^k_j = \delta^i_j \cdot 1 \; .$$

Then this algebra becomes a Hopf algebra with structure mappings

$$\Delta(T^i_k) = T^i_j \otimes T^j_k \; , \;\; \epsilon(T^i_j) = \delta^i_j \; , \;\; S(T^i_j) = (T^{-1})^i_j \; , \qquad (3.2.2)$$

which, as is readily verified, satisfy the standard axioms (see the previous section):

$$\begin{array}{c} (id \otimes \Delta)\Delta(T^i_j) = (\Delta \otimes id)\Delta(T^i_j) \; , \\ (\epsilon \otimes id)\Delta(T^i_j) = (id \otimes \epsilon)\Delta(T^i_j) = T^i_j \; , \\ m(S \otimes id)\Delta(T^i_j) = m(id \otimes S)\Delta(T^i_j) = \epsilon(T^i_j)1 \; . \end{array} \qquad (3.2.3)$$

The antipode S is not an involution, since instead of $S^2 = id$ we have equation

$$S^2(T^i_j) \, D^j_l = D^i_k \, T^k_l \; , \qquad (3.2.4)$$

which can be rewritten in the form

$$D^j_l T^l_k S(T^i_j) = D^i_k \; , \qquad (3.2.5)$$

and the matrix D has been defined in (3.1.12). The relations (3.2.4) and (3.2.5) can be interpreted as the rules of permutation of the operations of taking the inverse matrix and the transposition (t):

$$D^t (T^{-1})^t = (T^t)^{-1} D^t \qquad (3.2.6)$$

To prove (3.2.4) we note that the RTT relations (3.2.1) can be represented in the form

$$T_1^{-1} \, \hat{R}_{12} \, T_1 = T_2 \, \hat{R}_{12} \, T_2^{-1} \; .$$

We multiply this relation by $\hat{\Psi}_{01}$ from the left and by $\hat{\Psi}_{23}$ from the right and take the traces $Tr_{12}(\dots)$. After this, taking into account eqs. (3.1.10), we arrive at the relations:

$$Tr_1\left(\hat{\Psi}_{01}\, T_1^{-1}\, P_{13}\, T_1\right) = Tr_2\left(T_2\, P_{02}\, T_2^{-1}\, \hat{\Psi}_{23}\right) \; .$$

Acting on this relations by $Tr_3(\dots)$ and $Tr_0(\dots)$ we obtain, respectively

$$D_0 = Tr_2 T_2\, P_{02}\, T_2^{-1}\, D_2 \; , \quad Tr_1 Q_1\, T_1^{-1}\, P_{13}\, T_1 \; = Q_3 \; . \tag{3.2.7}$$

The first equation in (3.2.7) is identical to (3.2.4), (3.2.5) while the second one gives

$$S(T^j_i)\, T^k_l\, Q^l_j = Q^k_i \; . \tag{3.2.8}$$

The matrices D^i_j and Q^i_j (3.1.12) entering the conditions (3.2.7) define *the quantum traces* [17],[36]. To explain the concept of the quantum trace, we consider the N^2-dimensional adjoint $\mathcal{A}^*$-comodule E (in what follows we continue to use the concise notation $\mathcal{A}^*$ for the RTT algebra). We represent its basis elements in the form of an $N \times N$ matrix $E = ||E^i_j||$, $i, j = 1, ..., N$. The adjoint coaction is

$$E^i_j \to T^i_{i'} S(T^{j'}_j) \otimes E^{i'}_{j'} =: (TET^{-1})^i_j \; , \tag{3.2.9}$$

where in the final part of the expression (3.2.9) we have introduced abbreviations that we shall use in what follows (we should only remember that the elements E^i_j commute with the elements T^k_m). We note that there is a different form of the adjoint coaction:

$$E^i_j \to E^{i'}_{j'} \otimes S(T^i_{i'}) T^{j'}_j \; =: (T^{-1}ET)^i_j \; . \tag{3.2.10}$$

One can check that the elements E^i_j (3.2.9) and (3.2.10) form, respectively, left and right comodules. The matrix $||T^i_j||$ is interpreted now as the matrix of linear noncommutative transformations. Both left and right comodules E are reducible, and the irreducible subspaces in E can be distinguished by means of the quantum traces. For the case (3.2.9), the quantum trace has the form

$$Tr_D E := Tr(DE) \equiv \sum_{i,j=1}^{N} D^i_j E^j_i \tag{3.2.11}$$

and satisfies the following invariance property, which follows from Eqs. (3.2.4), (3.2.5) and first relation in (3.2.7):

$$Tr_D(TET^{-1}) = Tr_D(E) \; , \tag{3.2.12}$$

For the case (3.2.10), the definition of the quantum trace must be changed to

$$Tr_Q E := Tr(Q\, E) \equiv \sum_{i,j=1}^{N} Q^i_j\, E^j_i \; , \;\; Tr_Q(T^{-1}ET) = Tr_Q(E) \; , \tag{3.2.13}$$

this follows from the second relation in (3.2.7). Thus, $Tr_D(E)$ and $Tr_Q(E)$ are, respectively, the scalar parts of the comodules E (3.2.9) and (3.2.10), whereas the q-traceless part of E generates (N^2-1)-dimensional (reducible in the general case and irreducible in the case of linear quantum groups) $\mathcal{A}^*$-adjoint comodules. Note that, if the matrix D is invertible, one can substitute $Q \to const \cdot D^{-1}$ in (3.2.13) since eq. (3.2.5) is rewritten in the form $(D^{-1})^k_i = S(T^j_i)\, T^k_l\, (D^{-1})^l_j$ (cf. (3.2.8)).

An important consequence of the definition of the quantum trace (3.2.12)-(3.2.13) and the RTT relations (3.2.1) is the fact that

$$T_1^{-1} Tr_{D(2)}(X(\hat{R}))T_1 = Tr_{D(2)}(X(\hat{R}))\ , \ \ T_2\, Tr_{Q(1)}(X(\hat{R}))\, T_2^{-1} = Tr_{Q(1)}(X(\hat{R})) \tag{3.2.14}$$

[here $X(.)$ is an arbitrary function, and $Tr_{Q(1)}$, $Tr_{D(2)}$ are the quantum traces over the first and second space, respectively]. Eq. (3.2.14) indicates that the matrices $Y_2 = Tr_{D(3)}(X(\hat{R}_2)) = Tr_{Q(1)}(X(\hat{R}_1))$ (see (3.1.35)) must be proportional to the identity matrix if T^i_j defines an irreducible representation of the quantum group $\mathcal{A}$ (otherwise, the projectors constructed from the matrices Y will define the invariant subspaces for the representation T^i_j, see (3.2.14)). In particular, we must have

$$Tr_{D(3)}(\hat{R}^k_{23}) = Tr_{Q(1)}(\hat{R}^k_{12}) = c_k I_2\ , \tag{3.2.15}$$

where c_k are certain constants, e.g. $c_1 = 1$ (3.1.14), (3.1.15), and I_k is the identity matrix in the k-th space. Note that a direct consequence of (3.1.20) is

$$Tr_{D(2)}(\hat{R}^{-1}_{12}) = c_{-1} \cdot I_1 = D_1\, Q_1\ , \tag{3.2.16}$$

and for $c_{-1} \neq 0$ matrices D, Q are invertible. As we will see below, for the quantum groups of the classical series the fact (3.2.15) does indeed hold. In what follows, we shall attempt to restrict consideration to either left or right adjoint comodules with quantum traces (3.2.11) or (3.2.13). The analogous relations for right or left comodules, respectively, can be considered in exactly the same way.

It can be seen from comparison of the relations (3.2.1) and (3.1.1), (3.1.3) that for the generators T^i_j it is possible to choose the following finite-dimensional matrix representations:

$$(T^i_j)^k_l = R^{ik}_{jl}\ , \quad (T^i_j)^k_l = (R^{-1})^{ki}_{lj}\ . \tag{3.2.17}$$

Since the R matrix satisfies the Yang-Baxter equation, there exist linear functionals $(L^{\pm})^i_j \in \mathcal{A}$ that realize the homomorphisms (3.2.17), i.e., we have

$$\langle L^+_2, T_1 \rangle = R_{12} := R^{(+)}_{12}\ , \ \ \langle L^-_2, T_1 \rangle = R^{-1}_{21} := R^{(-)}_{12}\ , \tag{3.2.18}$$

For the case $R_{12} = \langle \mathcal{R}, T_1 \otimes T_2 \rangle$ we immediately construct the mapping from $\mathcal{A}^*$ to $\mathcal{A}$ (see, for example, [36, 46])

$$\begin{gathered} \langle \mathcal{R}\, ,\, id \otimes T^i_j \rangle = (L^+)^i_j\ , \quad \langle \mathcal{R}\, ,\, S(T^i_j) \otimes id \rangle = (L^-)^i_j\ , \\ \langle \mathcal{R}\, ,\, T^i_j \otimes id \rangle = S((L^-)^i_j)\ . \end{gathered} \tag{3.2.19}$$

Eqs. (3.2.18) are generalized in the following form

$$\langle L^{\pm}_f, T_1\, T_2 \dots T_k \rangle = R^{(\pm)}_{1f}\, R^{(\pm)}_{2f} \dots R^{(\pm)}_{kf}\ .$$

The Yang-Baxter equation (3.1.3) can now be reproduced from the RTT relations (3.2.1) by averaging them with the $L^{\pm}$ operators.

From the requirement that the $(L^{\pm})^i_j \in \mathcal{A}$ form the algebra that is the dual of the algebra $\mathcal{A}^*$ (the definition of the dual algebra is given in Def. 6, Subsec. 2.3), we obtain the following commutation relations for the generators $L^{(\pm)}$:

$$\hat{R}_{12}\, L_2^{\pm}\, L_1^{\pm} = L_2^{\pm}\, L_1^{\pm}\, \hat{R}_{12} \; , \tag{3.2.20}$$

$$\hat{R}_{12}\, L_2^{+}\, L_1^{-} = L_2^{-}\, L_1^{+}\, \hat{R}_{12} \; . \tag{3.2.21}$$

The same equations are obtained from the universal Yang-Baxter equation (2.3.11) by the averaging it with $(T_1 \otimes T_2 \otimes id)$, $(id \otimes T_1 \otimes T_2)$, $(T_1 \otimes id \otimes T_2)$ and using (3.2.19). The algebra (3.2.20), (3.2.21) is obviously a Hopf algebra with comultiplication, antipode, and coidentity:

$$\triangle (L^{\pm})^i_j = (L^{\pm})^i_k \otimes (L^{\pm})^k_j \; , \;\; S(L^{\pm}) = (L^{\pm})^{-1} \; , \tag{3.2.22}$$

$$\epsilon((L^{\pm})^i_j) = \langle (L^{\pm})^i_j , 1 \rangle = \delta^i_j \; , \tag{3.2.23}$$

where we have assumed that the matrices $L^{\pm}$ are invertible. As was shown in Ref. [17], for the R matrices of the quantum groups of the classical series A_N, B_n, C_n, D_n (respectively, $SL_q(N)$, $SO_q(2n+1)$, $Sp_q(2n)$, $SO_q(2n)$), the relations (3.2.20), (3.2.21) define quantum universal enveloping Lie algebras in which the elements $(L^{\pm})^i_j$ play the role of the quantum analog of the Cartan-Weyl generators (the case of $SL_q(N)$ will be analyzed in Sec. 3.4).

One can consider (see, e.g., [38]) the Hopf algebra (3.2.20) – (3.2.23) as a Drinfeld double of two dual Hopf subalgebras $\mathcal{B}^{\pm}$ with generators $(L^{\pm})^i_j$, defining relations (3.2.20) and structure maps (3.2.23), and (cf. (3.2.22))

$$\begin{gathered} \triangle (L^{+})^i_j = (L^{+})^i_k \otimes (L^{+})^k_j \; , \quad \triangle_{\text{op}} (L^{-})^i_j = (L^{-})^k_j \otimes (L^{-})^i_k \; , \\ S(L^{+}) = (L^{+})^{-1} \; , \;\; S_{\text{op}}^{-1}(L^{-}) = (L^{-})^{-1} \; , \end{gathered} \tag{3.2.24}$$

In this case, the Hopf algebras $\mathcal{B}^{+}$ and $\mathcal{B}^{-}$ are dual to one another with respect to the pairing [38]

$$\langle\langle L_1^{-} , L_2^{+} \rangle\rangle = R_{12}^{-1} \; . \tag{3.2.25}$$

We denote by $\mathcal{B}^{-O}$ the Hopf algebra with generators $(L^{-})^i_j$, and with comultiplication and antipode (3.2.22) opposite to that of (3.2.24). The algebras $\mathcal{B}^{+}$ and $\mathcal{B}^{-O}$ are antidual with respect to the pairing (3.2.25). As was shown in Sec. 2.4, from antidual Hopf algebras $\mathcal{B}^{+}$ and $\mathcal{B}^{-O}$ it is possible to construct a Drinfeld quantum double $\mathcal{B}^{+} \bowtie \mathcal{B}^{-O}$, for which the crossed commutation relations have the form (3.2.21). Thus, for the algebras $\mathcal{B}^{\pm}$ in (3.2.20), one can propose a special cross-product (quantum Drinfeld double), given by (3.2.21), which is again a Hopf algebra (with structure mapings (3.2.22), (3.2.23)), and which was used in Ref. [17] for R-matrix formulation of quantum deformations of the universal enveloping Lie algebras.

Note that the algebra (3.2.20), (3.2.21) is a covariant algebra with respect to the left and right cotransformations

$$\begin{gathered} (L^{\pm})^i_j \to (T^{-1})^k_j \otimes (L^{\pm})^i_k \equiv (L^{\pm}\, T^{-1})^i_j \; , \\ (L^{\pm})^i_j \to (L^{\pm})^k_j \otimes (T^{-1})^i_k \equiv (T^{-1}\, L^{\pm})^i_j \; , \end{gathered} \tag{3.2.26}$$

(we forget here for a moment that the matrices T and $L^{\pm}$ could have the different triangular properties). Thus, the matrices

$$L^i_j = (S(L^-)L^+)^i_j \; , \quad \bar{L}^i_j = (L^+ S(L^-))^i_j \; , \tag{3.2.27}$$

realize, respectively, the left and right adjoint comodules (3.2.9) and (3.2.10). It is readily verified that the coinvariants

$$C_M = Tr_D\left(L^M\right) \; , \quad \overline{C}_M = Tr_Q\left(\bar{L}^M\right) \tag{3.2.28}$$

are central elements (see also [17]) for the algebra (3.2.20), (3.2.21) (below we show that $C_M = \overline{C}_M$ for the realizations (3.2.27)). Indeed, one can obtain from (3.2.20), (3.2.21) the relations

$$L_2^M \, L_1^{\pm} = L_1^{\pm} \, \hat{R}^{\pm 1} \, L_1^M \, \hat{R}^{\mp 1} \; , \quad L_2^{\mp} \, \bar{L}_1^M = \hat{R}^{\pm 1} \, \bar{L}_2^M \, \hat{R}^{\mp 1} \, L_2^{\mp} \; , \tag{3.2.29}$$

where $\hat{R} := \hat{R}_{12}$. Then, by taking the traces $Tr_{D(2)}$, $Tr_{Q(1)}$ of the first and second relations (3.2.29) and using (3.1.30) we demonstrate the centrality of the elements (3.2.28)

The equality $C_M = \overline{C}_M$ for the elements (3.2.28) (where L and $\overline{L}$ are composed from $L^{\pm}$ (3.2.27)) is proved as follows:

$$Tr_D\left(L^M\right) = Tr_{D(2)}\left(S(L_2^-)\, \bar{L}_2^M \, L_2^-\right) = Tr_{Q(1)} Tr_{D(2)}\left(S(L_2^-)\, \hat{R}\, \bar{L}_2^M \, L_2^-\right) =$$

$$= Tr_{Q(1)} Tr_{D(2)}\left(\bar{L}_1^M \, S(L_2^-)\, \hat{R}\, L_2^-\right) = Tr_{Q(1)} Tr_{D(2)}\left(\bar{L}_1^M \, L_1^- \, \hat{R}\, S(L_1^-)\right) = Tr_Q\left(\bar{L}^M\right) \; ,$$

where we have used Eqs. (3.1.14), (3.1.15), (3.2.20), (3.2.21), (3.2.29).

Note also that the generators L^i_j and $\bar{L}^i_j$ satisfy equations

$$\hat{R}_{12}\, L_1 \, \hat{R}_{12}\, L_1 = L_1 \, \hat{R}_{12}\, L_1 \, \hat{R}_{12} \; , \tag{3.2.30}$$

$$\hat{R}_{12}\, \bar{L}_2 \, \hat{R}_{12}\, \bar{L}_2 = \bar{L}_2 \, \hat{R}_{12}\, \bar{L}_2 \, \hat{R}_{12} \; , \tag{3.2.31}$$

(they are the special limits of the reflection equations which will be considered in the last Section of this paper). Algebras with generators L^i_j and $\bar{L}^i_j$ are called left and right reflection equation algebras, since the defining relations (3.2.30), (3.2.31) of these algebras are covariant under the left and right co-transformations (3.2.9), (3.2.10). A set (which is incomplete in general; see below) of central elements for these algebras are represented by the same formulas as in (3.2.28). Indeed, one can deduce from (3.2.30), (3.2.31) the relations

$$L_1 \, \hat{R}_{12}\, L_1^M \, \hat{R}_{12}^{-1} = \hat{R}_{12}^{-1}\, L_1^M \, \hat{R}_{12}\, L_1 \; , \quad \bar{L}_2 \, \hat{R}_{12}\, \bar{L}_2^M \, \hat{R}_{12}^{-1} = \hat{R}_{12}^{-1}\, \bar{L}_2^M \, \hat{R}_{12}\, \bar{L}_2 \; .$$

Then, taking the quantum traces $Tr_{D(2)}(\ldots)$ and $Tr_{Q(1)}(\ldots)$ of the first and second relations and using (3.1.30) we prove the centrality of the elements (3.2.28).

The algebra (3.2.30) (and similarly the second algebra (3.2.31)) decomposes into the direct sum of two subalgebras, namely, into an abelian algebra with generator $C_1 = Tr_D(L)$ and an algebra with (N^2-1) traceless generators $\tilde{L}^i_j$ (we assume that the RE algebra is extended by the element C_1^{-1}):

$$L^i_j = \frac{C_1}{Tr_D(I)}\left(\delta^i_j + \lambda\, \tilde{L}^i_j\right) \;\Rightarrow\; \tilde{L}^i_j = \frac{1}{\lambda}\left[\frac{Tr_D(I)}{C_1} L^i_j - \delta^i_j\right] \; , \tag{3.2.32}$$

where the factor $\lambda = q - q^{-1}$ is introduced to ensure that the operators $\tilde{L}$ have the correct classical limit when the deformation parameter $q \to 1$. For the last algebra, it is easy to obtain the commutation relations

$$\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12} = \frac{1}{\lambda}(\hat{R}_{12}^2\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}^2)\;, \tag{3.2.33}$$

which can be regarded (for an arbitrary R matrix satisfying the Yang-Baxter equation) as a deformation of the commutation relations for Lie algebras. For the Hecke type R-matrix (3.1.46), the relations (3.2.33) are equivalent to

$$\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\,\tilde{L}_1\,\hat{R}_{12} = \hat{R}_{12}\,\tilde{L}_1 - \tilde{L}_1\,\hat{R}_{12}\;, \tag{3.2.34}$$

and corresponding algebra has a projector type representation: $(\tilde{L}^i_j)^k_m = A^{ik}B_{jm}$, where numerical matrices A and B are such that $B_{jk}A^{ik} = Q^i_j$ (for any Q-matrix which satisfies $Tr_1 Q_1 \hat{R}_{12} = I_2$; see (3.1.14)).

The relations (3.2.30), (3.2.31) and (3.2.33) are extremely important and arise, for example, in the construction of a differential calculus on quantum groups as the commutation relations for invariant vector fields (see [39] – [46] and refs. therein).

Note that, instead of (3.2.19), one can use a somewhat different linear mapping from $\mathcal{A}^*$ to $\mathcal{A}$ [23, 36, 46, 48, 49] (which is completely determined by (3.2.19)):

$$\langle \sigma(\mathcal{R})\,\mathcal{R}\,,\,id\otimes a\rangle = \alpha \quad (a \in \mathcal{A}^*\;, \alpha \in \mathcal{A})\;, \tag{3.2.35}$$

where $\sigma(a\otimes b) = (b\otimes a)$, $\forall a,b \in \mathcal{A}$. The explicit calculations give

$$\begin{gathered}
\langle \sigma(\mathcal{R})\,\mathcal{R}\,,\,id\otimes T^i_j\rangle = L^i_j\;,\\
\langle \sigma(\mathcal{R})\,\mathcal{R}\,,\,id\otimes T_1\,T_2\rangle = S(L^-_1)\,L_2\,L^+_1 = L_1\,\hat{R}_1\,L_1\,\hat{R}_1^{-1}\;,\\
\langle \sigma(\mathcal{R})\,\mathcal{R}\,,\,id\otimes T_1\,T_2\,T_3\rangle = S(L^-_1)\,S(L^-_2)\,L_3\,L^+_2\,L^+_1 = L_{\underline{1}}\,L_{\underline{2}}\,L_{\underline{3}} \equiv L_{\overline{3}}\,L_{\overline{2}}\,L_{\overline{1}}\;,\\
\dots\dots\dots\dots\;,\\
\langle \sigma(\mathcal{R})\,\mathcal{R}\,,\,id\otimes T_1\dots\,T_k\rangle = L_{\underline{1}}\,L_{\underline{2}}\dots L_{\underline{k}} \equiv L_{\overline{k}}\dots L_{\overline{2}}\,L_{\overline{1}}\;,
\end{gathered} \tag{3.2.36}$$

where $L_{\underline{k+1}} = \hat{R}_k\,L_{\underline{k}}\,\hat{R}_k^{-1}$, $L_{\overline{k+1}} = \hat{R}_k^{-1}\,L_{\overline{k}}\,\hat{R}_k$, $(L_{\underline{1}} = L_{\overline{1}} = L_1)$ and we have used eqs. (2.3.8), (3.2.19) and (3.2.29). If we confine ourselves to the fairly general case of quasitriangular Hopf algebras $\mathcal{A}$, for which the mapping (3.2.35) is invertible (such Hopf algebras are said to be factorizable [48]), one can rewrite (using relations (3.2.35)) the identities for the RTT algebra into the identities for the reflection equation algebra and vice versa (for more details see [49]).

In view of (3.2.36) one can represent the reflection equation algebra (3.2.30) in the "universal" form

$$\mathcal{R}_{32}\,(\mathcal{R}_{31}\mathcal{R}_{13})\,\mathcal{R}_{23}\,(\mathcal{R}_{21}\,\mathcal{R}_{12}) = (\mathcal{R}_{21}\,\mathcal{R}_{12})\,\mathcal{R}_{32}\,(\mathcal{R}_{31}\mathcal{R}_{13})\,\mathcal{R}_{23}\;,$$

where the notation $\mathcal{R}_{ij}$ has been introduced in (2.3.10). The pairing of this relation with $(id\otimes T\otimes T)$ gives (3.2.30). The algebra (3.2.31) has an analogous representation.

We note that the identity (which has been obtained in (3.2.36))

$$L_{\underline{1}}\,L_{\underline{2}}\dots L_{\underline{k}} = L_{\overline{k}}\dots L_{\overline{2}}\,L_{\overline{1}}\;, \tag{3.2.37}$$

valid in more general case of any reflection equation algebra (3.2.30) (even not realized in the form (3.2.27)). Below we also use the following identity (which can be proved by induction)

$$L_{\underline{k+1}}\, L_{\underline{k+2}} \dots L_{\underline{k+n}} = U_{(k,n)}\, L_{\underline{1}}\, L_{\underline{2}} \dots L_{\underline{n}}\, U^{-1}_{(k,n)} \; . \tag{3.2.38}$$

where the operator $U_{(k,n)}$ is represented as a product of k or n factors:

$$U_{(k,n)} = \hat{R}_{(k\to n+k-1)} \dots \hat{R}_{(2\to n+1)}\, \hat{R}_{(1\to n)} \equiv \hat{R}_{(k\leftarrow 1)}\, \hat{R}_{(k+1\leftarrow 2)} \dots \hat{R}_{(n+k-1\leftarrow n)} \; , \tag{3.2.39}$$

$$\hat{R}_{(k\leftarrow m)} := \hat{R}_k\, \hat{R}_{k-1} \cdots \hat{R}_m \; , \quad \hat{R}_{(m\to k)} := \hat{R}_m\, \hat{R}_{m+1} \cdots \hat{R}_k \; .$$

Now the description of more general set of central elements is in order.

Proposition 5. *Let X be an arbitrary element of the group algebra of the braid group $\mathcal{B}_m$ in R-matrix representation (see (3.1.6)) generated by a skew-invertible R-matrix. Then, the elements*

$$\sigma_m(X) = Tr_{\mathcal{D}(1\dots m)} \Big(X_{(1\to m)}\, L_{\underline{1}}\, L_{\underline{2}} \dots L_{\underline{m}} \Big) \; , \tag{3.2.40}$$

belong to a center $Z(L)$ of the RE-algebra (3.2.30).

Proof. First of all we note that $\sigma_m(X)$ (3.2.40) are represented in the forms

$$\begin{array}{c} \sigma_m(X)\, I_1 = Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, L_{\underline{2}}\, L_{\underline{3}} \dots L_{\underline{m+1}} \Big) = \\ = Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, L_{\overline{m+1}} \dots L_{\overline{3}}\, L_{\overline{2}} \Big) \; . \end{array} \tag{3.2.41}$$

The first representation follows from the chain of relations

$$\begin{array}{c} Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, L_{\underline{2}} \dots L_{\underline{m+1}} \Big) = \\ = Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, \hat{R}_1 \cdots \hat{R}_m\, L_{\underline{1}} \dots L_{\underline{m}}\, \hat{R}^{-1}_m \cdots \hat{R}^{-1}_1 \Big) = \\ = Tr_{\mathcal{D}(2\dots m+1)} \Big(\hat{R}_1 \cdots \hat{R}_m\, (X_{(1\to m)}\, L_{\underline{1}} \dots L_{\underline{m}})\, \hat{R}^{-1}_m \cdots \hat{R}^{-1}_1 \Big) = \\ = Tr_{\mathcal{D}(2\dots m)} \Big(\hat{R}_1 \cdots \hat{R}_{m-1}\, [Tr_{\mathcal{D}(m)}(X_{(1\to m)}\, L_{\underline{1}} \dots L_{\underline{m}})]\, \hat{R}^{-1}_{m-1} \cdots \hat{R}^{-1}_1 \Big) = \\ = \dots = Tr_{\mathcal{D}(1\dots m)} \Big(X_{(1\to m)}\, L_{\underline{1}}\, L_{\underline{2}} \dots L_{\underline{m}} \Big) \; , \end{array}$$

where we have applied (3.1.30) many times. The second form in (3.2.41) is proved in the same way, or by using the generalization of the identity (3.2.37)

$$L_{\underline{m}}\, L_{\underline{m+1}} \dots L_{\underline{k}} = L_{\overline{k}} \dots L_{\overline{m+1}}\, L_{\overline{m}} \quad (m < k) \; .$$

Then, the proof of the commutativity of the arbitrary generator of the RE algebra (3.2.30) with elements $\sigma_m(X)$ is straightforward

$$L_1\, \sigma_m(X) = Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, L_{\underline{1}} L_{\underline{2}}\, L_{\underline{3}} \dots L_{\underline{m+1}} \Big) =$$

$$= Tr_{\mathcal{D}(2\dots m+1)} \Big(X_{(2\to m+1)}\, L_{\overline{m+1}} \dots L_{\overline{2}}\, L_{\overline{1}} \Big) = \sigma_m(X)\, L_1 \; .$$

■

Remark. If elements X, in the definition of central generators (3.2.40), are idempotents in the group algebra of $\mathcal{B}_m$, then, these central elements form a basis in $Z(L)$ and $\sigma_m(X)$ are interpreted as characters of the irreducible representations which correspond to the idempotents X.

Note, that "power sums" (3.2.28) generate a subset in $Z(L)$. Indeed, the substitution of $X = \hat{R}_{(m-1\leftarrow 1)} := \hat{R}_{m-1} \dots \hat{R}_1$ in (3.2.40) gives

$$\sigma_m(X) = Tr_{\mathcal{D}(1\dots m)} \Big(L_{\underline{1}} \dots L_{\underline{m-1}} \, (\hat{R}_{(m-1\leftarrow 1)} L_1 \hat{R}^{-1}_{(m-1\leftarrow 1)}) \, \hat{R}_{(m-1\leftarrow 1)} \Big) =$$

$$= Tr_{\mathcal{D}(1\dots m)} \Big(L_{\underline{1}} \dots L_{\underline{m-2}} \, (\hat{R}_{(m-2\leftarrow 1)} L_1 \hat{R}^{-1}_{(m-2\leftarrow 1)}) \hat{R}_{m-1} \hat{R}_{(m-2\leftarrow 1)} L_1 \Big) = \qquad (3.2.42)$$

$$= Tr_{\mathcal{D}(1\dots m-1)} \Big(L_{\underline{1}} \dots L_{\underline{m-2}} \, \hat{R}_{(m-2\leftarrow 1)} L_1^2 \Big) = \dots = Tr_{D(1)}(L_1^m) = s_m \ ,$$

where in the first line we have used the cyclic property of the quantum trace (3.1.31) and in the second line we apply (3.1.14).

Now we discuss the set of commuting elements in the RTT algebra (3.2.1). For this algebra one can construct [47] the following mutually commuting elements:

$$Q_k = Tr_{Y(1\dots k)}(\hat{R}_{(k-1\leftarrow 1)} \, T_1 \, T_2 \cdots T_k) = Tr_{Y(1\dots k)}(\hat{R}_{(1\to k-1)} \, T_1 \, T_2 \cdots T_k) \ , \qquad (3.2.43)$$

where

$$Tr_{Y(1\dots k)}(X) := Tr_1 \dots Tr_k(Y_1 \dots Y_k \, X) \ ,$$

and the matrices Y are such that $Y_1 Y_2 \hat{R}_1 = \hat{R}_1 Y_1 Y_2$ (e.g. $Y = D$). The second equality in (3.2.43) is obtained as follows:

$$\begin{array}{c} Tr_{Y(1\dots k)}(\hat{R}_1 \cdots \hat{R}_{k-1} \, T_1 \cdots T_k) = Tr_{Y(1\dots k)}(\hat{R}_{k-1} \, T_1 \cdots T_k \, \hat{R}_1 \cdots \hat{R}_{k-2}) = \\ = Tr_{Y(1\dots k)}(\hat{R}_1 \cdots \hat{R}_{k-3} \, \hat{R}_{k-1} \, \hat{R}_{k-2} \, T_1 \cdots T_k \,) = \\ = Tr_{Y(1\dots k)}(\hat{R}_{k-1} \, \hat{R}_{k-2} \, T_1 \cdots T_k \, \hat{R}_1 \cdots \hat{R}_{k-3}) = \\ = \dots = Tr_{Y(1\dots k)}(\hat{R}_{k-1} \, \hat{R}_{k-2} \cdots \hat{R}_1 \, T_1 \cdots T_k) \ . \end{array} \qquad (3.2.44)$$

Note that by means of (3.2.35) we map the elements Q_k (3.2.43) (for $Y = D$) to the central elements C_k (3.2.28) of the reflection equation algebra.

Our proof of the commutativity of the elements Q_k is based (see [50]) on the fact that there exists the operator $U_{(k,n)}$ (3.2.39) which satisfies

$$U_{(k,n)} \, \hat{R}_i \, U^{-1}_{(k,n)} = \hat{R}_{i+k} \ , \quad (i = 1, \dots, n-1) \ ,$$

$$U_{(k,n)} \, \hat{R}_{n+j} \, U^{-1}_{(k,n)} = \hat{R}_j \ , \quad (j = 1, \dots, k-1) \ .$$

Using the operator $U_{(k,n)}$ we obtain the commutativity of Q_k:

$$Q_k \, Q_n = Tr_{Y(1\dots k)}(\hat{R}_{(1\to k-1)} \, T_1 \cdots T_k) \, Tr_{Y(1\dots n)}(\hat{R}_{(1\to n-1)} \, T_1 \cdots T_n) =$$

$$= Tr_{Y(1\dots k+n)}(\hat{R}_{(1\to k-1)} \, \hat{R}_{(k+1\to k+n-1)} \, T_1 \cdots T_{k+n}) =$$

$$= Tr_{Y(1\dots k+n)}(U_{(k,n)} \, \hat{R}_{(n+1\to n+k-1)} \, \hat{R}_{(1\to n-1)} \, U^{-1}_{(k,n)} \, T_1 \cdots T_{k+n}) = \qquad (3.2.45)$$

$$= Tr_{Y(1\dots k+n)}(\hat{R}_{(1\to n-1)} \, \hat{R}_{(n+1\to n+k-1)} \, U^{-1}_{(k,n)} \, T_1 \cdots T_{k+n} \, U_{(k,n)}) = Q_n \, Q_k$$

In fact, using the same method as in (3.2.45), one can prove [50] that the set of commuting elements in the RTT algebra is wider then the set (3.2.43) and consist of all elements of the form

$$Q_k(X) = Tr_{Y(1\dots k)} \Big(X(\hat{R}_1, \dots \hat{R}_{k-1})\, T_1\, T_2 \cdots T_k \Big) \tag{3.2.46}$$

where $X(\dots)$ are basis elements of the braid group algebra with generators $\{\hat{R}_i\}$ $(i = 1, \dots, k-1)$. However, for the Hecke type R matrices (3.1.46) the set of elements (3.2.43) is complete and all $Q_k(X)$ are expressed as the polynomials of the commuting variables $\{Q_1, \dots, Q_k\}$. These polynomials (with some additional constraints related to Markov moves for the braids X, see Sect. 1 in [113]) could be considered as link polynomials. On the other hand, eq. (3.2.46) defines characters for representations of $\mathcal{A}$ (and corepresentations of $\mathcal{A}^*$) characterized by special choices of the elements $X(\dots)$ being central idempotents in the braid group algebra generated by $\{\hat{R}_i\}$ $(i = 1, \dots, k-1)$.

Since the RTT algebra $\mathcal{A}^*$ (3.2.1), (3.2.2) and the quantum algebra $\mathcal{A}$ (3.2.20), (3.2.21), (3.2.22), (3.2.23) are Hopf dual to each other (with respect to the pairing (3.2.18)) one can define the left and right Heisenberg doubles (HD) of these algebras. Their cross-multiplication rules (2.5.1), (2.5.2) are written in the form

$$L^+_1\, T_2 = T_2 R_{21} L^+_1 \;, \quad L^-_1\, T_2 = T_2 R^{-1}_{12} L^-_1 \;, \tag{3.2.47}$$

for the left HD and for the right one we have

$$T_1 L^+_2 = L^+_2 R_{12} T_1 \;, \quad T_1 L^-_2 = L^-_2 R^{-1}_{21} T_1 \;. \tag{3.2.48}$$

The corresponding cross products of the RTT algebra and the reflection equation algebras (3.2.27), (3.2.30), (3.2.31) are described by the cross-multiplication rules

$$\overline{L}_1\, T_2 = T_2 \hat{R}_{12} \overline{L}_2 \hat{R}_{12} \;, \quad T_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12} T_1 \tag{3.2.49}$$

in the case of the left (3.2.47) and right (3.2.48) HD, respectively. A remarkable property [51] of these cross products is the existence of automorphisms of the HD algebras

$$\{T\,,\, \overline{L}\} \stackrel{\overline{m}_n}{\rightarrow} \{T\, \overline{L}^n\,,\, \overline{L}\} \;, \quad \{T\,,\, L\} \stackrel{m_n}{\rightarrow} \{L^n\, T\,,\, L\} \;, \tag{3.2.50}$$

(one can check it by induction using (3.2.1), (3.2.30), (3.2.31) and (3.2.49)). The maps $m_n, \overline{m}_n$ define discrete time evolutions on the RTT algebra. For the Hecke type R-matrices (3.1.46) the automorphisms (3.2.50) can be generalized in the form

$$\{T\,,\, \overline{L}\} \stackrel{\overline{m}'_n}{\rightarrow} \{T\, (\sum_{m=0}^{n} \bar{x}_m \overline{L}^m)\,,\, \overline{L}\} \;, \quad \{T\,,\, L\} \stackrel{m'_n}{\rightarrow} \{(\sum_{m=0}^{n} x_m L^m)\, T\,,\, L\} \;, \tag{3.2.51}$$

for any vectors $x, \bar{x} \in \mathbf{C}^{n+1}$ (it follows from the fact that any symmetric function of two variables L_1 and $\hat{R}_1 L_1 \hat{R}_1$ commutes with $\hat{R}_1$).

For the left (3.2.47) and right (3.2.48) HD one can define the adjoint reflection equation algebras, generated by the elements of transformed matrices $\overline{Y} = T\, \overline{L}^{-1}\, T^{-1}$, $Y = T^{-1}\, L^{-1}\, T$ (cf. (3.2.30), (3.2.31), (3.2.49)):

$$\hat{R}_{12}\, \overline{Y}_1\, \hat{R}_{12}\, \overline{Y}_1 = \overline{Y}_1\, \hat{R}_{12}\, \overline{Y}_1\, \hat{R}_{12} \;, \quad \hat{R}_{12}\, Y_2\, \hat{R}_{12}\, Y_2 = Y_2\, \hat{R}_{12}\, Y_2\, \hat{R}_{12} \;,$$

$$T_1 \overline{Y}_2 = \hat{R}_{12} \overline{Y}_1 \hat{R}_{12} T_1 \; , \quad Y_1 T_2 = T_2 \hat{R}_{12} \overline{Y}_2 \hat{R}_{12} \; .$$

The elements of these matrices satisfy: $[\overline{Y}_2, \, \overline{L}_1] = 0 = [Y_1, \, L_2]$.

The cross-multiplication rules (3.2.49) for the HD of the RTT and reflection equation algebras have been extensively exploited in the context of the R matrix approach to the differential calculus on quantum groups [40] – [46]. Another cross-multiplications (of the RTT and reflection equation matrix algebras) which characterized by the relations

$$\overline{L}_1 T_2 = T_2 \hat{R}_{12} \overline{L}_2 \hat{R}_{12}^{-1} \; , \quad T_1 L_2 = \hat{R}_{12} L_1 \hat{R}_{12}^{-1} T_1$$

have been also considered in various investigations [41], [43], [45] of a noncommutative differential geometry on quantum groups.

Definition 9. *Let N-dimensional vector spaces $\bar{V}_N$ and V_N be dual to each other. Let R be a Yang-Baxter numerical matrix $R \in End(V_N \otimes V_N)$ and $\mathcal{A}^*$ is corresponding RTT algebra (3.2.1). A noncommutative algebra $\mathcal{M}$ generated by components of a (k,m)-tensor $M \in V_N^{\otimes k} \otimes \bar{V}_N^{\otimes m}$ is called quantum (k,m)-tensor algebra (or $\mathcal{A}^*$-coideal algebra) if it is $\mathcal{A}^*$-comodule algebra with respect to any matrix coactions (homomorphisms) of $\mathcal{A}^*$: $\mathcal{M} \to \mathcal{M} \otimes \mathcal{A}^*$ (or $\mathcal{M} \to \mathcal{A}^* \otimes \mathcal{M}$). Quantum $(1,1)$-tensor algebra is called quantum matrix algebra.*

For example, the RTT algebra (3.2.1) and reflection equation algebras (3.2.30), (3.2.31) are quantum matrix algebras in view of the coactions (3.2.2) and (3.2.9), (3.2.10).

Now we present a definition of a more general quantum matrix algebra $\mathcal{M}(\hat{R}, \hat{F})$ generated by $(N \times N)$ matrix components M^i_j subject to the relation

$$\hat{R}_1 M_1 \hat{F}_1 M_1 \hat{F}_1 = M_1 \hat{F}_1 M_1 \hat{F}_1 \hat{R}_1 \; , \tag{3.2.52}$$

where the pair of the Yang-Baxter operators $\{\hat{R}, \hat{F}\}$ is compatible, i.e. they satisfy the conditions (cf. eqs. (2.6.8))

$$\hat{R}_1 \hat{F}_2 \hat{F}_1 = \hat{F}_2 \hat{F}_1 \hat{R}_2 \; , \quad \hat{R}_2 \hat{F}_1 \hat{F}_2 = \hat{F}_1 \hat{F}_2 \hat{R}_1 \; . \tag{3.2.53}$$

Specializing to $\hat{F} = P$ or $\hat{F} = \hat{R}$ one reproduces the RTT or reflection equation algebras, respectively. The algebras $\mathcal{M}(\hat{R}, \hat{F})$ and their modifications were discussed in [50], [52].

At the end of this subsection we introduce a notion of a coideal subalgebra of the quantum algebra (3.2.20), (3.2.21). Let R_{12} be a Yang-Baxter R matrix and there are numerical matrices G^i_j, $\overline{G}^i_j$ which satisfy the conditions

$$\begin{aligned} & R_{12} G_2 R_{12}^{t_2} G_1 = G_1 R_{21}^{t_1} G_2 R_{21}^{t_1 t_2} \; , \\ & R_{12} \overline{G}_1 R_{12}^{t_1} \overline{G}_2 = \overline{G}_2 R_{21}^{t_2} \overline{G}_1 R_{21}^{t_1 t_2} \; . \end{aligned} \tag{3.2.54}$$

Using relations (3.2.20), (3.2.21) and conditions (3.2.54) it can be shown directly that the elements of quantum matrices

$$K = L^- G \, (L^+)^t \; , \quad \overline{K} = S(L^+) \, \overline{G} \, (S(L^-))^t \; ,$$

obey the following commutation relations

$$
\begin{aligned}
R_{12}\,K_2\,R_{12}^{t_2}\,K_1 &= K_1\,R_{21}^{t_1}\,K_2\,R_{21}^{t_1 t_2}\ ,\\
R_{12}\,\overline{K}_1\,R_{12}^{t_1}\,\overline{K}_2 &= \overline{K}_2\,R_{21}^{t_2}\,\overline{K}_1\,R_{21}^{t_1 t_2}\ ,
\end{aligned}
\tag{3.2.55}
$$

which we consider as the defining relations for a new type of quantum matrix algebras $\mathcal{K}(\hat{R},G)$ and $\overline{\mathcal{K}}(\hat{R},\overline{G})$. The defining relations (3.2.55) are covariant[6] under the left and right $\mathcal{A}$-coactions:

$$
K^i_j \longrightarrow (L^-)^i_k\,(L^+)^j_n \otimes K^k_n\ ,\quad \overline{K}^i_j \longrightarrow \overline{K}^k_n \otimes S(L^+)^i_k\,S(L^-)^j_n\ . \tag{3.2.56}
$$

Thus, the unital algebras $\mathcal{K}$ and $\overline{\mathcal{K}}$ (with generators K^i_j and $\overline{K}^i_j$, respectively) are left and right $\mathcal{A}$-comodule algebras and these algebras are called coideal subalgebras of $\mathcal{A}$.

One can consider two more such algebras with generators $K' = L^+ G' (L^-)^t$ and $\overline{K}' = S(L^-)\overline{G}' S(L^+)^t$ which obey the following defining relations

$$
R_{12}^{-1}\,K'_1\,(R_{12}^{-1})^{t_1}\,K'_2 = K'_2\,(R_{21}^{-1})^{t_2}\,K'_1\,(R_{21}^{-1})^{t_1 t_2}\ ,
$$

$$
R_{12}^{-1}\,\overline{K}'_2\,(R_{12}^{-1})^{t_2}\,\overline{K}'_1 = \overline{K}'_1\,(R_{21}^{-1})^{t_1}\,\overline{K}'_2\,(R_{21}^{-1})^{t_1 t_2}\ .
$$

Note, that these relations can be obtained from (3.2.55) by the substitution $R_{12} \rightarrow R_{12}^{-1}$.

For the special case of $GL_q(N)$ R-matrices (see Sec. 3.4) the algebras (3.2.55) have been considered in [53], [55] (see also references therein). In this case the coideal subalgebras coincide with quantized enveloping algebras introduced earlier by Gavrilik and Klimyk [54].

Representation theory for compact quantum groups has been considered in [65]. In [66] a universal solution to the reflection equation has been introduced and general problems of the representation theory for the reflection equation algebra were discussed (representations and characters for some special reflection equation algebras were considered in [67]).

3.3 The semiclassical limit (Sklyanin brackets and Lie bialgebras)

We assume that the R matrix introduced in (3.2.1) has the following expansion in the limit $h \to 0$ ($q = e^h \to 1$):

$$
R_{12} = \mathbf{1} + h r_{12} + O(h^2)\ . \tag{3.3.1}
$$

Here $\mathbf{1} = I \otimes I$ denotes the $(N^2 \times N^2)$ unit matrix. One says that such R matrices have quasiclassical behavior, and r_{12} is called a classical r matrix. It is readily found from the quantum Yang-Baxter equation (3.1.6) that r_{12} satisfies the so-called classical Yang-Baxter equation

$$
[r_{12},\ r_{13} + r_{23}] + [r_{13},\ r_{23}] = 0\ . \tag{3.3.2}
$$

[6]Here the notion *covariant* is equivalent to the statement that (3.2.56) are homomorphisms for the algebras defined by (3.2.55).

Substituting the expansion (3.3.1) in the RTT relations (3.2.1), we obtain

$$[T_1, T_2] = h[T_1 T_2, r_{12}] + O(h^2) \ . \tag{3.3.3}$$

This equation demonstrates the fact that the RTT relations (3.2.1) can be interpreted as a quantization (deformation) of the classical Poisson bracket (Sklyanin bracket [18]):

$$\{T_1, T_2\} = [T_1 T_2, r_{12}] \ , \tag{3.3.4}$$

(here the elements T^i_j are commutative coordinates of some Poisson manifold). The classical Yang-Baxter equation (3.3.2) guarantees fulfillment of the Jacobi identity for the bracket (3.3.4). From the requirement of antisymmetry of the Poisson bracket (3.3.4), we obtain

$$\{T_1, T_2\} = [T_1 T_2, -r_{21}] \ . \tag{3.3.5}$$

Thus, the classical r matrix $r^{(-)}_{12} = -r_{21}$ corresponding to the representation $R^{(-)}$ (3.2.17) must also be a solution of Eq. (3.3.2), as is readily shown by making the substitution $3 \leftrightarrow 1$ in (3.3.2). On the other hand, comparing (3.3.4) and (3.3.5), we obtain

$$T_1 T_2 (r_{12} + r_{21}) = (r_{12} + r_{21}) T_1 T_2 \ . \tag{3.3.6}$$

Thus,

$$t_{12} = \frac{1}{2}(r_{12} + r_{21}) \tag{3.3.7}$$

is an invariant with respect to the adjoint action of the matrix $T_1 T_2$ (it is an ad-invariant). We introduce the new classical r matrix

$$\tilde{r}_{12} = \frac{1}{2}(r_{12} - r_{21}) \ . \tag{3.3.8}$$

Then the Sklyanin bracket can be represented in the manifestly antisymmetric form

$$\{T_1, T_2\} = [T_1 T_2, \tilde{r}_{12}] \ , \tag{3.3.9}$$

and the matrix $\tilde{r}$ (3.3.8) satisfies the modified classical Yang-Baxter equation

$$[\tilde{r}_{12}, \ \tilde{r}_{13} + \tilde{r}_{23}] + [\tilde{r}_{13}, \ \tilde{r}_{23}] = \frac{1}{4} [r_{23} + r_{32}, \ r_{13} + r_{31}] = [t_{23}, \ t_{13}] \ . \tag{3.3.10}$$

Note that the reflection equation algebras (3.2.30), (3.2.31) can also be regarded as the result of quantization of a certain Poisson structure. For example, for these algebras we have [68] (see also [45])

$$\{L_2, \ L_1\} = [L_1, \ [L_2, \ \tilde{r}_{12}]] + L_1 t_{12} L_2 - L_2 t_{12} L_1 \ ,$$

$$\{\overline{L}_2, \ \overline{L}_1\} = -[\overline{L}_1, \ [\overline{L}_2, \ \tilde{r}_{12}]] + \overline{L}_1 t_{12} \overline{L}_2 - \overline{L}_2 t_{12} \overline{L}_1 \ ,$$

where again we must assume that $[L_1 L_2, \ t_{12}] = 0 = [\overline{L}_1 \overline{L}_2, \ t_{12}]$ [cf. (3.3.6)]. On the other hand, the relations (3.2.33) in the zeroth order in h give the equations

$$[\tilde{L}_1, \ \tilde{L}_2] = [t_{12}, \ \tilde{L}_1] \ , \quad \Big([t_{12}, \ \tilde{L}_1 + \tilde{L}_2] = 0\Big) \ ,$$

and this enables us to regard (3.2.33) as a deformation of the defining relations of a Lie algebra.

Now we consider the universal enveloping U_g of a Lie algebra g with defining relations (2.2.4) as a bialgebra (2.2.5) and assume that the cocommutative comultiplication Δ (2.2.5) is quantized $\Delta \to \Delta_h$ in such a way that Δ_h is noncocommutative. The semiclassical expansion of Δ_h is[7]

$$\Delta_h(J_\alpha) = J^1_\alpha + J^2_\alpha + h\,\mu^{\beta\gamma}_\alpha\, J^1_\beta\, J^2_\gamma + h^2\,\left(\mu^{\beta_1\beta_2,\gamma}_\alpha\, J^1_{\beta_1}\, J^1_{\beta_2}\, J^2_\gamma + \mu^{\beta,\gamma_1\gamma_2}_\alpha\, J^1_\beta\, J^2_{\gamma_1}\, J^2_{\gamma_2}\right) + h^3 \dots \tag{3.3.11}$$

where $J^1_\alpha = J_\alpha \otimes 1$, $J^2_\alpha = 1 \otimes J_\alpha$, the term of zeroth order in h in (3.3.11) is the classical comultiplication (2.2.5) and $\mu^{\beta\gamma}_\alpha, \mu^{\beta_1\beta_2,\gamma}_\alpha, \dots$ are some constants. The comultiplication map (3.3.11) (as well as the opposite comultiplication Δ'_h; see (2.2.2)) should be a homomorphic map for the Lie algebra (2.2.4):

$$[\Delta_h(J_\alpha),\, \Delta_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta_h(J_\gamma)\;, \quad [\Delta'_h(J_\alpha),\, \Delta'_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta'_h(J_\gamma)\;. \tag{3.3.12}$$

Then, the subtraction of the second relation of (3.3.12) from the first one gives the following equation

$$[\Delta^-_h(J_\alpha),\, \Delta^+_h(J_\beta)] + [\Delta^+_h(J_\alpha),\, \Delta^-_h(J_\beta)] = t^\gamma_{\alpha\beta}\, \Delta^-_h(J_\gamma)\;,$$

(here we define $\Delta^-_h := \Delta_h - \Delta'_h$ and $\Delta^+_h := \frac{1}{2}(\Delta_h + \Delta'_h)$) which is rewritten (in the first order of h) as

$$[\delta(J_\alpha),\, J^1_\beta + J^2_\beta] + [J^1_\alpha + J^2_\alpha,\, \delta(J_\beta)] = t^\gamma_{\alpha\beta}\, \delta(J_\gamma)\;, \tag{3.3.13}$$

where the map δ: $g \to g \wedge g$ is

$$\delta(J_\alpha) = \delta^{\beta\gamma}_\alpha\, J_\beta \otimes J_\gamma\;, \quad \delta^{\beta\gamma}_\alpha := \mu^{\beta\gamma}_\alpha - \mu^{\gamma\beta}_\alpha\;. \tag{3.3.14}$$

Eq. (3.3.13) is nothing but the cocycle condition for $\delta^{\beta\gamma}_\alpha$:

$$\left(\delta^{\rho\mu}_\alpha\, t^\kappa_{\beta\rho} - \delta^{\rho\kappa}_\alpha\, t^\mu_{\beta\rho}\right) - \left(\delta^{\rho\mu}_\beta\, t^\kappa_{\alpha\rho} - \delta^{\rho\kappa}_\beta\, t^\mu_{\alpha\rho}\right) = t^\gamma_{\alpha\beta}\, \delta^{\mu\kappa}_\gamma\;.$$

On the other hand the structure constants $(\Delta^-)^{ij}_k = \Delta^{ij}_k - \Delta^{ji}_k$ satisfy the co-Jacobi identity

$$(\Delta^-)^{jk}_i\, (\Delta^-)^{nm}_j + (\Delta^-)^{jn}_i\, (\Delta^-)^{mk}_j + (\Delta^-)^{jm}_i\, (\Delta^-)^{kn}_j = 0\;,$$

as it is evident from the co-associativity condition (2.1.7). This identity for the comultiplication (3.3.11) in the order h^2 reduces to the co-Jacoby identity for the structure constants $\delta^{\beta\gamma}_\alpha$ (3.3.14):

$$\delta^{\beta\gamma}_\alpha\, \delta^{\rho\xi}_\beta + (\text{cycle}\;\; \gamma, \rho, \xi) = 0\;. \tag{3.3.15}$$

Thus, we have arrived to the following definition [22]:

Definition 10. *The vector space g with the basis $\{J_\alpha\}$ equipped with a linear map δ: $g \to g \wedge g$ (3.3.14) satisfying the co-Jacobi identity (3.3.15) is called Lie coalgebra. A Lie bialgebra is a Lie algebra (2.2.4) which is in the same time is a Lie coalgebra with the map δ: $g \to g \wedge g$ (3.3.14), (3.3.15) satisfying the cocycle condition (3.3.13).*

Let g be a Lie bialgebra. If there exists an element $r \in g \otimes g$ such that the map δ has the form

$$\delta(J) = [J \otimes 1 + 1 \otimes J,\, r] \quad \forall J \in g\;,$$

then the Lie bialgebra g is called co-boundary or r-matrix bialgebra.

---

[7]The terms $h\phi^{\beta\gamma}_\alpha J^1_\beta J^1_\gamma$ and $h\phi^{\beta\gamma}_\alpha J^2_\beta J^2_\gamma$ are gauged out by triviality transformation from this expansion (see, e.g., [37]).

3.4 The quantum groups $GL_q(N)$, $SL_q(N)$ and corresponding quantum algebras and hyperplanes

In this subsection, we discuss the simplest quantum groups, which are the quantizations (deformations) of the linear Lie groups $GL(N)$ and $SL(N)$. We begin with the definition of a quantum hyperplane. We recall that the Lie group $GL(N)$ is the set of nondegenerate $N \times N$ matrices T^i_j that act on an N-dimensional vector space, whose coordinates we denote by x^i, $(i = 1, \dots N)$. Thus, we have the transformations

$$x^i \to \tilde{x^i} = T^i_j x^j , \tag{3.4.1}$$

which we can regard from a different point of view. Namely, let $\{T^i_j\}$ and $\{x^i\}$ $(i, j = 1, \dots, N)$ be the generators of two Abelian (commuting) algebras

$$[x^i, x^j] = [T^i_j, T^k_l] = [T^i_j, x^k] = 0 \; . \tag{3.4.2}$$

Then the transformation (3.4.1) can be regarded as an action (more precisely, it is a coaction) of the algebra $\{T\}$ on the algebra $\{x\}$

$$x^i \to \tilde{x^i} = T^i_j \otimes x^j , \tag{3.4.3}$$

that preserves the Abelian structure of the latter, i.e., we have $[\tilde{x}^i, \, \tilde{x}^j] = 0$. We introduce a deformed N-dimensional "vector space" whose coordinates $\{x^i\}$ commute as follows:

$$x^i x^j = q x^j x^i , \; i < j \tag{3.4.4}$$

where q is some number (the deformation parameter). In other words, we now have a noncommutative associative algebra with N generators $\{x^i\}$. In accordance with (3.4.4), any element of this algebra, which is a monomial of arbitrary degree

$$x^{i_1} x^{i_2} \cdots x^{i_K} , \tag{3.4.5}$$

can be uniquely ordered lexicographically, i.e., in such a way that $i_1 \leq i_2 \leq \ldots \leq i_K$. Of such algebras, one says that they possess the Poincare-Birkhoff-Witt (PBW) property. An algebra with N generators satisfying (3.4.4) is called an N-dimensional quantum hyperplane [56],[57]. The relations (3.4.4) can be written in the matrix form

$$R^{i_1 i_2}_{j_1 j_2} x^{j_1} x^{j_2} = q x^{i_2} x^{i_1} \Leftrightarrow R_{12} x_1 x_2 = q x_2 x_1 \Leftrightarrow \hat{R} \, x_1 \, x_2 = q \, x_1 \, x_2 \; . \tag{3.4.6}$$

Here the indices 1 and 2 label the vector spaces on which the R matrix, realized in the tensor square $Mat(N)_1 \otimes Mat(N)_2$, acts. Thus, the indices 1 and 2 of the R matrix show how the R matrix acts on the direct product of the first and second vector spaces. We emphasize that the R matrix depends on the parameter q and, generally speaking, its explicit form is recovered nonuniquely from the relations (3.4.4). However, if we require that the R matrix (3.4.6) be constructed by means of two $GL(N)$-invariant tensors $\mathbf{1}_{12}$ and P_{12},[8] i.e.,

$$R^{i_1 i_2}_{j_1 j_2} = (\delta^{i_1}_{j_1} \delta^{i_2}_{j_2}) \cdot a_{i_1 i_2} + (\delta^{i_1}_{j_2} \delta^{i_2}_{j_1}) \cdot b_{i_1 i_2} \; , \tag{3.4.7}$$

[8] The form of R- matrix (3.4.7) proves to be very fruitful for the construction of solutions for the dynamical Yang-Baxter equations (see [58], [59] and references therein).

and also satisfy the Yang-Baxter equation (3.1.1) and have lower-triangular block form ($R^{i_1 i_2}_{j_1 j_2} = 0$, $i_1 < j_1$), then we obtain the explicit expression

$$R_{12} = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \lambda \sum_{i>j} e_{ij} \otimes e_{ji} \; , \tag{3.4.8}$$

where e_{ij} are matrix units: $(e_{ij})^k_l = \delta^{ik}\delta_{jl}$, $(i,j = 1,\dots N)$ and $\lambda = q - q^{-1}$. In the components eq. (3.4.8) is represented in the form

$$\begin{gathered} R^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}(1 + (q-1)\delta^{i_1 i_2}) + \lambda \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\Theta_{i_1 i_2} \; , \\ \hat{R}^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\, q^{\delta_{i_1 i_2}} + \lambda \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\Theta_{i_2 i_1} \; , \\ \Theta_{ij} = \{1 \;\text{ if }\; i > j, \;\; 0 \;\text{ if }\; i \leq j\}. \end{gathered} \tag{3.4.9}$$

It can be verified (using e.g. the diagrammatic technique of Sec. 3.5) that this R matrix satisfies the Hecke relation (3.1.46) [a special case of (3.1.41)]

$$R_{12} - \lambda \, P_{12} - R^{-1}_{21} = 0 \;\Rightarrow \hat{R} - \lambda - \hat{R}^{-1} = 0 \; . \tag{3.4.10}$$

The following helpful relations also follow from the explicit form (3.4.8), (3.4.9) for the $GL_q(N)$ R matrix:

$$R^{t_1 t_2}_{12} = R_{21} \; , \quad R_{12}(\frac{1}{q}) = R^{-1}_{12}(q) \; , \quad R^{t_1}_{12}\, R_{12} = R_{12}\, R^{t_1}_{12} \; .$$

For the R matrix (3.4.7) one can define skew-inverse matrix Ψ_{12} (3.1.10):

$$\hat{\Psi}^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\, \frac{1}{a_{i_2 i_1}} - \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, d_{i_2 i_1} \; , \tag{3.4.11}$$

where coefficients d_{ij} are defined from the matrix equation

$$d = A^{-1}\, b\, (A + b)^{-1} \; , \quad A_{ij} := a_{ii}\delta_{ij} \; .$$

Thus, for the given R matrix (3.4.9), the matrices $\hat{\Psi}_{12}$ (3.1.10), D, Q (3.1.12) and the quantum traces (3.2.11) can be chosen in the form

$$\hat{\Psi}^{i_1 i_2}_{j_1 j_2} = q^{-\delta_{i_1 i_2}}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1} - \lambda \Theta_{i_2 i_1} q^{2(i_1 - i_2)}\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \; , \tag{3.4.12}$$

$$\begin{gathered} D_1 \equiv Tr_2\left(\hat{\Psi}_{12}\right) = diag\{q^{-2N+1}, q^{-2N+3}, \dots, q^{-1}\} \; , \quad D^i_j = q^{2(i-N)-1}\delta^i_j \; , \\ Q_2 \equiv Tr_1\left(\hat{\Psi}_{12}\right) = diag\{q^{-1}, \dots, q^{-2N+3}, q^{-2N+1}\} \; , \quad Q^i_j = q^{1-2i}\delta^i_j \; , \end{gathered} \tag{3.4.13}$$

$$Tr_D A := Tr(DA) \equiv \sum_{i=1}^N q^{2(i-N)-1} A^i_i \; , \;\; Tr_Q A := Tr(QA) \equiv \sum_{i=1}^N q^{1-2i} A^i_i \; ,$$

(the $GL_q(N)$ matrix Ψ_{12} (3.4.13) has been presented in [37]). We also note the useful relations [cf. (3.1.14), (3.2.16)]

$$\begin{gathered} Tr_D(I) = Tr(D) = q^{-N}\, [N]_q = Tr(Q) = Tr_Q(I) \; , \\ q^N\, Tr_{D(3)} \hat{R}^{\pm 1}_{23} = q^{\pm N} \cdot I_{(2)} = q^N\, Tr_{Q(1)} \hat{R}^{\pm 1} \; , \end{gathered} \tag{3.4.14}$$

where $[N]_q = \frac{q^N - q^{-N}}{q - q^{-1}}$. One can readily prove the cyclic property of the quantum trace (the same property is valid for the quantum trace Tr_Q)

$$Tr_{D(12)}(\hat{R}E_{12}) = Tr_{12}(D_1 D_2 \hat{R} E_{12}) = Tr_{12}(\hat{R} D_1 D_2 \, E_{12}) = Tr_{q(12)}(E_{12}\hat{R}) \ , \quad (3.4.15)$$

for any quantum matrix $E_{12} \in Mat(N) \otimes Mat(N)$, since the matrix D obeys eq. $[\hat{R}, \, D_1 D_2] = 0$ (3.1.28) by definition (in fact, all diagonal matrices satisfy this equation with R-matrices of the type (3.4.7)).

In the semiclassical limit (3.3.1), the relation (3.4.10) can be rewritten in the form

$$r_{12} + r_{21} = 2 P_{12} \ . \quad (3.4.16)$$

Thus, for the Lie-Poisson structure on the group $GL(N)$ the transposition matrix P_{12} is taken as the ad-invariant tensor t_{12}. For the $\tilde{r}$ matrix (3.3.8) determining the Sklyanin bracket, we obtain from (3.4.9) the expression

$$\tilde{r}_{12} = \sum_{i>j} [e_{ij} \otimes e_{ji} - e_{ji} \otimes e_{ij}] \in gl(N) \wedge gl(N) \ . \quad (3.4.17)$$

In accordance with (3.1.46), (3.1.43), and (3.1.44) for $q^2 \neq -1$ the matrix $\hat{R}$ has the spectral decomposition

$$\hat{R} = q \mathbf{P}^+ - q^{-1} \mathbf{P}^- \ , \quad (3.4.18)$$

with projectors

$$\mathbf{P}^{\pm} = (q + q^{-1})^{-1} \{ q^{\mp 1} \mathbf{1} \pm \hat{R} \} \ , \quad (3.4.19)$$

which are the quantum analogs of the symmetrizer ($\mathbf{P}^+$) and antisymmetrizer ($\mathbf{P}^-$), as can be seen by setting $q = 1$ in (3.4.19). Using the projector $\mathbf{P}^-$, we can represent the definition (3.4.4) of the quantum hyperplane in the form

$$\mathbf{P}^- \, x_1 \, x_2 = 0. \quad (3.4.20)$$

Note that the relations

$$\mathbf{P}^+ \, x_1 \, x_2 = 0 \Leftrightarrow (x^i)^2 = 0 \ , \quad x^i x^j = -q^{-1} x^j x^i \ \ (i < j) \quad (3.4.21)$$

define a fermionic N-dimensional quantum hyperplane that is a deformation of the algebra of N fermions: $x^i x^j = -x^j x^i$.

A natural question now is that of the properties of the $N \times N$ matrix elements T^i_j that determine the transformations (3.4.3) of the quantum bosonic (3.4.4), (3.4.20) and fermionic (3.4.21) hyperplanes. These properties should be such that the transformed coordinates $\tilde{x}^i$ form the same quantum algebras (q- hyperplanes) (3.4.20) and (3.4.21). It is readily seen that the elements of the $N \times N$ matrix T^i_j must satisfy the conditions

$$\mathbf{P}^{\pm} \, T_1 \, T_2 \, \mathbf{P}^{\mp} = 0 \ . \quad (3.4.22)$$

Indeed, we have for bosonic x^- and fermionic x^+ hyperplanes (the symbol $\otimes$ is omitted)

$$0 = \mathbf{P}^{\pm} \, \tilde{x}^{\pm}_1 \, \tilde{x}^{\pm}_2 = \mathbf{P}^{\pm} \, T_1 \, T_2 \, x^{\pm}_1 \, x^{\pm}_2 =$$
$$= \mathbf{P}^{\pm} \, T_1 \, T_2 \, (\mathbf{P}^+ + \mathbf{P}^-) \, x^{\pm}_1 \, x^{\pm}_2 = \mathbf{P}^{\pm} \, T_1 \, T_2 \, \mathbf{P}^{\mp} \, x^{\pm}_1 \, x^{\pm}_2 \ ,$$

and we deduce (3.4.22) if new quadratic relations on the coordinates $x^{\pm}$ are not imposed. Eqs. (3.4.22) are equivalent to the RTT relations (3.2.1).

Definition 11. *A Hopf algebra generated by unit element and N^2 elements T^i_j $(i,j = 1,\dots,N)$ which satisfy RTT relations (3.2.1) with R matrix (3.4.9) is called the algebra of functions on the quantum group $GL_q(N)$ and denoted by $Fun(GL_q(N))$.*

For $Fun(GL_q(N))$ case the RTT relations (3.2.1) and (3.4.22) can be written in the component form as

$$\begin{gathered} T^i_k\, T^j_k = q\, T^j_k\, T^i_k \; , \quad T^k_i\, T^k_j = q\, T^k_j\, T^k_i \; , \;\; (i<j,\, k=1,\dots,N) \; , \\ [T^{i_1}_{j_1},\, T^{i_2}_{j_2}] = (q-q^{-1})\, T^{i_1}_{j_2}\, T^{i_2}_{j_1} \; , \quad [T^{i_1}_{j_2},\, T^{i_2}_{j_1}] = 0 \; , \;\; (i_1<i_2,\; j_1<j_2) \; . \end{gathered} \tag{3.4.23}$$

The simplest special case of the algebra $Fun(GL_q(2))$ is

$$\begin{gathered} T^1_k\, T^2_k = q\, T^2_k\, T^1_k \; , \quad T^k_1\, T^k_2 = q\, T^k_2\, T^k_1 \; , \;\; (k=1,2) \; , \\ [T^1_1,\, T^2_2] = (q-q^{-1})\, T^1_2\, T^2_1 \; , \quad [T^1_2,\, T^2_1] = 0 \; . \end{gathered} \tag{3.4.24}$$

One can directly check that $det_q(T) := T^1_1\, T^2_2 - qT^1_2\, T^2_1$ is a central element for the algebra $Fun(GL_q(2))$. This element is called quantum determinant (since for $q=1$ we obtain the usual determinant) for (2×2) quantum matrix T.

Now we generalize this definition for the case of $(N\times N)$ quantum matrices T and introduce the quantum determinant $det_q(T)$, which is a deformation of the ordinary determinant and also is a central element for the algebra $Fun(GL_q(N))$. For this aim we introduce the q-deformed antisymmetric tensors $\mathcal{E}_{j_1 j_2\dots j_N}$ and $\mathcal{E}^{j_1 j_2\dots j_N}$ $(j_k = 1,\dots,N)$ as follows:

$$\begin{gathered} \sum_{j_1\dots j_N=1}^{N} \mathcal{E}_{j_1 j_2\dots j_N}\, \mathcal{E}^{j_1 j_2\dots j_N} = \mathcal{E}_{\langle 12\dots N}\mathcal{E}^{12\dots N\rangle} = 1 \; , \\ \mathcal{E}_{\langle 12\dots N}\mathbf{P}^+_{k,k+1} = \mathcal{E}_{\langle 12\dots N}(\hat{R}_{k,k+1} + q^{-1}) = 0 \; , \;\; 1\le k<N \; , \\ \mathbf{P}^+_{k,k+1}\mathcal{E}^{12\dots N\rangle} = (\hat{R}_{k,k+1} + q^{-1})\mathcal{E}^{12\dots N\rangle} = 0 \; , \;\; 1\le k<N \; , \end{gathered} \tag{3.4.25}$$

where we have used concise matrix notations. Namely, we have denoted by $12\dots N\rangle$ and $\langle 12\dots N$ – sets of incomming and outgoing indices, where $1,2,\dots N$ are numbers of the vector spaces, and $\mathbf{P}^+_{k,k+1} = I^{\otimes(k-1)} \otimes \mathbf{P}^+ \otimes I^{\otimes(N-k-1)}$ are the symmetrizers (3.4.18) acting in the vector spaces labeled by numbers k and $k{+}1$. Note that, in view of the RTT relations (3.2.1), the tensors $\mathcal{E}_{\langle 12\dots N}(T_1 T_2\cdots T_N)$, $(T_1 T_2\cdots T_N)\mathcal{E}^{12\dots N\rangle}$ possess the same symmetry (3.4.25) as tensors $\mathcal{E}_{\langle 12\dots N}$, $\mathcal{E}^{12\dots N\rangle}$, respectively. Supposing that the $\mathcal{E}$-tensors are unique (up to normalization), one can write

$$\begin{gathered} det_q(T)\, \mathcal{E}_{j_1 j_2\dots j_N} = \mathcal{E}_{i_1 i_2\dots i_N} T^{i_1}_{j_1} \cdot T^{i_2}_{j_2} \cdots T^{i_N}_{j_N} \; , \\ \mathcal{E}^{i_1 i_2\dots i_N}\, det_q(T) = T^{i_1}_{j_1} \cdot T^{i_2}_{j_2} \cdots T^{i_N}_{j_N} \mathcal{E}^{j_1 j_2\dots j_N} \; , \end{gathered} \tag{3.4.26}$$

or in concise matrix notations we have

$$\begin{gathered} det_q(T)\, \mathcal{E}_{\langle 12\dots N} = \mathcal{E}_{\langle 12\dots N}\, T_1 \cdot T_2 \cdots T_N \; , \\ \mathcal{E}^{12\dots N\rangle}\, det_q(T) = T_1 \cdot T_2 \cdots T_N\, \mathcal{E}^{12\dots N\rangle} \; , \end{gathered} \tag{3.4.27}$$

where $T_m := I^{\otimes(m-1)} \otimes T \otimes I^{\otimes(N-m)}$ and the scalar coefficient $det_q(T)$:

$$det_q(T) = \mathcal{E}_{\langle 12\dots N} \, (T_1 T_2 \cdots T_N) \, \mathcal{E}^{12\dots N\rangle} = Tr_{12\dots N}(A_{1\to N} \, T_1 T_2 \cdots T_N) \; , \qquad (3.4.28)$$

is called the quantum determinant for the $(N \times N)$ quantum matrix T: $T^i_j \in Fun(GL_q(N))$. In (3.4.28) we have used the rank 1 projector

$$\begin{array}{c} A_{1\to N} := \mathcal{E}^{12\dots N\rangle} \mathcal{E}_{\langle 12\dots N} \; , \quad A_{1\to N} \, A_{1\to N} = A_{1\to N} \; , \\ A_{1\to N} \, \mathbf{P}^+_{k,k+1} = \mathbf{P}^+_{k,k+1} \, A_{1\to N} = 0 \; , \;\; 1 \leq k < N \; , \end{array} \qquad (3.4.29)$$

which acts as an q-antisymmetrizer in the tensor product of N vector spaces. It is interesting to note that the q-antisymmetrizers $A_{1\to 2} := \mathbf{P}^-_{1,2}$ (3.4.19) and $A_{1\to N}$ are special cases of antisymmetrizers $A_{1\to m}$

$$\begin{array}{c} A_{1\to m} \, A_{1\to m} = A_{1\to m} \; , \\ A_{1\to m} \, \mathbf{P}^+_{k,k+1} = \mathbf{P}^+_{k,k+1} \, A_{1\to m} = 0 \; , \;\; 1 \leq k < m \; , \end{array} \qquad (3.4.30)$$

(acting in the tensor product of m vector spaces, $m = 2, 3, \dots, N$) and all of them can be explicitly constructed in terms of the R matrix (3.4.8), (3.4.9) (see, e.g., [62] and Subsection 3.5 below).

The fact that $det_q(T)$ is indeed a central element in the algebra $Fun(GL_q(N))$ can be obtained as follows

$$\mathcal{E}_{\langle 12\dots N} \, det_q(T) \, T_{N+1} = \mathcal{E}_{\langle 12\dots N} \, T_1 \, T_2 \cdots T_N \, T_{N+1} = \qquad (3.4.31)$$

$$= \mathcal{E}_{\langle 12\dots N} \, (R_{1,N+1} \cdots R_{N,N+1})^{-1} T_{N+1} \, T_1 \, T_2 \cdots T_N \, (R_{1,N+1} \cdots R_{N,N+1}) =$$
$$= q^{-1} \, T_{N+1} \mathcal{E}_{\langle 12\dots N} \, T_1 \, T_2 \cdots T_N \, (R_{1,N+1} \cdots R_{N,N+1}) = T_{N+1} \, det_q(T) \, \mathcal{E}_{\langle 12\dots N} \; ,$$

where the indices $1, 2, \dots, N+1$ are the numbers of the vector spaces and we have used the definition (3.4.27), the RTT relations, and the equations

$$\begin{array}{c} qI_{N+1} \, \mathcal{E}_{\langle 12\dots N} = \mathcal{E}_{\langle 12\dots N} \, R_{1,N+1} \cdot R_{2,N+1} \cdots R_{N,N+1} \\ q^{-1} I_{N+1} \, \mathcal{E}_{\langle 12\dots N} = \mathcal{E}_{\langle 12\dots N} \, R^{-1}_{N+1,1} \cdot R^{-1}_{N+1,2} \cdots R^{-1}_{N+1,N} \end{array} \qquad (3.4.32)$$

In fact we have only used the first eq. (3.4.32). The second one is needed if we apply RTT relations in (3.4.31) in different manner.

The relations (3.4.32) are deduced from the expressions for the quantum determinants

$$det_q(R^{(\pm)}_{N+1}) = \mathcal{E}_{\langle 12\dots N} \, R^{(\pm)}_{1,N+1} \cdots R^{(\pm)}_{N,N+1} \mathcal{E}^{12\dots N\rangle} = q^{\pm 1} \, I_{N+1} \; , \qquad (3.4.33)$$

where the $R^{(\pm)}$-matrix representations for T^i_j are given in (3.2.17), (3.2.18). In their turn, the relations (3.4.33) follow from the fact that $R^{(+)}$ and $R^{(-)}$ are, respectively, upper and lower triangular block matrices with diagonal blocks of the form

$$(R^{(\pm)}{}^i_i)^k_l = \delta^k_l q^{\pm \delta_{ik}} \; .$$

The algebra $Fun(SL_q(N))$ can be obtained from the algebra $Fun(GL_q(N))$ by imposing the additional condition $det_q(T) = 1$ and, in accordance with (3.4.33), the matrix representations (3.2.18) for $T^i_j \in Fun(SL_q(N))$ have the form

$$\langle L_2^+, T_1 \rangle = \frac{1}{q^{1/N}} R_{12} \; , \;\; \langle L_2^-, T_1 \rangle = q^{1/N} R_{21}^{-1}, \tag{3.4.34}$$

(these formulas are interpreted as matrix representations of the universal enveloping algebra generators $L^\pm$ as well).

The quantum universal enveloping algebra $U_q(gl(N))$ appears in the R matrix approach [17] as the algebra (3.2.20), (3.2.21). Indeed, we consider the upper and lower triangular matrices L^+, L^- in the form [17] (see also [61])

$$L^+ = \begin{pmatrix} q^{H_1} & 0 & \dots & 0 \\ 0 & q^{H_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q^{H_N} \end{pmatrix} \begin{pmatrix} 1 & \lambda e_1 & \lambda e^1_3 & \dots & * \\ 0 & 1 & \lambda e_2 & \dots & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \lambda e_{N-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} , \tag{3.4.35}$$

$$L^- = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \lambda f_1 & 1 & 0 & \dots & 0 \\ \lambda f^3_1 & \lambda f_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & \dots & \dots & \lambda f_{N-1} & 1 \end{pmatrix} \begin{pmatrix} q^{-\tilde{H}_1} & 0 & \dots & 0 \\ 0 & q^{-\tilde{H}_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & q^{-\tilde{H}_N} \end{pmatrix} , \tag{3.4.36}$$

and, in particular, we have

$$(L^+)^i_i = q^{H_i} \; , \quad (L^-)^i_i = q^{-\tilde{H}_i} \; , \quad (L^+)^i_{i+1} = \lambda q^{H_i} e_i \; , \quad (L^-)^{i+1}_i = \lambda f_i q^{-\tilde{H}_i} \; . \tag{3.4.37}$$

In the case of R matrix (3.4.8), (3.4.9)) the relations (3.2.20), (3.2.21) are represented in the components as (no summation over repeated indices)

$$(L^\pm)^i_k \, (L^\pm)^j_k = q \, (L^\pm)^j_k \, (L^\pm)^i_k \; , \quad (L^\pm)^k_i \, (L^\pm)^k_j = q \, (L^\pm)^k_j \, (L^\pm)^k_i \; , \;\; (i > j) \; , \tag{3.4.38}$$

$$[(L^\pm)^{i_1}_{j_1}, \, (L^\pm)^{i_2}_{j_2}] = \lambda \, (L^\pm)^{i_1}_{j_2} \, (L^\pm)^{i_2}_{j_1}, \;\; [(L^\pm)^{i_1}_{j_2}, \, (L^\pm)^{i_2}_{j_1}] = 0, \;\; (i_1 > i_2, \, j_1 > j_2), \tag{3.4.39}$$

$$(L^+)^i_k \, (L^-)^j_k = q \, (L^-)^j_k \, (L^+)^i_k \; , \quad (L^-)^k_i \, (L^+)^k_j = q \, (L^+)^k_j \, (L^-)^k_i \; , \;\; (i < j) \; , \tag{3.4.40}$$

$$[(L^\mp)^{i_1}_{j_1}, \, (L^\pm)^{i_2}_{j_2}] = 0 \; , \;\; (i_1 > i_2, \; j_1 > j_2) \; , \quad [(L^+)^i_i, \, (L^-)^i_i] = 0 \; , \tag{3.4.41}$$

$$[(L^-)^{i_1}_{j_1}, \, (L^+)^{i_2}_{j_2}] = \lambda \left((L^+)^{i_1}_{j_2} \, (L^-)^{i_2}_{j_1} - (L^-)^{i_1}_{j_2} \, (L^+)^{i_2}_{j_1} \right) \; , \;\; (i_1 > i_2, \; j_1 < j_2) \; , \tag{3.4.42}$$

where we have written only the terms and relations which survive under the condition that $(L^+)^i_j = 0 = (L^-)^j_i$, $i > j$. Now the substitution of (3.4.37) into eqs. (3.4.38) - (3.4.42) gives the Drinfeld-Jimbo [62] formulation of $U_q(gl(N))$. From eqs. (3.4.38), (3.4.40) and (3.4.41) one can obtain that $q^{H_i - \tilde{H}_i}$ are the central elements. Thus, the matrices $L^\pm$ can be renormalized (by multiplying them with diagonal matrices) in such a way that elements $q^{H_i - \tilde{H}_i}$ are fixed as units. Then, from eq. (3.4.40) we find

$$e_i q^{H_j} = q^{\delta_{j,i} - \delta_{j,i+1}} \, q^{H_j} e_i \; , \quad f_i q^{H_j} = q^{\delta_{j,i+1} - \delta_{j,i}} \, q^{H_j} f_i \; . \tag{3.4.43}$$

The first eq. in (3.4.41) gives $e_i f_j = f_j e_i$ for $i \neq j$ and taking into account (3.4.42) we derive

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{q^{H_i - H_{i+1}} - q^{H_{i+1} - H_i}}{\lambda} \; . \tag{3.4.44}$$

The first eq. in (3.4.39)) yields a part of the Serre relations

$$e_i e_j = e_j e_i \; , \quad f_i f_j = f_j f_i \; , \quad (|i - j| \geq 2) \; . \tag{3.4.45}$$

and gives the definitions of the composite roots via the simple roots $\{e_i, f_j\}$:

$$\begin{aligned} e^{i-1}_{i+1} &= (e_i e_{i-1} - q^{-1} e_{i-1} e_i) = \lambda^{-1} \, q^{-H_{i-1}} \, (L^+)^{i-1}_{i+1} \; , \\ f^{i+1}_{i-1} &= (q f_i f_{i-1} - f_{i-1} f_i) = \lambda^{-1} \, (L^-)^{i-1}_{i+1} \, q^{H_{i-1}} \; . \end{aligned} \tag{3.4.46}$$

Using these definitions and eqs. (3.4.38) we deduce another part of Serre relations

$$\begin{aligned} & e_i^2 e_{i\pm 1} - (q + q^{-1}) e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0 \;\; (1 \leq i, i \pm 1 \leq N) \; , \\ & f_i^2 f_{i\pm 1} - (q + q^{-1}) f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0 \;\; (1 \leq i, i \pm 1 \leq N) \; . \end{aligned} \tag{3.4.47}$$

So, we see that equations (3.2.20), (3.2.21) (with the form of $L^{\pm}$ given in (3.4.35), (3.4.36)) yield not only the commutation relations (3.4.43), (3.4.44) for the elements of the Chevalley basis, but also present the Serre relations (3.4.45), (3.4.47) and define the composite roots (3.4.46) as the q- commutators of the simple roots. In this sense the generators $(L^{\pm})^i_j$ (3.4.35), (3.4.36)) play the role of a quantum analog of the Cartan-Weyl basis for $U_q(gl(N))$ (q^{H_i}, $(L^+)^i_j$, $(L^-)^i_j$ are Cartan elements, positive and negative roots, respectively). The quantum Casimir operators are given by eqs. (3.2.27) and (3.2.28). The co-multiplication, antipode and coidentity in terms of the generators $\{H_i,\, e_i,\, f_i\}$ can be deduced from (3.2.22), (3.2.23). For example the co-multiplication is

$$\Delta(q^{H_i}) = q^{H_i} \otimes q^{H_i} \; , \quad \Delta(e_i) = 1 \otimes e_i + e_i \otimes q^{H_{i+1} - H_i} \; , \quad \Delta(f_i) = f_i \otimes 1 + q^{H_i - H_{i+1}} \otimes f_i \; ,$$

Note that $\sum_i H_i$ is a central element in the algebra $U_q(gl(N))$ and the condition $\sum_i H_i = 0$ reduces $U_q(gl(N))$ to the algebra $U_q(sl(N))$ with generators $\{h_i := H_i - H_{i+1},\; e_i,\; f_i\}$ subject to the relations

$$[q^{h_i}, \, q^{h_j}] = 0 \; , \quad e_i \, q^{h_j} = q^{a_{ij}} \, q^{h_j} \, e_i \; , \quad f_i \, q^{h_j} = q^{-a_{ij}} \, q^{h_j} \, f_i \; , \tag{3.4.48}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}} \; , \tag{3.4.49}$$

and Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q^{d_i}} (e_i)^k e_j (e_i)^{1-a_{ij}-k} = 0 \; , \quad (e_i \rightarrow f_i) \; , \tag{3.4.50}$$

where

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} \; , \;\; [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}} \; , \;\; [k]_q! := [1]_q [2]_q \cdots [k]_q \; , \;\; [0]_q! := 1 \; , \tag{3.4.51}$$

$a_{ij} = 2\delta_{ij} - \delta_{ji+1} - \delta_{ij+1}$ is Cartan matrix for $sl(N)$, d_i are smallest positive integers (from the set 1,2,3) such that $d_i a_{ij}$ is symmetric matrix (for $sl(N)$ case $d_i = 1$).

The relations (3.4.48) - (3.4.50) are used for the Drinfeld - Jimbo [22], [63] formulation of the quantum universal enveloping algebra $U_q(g)$ for any semisimple Lie algebra g. Using this formulation of the quasitriangular Hopf algebra $U_q(g)$, one can explicitly construct the corresponding universal R matrix (the definition via cannonical element is given in (2.5.9)). In the case of algebra $U_q(sl_N)$, the explicit multiplicative formula for the universal R matrix has been found in [69]. This result was generalized in [70] for the case of $U_q(g)$, where g is any semisimple Lie algebra. For the case of quantum Lie superalgebras the universal R matrix has been found in [98]. Finite dimensional representations for the quantum simple Lie algebras $U_q(g)$ (3.4.48) - (3.4.50) has been constructed in [64].

Remark 1. The complexification of the linear quantum groups can be introduced as follows. We first consider the case of the group $GL_q(N)$ and assume that q is a real number. We must define an involution $*$-operation, or simply $*$-involution, (which is the antihomomorphism) on the algebra $Fun(GL_q(N))$ or, in other words, we must define the conjugated algebra $Fun(\widetilde{GL_q}(N))$ with generators[9]

$$\tilde{T} = (T^\dagger)^{-1} \; , \quad (T^\dagger := (T^*)^t \Leftrightarrow (T^\dagger)^i_j := (T^j_i)^*) \; , \tag{3.4.52}$$

and defining relations identical to (3.2.1):

$$R_{12} \, \tilde{T}_1 \, \tilde{T}_2 = \tilde{T}_2 \, \tilde{T}_1 \, R_{12} \;\Rightarrow\; \hat{R}_{12} \, \tilde{T}_1 \, \tilde{T}_2 = \tilde{T}_1 \, \tilde{T}_2 \, \hat{R}_{12} \; . \tag{3.4.53}$$

We now introduce the extended algebra with generators $\{T^i_j, \; \tilde{T}^k_l\}$ that is the cross (smash) product of the algebras (3.2.1) and (3.4.53) with subsidiary cross commutation relations (see, for example, Refs. [16] and [17])

$$\hat{R} \, T_1 \, \tilde{T}_2 = \tilde{T}_1 \, T_2 \, \hat{R} \; . \tag{3.4.54}$$

It is natural to relate this extended algebra to $Fun(GL_q(N, \mathbf{C}))$.

The case of $SL_q(N, \mathbf{C})$ can be obtained from $GL_q(N, \mathbf{C})$ by imposing two subsidiary conditions on the central elements:

$$det_q(T) = 1, \; det_q(\tilde{T}) = 1 \; . \tag{3.4.55}$$

The real form $U_q(N)$ is separated from $GL_q(N, \mathbf{C})$ if we require

$$T = \tilde{T} = (T^\dagger)^{-1} \tag{3.4.56}$$

and if in addition to this we impose the conditions (3.4.55), then the group $SU_q(N)$ is distinguished.

The $*$-involution on the algebra $U_q(sl(N))$ (3.4.48) – (3.4.50) for real q is defined if we note that the algebra with generators $T, \widetilde{T}$ (3.4.1), (3.4.53), (3.4.54) coinsides to the $L^\pm$-algebra (3.2.20), (3.2.21) after an identification: $L^- = T^{-1}$, $L^+ = \widetilde{T}^{-1}$. Then, according to (3.4.52) we require $(L^+)^\dagger = (L^-)^{-1}$. In terms of the Shevalley generators it means that

$$h_i^* = h_i \; , \quad e_i^* = -q \, q^{-h_i} \, f_i \; , \quad f_i^* = -q^{-1} \, e_i \, q^{-h_i} \; . \tag{3.4.57}$$

[9] We recall that $(T^{-1})^t \neq (T^t)^{-1}$ in the case of the quantum matrices (see (3.2.6)).

One can directly check that the relations (3.4.48) – (3.4.50) respect the antihomomorphism (3.4.57).

In the case $|q| = 1$, the definition of $*$-involutions on the linear quantum groups is a nontrivial problem that can be solved [71] only after extension of the algebra of functions on the quantum groups to the algebra of functions on their cotangent bundles, i.e., to the algebra which is a Heisenberg double of $Fun(GL_q(N))$ and $U_q(gl(N))$ with cross-multiplication rules (3.2.47) - (3.2.49).

3.5 Hecke type R matrices and related quantum matrix algebras

The $GL_q(N)$ matrices (3.4.8), (3.4.9) satisfy Hecke condition (3.4.10). We call the R matrices which obey Hecke condition as *Hecke type R matrices*. In this subsection we present some general facts about Hecke type R matrices and related quantum algebras.

The antisymmetrizers $A_{1\to m}$ (3.4.30) can be explicitly constructed in terms of the Hecke type R matrices by using the following inductive procedure [62] (the same procedure is used for the construction of antisymmetrizers in a Hecke algebra [60]; see also Sec. 4 below):

$$A_{1\to k} = A_{2\to k}\left(\frac{\hat{R}_1(q^{k-1})}{[k]_q}\right)A_{2\to k} = A_{1\to k-1}\left(\frac{\hat{R}_{k-1}(q^{k-1})}{[k]_q}\right)A_{1\to k-1} = \tag{3.5.1}$$

$$= \frac{1}{[k]_q!}\, A_{1\to k-1}\, \hat{R}_{k-1}\left(q^{k-1}\right)\dots \hat{R}_2\left(q^2\right)\, \hat{R}_1\left(q\right) \;\; , \quad (k = 2, 3, \dots N) \; ,$$

where $A_{1\to 1} = 1$, $\hat{R}(x) = (x^{-1}\hat{R} - x\hat{R}^{-1})/\lambda$ – Baxterized R-matrix (see below Subsect. 3.7), $\hat{R}$ is a Hecke type R-matrix, $[k]_q = (q^k - q^{-k})/\lambda$ and as usual

$$\hat{R}_k = I^{\otimes(k-1)} \otimes \hat{R} \otimes I^{\otimes(N-k)} \in Mat(N)^{\otimes(N+1)} \; . \tag{3.5.2}$$

Definition 12. *We say that the Hecke type R-matrix is of the height N, if $A_{1\to M} = 0$ $\forall M > N$ and $rank(A_{1\to N}) = 1$.*

Note that $A_{1\to N+1} = 0$ and the operator $A_{1\to N}$ is the highest q- antisymmetrizer in the sequence of the antisymmetrizers (3.5.1) for the $GL_q(N)$ type R-matrix (3.4.8), (3.4.9). Moreover we have $rank(A_{1\to N}) = 1$ in this case. It can easily be understood from the considering of the fermionic quantum hyperplane (3.4.21). Since the operators $A_{1\to k}$ (3.5.1) satisfy (cf. (3.4.25))

$$\hat{R}_j\, A_{1\to k} = A_{1\to k}\, \hat{R}_j = -q^{-1}\, A_{1\to k} \;\; (j = 1, \dots, k-1) \; , \tag{3.5.3}$$

they are symmetry operators for k-th order monomials $x^{i_1}\dots x^{i_k}$ in the q-fermionic algebra (3.4.21). In view of the explicit relations (3.4.21) one can conclude that there is only one independent monomial of the order N and all monomials $x^{i_1}\dots x^{i_k} = 0$ for $k > N$. This statement is equivalent to the conditions $rank(A_{1\to N}) = 1$ and $A_{1\to N+1} = 0$.

In view of the definition (3.5.1), the condition $A_{1\to N+1} = 0$ leads to (for arbitrary Hecke $\hat{R}$-matrix):

$$A_{1\to N}\, \hat{R}_N^{\pm 1}\, A_{1\to N} = \frac{q^{\pm N}}{[N]_q}\, A_{1\to N}\, I_{N+1}\ , \quad A_{2\to N+1}\, \hat{R}_1^{\pm 1}\, A_{2\to N+1} = \frac{q^{\pm N}}{[N]_q}\, A_{2\to N+1}\, I_1\ . \tag{3.5.4}$$

In the case of skew-invertible (3.1.10) Hecke $\hat{R}$-matrices eqs. (3.5.4) give

$$A_{1\to N}\, P_{_{N\,N+1}}\, A_{1\to N} = \frac{q^N}{[N]_q}\, A_{1\to N}\, Q_{_{N+1}}\ , \quad A_{2\to N+1}\, P_{_{1\,2}}\, A_{2\to N+1} = \frac{q^N}{[N]_q} D_1\, A_{2\to N+1}\ , \tag{3.5.5}$$

and for completely invertible $\hat{R}$-matrices we have in addition $\overline{Q} = q^{2N}\, Q$, $\overline{D} = q^{2N}\, D$. Applying $Tr_{_{N+1}}$ and Tr_1 to the first and second eqs. of (3.5.5), respectively, we deduce (cf. (3.4.14))

$$Tr(Q) = Tr(D) = q^{-N}[N]_q \;\;\Rightarrow\;\; Tr(\overline{Q}) = Tr(\overline{D}) = q^N [N]_q\ , \tag{3.5.6}$$

while applying $Tr_{(1\ldots N)}$ and $Tr_{(2\ldots N+1)}$ to the same eqs. we obtain

$$Tr_{(1\ldots N-1)} A_{1\to N} = \frac{\mathrm{rank}(A_{1\to N})}{Tr(Q)} Q_N\ , \;\; Tr_{(2\ldots N)} A_{1\to N} = \frac{\mathrm{rank}(A_{1\to N})}{Tr(D)} D_1\ .$$

On the other hand applying quantum traces $Tr_{D(N-k+1\ldots N)}$ and $Tr_{Q(1\ldots k)}$ to the antisymmetrizers $A_{1\ldots N}$ one can deduce $(0 \leq k \leq N)$

$$\begin{bmatrix} N \\ k \end{bmatrix}_q Tr_{D(k+1\ldots N)}\, (A_{1\ldots N}) = q^{(k-N)\,N}\, A_{1\ldots k} \quad (A_{1\ldots k}|_{k=0} \, := 1)\ , \tag{3.5.7}$$

$$\begin{bmatrix} N \\ k \end{bmatrix}_q Tr_{Q(1\ldots k)}\, (A_{1\ldots N}) = q^{-k\,N}\, A_{k+1\ldots N} \quad (A_{k+1\ldots N}|_{k=N} \, := 1)\ , \tag{3.5.8}$$

where q-binomial coefficients are defined in (3.4.51) and we have used eqs. (3.5.1) and identities

$$Tr_{D(k+1)} \hat{R}_k(x) = Tr_{Q(k-1)} \hat{R}_{k-1}(x) = \frac{x^{-1} - x\,(1 - \lambda Tr(D))}{\lambda}\, I_k = \frac{x^{-1} - x q^{-2\,N}}{\lambda}\, I_k\ , \tag{3.5.9}$$

which follow from (3.1.14), (3.5.6). In view of eqs. (3.1.28) matrices D and Q realize one dimensional representations of RTT algebra (3.2.1): $\rho_D(T^i_j) = D^i_j$, $\rho_Q(T^i_j) = Q^i_j$. Thus, we have

$$A_{1\ldots N} D_1 D_2 \ldots D_N = det_q(D)\, A_{1\ldots N}\ , \quad A_{1\ldots N} Q_1 Q_2 \ldots Q_N = det_q(Q)\, A_{1\ldots N}\ . \tag{3.5.10}$$

and taking $k = 0$ in (3.5.7) and $k = N$ in (3.5.8) we obtain

$$det_q(D) = q^{-N^2}\ , \quad det_q(Q) = q^{-N^2}\ . \tag{3.5.11}$$

For the Hecke type R-matrix one can construct (in addition to the q- antisymmetrizer $A_{1\to k}$ (3.5.1)) the q-symmetrizer $S_{1\to k}$:

$$S_{1\to k} = S_{2\to k} \left(\frac{\hat{R}_1(q^{1-k})}{[k]_q} \right) S_{2\to k} = S_{1\to k-1} \left(\frac{\hat{R}_{k-1}(q^{1-k})}{[k]_q} \right) S_{1\to k-1}\ , \tag{3.5.12}$$

(see also Sec 4 below). Using identities (3.5.1), (3.5.12) and (3.5.9) one can calculate q-ranks for the projectors $A_{1\to k}$ and $S_{1\to k}$:

$$Tr_{D(1\dots k)} A_{1\to k} = \frac{(q^{k-1}Tr(D) - [k-1]_q)}{[k]_q} Tr_{D(1\dots k-1)} A_{1\to k-1} = \dots =$$

$$= \frac{1}{[k]_q!} \prod_{m=1}^{k} \left(q^{m-1} Tr(D) - [m-1]_q \right) \; , \qquad (3.5.13)$$

and analogously

$$Tr_{D(1\dots k)} S_{1\to k} = \frac{1}{[k]_q!} \prod_{m=1}^{k} \left(q^{1-m} Tr(D) + [m-1]_q \right) \; , \qquad (3.5.14)$$

Substituting (3.5.6) in (3.5.13), (3.5.14) we deduce for the Hecke type R matrix (of the height N) the following "q-dimensions":

$$Tr_{D(1\dots k)} A_{1\to k} = q^{-kN} \begin{bmatrix} N \\ k \end{bmatrix}_q \;\; (k \leq N) \; , \quad Tr_{D(1\dots k)} A_{1\to k} = 0 \;\; (k > N) \; ,$$

$$Tr_{D(1\dots k)} S_{1\to k} = q^{-kN} \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q \; .$$

The general formula for q-dimensions of any Young projectors, (constructed from the Hecke type R-matrices, see Subsect. 4.2 below) is known and can be found in [116], [119].

Sometimes it is convenient to use Eqs. (3.4.32) in more general form which is valid for any Hecke R-matrix (see e.g. [126]):

$$A_{2\to N+1} \, \hat{R}_1^{\pm 1} \cdot \hat{R}_2^{\pm 1} \cdots \hat{R}_N^{\pm 1} = (-1)^{N-1} \, q^{\pm 1} \, [N]_q \, A_{2\to N+1} \, A_{1\to N} \; . \qquad (3.5.15)$$

(we demonstrate a connection of (3.4.32) with (3.5.15) below). The mirror counterpart of the relations (3.5.15) are also valid

$$\hat{R}_N^{\pm 1} \cdots \hat{R}_2^{\pm 1} \cdot \hat{R}_1^{\pm 1} \, A_{2\to N+1} = (-1)^{N-1} \, q^{\pm 1} \, [N]_q \, A_{1\to N} \, A_{2\to N+1} \; . \qquad (3.5.16)$$

Eqs. (3.5.15) and (3.5.16) can be readily deduced from equation

$$A_{2\to N+1} \, \hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1} = \hat{R}_1^{\pm 1} \dots \hat{R}_N^{\pm 1} \, A_{1\to N} \; , \qquad (3.5.17)$$

which follows from the representation of the antisymmetrizers in terms of R-matrices (3.5.1) and braid relations (3.1.6), (3.1.8). Acting on (3.5.17) by $A_{1\to N}$ from the left and using eq. (3.5.3) and representation (3.5.4) we deduce (3.5.16). Eq. (3.5.15) can be proved in the same way. Multiplying identities (3.5.15) and (3.5.16) we deduce

$$A_{1\to N} \, A_{2\to N} \, A_{1\to N} = [N]_q^{-2} \, A_{1\to N} \; , \quad A_{2\to N} \, A_{1\to N} \, A_{2\to N} = [N]_q^{-2} \, A_{2\to N} \; , \quad (3.5.18)$$

and eqs. which are equivalent to $A_{1\to N+1} = 0$:

$$A_{1\to N}(\hat{R}_N \dots \hat{R}_2 \hat{R}_1^2 \hat{R}_2 \dots \hat{R}_N - q^2) = 0 \; , \quad A_{2\to N+1}(\hat{R}_1 \hat{R}_2 \dots \hat{R}_N^2 \dots \hat{R}_2 \hat{R}_1 - q^2) = 0 \; .$$

The identity (3.5.15) which is valid for any Hecke R-matrix can be transformed into eq. (3.4.32). Indeed, for the case when $\mathrm{rank}(A_{1\to N}) = 1$ and, thus, $A_{1\to N}$ is given by (3.4.29), one can act on (3.5.15) by $\mathcal{E}_{\langle 2\dots N+1}$ from the left and as a result the counterpart of (3.4.32) is obtained

$$\mathcal{E}_{\langle 23\dots N+1}\, \hat{R}_1^{\pm 1} \cdot \hat{R}_2^{\pm 1} \cdots \hat{R}_N^{\pm 1} = q^{\pm 1}\, N_{\langle N+1}^{\;1\rangle}\, \mathcal{E}_{\langle 12\dots N}\ . \tag{3.5.19}$$

Here we have introduced the matrix:

$$N_{\langle N+1}^{\;1\rangle} := (-1)^{N-1}\, [N]_q\, \mathcal{E}_{\langle 2\dots N+1}\, \mathcal{E}^{1\dots N\rangle}\ , \tag{3.5.20}$$

which, for the case of $GL_q(N)$ R-matrix (3.4.8), is equal to the unit matrix $N^i_j = \delta^i_j$ (cf. (3.4.32)). Analogously acting by $\mathcal{E}^{2\dots N+1\rangle}$ on (3.5.16) from the right we deduce

$$\hat{R}_1^{\pm 1} \cdot \hat{R}_2^{\pm 1} \cdots \hat{R}_N^{\pm 1}\, \mathcal{E}^{23\dots N+1\rangle} = q^{\pm 1}\, \mathcal{E}^{12\dots N\rangle}\, (N^{-1})_{\langle 1}^{\;N+1\rangle}\ , \tag{3.5.21}$$

where matrix

$$(N^{-1})_{\langle 1}^{\;N+1\rangle} := (-1)^{N-1}\, [N]_q\, \mathcal{E}_{\langle 1\dots N}\, \mathcal{E}^{2\dots N+1\rangle}\ , \tag{3.5.22}$$

is inverse to the matrix (3.5.20) in view of (3.5.18).

In the case when matrices N^i_j are not proportional to the unit matrix the chain of relations (3.4.31) gives [126]:

$$\begin{array}{c}
\mathcal{E}_{\langle 1\dots N}\, det_q(T)\, T_{N+1} = \mathcal{E}_{\langle 1\dots N}\, T_1 \dots T_N\, T_{N+1} = \\
= \mathcal{E}_{\langle 1\dots N}\, \hat{R}_N^{-1} \dots \hat{R}_1^{-1}\, T_1 \dots T_N\, T_{N+1}\, \hat{R}_1 \dots \hat{R}_N = \\
= q^{-1} (N^{-1})_{\langle 1}^{N+1\rangle}\, T_1\, \mathcal{E}_{\langle 2\dots N+1}\, T_2 \dots T_{N+1}\, \hat{R}_1 \dots \hat{R}_N = \\
= q^{-1} (N^{-1})_{\langle 1}^{N+1\rangle}\, T_1\, det_q(T)\, \mathcal{E}_{\langle 2\dots N+1}\, \hat{R}_1 \dots \hat{R}_N = (N^{-1}\, T N)_{N+1}\, det_q(T) \mathcal{E}_{\langle 1\dots N}\ ,
\end{array} \tag{3.5.23}$$

where an explicit form of N^{-1} can be extracted from (3.5.18). It means that for RTT algebras defined by general Hecke type R-matrix the element $det_q(T)$ is not necessary central.

The structure (3.2.49) of the double for the RTT and reflection equation algebras (defined by the Hecke type R-matrix of the height N) helps to introduce the notion of the quantum determinants $Det_q(L)$, $Det_q(\overline{L})$ for the corresponding reflection equation algebras (3.2.30), (3.2.31). It can be done by using the definition (3.4.27), (3.4.28) of the quantum determinant $det_q(.)$ for $Fun(GL_q(N))$. This definition is valid for any RTT algebra with the Hecke type R-matrix of the height N. In view of the automorphism (3.2.50), the quantum matrix LT satisfies the same RTT relation (3.2.1) and, thus, one can consider the same quantum determinant $det_q(.)$ for the quantum matrix $L\,T$ as for the matrix T. This determinant is divisible from the right by $det_q(T)$ and the quotient depends only on the matrix L. This quotient is called the quantum determinant for the reflection equation algebra (3.2.30):

$$\begin{array}{l}
Det_q(L) := det_q(L\,T) \dfrac{1}{det_q(T)} = \mathcal{E}_{\langle 1\dots N}(L_1 T_1\, L_2 T_2 \dots L_N T_N) \mathcal{E}^{1\dots N\rangle}\, \dfrac{1}{det_q(T)} = \\[2ex]
= \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}}) T_1 \dots T_N \mathcal{E}^{1\dots N\rangle}\, \dfrac{1}{det_q(T)} = \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}}) \mathcal{E}^{1\dots N\rangle} =
\end{array} \tag{3.5.24}$$

$$= \mathcal{E}_{\langle 1\dots N}(L_{\widetilde{N}} \dots L_{\widetilde{2}} L_{\widetilde{1}})\mathcal{E}^{1\dots N\rangle} = Tr_{1\dots N}\left(A_{1\dots N}\, L_{\widetilde{1}}\, L_{\widetilde{2}} \dots L_{\widetilde{N}}\right)\ ,$$

where $L_{\widetilde{k+1}} = \hat{R}_k L_{\widetilde{k}} \hat{R}_k$ ($L_{\widetilde{1}} = L_1$) are commutative operators for all k. For the second type algebra (3.2.31) the definition is analogous: $Det_q(\overline{L}) = det_q^{-1}(T) det_q(T\overline{L})$.

Below we consider only the case of the left reflection equation algebra (3.2.30), since the case of the algebra (3.2.31) is investigated analogously.

An interesting property of the determinant $Det_q(L)$ is of its multiplicativity

$$Det_q(L^n) = det_q(L^n\, T)\frac{1}{det_q(T)} = \tag{3.5.25}$$

$$= det_q(L^n\, T)\frac{1}{det_q(L\,T)} det_q(L\,T)\frac{1}{det_q(T)} = Det_q(L^{n-1})\, Det_q(L) = (Det_q(L))^n\ .$$

In view of the discrete map (3.2.51) for $n = 1$ one can define (in the same way as in (3.5.24)) a quantum determinant $Det_q(L; x)$ [41]:

$$Det_q(L; x) := det_q((L + x)\, T)\frac{1}{det_q(T)} = Tr_{1\dots N}\left(A_{1\dots N}(L_{\widetilde{1}} + x) \dots (L_{\widetilde{N}} + x)\right)\ , \tag{3.5.26}$$

where $x \in \mathbf{C}$ is a parameter, and, thus, introduce the characteristic polynomial for the quantum matrix L. Here we prefer to use the notation $Det_q(L; x)$ instead of $Det_q(L+x)$, since the dependence on $(L+x)$ fails in the last expression of (3.5.26). Taking into account (3.4.27), the determinant (3.5.26) can be also written in the form

$$\mathcal{E}^{1\dots N\rangle}\, Det_q(L; x) = (L_1 + x)T_1\, (L_2 + x)T_2 \dots (L_N + x)T_N\, \mathcal{E}^{1\dots N\rangle}\ \frac{1}{det_q(T)} =$$

$$= \Big((L_{\widetilde{1}} + x)\, (L_{\widetilde{2}} + x) \dots (L_{\widetilde{N}} + x)\Big)\, \mathcal{E}^{1\dots N\rangle}\ . \tag{3.5.27}$$

The expansion of (3.5.26) over the parameter x gives

$$Det_q(L; x) = \sum_{k=0}^{N} x^k\, \sigma_{N-k}(L)\ . \tag{3.5.28}$$

Here $\sigma_0(L) = 1$, $\sigma_N(L) = Det_q(L)$,

$$\sigma_m(L) = Tr_{1\dots N}\left(A_{1\dots N} \sum_{\substack{k_1<k_2<\dots<k_m \\ k_r=1}}^{N} L_{\widetilde{k_1}} L_{\widetilde{k_2}} \dots L_{\widetilde{k_m}}\right) = \alpha_N^m\, Tr_{1\dots N}\left(A_{1\dots N} L_{\widetilde{1}}\, L_{\widetilde{2}} \cdots L_{\widetilde{m}}\right)\ , \tag{3.5.29}$$

and

$$\alpha_N^m = q^{m(m+1)} \sum_{\substack{k_1<k_2<\dots<k_m \\ k_r=1}}^{N} q^{-2(k_1+k_2+\dots k_m)} = q^{m(m-N)}\ \begin{bmatrix} N \\ m \end{bmatrix}_q\ , \tag{3.5.30}$$

where q-binomial coefficients $\begin{bmatrix} N \\ m \end{bmatrix}_q$ have been introduced in (3.4.50). The sums in (3.5.30) are readily calculated by means of their generating function

$$\alpha(t) = \sum_{m=0}^{N} t^{N-m} q^{-m(m+1)} \alpha_N^m = \prod_{m=1}^{N} (t + q^{-2m})\ .$$

We note that using (3.5.7), (3.5.10), (3.5.11) the elements $\sigma_m(L)$ can be also written in the form

$$\sigma_m(L) = \frac{\alpha_N^m}{det_q(D)}\, Tr_{D(1\dots N)} \left(A_{1\dots N} L_{\widetilde{1}} \cdots L_{\widetilde{m}} \right) = q^{m^2}\, Tr_{D(1\dots m)} \left(A_{1\dots m} L_{\widetilde{1}} \cdots L_{\widetilde{m}} \right) \; . \tag{3.5.31}$$

Then we have

$$\begin{array}{c} L_{\widetilde{1}} L_{\widetilde{2}} \cdots L_{\widetilde{m}} = [L_1 (\hat{R}_1 L_1)(\hat{R}_2 \hat{R}_1 L_1) \dots (\hat{R}_{m-1} \dots \hat{R}_1 L_1)] \hat{R}_{(1 \to m-1)} \dots \hat{R}_{(1\to 2)} \hat{R}_1 = \\ = L_{\underline{1}} L_{\underline{2}} \cdots L_{\underline{m}} (\hat{R}_1 \dots \hat{R}_{(m-1 \leftarrow 1)})(\hat{R}_{(1\to m-1)} \dots \hat{R}_1) = L_{\underline{1}} L_{\underline{2}} \cdots L_{\underline{m}}\, y_2\, y_3 \dots y_m \; , \end{array} \tag{3.5.32}$$

and analogously

$$L_{\widetilde{1}} L_{\widetilde{2}} \cdots L_{\widetilde{m}} = L_{\widetilde{m}} \cdots L_{\widetilde{2}} L_{\widetilde{1}} = y_2\, y_3 \dots y_m\, L_{\overline{m}} \dots L_{\overline{2}} L_{\overline{1}} \; , \tag{3.5.33}$$

where $y_2 = \hat{R}_1^2$, $y_{k+1} = \hat{R}_k y_k \hat{R}_k$ and elements $L_{\underline{k+1}} = \hat{R}_k L_{\underline{k}} \hat{R}_k^{-1}$, $L_{\overline{k+1}} = \hat{R}_k^{-1} L_{\overline{k}} \hat{R}_k$ have been introduced in (3.2.36). According to the identities (3.5.32), (3.5.33) and taking into account (3.2.37) one can write (3.5.31) in the form

$$\sigma_m(L) = q^m\, Tr_{D(1\dots m)} \left(A_{1\to m} L_{\underline{1}} L_{\underline{2}} \cdots L_{\underline{m}} \right) = q^m\, Tr_{D(1\dots m)} \left(A_{1\to m} L_{\overline{m}} \cdots L_{\overline{2}} L_{\overline{1}} \right) \; . \tag{3.5.34}$$

The elements $\sigma_m(L)$ (3.5.29), (3.5.31), (3.5.34) are central elements for the reflection equation algebra (3.2.30). Indeed, these elements are obtained from the general center elements (3.2.40) by substitution $X = A_{1\dots m}$.

Now we calculate the commutation relations of the elements $\sigma_m(L)$ with generators T^i_j of the RTT algebra defined by the Hecke type R-matrix. Note that in the case of the Heisenberg double of $Fun(SL_q(N))$ and $U_q(sl(N))$ we need to renormalize the R-matrix: $\hat{R} \to q^{-1/N} \hat{R}$ according to (3.4.34). This leads to the following generalization of the cross-multiplication rules (3.2.49) (we consider only the right HD)

$$T_1 L_2 = \alpha\, \hat{R}_{12} L_1 \hat{R}_{12} T_1 \; , \tag{3.5.35}$$

where for the case of the HD of $SL_q(N)$-type we have to put $\alpha = q^{-2/N}$ (but generally the constant $\alpha \neq 0$ is not fixed) and $\hat{R}$ is a Hecke R-matrix (3.4.10) of the height N. Note that the automorphism (3.2.51) is correct only for the choice $\alpha = 1$ in (3.5.35). For example, the quantum matrices $(L + x)T$ start to obey the modified RTT relations

$$\hat{R}_1\, (L_1 + x) T_1\, (\alpha^{-1} L_2 + x) T_2 = (L_1 + x) T_1\, (\alpha^{-1} L_2 + x) T_2\, \hat{R}_1 \; . \tag{3.5.36}$$

However, for the general choice of α in (3.5.35), the definition of the characteristic polynomial (3.5.28) is not changed, since instead of (3.5.27) we have

$$\begin{array}{c} (L_1 + x) T_1\, (\alpha^{-1} L_2 + x) T_2 \dots (\alpha^{1-N} L_N + x) T_N\, \mathcal{E}^{1\dots N\rangle}\, det_q^{-1}(T) = \\ = \Big((L_{\widetilde{1}} + x)\, (L_{\widetilde{2}} + x) \dots (L_{\widetilde{N}} + x) \Big)\, \mathcal{E}^{1\dots N\rangle} = \mathcal{E}^{1\dots N\rangle}\, Det_q(L; x) \; . \end{array} \tag{3.5.37}$$

(according to (3.5.36) we modify the first line in (3.5.27) but it does not affect on the final expression for $Det_q(L;x)$). Using (3.5.37), (3.5.36), (3.5.21) and (3.5.23) we obtain

$$
\begin{aligned}
&(L_1+x)T_1\, Det_q(L;\alpha x)\, \mathcal{E}^{2\ldots N+1\rangle} = \\
&= (L_1+x)T_1\, (\alpha^{-1}L_2+x)T_2 \ldots (\alpha^{-N}L_{N+1}+x)T_{N+1}\, \mathcal{E}^{2\ldots N+1\rangle}\, \tfrac{\alpha^N}{det_q(T)} = \\
&= \hat{R}_1 \ldots \hat{R}_N (L_1+x)T_1 \ldots (\alpha^{-N}L_{N+1}+x)T_{N+1}\, \hat{R}_N^{-1} \ldots \hat{R}_1^{-1}\, \mathcal{E}^{2\ldots N+1\rangle}\, \tfrac{\alpha^N}{det_q(T)} = \\
&= \mathcal{E}^{2\ldots N+1\rangle}\, Det_q(L;x)\, det_q(T)\, [N(L+\alpha^N x)TN^{-1}]_1\, det_q^{-1}(T) = \\
&= \alpha^N\, \mathcal{E}^{2\ldots N+1\rangle}\, Det_q(L;x)\, (q^2\, L_1 + x)\, T_1 \ ,
\end{aligned}
\tag{3.5.38}
$$

where we have taken into account the commutation relations of $det_q(T)$ and L^i_j deduced by the standard method

$$
\begin{aligned}
\mathcal{E}_{\langle 1\ldots N}\, det_q(T)\, L_{N+1} = \mathcal{E}_{\langle 1\ldots N}\, T_1 \ldots T_N\, L_{N+1} = \\
= \alpha^N\, \mathcal{E}_{\langle 1\ldots N}\, \hat{R}_N \ldots \hat{R}_1 L_1 \hat{R}_1 \ldots \hat{R}_N T_1 \ldots T_N = q^2\, \alpha^N\, (N^{-1}\, L\, N)_{N+1}\, det_q(T) \mathcal{E}_{\langle 1\ldots N} \ ,
\end{aligned}
\tag{3.5.39}
$$

(eqs. (3.5.35) and (3.5.19) were applied). Thus, we have the following relations (see (3.5.38))

$$
(L_1+x)T_1\, Det_q(L;\alpha x) = \alpha^N\, Det_q(L;x)\, (q^2\, L_1 + x)\, T_1 \ . \tag{3.5.40}
$$

The expansion of (3.5.40) over x gives the recurrent equation for desired commutation relations of $\sigma_k(L)$ with T^i_j ($k \geq 0$):

$$
\alpha^{-k}\, LT\, \sigma_k + \alpha^{-1-k}\, T\, \sigma_{k+1} = q^2\, \sigma_k\, LT + \sigma_{k+1} T \ , \quad T\, \sigma_0 = \sigma_0\, T \ .
$$

It is not hard to check that the solution of this equation is

$$
\alpha^{-k}\, T\sigma_k = \sigma_k T - (q^2-1) \sum_{m=1}^{k} (-1)^m \sigma_{k-m} L^m\, T \ . \tag{3.5.41}
$$

Using the fact that the matrix T is invertible we write this matrix equation in the form

$$
\alpha^{-k}\, T\sigma_k\, T^{-1} = \sigma_k - (q^2-1) \sum_{m=1}^{k} (-1)^m \sigma_{k-m} L^m \ . \tag{3.5.42}
$$

For the left hand side of (3.5.42) we deduce

$$
\begin{aligned}
\alpha^{-k}\, q^{-k}\, T_1 \sigma_k\, T_1^{-1} = Tr_{D(2\ldots k+1)} \left(A_{2\ldots k+1}\, L_{\underline{2}} \ldots L_{\underline{k+1}} \right) = \\
= Tr_{D(2\ldots k+1)} \left(A_{2\ldots k+1}\, \hat{R}_1 \ldots \hat{R}_k\, L_{\underline{1}} \ldots L_{\underline{k}}\, \hat{R}_k \ldots \hat{R}_1\, A_{2\ldots k+1} \right) = \\
= Tr_{D(2\ldots k+1)} \left(\hat{R}_{(1\to k)}\, A_{1\ldots k}\, L_{\underline{1}} \ldots L_{\underline{k}}\, A_{1\ldots k}\, \hat{R}_{(k\leftarrow 1)} \right) = \\
= Tr_{D(2\ldots k+1)} \left(\hat{R}_1 \ldots \hat{R}_k\, A_{1\ldots k}\, L_{\underline{1}} \ldots L_{\underline{k}}\, A_{1\ldots k} (\hat{R}_k^{-1} + \lambda) \ldots (\hat{R}_1^{-1} + \lambda) \right) = \\
= Tr_{D(2\ldots k)} \left(\hat{R}_{(1\to k-1)} \left[Tr_{D(k)} (A_{1\ldots k} L_{\underline{1}} \ldots L_{\underline{k}}\, A_{1\ldots k}) \right] \hat{R}_{(k-1\leftarrow 1)} \right) + \\
+ \lambda\, q^{2(1-k)}\, Tr_{D(2\ldots k)} \left(A_{1\ldots k} L_{\underline{1}} \ldots L_{\underline{k}} \right) = \ldots = \\
= \sigma_k + \lambda (1 + q^{-2} + \ldots q^{2(1-k)}) Tr_{D(2\ldots k)} \left(A_{1\ldots k} L_{\underline{1}} \ldots L_{\underline{k}} \right) =
\end{aligned}
$$

$$= \sigma_k + q(1-q^{-2k}) Tr_{D(2\dots k)} \left(A_{1\dots k} L_{\underline{1}} \dots L_{\underline{k}} \right) . \qquad (3.5.43)$$

Comparing (3.5.42) and (3.5.43) we obtain the identities for quantum matrices L (the so-called Cayley-Hamilton-Newton identities)

$$[k]_q \, Tr_{D(2\dots k)}(A_{1\to k} L_{\underline{1}} \dots L_{\underline{k}}) = - \sum_{m=1}^{k} (-1)^m \sigma_{k-m} L_1^m . \qquad (3.5.44)$$

It follows from (3.5.44) that the two basic sets (3.2.28), (3.5.34) of central elements for RE algebra (defined by Hecke type R-matrix) are related by the q-analogue of the Newton relations:

$$\frac{[k]_q}{q^k} \sigma_k + \sum_{m=1}^{k} (-1)^m \sigma_{k-m} \, C_m = 0 , \qquad (3.5.45)$$

where $C_m = Tr_D(L^m)$, $m = 1, \dots, N$, and we imply $\sigma_0 = 1$. For $k = N$ the relation (3.5.44) provides the characteristic identity for the quantum matrix L (q-analogue of the Cayley-Hamilton theorem):

$$\sum_{k=0}^{N} (-L)^k \, \sigma_{N-k}(L) = 0 . \qquad (3.5.46)$$

This identity can be formally obtained by the substitution of $x = -L$ in the characteristic polynomial (3.5.28). Thus, in view of (3.5.45) and (3.5.46), the elements $\sigma_m(L)$ can be interpreted as noncommutative analogs of elementary symmetric functions for eigenvalues of the quantum matrix L.

The Cayley-Hamilton-Newton identities (3.5.44) have been invented in [72]. For the reflection equation algebra in the case $N = 2$ the identity (3.5.46) has been considered in [81] (see eq. (21)) and in [143]. For general N it was proved in [73], [74]. Newton's relations (3.5.45) have been obtained in [74].

3.6 Many-parameter deformations of linear groups

In this subsection, we consider a many-parameter deformation of the linear group $GL(N)$ (Refs. [28, 57], and [75]–[79]). A many-parameter quantum hyperplane is defined by the relations

$$x^i x^j = r_{ij} x^j x^i , \quad i < j , \qquad (3.6.1)$$

which can be written in the R -matrix form (3.4.6) if we introduce an additional parameter q. Thus, we have $N(N-1)/2+1$ deformation parameters: r_{ij}, $i < j$ and q. The corresponding R matrix has the form (see e.g. [28])

$$R_{12} = q \sum_i e_{i,i} \otimes e_{i,i} + \sum_{i \neq j} (e_{i,i} \otimes e_{j,j}) a_{ji} + (q - q^{-1}) \sum_{i>j} e_{i,j} \otimes e_{j,i} , \qquad (3.6.2)$$

where $a_{ij} = 1/a_{ji} = r_{ij}/q$ (for $i > j$) and in the components the R matrix (3.6.2) is represented as

$$R^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \left(q \delta^{i_1 i_2} + \Theta_{i_2 i_1} \frac{q}{r_{i_1 i_2}} + \Theta_{i_1 i_2} \frac{r_{i_2 i_1}}{q} \right) + (q - q^{-1}) \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \Theta_{i_1 i_2} , \qquad (3.6.3)$$

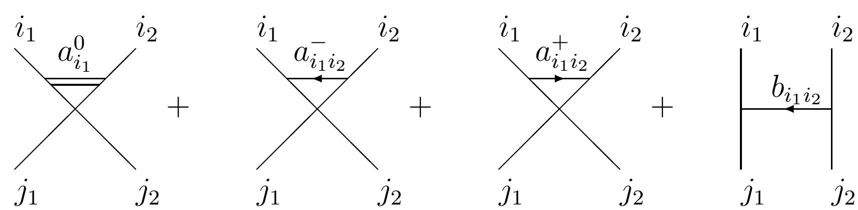
where Θ_{ij} is defined in (3.4.9). The R matrix (3.6.2) is obtained by twisting of the R matrix (3.4.8) (see Sec. 2.5)

$$R_{12} \to F_{21} R_{12} F_{12}^{-1} \Leftrightarrow \hat{R}_{12} \to F_{12} \hat{R}_{12} F_{12}^{-1} \; , \quad F_{12} = \sum_{i,j} (e_{i,i} \otimes e_{j,j}) \, f_{ij} \; , \tag{3.6.4}$$

where $a_{ij} = f_{ij}/f_{ji}$ and $\hat{F} = PF$ satisfies the conditions for twisting matrices (3.2.53). Thus, with the help of the appropriate twisting of the R matrix (3.6.3) the multi-parameter case is reduced to the one-parameter case (see also Refs. [28] and [79]).

By the construction, via twisting procedure, the R matrix (3.6.3) satisfies the Hecke condition (3.4.10), which is the same as in the one-parameter case, and the Yang-Baxter equation (3.1.6). Now we try to find the most general Yang-Baxter solution R_{12} of the form (3.4.7). We only require that the R matrix has the lower-triangular block form: $b_{ij} = 0$ for $i \geq j$ (as it has been recently shown in [37] this condition is not restrictive). In these calculations, it is convenient to use the diagrammatic technique,

$$\hat{R} = \hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \left(a^0_{i_1} \delta^{i_1 i_2} + \Theta_{i_2 i_1} a^-_{i_1 i_2} + \Theta_{i_1 i_2} a^+_{i_1 i_2} \right) + b_{i_1 i_2} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \Theta_{i_2 i_1} = \tag{3.6.5}$$



It turns out that not all solutions of the Yang-Baxter equation (3.1.6) that can be represented in the form (3.6.5) are exhausted by the many-parameter R matrices (3.6.3). Indeed, if we substitute the matrix (3.6.5) in the Yang-Baxter equation (3.1.6), we obtain the following general conditions on the coefficients a^0_i, $a^{\pm}_{ij}$, b_{ij}:

$$b_{ij} = b \; , \;\; a^+_{ij} a^-_{ji} = c \; , \;\; (a^0_i)^2 - b a^0_i - c = 0 \;\; (\forall i, j) \; . \tag{3.6.6}$$

We normalize (3.6.5) in such a way that $c = 1$ and choose for convenience, instead of the parameter b, a different parameter q, setting $b = q - q^{-1}$. Then a^0_i can take the two values $\pm q^{\pm 1}$. For such a normalization, the solution of the Yang-Baxter equation of the form (3.6.5) automatically satisfies the Hecke relation (3.4.10). If we set $a^0_i = q$ (or $a^0_i = -q^{-1}$) for all i, then we arrive at the many-parametric case $GL_{q,r_{ij}}(N)$ (3.6.3) (up to exchange $q \to -q^{-1}$ in the case $a^0_i = -q^{-1}$). If, however, we set

$$a^0_i = q \; (1 \leq i \leq M) \; , \;\; a^0_i = -q^{-1} \;\; (M+1 \leq i \leq N) \; , \tag{3.6.7}$$

then the R matrix (3.6.5) does not reduce to (3.6.3) and will correspond to a many-parameter deformation of the supergroup $GL(M|N-M)$ (we consider this case below in Sec. 3.6).

By virtue of the fulfillment of the Hecke identity (3.4.10) for the many-parameter case, we can introduce the same projectors $\mathbf{P}^-$ and $\mathbf{P}^+$ as in the one-parameter case (3.4.18), the first of them defining the bosonic quantum hyperplane (3.6.1) [the relations (3.4.6) with R matrix (3.6.3)], and the second defining the fermionic quantum hyperplane

$$\mathbf{P}^{+} x_1 x_2 = 0 \Leftrightarrow (x^i)^2 = 0 \ , \ \ q^2 x^i x^j = -r_{ij} x^j x^i \ (i > j) \ . \tag{3.6.8}$$

Regarding (3.6.1) and (3.6.8) as comodules for the many-parameter quantum group $GL_{q,r_{ij}}(N)$, we find that the generators T^i_j of the algebra $Fun(GL_{q,r_{ij}}(N))$ satisfy the same RTT relations (3.2.1) but with $\hat{R}$ matrix (3.6.3). Note, however, that the quantum determinant $det_q(T)$ (3.4.26) is not central in the many-parameter case [77]. This is due to the fact that in general for the many-parameter R matrix we have $N_{\pm} \neq const \cdot I$ in the equations (3.5.15), (3.5.23). Therefore, reduction to the SL case by means of the condition $det_q(T) = 1$ is possible only under certain restrictions on the parameters q, r_{ij}. A detailed discussion of these facts can be found in Refs. [77] and [79].

The algebra (3.2.20), (3.2.21) (with the many-parameter R matrix (3.8.3)) which is dual to the algebra $Fun(GL_{q,r_{ij}}(N))$ can also be considered. It appears that this algebra is isomorphic to the one-parameter deformation of $gl(N)$ (3.4.43) – (3.4.47). One can find the detailed discussion of the dual algebras for the special case of $Fun(GL_{q,p}(2))$ in papers [81], [82].

3.7 The quantum supergroups $GL_q(N|M)$ and $SL_q(N|M)$

We choose the R matrix (3.6.5) in the form (cf. Ref. [90])

$$\hat{R} = \sum_i (-1)^{[i]} \, q^{1-2[i]} \, e_{ii} \otimes e_{ii} + \sum_{i \neq j} (-1)^{[i][j]} \, e_{ij} \otimes e_{ji} + \lambda \sum_{j>i} e_{ii} \otimes e_{jj} \ , \tag{3.7.1}$$

(here $i, j = 1, \dots, N+M$) or in the component presentation

$$\hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \, (-1)^{[i_1][i_2]} \, q^{\delta_{i_1 i_2} \, (1-2[i_1])} + \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \, \lambda \, \Theta_{i_2 i_1} \ , \tag{3.7.2}$$

i.e., we have set

$$a^0_i = (-)^{[i]} q^{1-2[i]} \ , \ \ a^+_{ij} = (a^-_{ij})^{-1} = (-1)^{[i][j]} \ , \ \ b = q - q^{-1} = \lambda \ .$$

Here $[i] = 0, 1$ and, therefore, a^0_i can take the two values $\pm q^{\pm 1}$. Thus, as we assumed in Sec. 3.5, the R matrix (3.7.2) must correspond to some supergroup. Indeed, in the limit $q \to 1$, we find that $\hat{R}$ tends to the supertransposition operator:

$$\hat{R}^{i_1 i_2}_{j_1 j_2} \to (-)^{[i_1][i_2]} \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \equiv \mathcal{P}_{12} \ . \tag{3.7.3}$$

Suppose that the R matrix acts in the space of the direct product of two supervectors $x^{j_1} \otimes y^{j_2}$ and $[i] = 0, 1$ denotes the parity of the components x^i of the supervector. For definiteness, we will assume that

$$[i] = 0 \quad (1 \leq i \leq N) \ , \quad [i] = 1 \quad (N+1 \leq i \leq N+M) \ . \tag{3.7.4}$$

As was noted in Sec. 3.5, the R matrix (3.7.2) satisfies the Yang-Baxter equation (3.1.6) and the Hecke relation (3.1.46). In place of the matrix $\hat{R}$, we introduce the new R matrix

$$R_{12} = \mathcal{P}_{12} \hat{R} \Rightarrow \hat{R} = \mathcal{P}_{12} R_{12} = (-)^{(1)(2)} P_{12} R_{12} =$$

$$= \sum_i q^{1-2[i]}\, e_{ii}\otimes e_{ii} + \sum_{i\neq j} e_{ii}\otimes e_{jj} + \lambda \sum_{i>j} (-1)^{[i][j]}\, e_{ij}\otimes e_{ji} \; .$$

with the quasiclassical behavior (3.3.1). Here we have used the matrix

$$\Big((-)^{(1)(2)}\Big)^{i_1 i_2}_{j_1 j_2} := (-1)^{[i_1][i_2]}\,\delta^{i_1}_{j_1}\,\delta^{i_2}_{j_2} \; . \tag{3.7.5}$$

Then from the Yang-Baxter equation (3.1.6) there follows the graded form of the Yang-Baxter equation [92] for the new R matrix:

$$R_{12}(-)^{(2)(3)}R_{13}(-)^{(2)(3)}R_{23} = R_{23}(-)^{(2)(3)}R_{13}(-)^{(2)(3)}R_{12} \; . \tag{3.7.6}$$

where we have taken into account the fact that R_{12} is an even R matrix, i.e.,

$$\begin{gathered} R^{i_1 i_2}_{j_1 j_2} \neq 0 \;\text{ if }\; [i_1]+[j_1]+[i_2]+[j_2] = 0\; (mod(2)) \;\;\Rightarrow \\ (-1)^{[i_3]([i_1]+[i_2])}\, R^{i_1 i_2}_{j_1 j_2}\delta^{i_3}_{j_3} = R^{i_1 i_2}_{j_1 j_2}\delta^{i_3}_{j_3}\,(-1)^{[j_3]([j_1]+[j_2])} \;\;\Leftrightarrow \\ (-)^{(3)((1)+(2))}\, R_{12}\, I_3 = R_{12}\, I_3\, (-)^{(3)((1)+(2))} \; . \end{gathered}$$

In the last relation we have set

$$\Big((-)^{(3)((1)+(2))}\Big)^{i_1 i_2 i_3}_{j_1 j_2 j_3} = (-1)^{[i_3]([i_1]+[i_2])}\,\delta^{i_1}_{j_1}\,\delta^{i_2}_{j_2}\,\delta^{i_3}_{j_3} \; .$$

Finally, the skew-inverse matrix Ψ_{12} for the $GL_q(N|M)$ R-matrix (3.7.1), (3.7.2) has the form

$$\begin{gathered} \hat{\Psi}^{i_1 i_2}_{j_1 j_2} = (-1)^{[i_1][i_2]}\, q^{\delta_{i_1 i_2}\,(2[i_1]-1)}\,\delta^{i_1}_{j_2}\delta^{i_2}_{j_1} - \\ -(-1)^{[i_1]+[i_2]}\lambda\, q^{(-1)^{[i_1]}(2i_1-2N-1)}\, q^{(-1)^{[i_2]}(1+2N-2i_2))}\,\Theta_{i_2 i_1}\,\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} \; , \end{gathered} \tag{3.7.7}$$

which follows from the general formula (3.4.11) (for the case $[i_1]=[i_2]=0$ we reproduce matrix (3.4.12)). The corresponding matrices of quantum supertraces are

$$\begin{gathered} D_1 \equiv Tr_2\left(\hat{\Psi}_{12}\right) \;\;\Rightarrow\;\; D^i_j = (-1)^{[i]}\, q^{2M+(-1)^{[i]}(2i-2N-1)}\,\delta^i_j \; , \\ Q_2 \equiv Tr_1\left(\hat{\Psi}_{12}\right) \;\;\Rightarrow\;\; Q^i_j = (-1)^{[i]}\, q^{-2N+(-1)^{[i]}(2N+1-2i)}\delta^i_j \; , \\ Tr(D) = Tr(Q) = (1-q^{2(M-N)})/\lambda = q^{(M-N)}\,[N-M]_q \; . \end{gathered} \tag{3.7.8}$$

Note that the quantum supertrace Tr_D which is constructed by means of the matrix D (3.7.8) coincides up to the factor $q^{(3M-N)/2}$ with the quantum trace presented in [81]. For $q \to 1$ quantum supertraces Tr_D, Tr_Q tend to usual supertraces.

The quantum multidimensional superplanes for the R matrices (3.7.2) have the form (see, for example, Refs. [91] and [81])

$$\begin{gathered} (\hat{R}-q)x_1x_2 = 0 \Leftrightarrow x^i x^j = (-)^{[i][j]} q x^j x^i \;\; (i<j)\; ,\;\; (x^i)^2 = 0 \text{ if } [i]=1 \; , \\ (\hat{R}+q^{-1})x_1x_2 = 0 \Leftrightarrow q x^i x^j = -(-)^{[i][j]} x^j x^i \;\; (i<j)\; ,\;\; (x^i)^2 = 0 \text{ if } [i]=0 \; . \end{gathered} \tag{3.7.9}$$

The second hyperplane can be interpreted as the exterior algebra of differentials dx^i of the coordinates x^i for the first hyperplane.

We consider the left coaction (3.4.3) of the quantum supergroup on the quantum superspaces (3.7.9) where $\otimes$ is understood as a graded tensor product

$$(T^{i_1}_{j_1} \otimes x^{j_1})\,(T^{i_2}_{j_2} \otimes x^{j_2}) = (-1)^{[j_1]([i_2]+[j_2])}\,(T^{i_1}_{j_1}\,T^{i_2}_{j_2}) \otimes (x^{j_1}\,x^{j_2})\ .$$

In [92] the right coactions have been considered which lead to another signs in the formulas, but this difference is not important.

From the condition of covariance of the supermodules (3.7.9), we deduce the graded form of the RTT equations:

$$\hat{R}^{i_1 i_2}_{k_1 k_2}\,T^{k_1}_{j_1}\,(-1)^{[j_1][k_2]}\,T^{k_2}_{j_2}\,(-1)^{[j_1][j_2]} = T^{i_1}_{k_1}\,(-1)^{[k_1][i_2]}\,T^{i_2}_{k_2}\,(-1)^{[k_1][k_2]}\,\hat{R}^{k_1 k_2}_{j_1 j_2}\ ,$$

represented in the matrix notations (3.7.5) as

$$\begin{gathered}\hat{R}T_1(-)^{(1)(2)}T_2(-)^{(1)(2)} = T_1(-)^{(1)(2)}T_2(-)^{(1)(2)}\,\hat{R} \Leftrightarrow \\ R_{12}T_1(-)^{(1)(2)}T_2(-)^{(1)(2)} = (-)^{(1)(2)}T_2(-)^{(1)(2)}T_1R_{12}\ .\end{gathered} \tag{3.7.10}$$

These relations are the defining relations for the generators T^i_j of the graded algebra $Fun(GL_q(N|M))$. Using (3.3.1) the semiclassical analog of (3.7.10) can readily be deduced

$$\left\{T_1\,,\ (-)^{(1)(2)}T_2(-)^{(1)(2)}\right\} = [T_1(-)^{(1)(2)}T_2(-)^{(1)(2)},\ r_{12}]\ ,$$

or in the component form

$$\begin{gathered}(-1)^{[j_1]([i_2]+[j_2])}\,\left\{T^{i_1}_{j_1},\,T^{i_2}_{j_2}\right\} = \\ = T^{i_1}_{k_1}\,(-1)^{[k_1][i_2]}\,T^{i_2}_{k_2}\,(-1)^{[k_1][k_2]}\,r^{k_1 k_2}_{j_1 j_2} - r^{i_1 i_2}_{k_1 k_2}\,T^{k_1}_{j_1}\,(-1)^{[j_1][k_2]}\,T^{k_2}_{j_2}\,(-1)^{[j_1][j_2]}\ ,\end{gathered}$$

where $\{.,.\}$ denotes the Poisson super-brackets

$$\{T^{i_1}_{j_1},\,T^{i_2}_{j_2}\} = -(-1)^{([i_1]+[j_1])([i_2]+[j_2])}\,\{T^{i_2}_{j_2},\,T^{i_1}_{j_1}\}\ .$$

The matrix $||T^i_j||$ can be represented in the block form

$$T^i_j = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{3.7.11}$$

where the elements of the $N \times N$ matrix A and of the $M \times M$ matrix D form the algebras $Fun(GL_q(N))$ and $Fun(GL_q(M))$, respectively. It follows from this that the noncommutative matrices D and $A - BD^{-1}C$ are invertible, and therefore so is the matrix $||T^i_j||$, as follows from the Gauss decomposition

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} 1 & BD^{-1} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} A - BD^{-1}C & 0 \\ \hline 0 & D \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline D^{-1}C & 1 \end{array}\right) \tag{3.7.12}$$

Thus, the algebra $Fun(GL_q(N|M))$ with defining relations (3.7.10) is a Hopf algebra with usual structure mappings (3.2.2):

$$\Delta(T^i_k) = T^i_j \otimes T^j_k \ , \ \ \epsilon(T^i_j) = \delta^i_j \ , \ \ S(T^i_j) = (T^{-1})^i_j \ ,$$

where in the definition of Δ the tensor product is understood as a graded tensor product.

We now compare the relations (3.7.10) with the graded Yang-Baxter equation (3.7.6). From this comparison we readily see that the finite-dimensional matrix representations for the generators T^i_j of the quantum algebra $Fun(GL_q(N|M))$ [the superanalogs of the representations (3.2.17)] can be chosen in the form

$$T_1 = (-)^{(1)(3)} R_{13} (-)^{(1)(3)} \equiv R^{(+)} \ , \ \ T_1 = (R^{-1})_{31} \equiv R^{(-)} \ . \tag{3.7.13}$$

From this we obtain in an obvious manner definitions of the quantum superalgebras of the dual algebras $Fun(GL_q(N|M))$ [cf. Eqs. (3.2.18)]:

$$< L^+_2, T_1 >= (-)^{(1)(2)} R_{12} (-)^{(1)(2)} \ , \ \ < L^-_2, T_1 >= R^{-1}_{21}. \tag{3.7.14}$$

Note that for $GL_q(N|M)$ we can define the quantum supertrace (see Ref. [81]) and the quantum superdeterminant [93], [94]. The algebra $Fun(SL_q(N|M))$ is distinguished by the relation $sdet_q(T) = 1$.

The quantum supergroup $GL_q(N|M)$ was studied in detail from somewhat different positions in Ref. [94]. The simplest example of a quantum supergroup, $GL_q(1|1)$, has been investigated in many studies (see, for example, Refs. [81] and [95]). The R matrices (3.6.5) can be used to construct the supersymmetric Baxterized solutions of the Yang-Baxter equation (3.8.5) obtained for the first time in Ref. [96]. The Yangian limits of these solutions (the corresponding RTT algebra defines the Yangian of the Lie superalgebra $gl(n|m)$ [93]) have been used to formulate integrable supersymmetric spin chains [97]. The universal R matrices for the linear quantum supergroups (and more generally for quantum deformations of finite dimensional contragredient Lie (super)algebras) were constructed in Ref. [98].

3.8 $GL_q(N)$- and $GL_q(N|M)$- invariant Baxterized R matrices

By Baxterization, we mean the construction of an R matrix that depends not only on a deformation parameter q but also on an additional complex spectral parameter x. If we wish to find a solution $\hat{R}(x)$ of the Yang-Baxter equation (with spectral parameter x) satisfying the condition of quantum invariance

$$T_1 T_2 \hat{R}(x) (T_1 T_2)^{-1} = \hat{R}(x) \ , \ \ \left(T^i_j \in Fun(GL_q(N)) \right) \ ,$$

then we must seek it in the form

$$\hat{R}(x) = b(x)(\mathbf{1} + a(x)\hat{R}) \ , \tag{3.8.1}$$

[here $a(x)$ and $b(x)$ are certain functions of x], since by virtue of the Hecke condition (3.4.10) there exist only two basis matrices 1 and R that are invariants in the sense of the relations (3.2.1). The Yang-Baxter equation with dependence on the spectral parameter is chosen in the form

$$\hat{R}_{12}(x)\,\hat{R}_{23}(xy)\,\hat{R}_{12}(y) = \hat{R}_{23}(y)\,\hat{R}_{12}(xy)\,\hat{R}_{23}(x)\ . \tag{3.8.2}$$

Only the function a(x) is fixed by this equation. Indeed, we substitute here (3.8.1) and take into account (3.1.6) and the Hecke condition (3.1.46); we then obtain the equation

$$a(x) + a(y) + \lambda a(x)a(y) = a(xy), \tag{3.8.3}$$

which is readily solved by the change of variables $a(x) = (1/\lambda)(\tilde{a}(x) - 1)$. After this, we obtain for a the general solution

$$a(x) = (1/\lambda)(x^{\xi} - 1), \tag{3.8.4}$$

where for simplicity the arbitrary parameter ξ can be set equal to -2. For convenience, we choose the normalizing function $b(x) = x$. Then the Baxterized R matrix satisfying the Yang-Baxter equation (3.8.2) will have the form

$$\hat{R}(x) = b(x)\left(\mathbf{1} + (1/\lambda)(x^{-2} - 1)\hat{R}\right) = \frac{1}{\lambda}\,(x^{-1}\hat{R} - x\hat{R}^{-1})\ . \tag{3.8.5}$$

For such a normalization we obtain

$$\hat{R}(1) = 1\ , \quad \mathbf{P}^{\pm} = \frac{1}{[2]_q}\,\hat{R}(q^{\mp 1})\ , \tag{3.8.6}$$

and the unitarity condition holds[10]

$$\hat{R}(x)\,\hat{R}(x^{-1}) = \left(1 - \frac{(x - x^{-1})^2}{\lambda^2}\right)\ , \tag{3.8.7}$$

which can be readily deduced from the spectral decomposition

$$\hat{R}(x) = \frac{(x^{-1}q - xq^{-1})}{\lambda}\mathbf{P}^{+} + \frac{(xq - (xq)^{-1})}{\lambda}\mathbf{P}^{-}\ ,$$

where projectors $\mathbf{P}^{\pm}$ have been defined in (3.4.18), (3.4.19). Note that we have obtained the Baxterized solution (3.8.5) of the Yang-Baxter equation (3.8.2) only using the braiding relations (3.1.4) and Hecke condition (3.4.10) for the constant matrix $\hat{R}$. Thus, any constant Hecke solution of (3.1.4) (e.g. the multi-parametric solution (3.6.2)) can be used for the construction of the Baxterized R matrices (3.8.5).

The cross-unitarity for the Hecke type R-matrix (3.8.5) can be written in the matrix form as

$$\begin{aligned} Tr_{D(2)}\left(\hat{R}_1(x)P_{01}\hat{R}_1(z)\right) &= \eta(x,z)\,D_0\,I_1\ , \\ Tr_{Q(1)}\left(\hat{R}_1(x)P_{23}\hat{R}_1(z)\right) &= \eta(x,z)\,Q_3\,I_2\ , \end{aligned} \tag{3.8.8}$$

[10]Strictly speaking we have to renormalize the R-matrix (3.8.5): $\hat{R}(x) \to \lambda(x^{-1}q - xq^{-1})^{-1}\hat{R}(x)$, to obtain the unitarity condition with the unit matrix in the right hand side of (3.8.7).

where the matrices D, Q have been defined in (3.1.12) and

$$\eta(x,z) = \frac{(x-x^{-1})(z-z^{-1})}{\lambda^2} \; , \quad (x\,z)^2 = \frac{1}{1-\lambda Tr(D)} =: b^2 \; .$$

We stress that for the $GL(N|M)$-type R-matrix we have $b^2 = q^{2(N-M)}$. Let $\hat{\Psi}_{12}$ be a skew-inverse matrix (3.1.10) for the constant Hecke type R-matrix (3.4.10). Then, one can define the skew-inverse matrix $\hat{\Psi}(x)$ for the Baxterized Hecke type R-matrix (3.8.5):

$$\hat{\Psi}_{12}(x) = \frac{\lambda}{x^{-1}-x} \left(\hat{\Psi}_{12} + \frac{\lambda}{b^{-2}-x^{-2}} \, D_1 Q_2 \right) \; , \tag{3.8.9}$$

such that

$$Tr_2 \left(\hat{\Psi}_{12}(x) \, \hat{R}_{23}(x) \right) = P_{13} = Tr_2 \left(\hat{R}_{12}(x) \, \hat{\Psi}_{23}(x) \right) \; . \tag{3.8.10}$$

It is a remarkable fact that the relations (3.2.20), (3.2.21) can be represented as follows:

$$\hat{R}_{12}(x) \, L_2(xy) \, L_1(y) = L_2(y) \, L_1(xy) \, \hat{R}_{12}(x), \tag{3.8.11}$$

where the spectral parameters x and y are arbitrary, and

$$L(x) = x^{-1} L^+ - x L^- . \tag{3.8.12}$$

Moreover, if we average the relation (3.8.11) with matrix T^i_j acting in the third space, we obtain the Yang-Baxter equation (3.8.2). Thus, in a certain sense (3.8.11) generalizes (3.8.2). One can consider eqs. (3.8.11) as defining relations for new infinite dimensional algebras $(L^\pm)^i_j(x) = (L^\pm)^i_j + \sum_{r>0} (L^\pm_{(r)})^i_j x^{\pm 2r}$ called quantum Yangians $Y^\pm_q(gl_N)$. These algebras are subalgebras in a quantum affine algebra $U_q(\hat{gl}_N)$ (the R matrix definition of $U_q(\hat{gl}_N)$ is given in [80], [61]). Formula (3.8.12) for the algebra (3.8.11) defines homomorphism $U_q(\hat{gl}_N) \to U_q(gl_N)$ which is called *evaluation homomorphism*. We also recall that (3.8.2) is the condition of unique ordering of the monomials of third degree $L_1(x)L_2(y)L_3(z)$ for the algebra (3.8.11) ("diamond" condition):

$$\begin{tikzcd}
 & L(y)L(x)L(z) \arrow[r] & L(y)L(z)L(x) \arrow[dr] & \\
L(x)L(y)L(z) \arrow[ur] \arrow[dr] & & & L(z)L(y)L(x) \\
 & L(x)L(z)L(y) \arrow[r] & L(z)L(x)L(y) \arrow[ur] &
\end{tikzcd}$$

The quantum determinants (the analogs of (3.4.28)) can also be constructed [62] for the algebra (3.8.11) associated with the R matrix (3.8.5):

$$\det{}_q(L(x)) \, \mathcal{E}_{<12\dots N} = \mathcal{E}_{<12\dots N} \, L_N(q^{N-1}x) \cdots L_2(qx) \, L_1(x) \quad \Leftrightarrow \tag{3.8.13}$$

$$\begin{aligned} \det{}_q(L(x)) &= Tr_{1\dots N} \left(A_{1\to N} \, L_N(q^{N-1}x) \cdots L_2(qx) \, L_1(x) \right) = \\ &= Tr_{1\dots N} \left(L_N(x) \, L_{N-1}(qx) \cdots L_1(q^{N-1}x) \, A_{1\to N} \right) \; , \end{aligned} \tag{3.8.14}$$

$$\det{}_q(L(x)) \, A_{1\to N} = L_N(x) \, L_{N-1}(qx) \cdots L_1(q^{N-1}x) \, A_{1\to N} \; , \tag{3.8.15}$$

where the rank 1 antisymmetrizer $A_{1\to N}$ has been introduced in (3.5.1). Eq. (3.8.13) is self-consistent since its right hand side has the same symmetry as the left hand side (the action on both sides of this equation by the projectors (3.8.6) $\mathbf{P}^+_k \sim \hat{R}_k(q^{-1})$ gives zero). The last form of the quantum determinant (3.8.14) is obtained with the help of (3.5.1) and (3.8.11).

Proposition 6. *The q-determinant* $\det_q(L(x))$ *is a generating function for central elements in the algebras (3.8.11) with the GL-type R matrices (3.8.5).*

Proof. We must prove that $[L^i_j(xy), \det_q(L(x))] = 0 \;\forall x, y$. Indeed,

$$L_{N+1}(xy)\, Tr_{1\dots N} \Big(L_N(x)\, L_{N-1}(qx) \cdots L_1(q^{N-1}x)\, A_{1\to N} \Big) = \tag{3.8.16}$$

$$\begin{gathered} Tr_{1\dots N} \Big(\hat{R}^{-1}_N(y) \dots \hat{R}^{-1}_1(q^{1-N}y)\, L_{N+1}(x) \cdots L_2(q^{N-1}x) L_1(xy) \cdot \\ \cdot \hat{R}_1(q^{1-N}y) \dots \hat{R}_N(y)\, A_{1\to N} \Big) \; . \end{gathered} \tag{3.8.17}$$

Using the Yang-Baxter eq. (3.8.2) and the representation of $A_{1\to N}$ in terms of the Baxterized elements (3.5.1) we deduce

$$\hat{R}_1(q^{1-N}y) \dots \hat{R}_N(y)\, A_{1\to N} = A_{2\to N+1}\, \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y)$$

By means of this relation and eq. (3.8.15) one can rewrite (3.8.17) in the form

$$\begin{gathered} \det_q(L(x))\, Tr_{1\dots N} \Big(\hat{R}^{-1}_N(y) \dots \hat{R}^{-1}_1(q^{1-N}y)\, L_1(xy) A_{2\to N+1} \cdot \\ \cdot \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y) A_{1\to N} \Big) = \\ = \det_q(L(x)) \, \Big(N^{-1}(y)\, L(xy)\, N(y) \Big)_{N+1} = \det_q(L(x))\, L_{N+1}(xy) \end{gathered} \tag{3.8.18}$$

where the matrices $N(y)$, $N(y)^{-1}$ are proportional to the unit matrix and defined by

$$(N(y))^{1\rangle}_{\langle N+1} = \mathcal{E}_{\langle 2\dots N+1}\, \hat{R}_1(y) \dots \hat{R}_N(q^{1-N}y)\, \mathcal{E}^{1\dots N\rangle} \; .$$

$$(N(y)^{-1})^{N+1\rangle}_{\langle 1} = \mathcal{E}_{\langle 1\dots N}\, \hat{R}^{-1}_N(y) \dots \hat{R}^{-1}_1(q^{1-N}y)\, \mathcal{E}^{2\dots N+1\rangle} \; .$$

Comparing (3.8.16) and (3.8.18) we obtain the statement of the Proposition. $\Box$

We now note that from the algebra (3.8.11), disregarding the particular representation (3.8.12) for the $L(x)$ operator, we can obtain a realization for the Yangian $Y(gl(N))$ [83], [22] (see also review paper [84]). Indeed, in (3.8.2) and (3.8.11) we make the change of spectral parameters

$$x = exp(-\frac{1}{2}\lambda(\theta - \theta')) \; , \;\; y = exp(-\frac{1}{2}\lambda\theta') \; . \tag{3.8.19}$$

Then the relations (3.8.2) and (3.8.11) can be rewritten in the form

$$\hat{R}_{12}(\theta - \theta')\, \hat{R}_{23}(\theta)\, \hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\, \hat{R}_{12}(\theta)\, \hat{R}_{23}(\theta - \theta') \Rightarrow \tag{3.8.20}$$

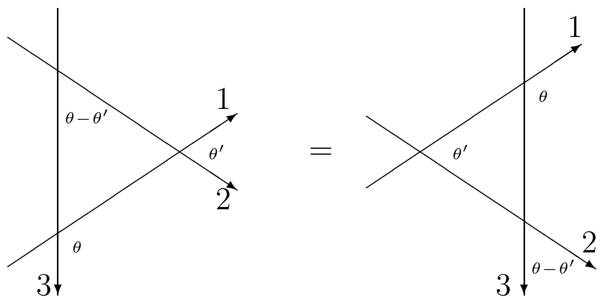
$$R_{23}(\theta - \theta')\, R_{13}(\theta)\, R_{12}(\theta') = R_{12}(\theta')\, R_{13}(\theta)\, R_{23}(\theta - \theta') \; , \tag{3.8.21}$$

$$\hat{R}_{12}(\theta - \theta')\, L_2(\theta)\, L_1(\theta') = L_2(\theta')\, L_1(\theta)\, \hat{R}_{12}(\theta - \theta'), \tag{3.8.22}$$

where we have made the redefinition $L(\theta) := L\left(\exp(-\frac{\lambda}{2}\theta)\right)$,

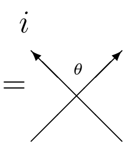
$$\hat{R}(\theta) := \hat{R}\left(e^{-\frac{\lambda}{2}\theta}\right) = \cosh\left(\lambda\theta/2\right) + \frac{1}{\lambda}\,\sinh\left(\lambda\theta/2\right)(\hat{R} + \hat{R}^{-1})\ . \tag{3.8.23}$$

Note that Eqs. (3.8.21) have a beautiful graphical representation in the form of the triangle equation [3]



(3.8.24)

where the arrowed lines show trajectories of point particles and

$R_{ij}(\theta) =$ 

is a single act of their scattering.

We now take the limit $\lambda = q - q^{-1} \to 0$ in eq. (3.8.22). On the basis of (3.8.5), (3.8.23) we readily find that in this limit the matrix $\hat{R}(\theta)$ is equal to the Yang matrix:

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, P_{12})\ . \tag{3.8.25}$$

For the operators $L(\theta)$, we shall assume the validity of the expansion

$$L(\theta)^i_j = \sum_{k=0}^{\infty} T^{(k)}{}^i_j \theta^{-k}\ , \tag{3.8.26}$$

where $T^{(0)}{}^i_j = \delta^i_j$ and $T^{(k)}{}^i_j$ $(k > 0)$ are the generators of the Yangian $Y(gl(N))$ (Ref. [22]). The defining relations for the Yangian $Y(gl(N))$ are obtained from (3.8.22) by substituting (3.8.25) and (3.8.26). The comultiplication for $Y(gl(N))$ obviously has the form

$$\Delta(L(\theta)^i_j) = L(\theta)^i_k \otimes L(\theta)^k_j\ . \tag{3.8.27}$$

The Yangian $Y(sl(N))$ can be obtained from $Y(gl(N))$ after the imposition of a subsidiary condition on the generators $T^{(k)}{}^i_j$:

$$\det_q(L(\theta)) = 1\ ,$$

where the Yangian quantum determinant [85]

$$\det_q(L(\theta)) = Tr_{1\ldots N}\left(A_{1\to N}\, L_N(\theta - N + 1)\cdots L_2(\theta - 1)\, L_1(\theta)\right)$$

(here $A_{1\to N}$ is a classical antisymmetrizer) is obtained from (3.8.14) if we substitute $q = e^h$, $x = \exp(-\frac{\lambda}{2}\theta) \sim e^{-h\theta}$ and take the limit $h \to 0$ (or $\lambda \to 0$).

Since the $\hat{R}$ matrix (3.7.1), (3.7.2) (for the group $GL_q(N|M)$) satisfies the Hecke condition (3.4.10), the same Baxterized R matrix (3.8.5) is appropriate for the supersymmetric case. All statements of this subsection can be readily reformulated for the supersymmetric case. In particular, the Yangian R matrix for $Y(gl(N|M))$ is deduced from (3.8.23) and has the form (cf. (3.8.25)):

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, \mathcal{P}_{12}) \,, \tag{3.8.28}$$

where $\mathcal{P}_{12}$ is a supertransposition operator introduced in (3.7.3). The defining relations (3.8.11) should be modified for the super Yangian $Y(gl(N|M))$ (cf. (3.7.10)):

$$\hat{R}_{12}(\theta-\theta')\,(-)^{(1)(2)}\,L_2(\theta)\,(-)^{(1)(2)}\,L_1(\theta') = (-)^{(1)(2)}\,L_2(\theta')\,(-)^{(1)(2)}\,L_1(\theta)\,\hat{R}_{12}(\theta-\theta'), \tag{3.8.29}$$

while the form of the comultiplication (3.8.27) (where $\otimes$ is the graded tensor product) is unchanged. Taking into account (3.8.26) and (3.8.28) we obtain the component form of the defining relations (3.8.29) for $Y(gl(N|M))$

$$[T^{(r)}{}^i_j,\, T^{(s+1)}{}^k_l\} - [T^{(r+1)}{}^i_j,\, T^{(s)}{}^k_l\} = (-1)^{[k][i]+[k][j]+[i][j]}\left(T^{(s)}{}^k_j\, T^{(r)}{}^i_l - T^{(r)}{}^k_j\, T^{(s)}{}^i_l\right)$$

where $r, s \geq 0$, $T^{(0)}{}^i_j = (-1)^{[i]}\,\delta^i_j$, the parities $[i] = 0, 1$ are given in (3.7.4) and $[a, b\} := a\,b - (-1)^{deg(a)deg(b)}b\,a$ is the supercommutator.

The relations (3.8.22), (3.8.29) play an important role in the quantum inverse scattering method [1]. Equations (3.8.21) are the conditions of factorization of the S matrices in certain exactly solvable two-dimensional models of quantum field theory (see Ref. [3]). The matrix representations for the operators (3.8.12) satisfying (3.8.11) lead to the formulation of lattice integrable systems (see, for example, Ref. [86]). These questions will be discussed in more detail in the final section of the review.

Another interesting presentations of quantum operator $L(x)$, which is intertwined by (3.8.11), are given in [87], [88]. These L operators were applied to the formulation of 3-dimensional integrable models. Note that in [88] the nonstandard many-parameter R matrix (3.6.3) has been used.

The super Yangians $Y(gl(N|M))$ and their representations have been discussed in [93], [89].

3.9 The quantum groups $SO_q(N)$ and $Sp_q(2n)$ (B, C, and D series)

In Ref. [17], quantum groups with the defining relations (3.2.1), which are quantum deformations of the Lie groups $SO(N)$, where $N = 2n+1$ (B_n series) and $Sp(N)$, $SO(N)$, where $N = 2n$ (C_n and D_n series), were studied. It was shown that the R matrices for the groups $SO_q(N)$ and $Sp_q(2n)$ (their explicit form [17] is given below in Sec. 3.9) satisfy not only the Yang - Baxter eq. (3.1.1), (3.1.4) but also the cubic characteristic equation (3.1.41):

$$(\hat{R} - q\mathbf{1})(\hat{R} + q^{-1}\mathbf{1})(\hat{R} - \nu\mathbf{1}) = 0 \tag{3.9.1}$$

where $\nu = \epsilon q^{\epsilon - N}$ is a "singlet" eigenvalue of $\hat{R}$, the case $\epsilon = +1$ corresponds to the orthogonal groups $SO_q(N)$ (B and D series), while the case $\epsilon = -1$ corresponds to the symplectic groups $Sp_q(2n)$ (C series). The projectors (3.1.43) corresponding to the characteristic equation (3.9.1) can be written as follows [17]

$$\mathbf{P}^{\pm} = \frac{(\hat{R} \pm q^{\mp 1}\mathbf{1})(\hat{R} - \nu \mathbf{1})}{(q + q^{-1})(q^{\pm 1} \mp \nu)} \equiv \frac{1}{q + q^{-1}} \left(\pm \hat{R} + q^{\mp 1}\mathbf{1} + \mu_{\pm}\mathbf{K} \right) \; , \tag{3.9.2}$$

$$\mathbf{P}^{0} = \frac{(\hat{R} - q\mathbf{1})(\hat{R} + q^{-1}\mathbf{1})}{(\nu - q)(q^{-1} + \nu)} \equiv \mu^{-1}\mathbf{K} \; .$$

Here

$$\mu = \frac{(q-\nu)(q^{-1}+\nu)}{\lambda \nu} = \frac{\lambda + \nu^{-1} - \nu}{\lambda} = (1 + \epsilon[N - \epsilon]_q) \; ,$$

$$\mu_{\pm} = \pm \frac{\lambda}{(1 \mp q^{\pm 1}\nu^{-1})} = \mp \frac{\nu \pm q^{\mp 1}}{\mu} .$$

We also give the relations between the parameters ν, μ, $\mu_{\pm}$ that we introduced:

$$q\mu_{+} - q^{-1}\mu_{-} = \nu(\mu_{+} + \mu_{-}) \; , \quad \mu_{+} + \mu_{-} = -\frac{q + q^{-1}}{\mu} \; ,$$

which are helpful in various calculations that use the projectors (3.9.2). For convenience, we have introduced in (3.9.2) the renormalized projector $\mathbf{K}^{i_1 i_2}_{j_1 j_2}$, which projects $\hat{R}$ onto the "singlet" eigenvalue ν:

$$\mathbf{K}\hat{R} = \hat{R}\mathbf{K} = \nu \mathbf{K} \; , \quad (\mathbf{K}^2 = \mu \mathbf{K}) \; . \tag{3.9.3}$$

Then, the characteristic equation (3.9.1) is rewritten in a different form [cf. (3.1.46)]

$$\hat{R} - \hat{R}^{-1} - \lambda + \lambda \mathbf{K} = 0 \; . \tag{3.9.4}$$

In the semiclassical limit, this characteristic equation is reduced to the relation:

$$\frac{1}{2}(r_{12} + r_{21}) = P_{12} - \epsilon \mathbf{K}^{(0)}_{12} \; . \tag{3.9.5}$$

Thus, as in the $GL_q(N)$ case (3.3.8), the semiclassical limit (3.9.5) of the characteristic equation fixes the ad-invariant part of the classical r matrix. Here we have used an expansion of the matrix $\mathbf{K} = \mathbf{K}^{(0)} + h\mathbf{K}^{(1)} + O(h^2)$, where the first term is

$$(\mathbf{K}^{(0)})^{i_1 i_2}_{j_1 j_2} = (C_0)^{i_1 i_2}(C_0^{-1})_{j_1 j_2} \Rightarrow \mathbf{K}^{(0)}_{12} = C_0^{12\rangle}\, (C_0^{-1})_{\langle 12} \; . \tag{3.9.6}$$

The matrices $(C_0)^{ij}$: $(C_0)^2 = \epsilon$, $(C_0)^t = \epsilon C_0$ are the metric (symmetric) and symplectic (antisymmetric) matrices, respectively, for the groups $SO(N)$ and $Sp(2n)$. The semiclassical expansion for the projectors (3.9.2) and (3.9.40) has the form

$$\begin{aligned} \mathbf{P}^{\pm}_{cl} &= \tfrac{1}{2}\left((\mathbf{1} \pm P) \pm hP\tilde{r} - (1 \pm \epsilon)\mathbf{P}^{0}_{cl} \right) \; , \\ \mathbf{P}^{0}_{cl} &= \tfrac{\epsilon}{N}\left(\mathbf{K}^{(0)} + h\mathbf{K}^{(1)} \right) \end{aligned} \tag{3.9.7}$$

where $(P)^{i_1 i_2}_{j_1 j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the premutation matrix, and the classical r matrix $\tilde{r}$ (3.3.8) (which satisfies the modified classical Yang-Baxter equation) is given by the formula:

$$\tilde{r} = r_{12} - P_{12} + \epsilon K^{(0)}_{12} = -r_{21} + P_{12} - \epsilon K^{(0)}_{12} \ .$$

The ranks of the quantum projectors (3.9.2) are equal (for q which is not the root of unity) to the ranks of the projectors (3.9.7), which are readily calculated in the classical limit $h = 0$. Accordingly, we have [17]:
1) for the groups $SO_q(N)$

$$\mathrm{rank}(P^{(+)}) = \frac{N(N+1)}{2} - 1, \ \mathrm{rank}(P^{(-)}) = \frac{N(N-1)}{2}, \ \mathrm{rank}(P^{(0)}) = 1 \ ; \quad (3.9.8)$$

2) for the groups $Sp_q(2n)$

$$\mathrm{rank}(P^{(+)}) = \frac{N(N+1)}{2}, \ \mathrm{rank}(P^{(-)}) = \frac{N(N-1)}{2} - 1, \ \mathrm{rank}(P^{(0)}) = 1 \ . \quad (3.9.9)$$

The number of generators for the algebras $Fun(SO_q(N))$ and $Fun(Sp_q(2n))$ coincides with the number of generators in the undeformed case, since for the quantum generators T^i_j (3.2.1) the following subsidiary conditions are imposed:

$$TCT^tC^{-1} = CT^tC^{-1}T = 1 \Rightarrow \quad (3.9.10)$$

$$T_1T_2\, C^{12\rangle} = C^{12\rangle} \ , \ \ C^{-1}_{\langle 12}\, T_1T_2 = C^{-1}_{\langle 12} \ . \quad (3.9.11)$$

These relations directly generalize the classical conditions for the elements of the groups $SO(N)$ and $Sp(2n)$. The matrices C^{ij}, C^{-1}_{kl}, which are understood in (3.9.11) as objects in $Vect(N) \otimes Vect(N)$ (1 and 2 label the spaces) are the q analogs of the metric and symplectic matrices C_0 for $SO(N)$ and $Sp(N)$, respectively. The explicit form of these matrices, which is given in Ref. [17] (see also Sec. 3.9) is not important for us, but we stress that the following equation holds

$$C^{-1} = \epsilon C \ . \quad (3.9.12)$$

Substituting the R matrix representations (3.2.17) for T^i_j in the relations (3.9.10), we obtain the following conditions on the R matrices:

$$R_{12} = C_1(R^{t_1}_{12})^{-1}C_1^{-1} = C_2(R^{-1}_{12})^{t_2}C_2^{-1} \ , \quad (3.9.13)$$

where, as usual, $C_1 = C \otimes I$ and $C_2 = I \otimes C$. As consequences of (3.9.13) we have the equation

$$R^{t_1 t_2}_{12} = C_1^{-1}C_2^{-1}R_{12}C_1C_2 \ , \quad (3.9.14)$$

and also subsidiary conditions on the generators of the dual algebra (3.2.20), (3.2.21):

$$L^{\pm}_2\, L^{\pm}_1\, C^{12\rangle} = C^{12\rangle} \ , \ \ C^{-1}_{\langle 12}\, L^{\pm}_2\, L^{\pm}_1 = C^{-1}_{\langle 12} \ . \quad (3.9.15)$$

The semiclassical analogs of the conditions (3.9.13) and (3.9.14) have the form

$$r_{12} = -(C_0)_1 r^{t_1}_{12}(C_0)^{-1}_1 = -(C_0)_2 r^{t_2}_{12}(C_0)^{-1}_2 = (C_0)_1(C_0)_2 r^{t_1 t_2}_{12}(C_0)^{-1}_1(C_0)^{-1}_2 \ .$$

It follows from Eqs. (3.9.10) and (3.9.12) that the antipode $S(T) = C\,T^t\,C^{-1}$ for the Hopf algebras $Fun(SO_q(N))$ and $Fun(Sp_q(N))$ satisfy the relation

$$S^2(T) = (CC^t)T(CC^t)^{-1} \; , \tag{3.9.16}$$

which is analogous to (3.2.4). Thus, the matrix D that defines the quantum trace for the quantum groups of the SO and Sp series can be chosen in the form

$$D = \epsilon\nu\, CC^t \Leftrightarrow D^i_j = \epsilon\nu\, C^{ik} C^{jk} \; . \tag{3.9.17}$$

Here we choose the numerical factor $\epsilon\nu$ in order to relate (3.9.17) with the general definitions of D-matrix (3.1.12), (3.1.14).

We now note that the matrix $C^{12\rangle}\, C^{-1}_{\langle 12} \in Mat(N) \otimes Mat(N)$ projects any vector $X^{12\rangle}$ onto the vector $C^{12\rangle}$, i.e., the rank of the projector $C^{12\rangle} C^{-1}_{\langle 12}$ is 1. In addition, from (3.9.11) we have eq.

$$C^{12\rangle} C^{-1}_{\langle 12}\, T_1\, T_2 = T_1\, T_2\, C^{12\rangle} C^{-1}_{\langle 12} \; ,$$

which means that the projector $C^{12\rangle} C^{-1}_{\langle 12}$ should be a polynomial in $\hat{R}$. Therefore $C^{12\rangle} C^{-1}_{\langle 12} \sim \mathbf{P}^0_{12}$, and, as it was established in Ref. [17],

$$C^{12\rangle} C^{-1}_{\langle 12} \equiv \mathbf{K}_{12} \; . \tag{3.9.18}$$

Using this relation, RTT relations (3.2.1) and equations (3.9.4) one can deduce

$$T_1\, T_2\, \mathbf{K}_{12} = \mathbf{K}_{12}\, T_1\, T_2 = \tau\, \mathbf{K}_{12} \; , \tag{3.9.19}$$

where the scalar element $\tau = \mu^{-1} C^{-1}_{\langle 12} T_1 T_2 C^{12\rangle}$ has been introduced. Comparing of Eq. (3.9.19) with Eqs. (3.9.10) and (3.9.11) we conclude that $\tau = 1$. Therefore, for the correct definition of the quantum groups $SO_q(N)$ and $Sp_q(N)$ we should require the centrality of the element τ in the RTT algebra.

We note that eqs. (3.9.3), (3.9.18) are equivalent to the relations

$$\hat{R}_{12}\, C^{12\rangle} = \nu\, C^{12\rangle} \; , \quad C^{-1}_{\langle 12}\, \hat{R}_{12} = \nu\, C^{-1}_{\langle 12} \tag{3.9.20}$$

which give the possibility to rewrite conditions (3.9.15) for the generators of the reflection equation algebras (3.2.27):

$$\begin{gathered}
L_1\, \hat{R}_{12}\, L_1\, C^{12\rangle} = \nu\, C^{12\rangle} \; , \quad C^{-1}_{\langle 12}\, L_1\, \hat{R}_{12}\, L_1 = \nu\, C^{-1}_{\langle 12} \; , \\
\overline{L}_2\, \hat{R}_{12}\, \overline{L}_2\, C^{12\rangle} = \nu\, C^{12\rangle} \; , \quad C^{-1}_{\langle 12}\, \overline{L}_2\, \hat{R}_{12}\, \overline{L}_2 = \nu\, C^{-1}_{\langle 12} \; .
\end{gathered}$$

We now present some important relations for the matrices $\hat{R}$ and $\mathbf{K}$; many of them are given, in some form or other, in Ref. [17]. We note first that in accordance with (3.1.8)

$$\mathbf{K}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \mathbf{K}_{23} \Leftrightarrow \hat{R}_{12}\, \hat{R}_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, \hat{R}_{12}\, \hat{R}_{23}. \tag{3.9.21}$$

Further, from Eqs. (3.9.13) and (3.9.18) [or substituting the matrix representations (3.2.17) in (3.9.19) for $\tau = 1$], we obtain

$$\begin{gathered}
\hat{R}^{\pm 1}_{12}\, \hat{R}^{\pm 1}_{23}\, \mathbf{K}_{12} = P_{12}\, P_{23}\, \mathbf{K}_{12} = \mathbf{K}_{23}\, P_{12}\, P_{23} \; , \\
\mathbf{K}_{12}\, \hat{R}^{\pm 1}_{23}\, \hat{R}^{\pm 1}_{12} = \mathbf{K}_{12}\, P_{23}\, P_{12} = P_{23}\, P_{12}\, \mathbf{K}_{23} \; .
\end{gathered} \tag{3.9.22}$$

A consequence of these relations is the equations

$$\hat{R}_{23}^{\pm 1}\,\mathbf{K}_{12}\,\hat{R}_{23}^{\pm 1} = \hat{R}_{12}^{\mp 1}\,\mathbf{K}_{23}\,\hat{R}_{12}^{\mp 1} \Leftrightarrow \hat{R}_{12}\,\hat{R}_{23}\,\mathbf{K}_{12} = \mathbf{K}_{23}\,\hat{R}_{12}^{-1}\,\hat{R}_{23}^{-1}\;, \qquad \hat{R}_{23}\,\hat{R}_{12}\,\mathbf{K}_{23} = \mathbf{K}_{12}\,\hat{R}_{23}^{-1}\,\hat{R}_{12}^{-1}\;. \tag{3.9.23}$$

In particular, taking into account the characteristic equation (3.9.4), we obtain the identity

$$(\hat{R}_{12}-\lambda)\,\mathbf{K}_{23}\,(\hat{R}_{12}-\lambda) = (\hat{R}_{23}-\lambda)\,\mathbf{K}_{12}\,(\hat{R}_{23}-\lambda)$$

or

$$\hat{R}_{12}\,\mathbf{K}_{23}\hat{R}_{12} = \hat{R}_{23}^{-1}\,\mathbf{K}_{12}\,\hat{R}_{23}^{-1} =$$

$$\hat{R}_{23}\,\mathbf{K}_{12}\,\hat{R}_{23} + \lambda(\hat{R}_{12}\,\mathbf{K}_{23} - \mathbf{K}_{12}\,\hat{R}_{23} - \hat{R}_{23}\,\mathbf{K}_{12} + \mathbf{K}_{23}\,\hat{R}_{12}) + \lambda^2(\mathbf{K}_{12}-\mathbf{K}_{23})\;, \tag{3.9.24}$$

which will be used in Sec. 3.10. Equation (3.9.18) leads to the identities

$$\mathbf{K}_{12}\,\mathbf{K}_{23} = \mathbf{K}_{12}\,P_{23}\,P_{12} = P_{23}\,P_{12}\,\mathbf{K}_{23}\;, \quad \mathbf{K}_{23}\,\mathbf{K}_{12} = P_{12}\,P_{23}\,\mathbf{K}_{12} = \mathbf{K}_{23}\,P_{12}\,P_{23}\;, \tag{3.9.25}$$

from which we immediately obtain

$$\mathbf{K}_{12}\,\mathbf{K}_{23}\,\mathbf{K}_{12} = \mathbf{K}_{12}\;, \quad \mathbf{K}_{23}\,\mathbf{K}_{12}\,\mathbf{K}_{23} = \mathbf{K}_{23}\;. \tag{3.9.26}$$

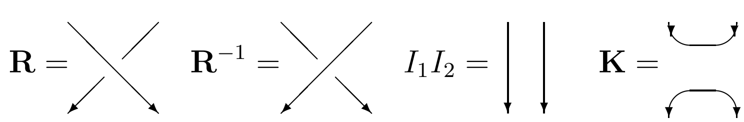
We now compare the relations (3.9.22) and (3.9.25). The result of this comparison is the equations

$$\hat{R}_{23}^{\pm 1}\,\hat{R}_{12}^{\pm 1}\,\mathbf{K}_{23} = \mathbf{K}_{12}\,\mathbf{K}_{23} = \mathbf{K}_{12}\,\hat{R}_{23}^{\pm 1}\hat{R}_{12}^{\pm 1}\;, \qquad \hat{R}_{12}^{\pm 1}\,\hat{R}_{23}^{\pm 1}\,\mathbf{K}_{12} = \mathbf{K}_{23}\,\mathbf{K}_{12} = \mathbf{K}_{23}\,\hat{R}_{12}^{\pm 1}\,\hat{R}_{23}^{\pm 1}, \tag{3.9.27}$$

We now apply to the first of the chain of equations in (3.9.27) the matrix $\mathbf{K}_{12}$ from the right (or $\mathbf{K}_{23}$ from the left) and take into account (3.9.3) and (3.9.26). We then obtain

$$\mathbf{K}_{23}\,\hat{R}_{12}^{\pm 1}\,\mathbf{K}_{23} = \nu^{\mp 1}\mathbf{K}_{23}\;, \quad \mathbf{K}_{12}\,\hat{R}_{23}^{\pm 1}\,\mathbf{K}_{12} = \nu^{\mp 1}\mathbf{K}_{12}\;. \tag{3.9.28}$$

The braid relation (3.1.6) and eqs. (3.9.3), (3.9.4), (3.9.28) define the R- matrix representation of the Birman - Murakami - Wenzl algebra [123]. Eqs. (3.9.21), (3.9.23), (3.9.24), (3.9.26) and (3.9.27) directly follow from this definition. As we shall see in Sec. 3.10, the relations for the Birman - Murakami - Wenzl algebra will be sufficient for the construction of $SO_q(N)$ and $Sp_q(2n)$-symmetric Baxterized $\hat{R}(x)$ matrices. The relations (3.9.21), (3.9.23), and (3.9.26)-(3.9.28) have a natural graphical representation in the form of relations for braids and links if we use the diagrammatic technique (only 3 of these operators are independent in view of (3.9.4))



(3.9.29)

We now give some important relations for the quantum trace (3.2.11) corresponding to the quantum groups $SO_q(N)$ and $Sp_q(N)$. Similar relations for the q-

trace (3.2.13) can be derived in exactly the same way. From the definitions of the matrix $\mathbf{K}$ (3.9.18) and the matrix D (3.9.17), we obtain

$$Tr_{q2}(\mathbf{K}_{12}) = \nu\, I_{(1)} \ . \tag{3.9.30}$$

We use the relations (3.9.14) and the definition of the quantum trace (3.2.11) with matrix D (3.9.17); then, for an arbitrary quantum matrix E^i_j, we obtain the relations:

$$\begin{gathered} \nu\, \hat{R}^n_{12}\, E_1\, \mathbf{K}_{12} = Tr_{q2}(\mathbf{K}_{12}\, E_1\, \hat{R}^n_{12})\mathbf{K}_{12} \ , \\ \nu\, \mathbf{K}_{12}\, E_1\, \hat{R}^n_{12} = \mathbf{K}_{12}\, Tr_{q2}(\hat{R}^n_{12}\, E_1\, \mathbf{K}_{12}) \ , \forall n \ , \end{gathered} \tag{3.9.31}$$

$$\nu\, \mathbf{K}_{12}\, E_1\, \mathbf{K}_{12} = Tr_q(E)\mathbf{K}_{12} \ . \tag{3.9.32}$$

Calculating the Tr_{q2} of (3.9.31), we deduce

$$Tr_{q2}(\hat{R}^n_{12}\, E_1\, \mathbf{K}_{12}) = Tr_{q2}(\mathbf{K}_{12}\, E_1\, \hat{R}^n_{12}) \ , \quad \forall n \ . \tag{3.9.33}$$

Further, from the first identity of (3.9.28), averaging it by means of Tr_{q2}, we readily obtain for the algebras $Fun(SO_q(N))$ and $Fun(Sp_q(N))$ analogs of (3.2.15). These take the form

$$Tr_{q2}(\hat{R}^{\pm 1}_{12}) \equiv \epsilon\, \nu\, Tr_2(CC^t \hat{R}^{\pm 1}_{12}) = \nu^{1 \mp 1} I_{(1)}. \tag{3.9.34}$$

Using this relation and Eq. (3.9.4), we can calculate

$$Tr_q(I) = Tr(D) = \nu(1 + \epsilon\, [N - \epsilon]_q) = \nu\, \mu \ . \tag{3.9.35}$$

We now separate irreducible representations for the left adjoint comodules (3.2.9). For an arbitrary $N \times N$ quantum matrix E^i_j, we have

$$E_1 = \nu^{-1} Tr_{q2}(E_1\, \mathbf{K}_{12}) = E^{(0)}_1 + E^{(+)}_1 + E^{(-)}_1 \ , \tag{3.9.36}$$

where $E^{(i)}_1 = \nu^{-1}\, Tr_{q2}(\mathbf{P}^i_{12}\, E_1\, \mathbf{K}_{12}) = \nu^{-1}\, Tr_{q2}(\mathbf{K}_{12}\, E_1\, \mathbf{P}^i_{12})$. It is obvious that the tensors $E^{(i)}$, $(i = \pm,\ 0)$ are invariant with respect to the adjoint coaction (3.2.9) and $Tr_{q2}(\mathbf{P}^{(j)} E^{(i)} \mathbf{K}) = 0$ (if $i \neq j$) by virtue of (3.9.31). Thus, (3.9.36) is the required decomposition of the adjoint comodule E into irreducible components. It is clear that the component $E^{(0)}$ is proportional to the unit matrix, $(E^{(0)})^i_j = e^{(1)} \cdot \delta^i_j$ ($e^{(1)}$ is a constant), and, thus, applying Tr_{q1} to (3.9.36), we obtain

$$Tr_q(E) = e^{(1)} Tr_q(I) = \nu \mu E^{(1)} \ , \tag{3.9.37}$$

where we have used the property (3.4.15), which also holds for the case of the quantum groups $SO_q(N)$ and $Sp_q(2n)$. To conclude this subsection, we note that, as in the case of the linear quantum groups, we can define fermionic and bosonic quantum hyperplanes covariant with respect to the coactions of the groups $SO_q(N)$ and $Sp_q(2n)$. Taking into account the ranks of the projectors (3.9.8) and (3.9.9), we can formulate definitions of the hyperplanes for $SO_q(N)$ ($\epsilon = 1$) and for $Sp_q(N)$ ($\epsilon = -1$) in the form

$$(\mathbf{P}^- + (\epsilon - 1)\mathbf{K}) x x' = 0 \tag{3.9.38}$$

for the bosonic hyperplane [number of relations $N(N-1)/2$] and

$$(\mathbf{P}^{+} + (\epsilon+1)\mathbf{K})xx' = 0 \qquad (3.9.39)$$

for the fermionic hyperplane [number of relations $N(N+1)/2$]. For all these algebras, the elements $\mathbf{K}xx'$ are central elements, and it is obvious that for $Sp_q(N)$ bosons and $SO_q(N)$ fermions we have $\mathbf{K}xx' = 0$. It is interesting that the projectors $\mathbf{P}^{\pm}$ (3.9.2) can be represented as

$$\mathbf{P}^{\pm} = \frac{1}{q+q^{-1}}\left(\pm\hat{R}' + q^{\mp 1}\mathbf{1}\right) - \frac{1}{2\mu}(1\pm\epsilon)\mathbf{K}\ , \qquad (3.9.40)$$

where the matrix

$$\hat{R}' = \hat{R} - \frac{1}{2}[\mu_{-}(1+\epsilon) + \mu_{+}(\epsilon-1)]\mathbf{K}\ ,$$

satisfies the Hecke condition (3.1.46). However, using (3.9.21) – (3.9.28) one can directly check that $\hat{R}'$ does not obey the Yang-Baxter eq. (3.1.4).

Note that the conditions (3.9.10) and (3.9.11) can be understood as conditions of invariance of the quadratic forms $x_{(1)}C^{-1}x_{(2)}$ and $y_{(1)}Cy_{(2)}$ with respect to left and right transformations of the hyperplanes $x_{(k)}$, $y_{(k)}$:

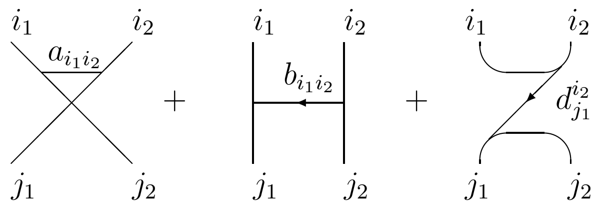
$$x^i_{(k)} \to T^i_j \otimes x^j_{(k)}\ ,\quad y_{(k)i} \to y_{(k)j} \otimes T^j_i\ .$$

3.10 The many-parameter case of the $SO_{q,a_{ij}}(N)$ and $Sp_{q,a_{ij}}(2n)$ groups and the quantum supergroups $Osp_q(N|2m)$

In this subsection we show that it is possible to define many-parameter deformations of the groups $SO(N)$ and $Sp(2n)$ and also the quantum supergroups $OSp_q(N|2m)$ if we consider for the R matrix the ansatz:

$$\hat{R} = \sum_{i,j=1}^{K} a_{ij}\, e_{ij}\otimes e_{ji} + \sum_{i<j} b_{ij}\, e_{ii}\otimes e_{jj} + \sum_{i>j} d^i_j\, e_{i'j}\otimes e_{ij'} \ \Rightarrow \qquad (3.10.1)$$

$$\hat{R}^{i_1,i_2}_{j_1,j_2} = \delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\, a_{i_1i_2} + \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\, b_{i_1i_2}\,\Theta_{i_2i_1} + \delta^{i_1i'_2}\delta_{j_1j'_2}\, d^{i_2}_{j_1}\,\Theta^{i_2}_{j_1} = \qquad (3.10.2)$$

$=$ 

where $\Theta^i_j = \Theta_{ij}$, $j' = K+1-j$, $K = N$ for the groups $SO(N)$, $K = 2n$ for the groups $Sp(2n)$, and $K = N+2m$ for the groups $Osp(N|2m)$. The expression (3.10.2) is a natural generalization of the expression (3.6.5) for the many-parameter R matrix corresponding to the linear quantum groups. Namely, the third term in (3.10.2) is constructed from the SO-invariant tensor $\delta^{i_1i'_2}\delta_{j_1j'_2}$, which takes into account the presence of the invariant metrics for the considered groups. The functions Θ are introduced in (3.10.2) in order to ensure that the matrix $R_{12} = P_{12}\hat{R}$ has lower triangular block form. This is necessary for the correct definition of the operators $L^{(\pm)}$ by means of the expressions (3.2.18). We demonstrate below that the ansatz (3.10.2) for the solution of the Yang-Baxter equation (3.1.4) automatically define the family of the Birman-Murakami-Wenzl R- matrices with fixed parameter ν which corresponds to the quantum groups $SO_q(N)$, $Sp_q(2n)$ and $Osp_q(N|2m)$.

We substitute the ansatz (3.10.2) for R matrix in the Yang-Baxter equation (3.1.4). It is obvious that the first two terms in (3.10.2) make contributions to the Yang-Baxter equation that are analogous to the contributions of the general R matrix ansatz in the case of the linear quantum groups (see Sec. 3.5). It is therefore clear that for the parameters a_{ij} and b_{ij} we reproduce almost the same conditions (3.6.6), which in the convenient normalization $c = 1,\ b = q - q^{-1}$ have the form

$$b_{ij} = b = \lambda \ \ (\forall i, j), \ \ a_{ii} = a^0_i = \pm q^{\pm 1} \ \ (i \neq i'), \ \ a_{ij}\, a_{ji} = 1 \ \ (i \neq j, \ i \neq j') \; . \quad (3.10.3)$$

Note that the conditions in (3.10.3) are somewhat weaker than in (3.6.6) (because of the restrictions $i \neq i'$, $i \neq j'$). This is due to the fact that the contributions to the Yang-Baxter equation proportional to $a_{ii'}$ begin to be canceled by the contributions from the third term in (3.10.2). The corresponding condition on $a_{ii'}$ fulfilling the Yang-Baxter equation can be expressed as follows:

$$\begin{gathered} a_{jj'} = \kappa_j^{-1}(a^0_j - b) \; , \quad a_{j'j} = \kappa_j(a^0_j - b) \; (j \neq j') \Leftrightarrow \\ a^0_j \, a_{jj'} = \kappa_j^{-1} \; , \quad a^0_j \, a_{j'j} = \kappa_j \; (j \neq j') \; , \end{gathered} \quad (3.10.4)$$

where in addition for the constants a^0_j and κ_i we have

$$\kappa_j \kappa_{j'} = 1 \; , \quad a^0_j = a^0_{j'} \; . \quad (3.10.5)$$

Taking into account Eqs. (3.10.3), the relations (3.10.4) are equivalent to the pair of possibilities ($j \neq j'$):

$$\begin{gathered} 1.) \ \ a^0_j = q \rightarrow a_{j'j}\, \kappa_j^{-1} = q^{-1} = a_{jj'}\, \kappa_j \; , \\ 2.) \ \ a^0_j = -q^{-1} \rightarrow a_{j'j}\, \kappa_j^{-1} = -q = a_{jj'}\, \kappa_j \; . \end{gathered} \quad (3.10.6)$$

We shall see below that if we restrict consideration to the first possibility for all j (or only the second possibility), then we obtain the R matrices for the quantum groups $SO_q(N)$ and $Sp_q(2n)$. If, however, we consider the mixed case, when both possibilities are satisfied (for different j), then we expect (by analogy with the linear quantum groups; see Sec. 3.6) that the corresponding R matrix will be associated with the supergroups $Osp_q(N|2m)$. The case $j = j'$ is obviously realized only for groups of the series B ($SO_q(2n+1)$) and for the supergroups $Osp_q(2n+1|2m)$, and it follows from the Yang-Baxter equation (3.1.6) that

$$a_{jj'} = 1 \; , \quad \kappa_j = 1 \; , \quad \text{for} \ \ j = j' = \frac{K+1}{2} \; . \quad (3.10.7)$$

For the groups $SO_q(2n)$, $Sp_q(2n)$ and $Osp_q(2n|2m)$, the parameter $a_{jj'}$ ($j = j'$) is simply absent. Further consideration of the contributions to the Yang-Baxter equation from the third term in (3.10.2) leads to the equations

$$a_{ij}\, a_{i'j} = \kappa_j \; , \quad a_{ji}\, a_{ji'} = \kappa_j^{-1} \quad (\forall i \neq i') \; , \quad (3.10.8)$$

$$\lambda \, d^j_k \, \kappa_i + d^j_i \, d^i_k = 0 \; , \qquad (3.10.9)$$

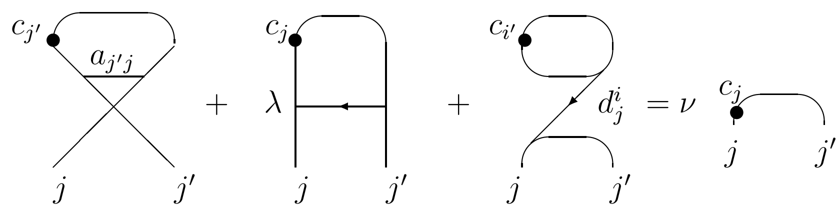
(there is no summation over repeated indices). The general solution of Eq. (3.10.9) has the form

$$d^i_j = -\lambda \, \kappa_i \, \frac{c_j}{c_i} \; , \qquad (3.10.10)$$

where c_i are arbitrary parameters. The remaining terms in the Yang-Baxter equation that do not cancel under the conditions (3.10.3)-(3.10.10) give recursion relations for the coefficients c_i:

$$c_{j'} a_{j'j} + \lambda c_j \Theta_{j'j} - \lambda c_j \sum_{i>j} \kappa_i \frac{c_{i'}}{c_i} = \nu c_j \; . \qquad (3.10.11)$$

These relations can be represented graphically in the form



Another equivalent forms of (3.10.11) are

$$\sum_{k>m} d^i_{k'} d^k_j = d^i_j \left(\nu - \frac{c_{m'}}{c_m} a_{m'm} - \lambda \Theta_{m'm} \right) \; , \qquad (3.10.12)$$

$$\sum_{k<m} d^i_{k'} d^k_j = d^i_j \left(-\nu^{-1} + \frac{c_{m'}}{c_m} \, a^{-1}_{m\,m'} - \lambda \Theta_{mm'} \right) \; , \qquad (3.10.13)$$

which are related to each other by the identity

$$\sum_k d^i_{k'} d^k_j = -\lambda \, \mu \, d^i_j \; , \quad (\mu := (\lambda - \nu + \nu^{-1})/\lambda) \; , \qquad (3.10.14)$$

using below. Now the R matrix (3.10.1) is represented in the form

$$\hat{R} = a_{ij} \, e_{ij} \otimes e_{ji} + \lambda \, \Theta_{ji} \, e_{ii} \otimes e_{jj} + \Theta_{ij} \, d^i_j \, e_{i'j} \otimes e_{ij'} \; , \qquad (3.10.15)$$

where the summation over the indices i, j is assumed and parameters a_{ij} and d^i_j are fixed by the conditions (3.10.3) – (3.10.8), (3.10.10) and (3.10.11). This R matrix satisfies the Yang-Baxter equation (3.1.4) and additional relations (cf. (3.9.3), (3.9.4) and (3.9.28))

$$\hat{R}^2 - \lambda \hat{R} - \mathbf{1} = -\lambda \nu \, \mathbf{K} \; , \quad \mathbf{K} \hat{R} = \hat{R} \mathbf{K} = \nu \mathbf{K} \; , \qquad (3.10.16)$$

$$\mathbf{K}_{12} \, \hat{R}^{\pm 1}_{23} \, \mathbf{K}_{12} = \nu^{\mp 1} \, \mathbf{K}_{12} \; , \quad \mathbf{K}^2 = \mu \, \mathbf{K} \; , \qquad (3.10.17)$$

where we have introduced the rank 1 matrix:

$$\mathbf{K} := -\lambda^{-1} \sum_{i,j} d^i_j \, e_{i'j} \otimes e_{ij'} = \sum_{i,j} \kappa_i \, \frac{c_j}{c_i} \, e_{i'j} \otimes e_{ij'} \;\; \Leftrightarrow \qquad (3.10.18)$$

$$\mathbf{K}^{i_1 i_2}_{j_1 j_2} = C^{i_1 i_2}\, C_{j_1 j_2} \; , \quad C^{ij} = \epsilon \delta^{ij'} \frac{\kappa_j}{c_j} \; , \;\; C_{ij} = \frac{1}{\epsilon} \delta_{ij'} c_i \; . \tag{3.10.19}$$

To prove relations (3.10.16), (3.10.17) we have used the definitions of a_{ij} (3.10.3) – (3.10.7), d^i_j (3.10.10) and take into account the identities (3.10.12) – (3.10.14) and

$$\Theta_{ki'}\Theta_{kj} = \Theta_{ki'}\Theta_{kj}(\Theta_{i'j} + \Theta_{ji'} + \delta_{i'j}) = \Theta_{i'j}\Theta_{ki'} + (\Theta_{ji'} + \delta_{i'j})\Theta_{kj} \; .$$

Thus, the R matrix (3.10.15) with constraints (3.10.3) – (3.10.8), (3.10.10) and (3.10.11) automatically leads to the R matrix representation of the Birman - Murakami - Wenzl algebra. In (3.10.19) we define the quantum metric (or symplectic) matrices C [cf. (3.9.18)]. The parameter ϵ (see Sec. 3.8) is introduced in (3.10.19) in order to match the definition of the matrices C to the study of Ref. [17], where $\epsilon = \pm 1$.

Note that the conditions (3.10.3) - (3.10.8) can be solved as:

$$a_{ij} = (a^0_i)^{(\delta_{ij} - \delta_{ij'})} \, \frac{f_{ij}}{f_{ji}} \; , \quad \frac{f_{ij} f_{i'j}}{f_{ji} f_{ji'}} = \kappa_j \;\; (\forall i \neq i') \;\; \Rightarrow \;\; \kappa_j = \frac{f_{j'j}}{f_{jj'}} \; , \tag{3.10.20}$$

and, after substitution of (3.10.20) in (3.10.15), one can observe that the R matrix (3.10.15) is

$$\hat{R} = \sum_{i,j} (a^0_i)^{(\delta_{ij} - \delta_{ij'})} \, \frac{f_{ij}}{f_{ji}} \, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{f_{i'i}}{f_{ii'}} \frac{c_j}{c_i} \, e_{i'j} \otimes e_{ij'} \; , \tag{3.10.21}$$

and produced by the twisting (3.6.4) from the matrix

$$\hat{R} = \sum_{i,j} (a^0_i)^{(\delta_{ij} - \delta_{ij'})} \, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{\tilde{c}_j}{\tilde{c}_i} \, e_{i'j} \otimes e_{ij'} \; , \tag{3.10.22}$$

where $\tilde{c}_i = f_{ii'} c_i$ and the parameters $a^0_j = a^0_{j'}$, c_j are determined in (3.10.6), (3.10.7) and (3.10.11). In this case the relations (3.2.53) lead to additional conditions on the twisting parameters f_{ij}:

$$f_{ij} f_{i'j} = \kappa_j v_j \; , \quad f_{ji} f_{ji'} = v_j \; , \;\; \forall i \; , \tag{3.10.23}$$

which are consistent with (3.10.8), (3.10.20). It is evident that for R matrix (3.10.22) the analogs of matrices (3.10.18), (3.10.19) are

$$\mathbf{K}^{i_1 i_2}_{j_1 j_2} = \sum_{i,j} \frac{\tilde{c}_j}{\tilde{c}_i} \, (e_{i'j})^{i_1}_{j_1} \otimes (e_{ij'})^{i_2}_{j_2} = \widetilde{C}^{i_1 i_2} \, \widetilde{C}_{j_1 j_2} \quad \Rightarrow$$

$$\widetilde{C}^{ij} = \epsilon \delta^{ij'} \frac{1}{\tilde{c}_j} \; , \quad \widetilde{C}_{ij} = \frac{1}{\epsilon} \delta_{ij'} \tilde{c}_i \quad \Rightarrow \quad \widetilde{C}^{ij} \, \widetilde{C}_{jk} = \delta^i_k \; . \tag{3.10.24}$$

Now we show that the constant ν is fixed by the relations (3.10.11) uniquely. We consider the solution of Eqs. (3.10.11), which we write in the form

$$\gamma_j \; a_{j'j} \; \kappa_j^{-1} + \lambda \; \Theta_{j'j} - \lambda \sum_{i=j+1}^{K} \gamma_i = \nu \; , \tag{3.10.25}$$

where

$$\gamma_j = \frac{c_{j'}}{c_j}\,\kappa_j = \frac{\tilde{c}_{j'}}{\tilde{c}_j} = \frac{1}{\gamma_{j'}}\ . \tag{3.10.26}$$

Equation (3.10.25) is readily solved by the changing of variables

$$X_j = q^{2j}\sum_{i=j+1}^{K}\gamma_i\ ,\ \ (X_K = 0)\ ,$$

where the inverse transformation is $\gamma_j = q^{-2j}(q^2 X_{j-1} - X_j)$ and we fix ν by taking into account the properties (3.10.26).

1. The case of groups $SO_q(N)$ and $Sp_q(N)$.

First, we consider the possibility 1 in (3.10.6). The possibility 2 gives, in view of a symmetry of equation (3.10.25), an analogous result except for the substitution $q \to -q^{-1}$. The corresponding form of eq. (3.10.26) for $j > j'$ is

$$q\,(X_{j-1} - X_j) = q^{2j}\,\nu\ ,$$

and we obtain the solution:

$$X_j = \nu q^{2K-1}\,\frac{1-q^{-2(K-j)}}{1-q^{-2}} \quad \Leftrightarrow \quad \gamma_j = \nu q^{2K-2j+1}\ ,\quad (j > j')\ . \tag{3.10.27}$$

For the case $K = 2n+1$ the possibility $j = j'(= K+1-j = n+1)$ is realized and Eq. (3.10.25) (in view of (3.10.7), (3.10.26), (3.10.27)) gives

$$\gamma_{n+1} = \nu q^{K-1} = 1 \ \ \Rightarrow \ \ \nu = q^{1-K}\ . \tag{3.10.28}$$

For the case $K = 2n$ we take $j = \frac{K}{2}$, $(j' = \frac{K}{2}+1 > j)$ in Eq. (3.10.25) and obtain $\gamma_{\frac{K}{2}} = \nu q^{K+1} - \lambda q$. On the other hand, eq. (3.10.27) gives $\gamma_{\frac{K}{2}+1} = \nu q^{K-1}$. Thus, in view of the condition $\gamma_{\frac{K}{2}} = \gamma^{-1}_{\frac{K}{2}+1}$ (3.10.26), we deduce the equation $1+\lambda q^K \nu = \nu^2 q^{2K}$ with two roots:

$$\nu_1 = q^{1-K}\ ,\quad \nu_2 = -q^{-1-K}\ . \tag{3.10.29}$$

We summarize the results (3.10.27) – (3.10.29), for the solution of (3.10.25), in the form

$$\gamma_j \equiv \frac{\tilde{c}_{j'}}{\tilde{c}_j} = \nu q^{2(N-j)+1}\ (j > j')\ ,\quad \gamma_j = 1\ (j = j')\ ,\quad \nu = \epsilon q^{\epsilon - N}\ , \tag{3.10.30}$$

(parameters $\tilde{c}_i$ have been introduced in (3.10.22)) and relate the cases ($\epsilon = +1$) and ($\epsilon = -1$) to the groups $SO_q(N)$ and $Sp_q(N)$, respectively.

In order to determine from the conditions (3.10.30) the parameters $\tilde{c}_j$ and, thus, to fix the matrices $\widetilde{C}$ (3.10.24), we require fulfillment of the relation $\widetilde{C}^{ij} = \epsilon \widetilde{C}_{ij}$ (cf. (3.9.12)). Substitution of (3.10.24) gives the equation $\tilde{c}_j \tilde{c}_{j'} = \epsilon$, which together with (3.10.30) enables us to choose $\tilde{c}_j$ in the form [17]

$$\tilde{c}_j = \epsilon\, q^{j-\frac{1}{2}(N+\epsilon+1)}\ \ (j > j') \Rightarrow \tilde{c}_j = \epsilon_j\, q^{-\rho_j}\ , \tag{3.10.31}$$

where

$$\begin{array}{c} \epsilon_i = +1 \; \forall i \;\; (\text{for } \; SO_q(N)) \; , \\ \epsilon_i = +1 \; (1 \leq i \leq n), \;\; \epsilon_i = -1 \; (n+1 \leq i \leq 2n) \;\; (\text{for } \; Sp_q(2n)), \end{array} \tag{3.10.32}$$

and

$$(\rho_1, \ldots, \rho_N) = \left\{ \begin{array}{l} (n-\frac{1}{2}, \, n-\frac{3}{2}, \ldots, \frac{1}{2}, \, 0, \, -\frac{1}{2}, \, \ldots, -n+\frac{1}{2}) \; B : (SO_q(2n+1)), \\ (n, \, n-1, \, \ldots, 1, \, -1, \ldots, \, 1-n, \, -n) \quad C : (Sp_q(2n)), \\ (n-1, \, n-2, \ldots, 1, \, 0, \, 0, \, -1, \ldots, 1-n) \; D : (SO_q(2n)). \end{array} \right. \tag{3.10.33}$$

Thus, the final expression for the R matrix (3.10.21) corresponding to the many-parameter deformation of the groups $SO(N)$ and $Sp(2n)$ (Ref. [78], [79]) is:

$$\hat{R}_{12} = \sum_{i,j} q^{(\delta_{ij} - \delta_{ij'})} \, \frac{f_{ij}}{f_{ji}} \, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} \frac{f_{i'i}}{f_{jj'}} \, q^{\rho_i - \rho_j} \, \epsilon_i \, \epsilon_j \, e_{i'j} \otimes e_{ij'} \; ,$$

where the parameters are defined in (3.10.23), (3.10.32), (3.10.33). The matrix $R = P\hat{R}$ is represented in the component form as

$$\begin{array}{c} R^{i_1, i_2}_{j_1, j_2} = \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \left[(q \, \delta^{i_1 i_2}|_{i_1 \neq i'_2} + q^{-1} \, \kappa_{i_1} \, \delta^{i_1 i'_2}|_{i_1 \neq i_2} + a_{i_2 i_1}|_{i_1 \neq i_2 \neq i'_1} + \delta^{i_1 i'_1} \delta^{i_2 i'_2} \right] + \\ + \lambda \, \delta^{i_1}_{j_2} \delta^{i_2}_{j_1} \, \Theta_{i_1 i_2} - \lambda \, \kappa_{i_1} \delta^{i_1 i'_2} \delta_{j_1 j'_2} \, \Theta^{i_1}_{j_1} \, \epsilon_{i_1} \, \epsilon_{j_1} q^{\rho_{i_1} - \rho_{j_1}} \; , \end{array} \tag{3.10.34}$$

where

$$\begin{array}{c} a_{ij} = 1/a_{ji} = \frac{f_{ij}}{f_{ji}} \;\; \forall j \neq i \neq j' \; , \quad a_{jj'} \kappa_j = a_{j'j} \kappa_j^{-1} = q^{-1} \;\; \forall j \neq j' \; , \\ a_{ij} \, a_{i'j} = \kappa_j \; , \quad a_{ji} \, a_{ji'} = \kappa_j^{-1} \; , \;\; \kappa_i = (\kappa_{i'})^{-1} = \frac{f_{i'i}}{f_{ii'}} \; . \end{array} \tag{3.10.35}$$

Now we clarify the role of the parameters κ_i. We show that for $\kappa_i \neq \pm 1$ the element τ introduced in (3.9.19) is not central [78]. Indeed, we take the identity $\mathbf{K}_{12} \mathbf{K}_{23} \mathbf{K}_{12} = \mathbf{K}_{12}$ (which is readily deduced from the explicit representation (3.10.18)) and multiply it by $T_1 T_2 T_3$ from the right. For the right-hand side we have

$$\mathbf{K}_{12} T_1 T_2 T_3 = \tau \, \mathbf{K}_{12} \, T_3 \; , \tag{3.10.36}$$

while for the left-hand side we obtain

$$\mathbf{K}_{12} \mathbf{K}_{23} \mathbf{K}_{12} T_1 T_2 T_3 = \mathbf{K}_{12} T_1 \mathbf{K}_{23} T_2 T_3 \, \mathbf{K}_{12} = \mathbf{K}_{12} T_1 \mathbf{K}_{23} \, \mathbf{K}_{12} \tau = X_3 T_3 X_3^{-1} \mathbf{K}_{12} \tau \; , \tag{3.10.37}$$

where $X^i_j = C_{jk} C^{ki} = \delta^i_j \, \kappa_i$, $(X^{-1})^i_j = C_{kj} C^{ik} = \delta^i_j \, \kappa_{i'}$ and we have used the identity $T_1 \mathbf{K}_{23} \, \mathbf{K}_{12} = \mathbf{K}_{23} \, \mathbf{K}_{12} X_3 T_3 X_3^{-1}$ which is obtained from the definition (3.10.19). Comparing (3.10.36) and (3.10.37) we obtain $\tau \, T^i_j = \kappa_i T^i_j \kappa_{j'} \, \tau$. Thus, only for $\kappa_i = \pm 1$ the element τ is central and one can relate the R matrices (3.10.34) with quantum deformations of the groups $SO(N)$ and $Sp(2n)$.

The conditions (3.10.35), show that for $\kappa_i = \pm 1$ the independent parameters are q and a_{ij} for $i < j \leq j'$ (the numbers of these parameters are $n(n-1)/2 + 1$ and $n(n+1)/2 + 1$, respectively, for the groups of the series C, D, $N = 2n$ and B, $N = 2n + 1$). Note that the last term in the square brackets in the expression (3.10.34) is present only for the groups of the series B. If we set $a_{ij} = 1$ $(j' \neq i \neq j)$,

$\kappa_i = 1$ then the R matrices (3.10.34) are identical to the one-parameter matrices $R = P\hat{R}$ deduced from (3.10.22) and given in Ref. [17]

$$R_{12} = \sum_{i,j} q^{(\delta_{ij}-\delta_{ij'})}\, e_{ii} \otimes e_{jj} + \lambda \sum_{i>j} e_{ij} \otimes e_{ji} - \lambda \sum_{i>j} q^{\rho_i - \rho_j}\, \epsilon_i\, \epsilon_j\, e_{ij} \otimes e_{i'j'} \; . \quad (3.10.38)$$

2. The case of supergroups $Osp_q(N|2m)$.

For the groups $Osp(N|2m)$ $(K = N + 2m)$, we choose a grading in accordance with the rules

$$\begin{array}{l} [j] = 0 \;\text{ for }\; m+1 \leq j \leq m+N \\ [j] = 1 \;\text{ for }\; 1 \leq j \leq m \; , \quad m+N+1 \leq j \leq N+2m \; . \end{array} \quad (3.10.39)$$

Thus, for $[j] = 0$ $[j \neq j']$ and $[j] = 1$ the possibilities 1 and 2 (3.10.6) are realized, respectively

$$a^0_j = (-1)^{[j]}\, q^{1-2[j]} = (-1)^{[j]}\, q^{(-1)^{[j]}} \; , \quad [j] = (j') \; . \quad (3.10.40)$$

In this case eq. (3.10.25) is represented as the system of equations

$$\gamma_j\, q + \lambda \sum_{i=j+1}^{K} \gamma_i = -\nu \; , \;\; (N+m+1 \leq j \leq N+2m) \; , \quad (3.10.41)$$

$$\gamma_j\, q - \lambda + \lambda \sum_{i=j+1}^{K} \gamma_i = -\nu \; , \;\; (1 \leq j \leq m) \; , \quad (3.10.42)$$

$$\gamma_j\, q^{(\delta_{j'j}-1)} + \lambda\, \Theta_{j'j} - \lambda \sum_{i=j+1}^{K} \gamma_i = \nu \; , \;\; (m+1 \leq j \leq m+N) \; . \quad (3.10.43)$$

In (3.10.43), for the case $j = j'$, we take into account (3.10.7). The solution of (3.10.41) is (cf. (3.10.27)):

$$\gamma_j = -\nu\, q^{2(j-K)-1} \; , \;\; (N+m+1 \leq j \leq N+2m) \; , \quad (3.10.44)$$

and we have $\lambda \sum_{i>m+N} \gamma_i = \nu(q^{-2m} - 1)$. Using this fact eq. (3.10.43) is written in the form:

$$\gamma_j q^{(\delta_{j'j}-1)} + \lambda\, \Theta_{j'j} - \lambda \sum_{i=j+1}^{m+N} \gamma_i = \nu q^{-2m} \; , \quad (3.10.45)$$

and its solution is:

$$\gamma_j = \nu\, q^{2(N-j)+1} \; , \;\; (j' < j \leq N+m) \; . \quad (3.10.46)$$

In addition, for $N = 2n+1$ and $j = j' = m+n+1$, we have

$$\gamma_{m+n+1} = \nu q^{N-2m-1} = 1 \; , \quad \nu = q^{2m-N+1} \; , \quad (3.10.47)$$

and for $N = 2n$ we obtain condition: $\gamma_{m+\frac{N}{2}} = \nu q^{N-2m+1} - \lambda q = \gamma^{-1}_{m+\frac{N}{2}+1}$ which is equivalent to the quadratic equation on ν:

$$(\nu q^{N-2m} - q)(\nu q^{N-2m} + q^{-1}) = 0 \; . \quad (3.10.48)$$

Accordingly, we summarize the results (3.10.44), (3.10.46) – (3.10.48) as

$$\gamma_j \equiv \frac{\tilde{c}_{j'}}{\tilde{c}_j} = (-1)^{[j]} \nu q^{(-1)^{[j]}(2N-2j+1)-[j]4m} \quad (j > j') \; , \quad \nu = \epsilon q^{\epsilon+2m-N} \; , \tag{3.10.49}$$

where $\epsilon = \pm 1$ and the case $\epsilon = +1$ corresponds $Osp_q(N|2m)$ while the case $\epsilon = -1$ we relate to a quantum group denoted as $Osp'_q(2m|2n)$. It is obvious that for the groups $Osp_q(2n+1|2m)$ (as well as for $SO_q(2n+1)$) we have $\gamma_j = \gamma_{j'} = 1$ for $j = j'$. Note that if in (3.10.49) we set $m = 0$ or $N = 0$, $q \to -q^{-1}$, then we reproduce (3.10.30).

The analog of the relation (3.9.12) for the groups $Osp_q(N|2m)$ is the equation $\widetilde{C}^{ij} = (-1)^{(i)} \epsilon \, \widetilde{C}_{ij}$ which is equivalent to $(-1)^{(i)} \tilde{c}_i \, \tilde{c}_{i'} = \epsilon$ and taking into account (3.10.49) we obtain

$$c_j = \epsilon \, q^{(-1)^{[j]}(j-m-N-\frac{1}{2})-\frac{\epsilon}{2}+\frac{N}{2}} \quad (j > j') \quad \Rightarrow \quad \tilde{c}_j = \epsilon_j q^{-\rho_j} \; ,$$

where $[j] = 0, 1$ is the grading (3.10.39). The parameters $(\rho_1, \dots, \rho_K)$, $(\epsilon_1, \dots, \epsilon_K)$ are fixed according to the following cases:

1.) The case $\epsilon = +1$, $\nu = q^{1+2m-N}$ for $Osp_q(N|2m)$ $(N = 2n+1)$:

$$\rho_i = (\underbrace{\frac{N}{2} - m, \dots, \frac{N}{2} - 1}_{m}; \underbrace{\frac{N}{2} - 1, \dots, \frac{1}{2}, 0, -\frac{1}{2}, \dots, 1 - \frac{N}{2}}_{2n+1}; \underbrace{1 - \frac{N}{2}, \dots, m - \frac{N}{2}}_{m})$$

$$\epsilon_i = (\underbrace{-1, \dots, -1}_{m}; \underbrace{+1, \dots, +1}_{2n+1}; \underbrace{+1, \dots, +1}_{m}) \tag{3.10.50}$$

2.) The case $\epsilon = +1$, $\nu = q^{1+2m-N}$ for $Osp_q(N|2m)$ $(N = 2n)$:

$$\rho_i = (\underbrace{n - m, \dots, n - 1}_{m}; \underbrace{n - 1, \dots, 1, 0, 0, -1, \dots, 1 - n}_{2n}; \underbrace{1 - n, \dots, m - n}_{m})$$

$$\epsilon_i = (\underbrace{-1, \dots, -1}_{m}; \underbrace{+1, \dots, +1}_{2n}; \underbrace{+1, \dots, +1}_{m}) \tag{3.10.51}$$

3.) The case $\epsilon = -1$, $\nu = -q^{-1+2m-2n}$ for $Osp'_q(2m|2n)$:

$$\rho_i = (\underbrace{n + 1 - m, \dots, n}_{m}; \underbrace{n, \dots, 1, -1, \dots, -n}_{2n}; \underbrace{-n, \dots, m - 1 - n}_{m})$$

$$\epsilon_i = (\underbrace{-1, \dots, -1}_{m}; \underbrace{+1, \dots, +1}_{n}, \underbrace{-1, \dots, -1}_{n}; \underbrace{-1, \dots, -1}_{m}) \tag{3.10.52}$$

To conclude this subsection, we give the final expression for the R matrix (3.10.21) corresponding to the quantum supergroups $Osp_q(N|2m)$ $(\epsilon = +1)$ and $Osp'_q(2m|2n)$ $(\epsilon = -1)$

$$\hat{R}_{12} = \sum_{i,j} (-1)^{(i)[j]} \, q^{(-)^{(i)}(\delta_{ij} - \delta_{ij'})} \, e_{ij} \otimes e_{ji} + \lambda \sum_{i<j} e_{ii} \otimes e_{jj} - \lambda \sum_{i>j} q^{\rho_i - \rho_j} \, \epsilon_i \, \epsilon_j \, e_{i'j} \otimes e_{ij'} \; , \tag{3.10.53}$$

where the parameters ϵ_i, ρ_j are defined in (3.10.50) - (3.10.52). To obtain (3.10.53) we have used the condition (3.10.40) and put $f_{ij}/f_{ji} = (-1)^{(i)[j]}$ $(\forall i \neq j \neq i')$, $f_{i'i} = f_{ii'} = 1$ in (3.10.21). This choice of the parameters f_{ij} is such that $\hat{R}_{12}$ tends to the supertransposition matrix $(-1)^{(1)(2)} P_{12}$ (for the notation see (3.7.5)) when $q \to 1$. The multi-parameter R matrices (for the supergroup case) are restored directly from (3.10.53) by the same twisting (3.6.4) if we take into account conditions (3.10.23).

We note that the parameters ν for the cases Osp_q (3.10.51) and Osp'_q (3.10.52) are related to each other by means of the transformation: $q \leftrightarrow -q^{-1}$, $n \leftrightarrow m$. However, this transformation does not relate the corresponding R matrices (3.10.53). Our conjecture is that for the cases Osp_q (3.10.51) and Osp'_q (3.10.52) the R matrices (3.10.53) and corresponding quantum groups are inequivalent.

The R matrices constructed in this subsection for the quantum supergroups realize R matrix representations of the Birman - Murakami - Wenzl algebra since they are the special examples of the R matrix (3.10.15) which satisfy (3.10.16), (3.10.17). Some of these R matrices can be obtained on the basis of the results of Ref. [96], in which Baxterized trigonometric solutions (see Sec. 3.10) of the Yang-Baxter equation associated with the classical supergroups $Osp(N|2m)$ were obtained. Rational solutions, some special cases, and other questions relating to the subject of the quantum supergroups $Osp_q(N|2m)$ are also discussed in Refs. [97], [99] and [101].

3.11 $SO_q(N)$-, $Sp_q(2n)$- and $Osp_q(N|2m)$- invariant Baxterized R matrices

Arguing as in Sec. 3.7, we conclude that the $SO_q(N)$- and $Sp_q(N)$ (as well as $Osp_q(N|2m)$) -invariant Baxterized matrices $\hat{R}(x)$ must be sought [by virtue of the fact that the characteristic equation (3.9.1) is cubic] in the form of a linear combination of the three basis matrices $\mathbf{1}$, $\hat{R}$, $\hat{R}^2$. Expressing $\hat{R}^2$ in terms of $\mathbf{K}$ and $\hat{R}$, we can represent $R(x)$ in the form

$$\hat{R}(x) = c(x)\left(\mathbf{1} + a(x)\hat{R} + b(x)\mathbf{K}\right) , \tag{3.11.1}$$

where $a(x)$, $b(x)$ and $c(x)$ are certain functions that depend on the spectral parameter x. We determine the functions $a(x)$, $b(x)$ from the Yang-Baxter equation (3.8.2). After substitution of (3.11.1) in (3.8.2) and using (3.9.21) – (3.9.28) the following relations arise:

$$\begin{gathered} a_1 + a_3 + \lambda a_1 a_3 = a_2 , \\ b_3 - b_2 - \lambda\nu a_1 a_3 + \nu a_1 b_3 - \lambda a_1 b_2 b_3 + b_1(1 + \nu a_3 - \lambda a_3 b_2 + \\ + \mu b_3 + \nu^{-1} a_2 b_3 + b_2 b_3) + \lambda^2 a_1 a_3 b_2 = 0 , \\ a_2 b_1 + a_3 b_1 b_2 = a_1 b_2 + \lambda a_1 a_3 b_2 , \;\; a_2 b_3 + a_1 b_2 b_3 = a_3 b_2 + \lambda a_1 a_3 b_2 . \end{gathered} \tag{3.11.2}$$

where we have put

$$a_1 = a(x),\; a_2 = a(xy),\; a_3 = a(y),\; b_1 = b(x),\; b_2 = b(xy),\; b_3 = b(y) .$$

The four relations (3.11.2) are equivalent to the three functional equations

$$a(x) + a(y) + \lambda a(x)a(y) = a(xy) \ , \tag{3.11.3}$$

$$\begin{gathered} b(y) - b(xy) + a(x)[\nu b(y) - \lambda\nu a(y) - \lambda b(xy)b(y) + \lambda^2 a(y)b(xy)] + \\ + b(x)[1 + \nu a(y) - \lambda a(y)b(xy) + \mu b(y) + \nu^{-1}a(xy)b(y) + b(xy)b(y)] = 0 \ , \end{gathered} \tag{3.11.4}$$

$$a(xy)b(y) + a(x)b(xy)b(y) = b(xy)(a(y) + \lambda a(x)a(y)) \ , \tag{3.11.5}$$

since the third and fourth relations in (3.11.2) give the same equation (3.11.5). As was to be expected, Eq. (3.11.3) is identical to Eq. (3.8.3) obtained in the $GL_q(N)$ case, and its general solution is given in (3.8.4). By means of (3.11.3), we can transform the right-hand side of Eq. (3.11.5) in such a way that (3.11.5) reduces to the equation

$$\frac{a(x)}{a(xy)} = \frac{b(xy) - b(y)}{b(xy)(b(y) + 1)} \equiv 1 - \frac{b(y)(1 + b(y))^{-1}}{b(xy)(1 + b(xy))^{-1}} \ , \tag{3.11.6}$$

We now note that Eq. (3.11.3) can be rewritten in the form

$$\frac{a(x)}{a(xy)} = 1 - \frac{a(y)(\lambda a(y) + 1)^{-1}}{a(xy)(\lambda a(xy) + 1)^{-1}} \tag{3.11.7}$$

and, comparing (3.11.6) and (3.11.7), we arrive at the result

$$\frac{a(y)(b(y) + 1)}{(\lambda a(y) + 1)b(y)} = const \equiv \frac{\alpha + 1}{\lambda} \ , \tag{3.11.8}$$

where α denotes an arbitrary parameter. The specific choice of the constant on the right-hand side of (3.11.8) is made for convenience in what follows. Substituting the solution (3.8.4) in (3.11.8), we obtain the following general expression for $b(y)$:

$$b(y) = \frac{y^\xi - 1}{\alpha y^\xi + 1} \ . \tag{3.11.9}$$

It is a remarkable fact that Eq. (3.11.4) is satisfied identically on the functions (3.8.4) and (3.11.9) if the constant α satisfies the quadratic equation

$$\alpha^2 - \frac{\lambda}{\nu}\alpha - \frac{1}{\nu^2} = 0 \ , \tag{3.11.10}$$

the two solutions of which are readily found:

$$\alpha_\pm = \pm\frac{q^{\pm 1}}{\nu} \ . \tag{3.11.11}$$

where

$$\begin{aligned} &\nu = \epsilon q^{\epsilon - N} \ , && \text{for groups } SO_q(N) \ (\epsilon = +1), \ Sp_q(N) \ (\epsilon = -1); \\ &\nu = \epsilon q^{\epsilon + 2m - N} \ , && \text{for supergroups } Osp_q(N|2m) \ (\epsilon = +1) \text{ and} \\ & && \qquad Osp'_q(2m|N) \ (N = 2n, \ \epsilon = -1). \end{aligned} \tag{3.11.12}$$

Thus, the solutions of the Yang-Baxter equation (3.8.2) can be represented in the form

$$\hat{R}(x) = c(x)\left(\mathbf{1} + \frac{1}{\lambda}(x^{\xi}-1)\hat{R} + \frac{x^{\xi}-1}{\alpha x^{\xi}+1}\mathbf{K}\right) . \tag{3.11.13}$$

and we have the two possibilities $\alpha = \alpha_{\pm}$ (3.11.11), which are inequivalent [both for all the cases $SO_q(N)$, $Sp_g(N)$ and $Osp_q(N|2m)$], since these solutions cannot be reduced to each other by any functional transformations of the spectral parameter. For convenience we choose $c(x) = x$ and $\xi = -2$ in (3.11.13); then for the R matrices (3.11.13) we can propose four equivalent forms of expression:

$$\hat{R}^{\pm}(x) := \frac{1}{\lambda}\left(x^{-1}\hat{R} - x\hat{R}^{-1}\right) + \frac{\alpha_{\pm}+1}{\alpha_{\pm}x^{-1}+x}\mathbf{K} = \tag{3.11.14}$$

$$= \frac{(q^{\pm 2}x^{-1}-x)}{(q^{\pm 2}-1)}\frac{(\hat{R}x^{-1} \pm q^{\pm 1}x)}{(\hat{R}x \pm q^{\pm 1}x^{-1})} =$$

$$= \frac{x-x^{-1}}{\lambda(x+\alpha_{\pm}x^{-1})}\left(-x\hat{R}^{-1} - \alpha_{\pm}x^{-1}\hat{R} + \frac{\lambda(\alpha_{\pm}+1)}{x-x^{-1}}\right) = \tag{3.11.15}$$

$$= \frac{(x^{-1}q - xq^{-1})}{\lambda}\mathbf{P}^{+} + \frac{(xq-(xq)^{-1})}{\lambda}\mathbf{P}^{-} + \frac{(q^{\pm 2}x^{-1}-x)}{(q^{\pm 2}-1)}\frac{(x^{-1}+x\,\alpha_{\pm})}{(x+x^{-1}\,\alpha_{\pm})}\mathbf{P}^{0} .$$

The last expression determines the spectral decomposition of $\hat{R}(x)$, from which, for example, we can readily obtain

$$\hat{R}^{+}(\pm q) = \pm(q+q^{-1})\,\mathbf{P}^{-} , \quad \hat{R}^{-}(\pm q^{-1}) = \pm(q+q^{-1})\,\mathbf{P}^{+} , \tag{3.11.16}$$

$$\hat{R}^{\pm}(1) = \mathbf{1} , \quad \hat{R}^{\pm}(i) = \pm\frac{i(q+q^{-1})}{\lambda}(\mathbf{1} - 2\mathbf{P}^{\pm}) ,$$

$$\hat{R}^{\pm}(x)\,\hat{R}^{\pm}(x^{-1}) = \left(1 - \frac{(x-x^{-1})^2}{\lambda^2}\right)\cdot\mathbf{1} . \tag{3.11.17}$$

The relations (3.11.16) agree with the Yang-Baxter equation (3.8.2).

The cross-unitarity for the BMW type R-matrix (3.11.14) can be written in the matrix form as (cf. (3.8.8))

$$\begin{gathered} Tr_{D(2)}\left(\hat{R}_1^{\pm}(x)P_{01}\hat{R}_1^{\pm}(z)\right) = \eta^{\pm}(x)\,\eta^{\pm}(z)\,D_0\,I_1 , \\ Tr_{Q(1)}\left(\hat{R}_1^{\pm}(x)P_{23}\hat{R}_1^{\pm}(z)\right) = \eta^{\pm}(x)\,\eta^{\pm}(z)\,Q_3\,I_2 , \end{gathered} \tag{3.11.18}$$

where the matrices D, Q have been defined in (3.1.12) and

$$\eta^{\pm}(x) = \frac{1}{\lambda}(x-x^{-1})\frac{(\alpha_{\pm}\nu x^2 + \nu^{-1})}{(x^2+\alpha_{\pm})} , \quad (x\,z)^2 = \alpha_{\pm}^2 .$$

The Baxterized $\hat{R}$ matrices (3.11.13) and (3.11.15) must determine algebras with the defining relations (3.8.11). However, a realization of the operators $L(x)$ in terms of the generators $L^{(\pm)}$ of the quantum algebras $U_q(so(N))$ and $U_q(sp(N))$ [analogous to (3.8.12)] is, unfortunately, not known to me (see, however, [53]). Such a realization would be extremely helpful for many applications.

To conclude this subsection, we give the expressions for the rational R matrices of the Yangians $Y(so(N))$, $Y(sp(N))$ and $Y(osp(N|2m))$. We make the ansatz for the spectral parameter $x = \exp(-\lambda\theta/2)$ in (3.11.14) and rewrite the R matrix in the form (cf. (3.8.23)):

$$\hat{R}(\theta) := \hat{R}\left(e^{-\frac{\lambda}{2}\theta}\right) = \cosh(\lambda\theta/2)\,[\mathbf{1} - \mathbf{K}] + \tfrac{1}{\lambda}\sinh(\lambda\theta/2)\,[\hat{R} + \hat{R}^{-1}] + \\ + [\cosh(\lambda\theta/2) + \beta_\pm \sinh(\lambda\theta/2)]^{-1}\,\mathbf{K} \tag{3.11.19}$$

where $\beta_\pm = \frac{\alpha_\pm - 1}{\alpha_\pm + 1}$. The Yangian R matrices can be obtained from (3.11.19) after the passage to the limit $h \to 0$ ($q = exp(h) \to 1$). Further, it is easy to see that the cases $\alpha = \alpha_+$, $\epsilon = 1$ ($SO_q(N)$) and $\alpha = \alpha_-$, $\epsilon = -1$ ($Sp_q(2n)$) reduce to $GL(N)$-symmetric Yang's R matrix (3.8.25). The nontrivial $SO(N)$- and $Sp(N)$-symmetric Yangian R matrices for $Y(so(N))$ and $Y(sp(N))$ correspond to the choice

$$\alpha = \alpha_-, \;\; \epsilon = 1 \;\; (SO_q(N)) \; ; \;\; \alpha = \alpha_+, \;\; \epsilon = -1 \;\; (Sp_q(N)) \; , \tag{3.11.20}$$

and have the form

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, P_{12}) + \frac{2\,\theta}{(2\epsilon - (N + 2\theta))}\, K_{12}^{(0)} \; . \tag{3.11.21}$$

The matrix $K_{12}^{(0)}$ is defined in (3.9.6). Nontrivial rational R matrices for super Yangians $Y(osp(N|2m))$ and $Y'(osp(2m|2n))$ can be obtained from (3.11.19) in the cases

$$\alpha = \alpha_-, \;\; \epsilon = 1 \;\; (Osp_q(N|2m)) \; ; \;\; \alpha = \alpha_+, \;\; \epsilon = -1 \; , \;\; N = 2n \;\; (Osp'_q(2m|2n)) \; .$$

The form of these R matrices is

$$\hat{R}(\theta) = (\mathbf{1} + \theta\, \mathcal{P}_{12}) + \frac{2\,\theta}{2\epsilon + 2m - (N + 2\,\theta)}\, \mathcal{K}_{12}^{(0)} \tag{3.11.22}$$

where $\mathcal{P}^{i_1 i_2}_{j_1 j_2} = (-1)^{[i_1][i_2]}\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the supertransposition operator (the parity $[j]$ is defined in (3.10.39)). The matrix $(\mathcal{K}^{(0)})^{i_1 i_2}_{j_1 j_2} = \mathcal{C}^{i_1 i_2}\mathcal{C}_{j_1 j_2}$ is a classical limit ($q \to 1$) of the rank 1 matrix $\mathbf{K}$ in the supersymmetric case and the ortho-symplectic matrices $\mathcal{C}^{ij} = \epsilon_j \delta^{ij'}$, $\mathcal{C}_{ij} = \epsilon_i \delta_{ij'}$ are determined by their parameters ϵ_i (3.10.50) - (3.10.52). Then, the defining relations for the generators (3.8.26) of the Yangians $Y(so(N))$, $Y(sp(N))$ and $Y(osp(N|2m))$, $Y'(osp(2m|2n))$ are identical to (3.8.22) and (3.8.29), respectively, while the comultiplication is given by (3.8.27).

The Yangian R matrix (3.11.21) for the $SO(N)$ case was found in Ref. [3], and that for the $Sp(2n)$ case in Ref. [100]. These R matrices were used in Ref. [86] to construct and investigate exactly solvable $SO(N)$- and $Sp(2n)$-symmetric magnets. Twisted Yangians for the $SO(N)$ and $Sp(2n)$ cases have been considered in [84]. The super Yangians of the type $Y(osp(N|2m))$ and corresponding spin chain models where discussed in [101]. Baxterized trigonometric R matrices (3.11.13) corresponding to the special parameter values (3.11.20) were first found by Bazhanov in 1984 and were published in Ref. [102]. The same R matrices were independently constructed in Ref. [103].

3.12 Quantum Knizhnik - Zamolodchikov equations

In Sections 3.7 and 3.10, using R matrix representations for the Hecke and Birman-Murakami-Wenzl algebras, we have found the trigonometric solutions $R(x)$ of the Yang - Baxter equations (Baxterized R matrices). In this subsection we show that, for every trigonometric solution $R(x)$, one can construct the set of difference equations which are called quantum Knizhnik - Zamolodchikov equations. These equations are important since their solutions are related (see e.g. [104], [105] and Refs. therein) to the correlators in spin chain models associated with the same trigonometric matrix $R(x)$.

In this subsection we follow the presentation of the papers [106], [107].

Consider a tensor function $\Psi^{1\dots N\rangle}(z_1,\dots,z_N) \in V^{\otimes N}$ $(z_i \in \mathbf{C},\ i = 1,\dots,N)$ which satisfies a system of difference equations

$$T_{(i)}\,\Psi^{1\dots N\rangle}(z_1,\dots,z_N) = A^{(i)}_{1\dots N}(z_1,\dots,z_N)\,\Psi^{1\dots N\rangle}(z_1,\dots,z_N)\ , \tag{3.12.1}$$

where operator $T_{(i)}$ is defined as

$$T_{(i)}\,\Psi^{1\dots N\rangle}(z_1,\dots,z_N) := \Psi^{1\dots N\rangle}(z_1,\dots,z_{i-1},pz_i,z_{i+1},\dots,z_N)\ , \tag{3.12.2}$$

$A^{(i)}_{1\dots N}(z_1,\dots,z_N) \in End(V^{\otimes N})$ is called discrete connection and indices $1,\dots,N$ denote the numbers of the vector spaces V in $V^{\otimes N}$. A consistence condition $T_{(i)}T_{(j)} = T_{(j)}T_{(i)}$ of the system (3.12.1) requires additional constraints on the discrete connection $A^{(j)}_{1\dots N}(z_1\dots z_N)$:

$$T_{(i)}A^{(j)}_{1\dots N}T^{-1}_{(i)}A^{(i)}_{1\dots N} = T_{(j)}A^{(i)}_{1\dots N}T^{-1}_{(j)}A^{(j)}_{1\dots N}\ . \tag{3.12.3}$$

Discrete connections which satisfy (3.12.3) are called flat (or integrable).

Now we introduce the following discrete connection

$$A^{(j)}_{1\dots N}(z_1,\dots,z_N) = T_{(j)}R_{j\,j-1}\dots R_{j\,2}\,R_{j\,1}T^{-1}_{(j)}\,D_j\,R^{-1}_{N\,j}R^{-1}_{N-1\,j}\dots R^{-1}_{j+1\,j}\ , \tag{3.12.4}$$

where $R_{i\,j} := R_{i\,j}(z_i/z_j)$ is the R matrix which acts nontrivially only in the vector spaces V with numbers i,j and satisfies the Yang-Baxter eq. (3.8.2) in the form $(R_{ij}(x) = P_{ij}\hat{R}_{ij}(x))$

$$R_{ij}(x)\,R_{ik}(xy)\,R_{jk}(y) = R_{jk}(y)\,R_{ik}(xy)\,R_{ij}(x)\ , \tag{3.12.5}$$

and the unitarity condition $R_{ij}(x)R_{ji}(x^{-1}) = \mathbf{1}$ (cf. (3.8.7), (3.11.17)). The constant matrix D_i acts in i-th vector space V_i and obeys $R_{ij}D_iD_j = D_iD_jR_{ij}$. Eqs. (3.12.1) with discrete connection (3.12.4) is called quantum Knizhnik - Zamolodchikov (q-KZ) equations. It is convenient to rewrite the definition of the discrete connection (3.12.4) in the form

$$\begin{aligned} &A^{(j)}_{1\dots N}(z_1,\dots,z_N) = \\ &= T_{(j)}\hat{\hat{R}}_{j-1}\dots,\hat{\hat{R}}_1T^{-1}_{(1)}\,D_1\,\mathbf{P}_{1,2}\mathbf{P}_{2,3}\dots\mathbf{P}_{N-1,N}\hat{\hat{R}}^{-1}_{N-1}\hat{\hat{R}}^{-1}_{N-2}\dots\hat{\hat{R}}^{-1}_{j}\ , \end{aligned} \tag{3.12.6}$$

where $\mathbf{P}_{j,k} = P_{j,k}{\cdot}P_{z_j,z_k}$, P_{z_j,z_k} is an operator which permutes the spectral parameters z_j and z_k:

$$P_{z_j,z_k}\cdot f(z_1,\dots,z_k,\dots,z_j,\dots,z_N) = f(z_1,\dots,z_j,\dots,z_k,\dots,z_N)\cdot P_{z_j,z_k}\ ,$$

and $\hat{\hat{R}}_j = \mathbf{P}_{j,j+1} R_{j,j+1}(z_j/z_{j+1})$ realize generators of a braid group (see eqs. (4.1.1) in Sect. 4).

Our statement is (see [106], [107]) that discrete connection (3.12.4) is the flat discrete connection (i.e. satisfies (3.12.3)) and therefore the system of equations (3.12.1) is consistent. Indeed, we have from (3.12.3) for $j > i$

$$T_{(i)} T_{(j)} R_{j\,j-1} \dots R_{j\,1} T_{(j)}^{-1} D_j R_{Nj}^{-1} \dots R_{j+1\,j}^{-1} R_{i\,i-1} \dots R_{i\,1} T_{(i)}^{-1} D_i R_{Ni}^{-1} \dots R_{i+1\,i}^{-1} =$$

$$= T_{(i)} R_{i\,i-1} \dots R_{i\,1} T_{(i)}^{-1} D_i T_{(j)} R_{Ni}^{-1} \dots R_{i+1\,i}^{-1} R_{j\,j-1} \dots R_{j\,1} T_{(j)}^{-1} D_j R_{Nj}^{-1} \dots R_{j+1\,j}^{-1} \, .$$

In the left hand side we obtain for $j > i$

$$T_{(i)} T_{(j)} R_{j\,j-1} \dots R_{j\,1} R_{i\,i-1} \dots R_{i\,1} T_{(i)}^{-1} T_{(j)}^{-1} D_i D_j R_{N\,j}^{-1} \dots R_{j+1\,j}^{-1} R_{Ni}^{-1} \dots R_{i+1\,i}^{-1} \, . \quad (3.12.7)$$

Then we use here identities for transfer-matrices

$$R_{j\,j-1} \dots R_{j\,1} (R_{i\,i-1} \dots R_{i\,1}) = (R_{i\,i-1} \dots R_{i\,1}) R_{j\,j-1} \dots R_{j\,i+1} R_{j\,i-1} \dots R_{j\,1} R_{ji}$$

$$(R_{Nj}^{-1} \dots R_{j+1\,j}^{-1}) R_{N\,i}^{-1} \dots R_{i+1\,i}^{-1} = R_{ji}^{-1} R_{Ni}^{-1} \dots R_{j+1\,i}^{-1} R_{j-1\,i}^{-1} \dots R_{i+1\,i}^{-1} (R_{Nj}^{-1} \dots R_{j+1\,j}^{-1})$$

and obvious relations $[T_{(i)}^{-1} T_{(j)}^{-1}, \, R_{i\,j}] = 0 = [D_i D_j, \, R_{i\,j}]$. As a result we obtain for l.h.s. (3.12.7)

$$\begin{gathered} T_{(i)} T_{(j)} (R_{i\,i-1} \dots R_{i\,1}) R_{j\,j-1} \dots R_{j\,i+1} R_{j\,i-1} \dots R_{j\,1} T_{(i)}^{-1} T_{(j)}^{-1} D_i D_j \cdot \\ \cdot R_{Ni}^{-1} \dots R_{j+1\,i}^{-1} R_{j-1\,i}^{-1} \dots R_{i+1\,i}^{-1} (R_{Nj}^{-1} \dots R_{j+1\,j}^{-1}) \end{gathered} \quad (3.12.8)$$

In the r.h.s. we use $[R_{i\,j} D_k] = 0 = [R_{i\,j} T_k]$ for $i, j \neq k$ and the identity

$$\begin{gathered} R_{Ni}^{-1} \dots R_{i+1\,i}^{-1} R_{j\,j-1} \dots R_{j\,1} = \\ = R_{j\,j-1} \dots R_{j\,i+1} R_{j\,i-1} \dots R_{j\,1} R_{Ni}^{-1} \dots R_{j+1\,i}^{-1} R_{j-1\,i}^{-1} \dots R_{i+1\,i}^{-1} \end{gathered}$$

which gives for the r.h.s. just the same answer (3.12.8) as for the l.h.s.

At the end of this subsection we present the definition of the q-KZ equations due to F.A. Smirnov [108]

$$\begin{gathered} \Psi^{1 \dots N \rangle}(z_1, \dots, z_{i+1}, z_i, \dots z_N) = \hat{R}_{i\,i+1}(z_i/z_{i+1}) \Psi^{1 \dots N \rangle}(z_1, \dots, z_i, z_{i+1}, \dots, z_N) \\ \Psi^{1 \dots N \rangle}(p\, z_1, z_2, \dots, z_N) = D_1 \, \Psi^{2 \dots N,1 \rangle}(z_2, z_3, \dots, z_N, z_1) \, . \end{gathered} \quad (3.12.9)$$

One can show explicitly that (3.12.9) lead to the eqs. (3.12.1), (3.12.2), (3.12.4). The self-consistence of eqs. (3.12.9) can be checked directly. It also follows from the self-consistence of the extended Zamolodchikov algebra with generators $\{A_i(z_i), Q\}$ $(i = 1, \dots, N)$:

$$\hat{R}_{1\,2}(z_1/z_2) \, A^{1\rangle}(z_1) \, A^{2\rangle}(z_2) = A^{1\rangle}(z_2) \, A^{2\rangle}(z_1) \, , \quad D_1 A^{1\rangle}(z_1) \, Q = Q \, A^{1\rangle}(p\, z_1) \, ,$$

and remark that eqs. (3.12.9) can be formally produced from the representation

$$\Psi^{1 \dots N\rangle}(z_1, \dots, z_N) = \mathrm{Tr} \left(Q A^{1\rangle}(z_1) \, A^{2\rangle}(z_2) \dots A^{N\rangle}(z_N) \right) \, .$$

The semiclassical limit of the q-KZ equations (if we take the trigonometric solutions (3.8.5) and (3.11.14) for R-matrices) gives the usual Knizhnik - Zamolodchikov equations.

3.13 Elliptic solutions of the Yang-Baxter equation

In this subsection, we consider $Z_N \otimes Z_N$-symmetric solutions of the Yang-Baxter equation (3.8.21) (Ref. [109]). The elements $R^{i_1 i_2}_{j_1 j_2}(\theta)$ of the corresponding R matrix will be expressed in terms of elliptic functions of the spectral parameter θ.

We construct this solution explicitly, following the method of Ref. [109]. We consider two matrices g and h such that $g^N = h^N = 1$:

$$g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix} \; , \;\; h = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \; , \tag{3.13.1}$$

where $\omega = exp(2\pi i/N)$ and $h\,g = \omega\,g\,h$. The matrices g and h are Z_N-graded generators of the algebra $Mat(N)$, the graded basis for which can be chosen in the form

$$I_{\vec{\alpha}} = I_{\alpha_1\alpha_2} = g^{\alpha_1} h^{\alpha_2} \; , \;\; \alpha_{1,2} = 0, 1, \dots, N-1 \; . \tag{3.13.2}$$

On the other hand, the matrices (3.13.2) realize a projective representation of the group $Z_N \otimes Z_N$: $I_{\vec{\alpha}} I_{\vec{\beta}} = \omega^{\alpha_2\beta_1} I_{\vec{\alpha}+\vec{\beta}}$. Any matrix $R_{12}(\theta) = R^{i_1 i_2}_{j_1 j_2}(\theta)$ can now be written in the form

$$R_{12}(\theta) = W_{\vec{\alpha},\vec{\beta}}(\theta)\, I_{\vec{\alpha}} \otimes I_{\vec{\beta}} \; ,$$

(the sum over α_i, β_j is assumed). We consider the $Z_N \otimes Z_N$-invariant subset of such matrices:

$$R_{12}(\theta) = W_{\vec{\alpha}}(\theta) I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}} \; , \tag{3.13.3}$$

where $I^{-1}_{\vec{\alpha}} = h^{-\alpha_2} g^{-\alpha_1} = \omega^{\alpha_1\alpha_2} I_{-\vec{\alpha}}$. The invariance of the matrices (3.13.3) is expressed by the relations

$$R_{12}(\theta) = (I_{\vec{\gamma}} \otimes I_{\vec{\gamma}})\, R_{12}(\theta)\, (I_{\vec{\gamma}} \otimes I_{\vec{\gamma}})^{-1} \quad \forall \vec{\gamma} \; , \tag{3.13.4}$$

which obviously follow from the identity

$$I_{\vec{\gamma}}\, I_{\vec{\alpha}}\, I^{-1}_{\vec{\gamma}} = \omega^{<\alpha,\gamma>} I_{\vec{\alpha}} \; , \;\; <\alpha,\, \gamma> = \alpha_1\gamma_2 - \alpha_2\gamma_1 \; .$$

It was noted in Ref. [109] that the relations

$$\begin{gathered} R_{12}(\theta+1) = g_1^{-1}\, R_{12}(\theta)\, g_1 = g_2\, R_{12}(\theta)\, g_2^{-1} \; , \\ R_{12}(\theta+\tau) = exp(-i\pi\tau) exp(-2\pi i\theta) h_1^{-1}\, R_{12}(\theta)\, h_1 = \\ exp(-i\pi\tau) exp(-2\pi i\theta) h_2\, R_{12}(\theta)\, h_2^{-1} \; , \end{gathered} \tag{3.13.5}$$

$$R_{12}(0) = I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}} \equiv P_{12} \; , \tag{3.13.6}$$

where τ is some complex parameter (period), are consistent with the Yang-Baxter equation (3.8.21) and can be regarded as subsidiary conditions to these equations (the last identity in (3.13.6) follows from $(I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}})\, I_{\vec{\beta}} \otimes I_{\vec{\gamma}} = I_{\vec{\gamma}} \otimes I_{\vec{\beta}}\, (I_{\vec{\alpha}} \otimes I^{-1}_{\vec{\alpha}})$). Moreover, for the $Z_N \otimes Z_N$-invariant R matrix (3.13.3) the conditions (3.13.5), (3.13.6) determine the solution of the Yang-Baxter equation uniquely. Indeed, substitution of (3.13.3) in (3.13.5), (3.13.6) leads to the equations

$$\begin{aligned} & W_{\vec{\alpha}}(\theta+1) = \omega^{\alpha_2}\, W_{\vec{\alpha}}(\theta) \; , \\ & W_{\vec{\alpha}}(\theta+\tau) = exp(-i\pi\tau)\, exp(-2\pi i\theta)\, \omega^{-\alpha_1}\, W_{\vec{\alpha}}(\theta) \; , \quad W_{\vec{\alpha}}(0) = 1 \; , \end{aligned} \tag{3.13.7}$$

the solution of which can be found by means of an expansion in a Fourier series and has the form

$$W_{\vec{\alpha}}(\theta) = \frac{\Theta_{\vec{\alpha}}(\theta+\eta)}{\Theta_{\vec{\alpha}}(\eta)} \; , \quad (W_{\vec{\alpha}+\vec{\nu}}(u) = W_{\vec{\alpha}+\vec{\nu}\,'}(u) = W_{\vec{\alpha}}(u)) \; , \tag{3.13.8}$$

where $\vec{\nu} = (N, 0)$, $\vec{\nu}\,' = (0, N)$,

$$\Theta_{\vec{\alpha}}(u) = \sum_{m=-\infty}^{\infty} exp\left[i\pi\tau(m + \frac{\alpha_2}{N})^2 + 2\pi i(m + \frac{\alpha_2}{N})(u + \frac{\alpha_1}{N}) \right] \; , \tag{3.13.9}$$

and we recall that $\alpha_{1,2} \in \mathbf{Z}_N$. The parameter η in (3.13.8) is arbitrary. For $N = 2$, the solution (3.13.8) is identical to the solution obtained by Baxter [2], [110] in connection with the investigation of the so-called eight-vertex lattice model.

Direct substitution of (3.13.3) in the Yang-Baxter equation (3.8.21) shows that the functions $W_{\vec{\alpha}}(\theta)$ must satisfy the relations

$$\sum_{\vec{\gamma}} W_{\vec{\gamma}}(\theta-\theta')\, W_{\vec{\alpha}-\vec{\gamma}}(\theta)\, W_{\vec{\beta}+\vec{\gamma}}(\theta') \left(\omega^{<\gamma,\beta>} - \omega^{<\alpha-\gamma,\beta>} \right) = 0 \; . \tag{3.13.10}$$

As it was proved in [111], these relations hold when the functions (3.13.8) and (3.13.9) are substituted. We will see later that the identities (3.13.10) are intimately related to a version of the Yang-Baxter equations appeared in Interaction Round Face models (see Sect. 5.3).

4 GROUP ALGEBRA OF BRAID GROUP AND ITS QUOTIENTS

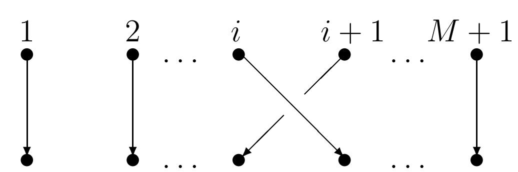
4.1 Group Algebra of Braid Group

A braid group $\mathcal{B}_{M+1}$ is generated by elements σ_i ($i = 1, \ldots M$) subject to the relations:

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1} \; , \quad [\sigma_i, \sigma_j] = 0 \;\; \text{for} \;\; |i-j| > 1 \; . \tag{4.1.1}$$

By definition the elements σ_i are supposed to be invertible and represented graphically as

$\sigma_i \;\; =$



(4.1.2)

Definition 13. *An extension of the braid group $\mathcal{B}_M$ by one invertible generator σ_M subject to relations*

$$\begin{gathered}\sigma_M\sigma_{M-1}\sigma_M = \sigma_{M-1}\sigma_M\sigma_{M-1}\ , \quad \sigma_1\sigma_M\sigma_1 = \sigma_M\sigma_1\sigma_M\ , \\ [\sigma_M\, ,\, \sigma_k] = 0 \ \ (k = 2, \dots, M-2)\ ,\end{gathered} \tag{4.1.3}$$

is called a periodic braid group $\overline{\mathcal{B}}_M$.

Definition 14. *An extension of the braid group $\mathcal{B}_{M+1}$ by one invertible generator y_1 which satisfies the relations*

$$y_1\sigma_1 y_1\sigma_1 = \sigma_1 y_1\sigma_1 y_1\ , \quad [\sigma_i,\, y_1] = 0 \ \ \forall i > 1\ ,$$

is called affine braid group $\hat{\mathcal{B}}_{M+1}$.

The elements $\{y_i\}$ $(i = 1, \dots M+1)$ defined by:

$$y_1\ , \quad y_2 = \sigma_1 y_1\sigma_1\ , \quad y_3 = \sigma_2\sigma_1 y_1\sigma_1\,\sigma_2\ , \ \dots\ , \ y_{i+1} = \sigma_i y_i\sigma_i\ , \tag{4.1.4}$$

generate an Abelian subgroup in $\hat{\mathcal{B}}_{M+1}$. For $y_1 = 1$ elements (4.1.4) generate an Abelian subgroup in the braid group $\mathcal{B}_{M+1}$. Note that $y_n y_{n+1} = y_{n+1} y_n$ is equivalent to the reflection equation for y_n:

$$y_n\,\sigma_n\,y_n\,\sigma_n = \sigma_n\,y_n\,\sigma_n\,y_n\ . \tag{4.1.5}$$

Proposition 7. *The product of m elements of $\hat{B}_{M+1}$: $y^{(m)}_{k+1} := y_{k+1}y_{k+2}\dots y_{k+m}$ $(k+m < M+1)$ satisfies the following relations*

$$y^{(m)}_{k+1} = U_{(k,m)}\ y^{(m)}_1\ U_{(m,k)}\ , \tag{4.1.6}$$

where (cf. (3.2.39))

$$U_{(k,m)} = \sigma_{(k\to m+k-1)}\dots\sigma_{(2\to m+1)}\,\sigma_{(1\to m)} \equiv \sigma_{(k\leftarrow 1)}\sigma_{(k+1\leftarrow 2)}\sigma_{(k+m-1\leftarrow m)}\ , \tag{4.1.7}$$

and $(k \leq n)$

$$\sigma_{(k\to n)} = \sigma_k\sigma_{k+1}\dots\sigma_n\ , \quad \sigma_{(n\leftarrow k)} = \sigma_n\dots\sigma_{k+1}\sigma_k\ .$$

Proof. First of all we show that

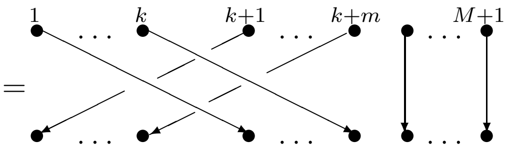
$$y^{(m)}_{k+1} = \sigma_{(k\to k+m-1)}\,y^{(m)}_k\,\sigma_{(k+m-1\leftarrow k)}\ . \tag{4.1.8}$$

This identity is proved by induction. For $m = 1$ we obviously have $y_{k+1} = \sigma_k y_k\sigma_k$. Let (4.1.8) be correct for some m. Then, for $y^{(m+1)}_{k+1}$ we have

$$\begin{aligned} y^{(m+1)}_{k+1} &= y^{(m)}_{k+1}\,y_{k+m+1} = \sigma_{(k\to k+m-1)}\,y^{(m)}_k\,\sigma_{(k+m-1\leftarrow k)}\,\sigma_{k+m}y_{k+m}\sigma_{k+m} = \\ &= \sigma_{(k\to k+m-1)}\,y^{(m)}_k\,\sigma_{k+m}y_{k+m}\sigma_{k+m}\,\sigma_{(k+m-1\leftarrow k)} = \\ &= \sigma_{(k\to k+m-1)}\sigma_{k+m}\,(y^{(m)}_k y_{k+m})\,\sigma_{k+m}\sigma_{(k+m-1\leftarrow k)}\ , \end{aligned}$$

which coincides with (4.1.8) for $m \to m+1$. Applying (4.1.8) several times we deduce (4.1.6). □

One can graphically represent elements $U_{(k,m)}$ (4.1.7) (by means of the rules (4.1.2)) in the following form

$U_{(k,m)} \;=$ 

(4.1.9)

From this representation it becomes clear that the following braid relations hold

$$U_{(k,m)}\,(T_m U_{(k,n)})\,U_{(m,n)} = (T_k U_{(m,n)})\,U_{(k,n)}\,(T_n U_{(k,m)}) \;, \tag{4.1.10}$$

where we have introduced jump operators T_m: $(\sigma_i \to \sigma_{i+m})$. One can check relations (4.1.10) by direct calculations.

Consider elements $\Sigma_{m\to n}$ $(n = m, m{+}1, \dots, M{+}1)$ of the group algebra of B_{M+1} which are defined inductively:

$$\Sigma_{m\to n} = f_{m\to n}\,\Sigma_{m\to n-1} = f_{m\to n}\,f_{m\to n-1}\cdots f_{m\to m+1}\,f_{m\to m} \;, \tag{4.1.11}$$

where $m \to n$ denotes the set of indices $(m, m+1, \dots, n)$, $\Sigma_{m\to m} = 1$, $f_{k\to k} = 1$ and

$$f_{k\to n} = 1 + \sigma_{n-1} + \sigma_{n-2}\,\sigma_{n-1} + \ldots + \sigma_k\,\sigma_{k+1}\cdots\sigma_{n-1} \;, \tag{4.1.12}$$

for $(k < n)$. The important properties of the elements $f_{k\to n}$ (4.1.12) of the group algebra of B_{M+1} are:

$$f_{1\to n} f_{1\to n-1} \ldots f_{1\to m+1} = \mathrm{E}^{(m,n-m)}_{1\to n}\,\Sigma_{m+1\to n} \;, \quad (0 \le m < n) \tag{4.1.13}$$

where $\mathrm{E}^{(n,0)}_{1\to n} = \mathrm{E}^{(0,n)}_{1\to n} = 1$, $\mathrm{E}^{(m-1,1)}_{1\to m} = f_{1\to m}$. The identities (4.1.13) are equivalent to the right factorization formula

$$\Sigma_{1\to n} = \mathrm{E}^{(m,n-m)}_{1\to n}\,\Sigma_{1\to m}\,\Sigma_{m+1\to n} \;, \tag{4.1.14}$$

and all elements $\mathrm{E}^{(m,n-m)}_{1\to n}$ are defined inductively using the recurrent relations

$$\mathrm{E}^{(m,n+1-m)}_{1\to n+1} = \mathrm{E}^{(m,n-m)}_{1\to n} + \mathrm{E}^{(m-1,n+1-m)}_{1\to n}\,\sigma_n\,\sigma_{n-1}\ldots\sigma_m \;.$$

In particular, we have $\mathrm{E}^{(n-1,2)}_{1\to n+1} = (f_{1\to n} + \mathrm{E}^{(n-2,2)}_{1\to n}\,\sigma_n\,\sigma_{n-1})$, $(n \ge 2)$ which gives

$$\begin{gathered}\mathrm{E}^{(m-1,2)}_{1\to m+1} = f_{1\to m} + f_{1\to m-1}\,(\sigma_m\,\sigma_{m-1}) + f_{1\to m-2}\,(\sigma_{m-1}\,\sigma_{m-2})\,(\sigma_m\,\sigma_{m-1}) \\ + \ldots + f_{1\to 2}\,(\sigma_3\,\sigma_2)\cdots(\sigma_m\,\sigma_{m-1}) + (\sigma_2\,\sigma_1)\cdots(\sigma_m\,\sigma_{m-1}) \;.\end{gathered} \tag{4.1.15}$$

The next relation is: $\mathrm{E}^{(n-2,3)}_{1\to n+1} = \mathrm{E}^{(n-2,2)}_{1\to n} + \mathrm{E}^{(n-3,3)}_{1\to n}\,\sigma_n\,\sigma_{n-1}\,\sigma_{n-2}$ for $(n \ge 3)$, etc.

Note that $\mathrm{E}^{(m,n-m)}_{1\to n}$ are sums over the braid group elements which can be considered as quantum analogs of $(m,\, n-m)$ shuffles of two piles with m and $n-m$ elements if we read words from right to left (the shuffles are obtained by projection $\sigma_i \to s_i$, where s_i are generators of symmetric group $\mathcal{S}_{M+1}$). As it follows from (4.1.13) the elements $f_{1\to m} = \mathrm{E}^{(m-1,1)}_{1\to m}$ are the sums of $(m-1, 1)$ shuffles. One can use the operators $\Sigma_{1\to n}$ (4.1.11) for the definition of the associative products (analogous to the wedge products proposed by S.Woronowicz in the theory of differential forms on quantum groups [39]). In view of (4.1.14) these products are related to the quantum shuffle products (about quantum shuffles and corresponding products see [112], [39]). The associativity of these products is provided by the identities:

$$\mathrm{E}_{1\to n}^{(n-m,m)}\,\mathrm{E}_{1\to m}^{(k,m-k)} = \mathrm{E}_{1\to n}^{(k,n-k)}\,\mathrm{E}_{k+1\to n}^{(m-k,n-m)} \quad (k<m<n)\;, \tag{4.1.16}$$

which are the consistence conditions for the definition of a 3-pile shuffles $(k, m-k, n-m)$:

$$\Sigma_{1\to n} = \mathrm{E}_{1\to n}^{(k,m-k,n-m)}\,\Sigma_{1\to k}\,\Sigma_{k+1\to m}\,\Sigma_{m+1\to n}\;,$$

Going further one can introduce m-pile shuffles $\mathrm{E}_{1\to n}^{(n_1,n_2,\dots,n_m)}$ of the pack of n cards $(n = n_1+n_2+\ldots+n_m)$. Then, we observe that the "symmetrizer" $\Sigma_{1\to n}$ (4.1.11) is nothing else but the n-pile shuffle $\mathrm{E}_{1\to n}^{(1,1,\dots,1)}$.

We note that there is a mirror set of relations for the elements $\Sigma_{1\to n}$ (4.1.11) in the group algebra of the braid group. In particular, we have

$$\Sigma_{k\to n} = \Sigma_{k\to n-1}\,\overline{f}_{k\to n} = \overline{f}_{k\to k}\,\overline{f}_{k\to k+1}\cdots\overline{f}_{k\to n-1}\,\overline{f}_{k\to n}\;, \tag{4.1.17}$$

where $\overline{f}_{k\to k} = 1$ and

$$\overline{f}_{k\to n} = 1+\sigma_{n-1}+\sigma_{n-1}\,\sigma_{n-2}+\ldots+\sigma_{n-1}\ldots\sigma_{k+1}\,\sigma_k\;, \tag{4.1.18}$$

for $(k<n)$. In addition we have a left factorization formula (cf. (4.1.14))

$$\Sigma_{1\to n} = \Sigma_{1\to m}\,\Sigma_{m+1\to n}\,\exists_{1\to n}^{(m,n-m)}\;, \tag{4.1.19}$$

where the elements $\exists_{1\to n}^{(m,n-m)}$ are defined by recurrence relations

$$\exists_{1\to n+1}^{(m,n-m+1)} = \exists_{1\to n}^{(m,n-m)} + \sigma_m\ldots\sigma_n \exists_{1\to n}^{(m-1,n-m+1)}$$

$(\exists_{1\to n}^{(n,0)} = \exists_{1\to n}^{(0,n)} = 1,\ \exists_{1\to n}^{(n-1,1)} = \overline{f}_{1\to n})$ and $\exists_{1\to n}^{(m,n-m)}$ is a sum over $(m, n-m)$ quantum shuffles (if we read words from left to right). The mirror analogs of the factorization identities (4.1.16) are also hold

$$\exists_{1\to m}^{(k,m-k)}\,\exists_{1\to n}^{(m,n-m)} = \exists_{k+1\to n}^{(m-k,n-m)}\,\exists_{1\to n}^{(k,n-k)}\;.$$

4.2 A-Type Hecke algebras

A-Type Hecke algebra $H_{M+1}(q)$ (see e.g. [113] and Refs. therein) is a quotient of the braid group algebra (4.1.1) in which the additional relation

$$\sigma_i^2 - 1 = \lambda\,\sigma_i\;,\;\; (i = 1,\ldots,M)\;, \tag{4.2.1}$$

is satisfied. Here $\lambda = (q-q^{-1})$ and $q \in \mathbf{C}\backslash\{0,\pm1\}$ is a deformation parameter. Note that the algebras $H_{M+1}(q)$ and $H_{M+1}(-q^{-1})$ are isomorphic to each other: $H_{M+1}(q) \simeq H_{M+1}(-q^{-1})$. The group algebra of $\mathcal{B}_{M+1}$ (4.1.1) has an infinite dimension while its quotient H_{M+1} is finite dimensional. It can be shown (see e.g. [116])

that H_{M+1} is spanned linearly by $(M+1)!$ elements which appear in the expansion of $\Sigma_{1\to M+1}$ (4.1.11) (or in the expansion of (4.1.17)).

A-Type Hecke algebra is a special case of a general Hecke algebra which is generated by a set of elements $\{\sigma_i\}$ satisfying conditions (4.2.1) and

$$\underbrace{\sigma_i\,\sigma_j\,\sigma_i\ \dots}_{m_{ij}\ \text{factors}} = \underbrace{\sigma_j\,\sigma_i\,\sigma_j\ \dots}_{m_{ij}\ \text{factors}}\,, \tag{4.2.2}$$

where $m_{ij} = m_{ji}$ are integers such that: $m_{ii} = 1$, $m_{ij} \geq 2$ for $i \neq j$. The set of data given by the matrix $||m_{ij}||$ is conveniently represented as the *Coxeter-Dynkin graph* with M nodes associated with generators σ_i and the nodes i and j are not connected if $m_{ij} = 2$ and connected by $m_{ij} - 2$ lines if $m_{ij} \geq 3$. Thus, the Coxeter-Dynkin graph for the braid group relations (4.1.1) is the A-type graph:

$$\begin{array}{ccccccccc} \circ & \text{—} & \circ & \text{—} & \circ & \text{—}\ \cdots\cdots\cdots\ \text{—} & \circ & \text{—} & \circ \\ \sigma_1 & & \sigma_2 & & \sigma_3 & & \dots & & \sigma_M \end{array}$$

That is why the Hecke algebra with defining relations (4.1.1) and (4.2.1) is called A-type Hecke algebra.

An essential information about a finite dimensional semisimple algebra $\mathcal{A}$ is contained in the structure of its regular bimodule, which decomposes into direct sums: $\mathcal{A} = \bigoplus_{\alpha=1}^{s} \mathcal{A}\cdot e_\alpha$, $\mathcal{A} = \bigoplus_{\alpha=1}^{s} e_\alpha \cdot \mathcal{A}$ of left and right submodules (ideals), respectively (left- and right- Peirce decompositions). Here the elements $e_\alpha \in \mathcal{A}$ $(\alpha = 1, \dots, s)$ are mutually orthogonal idempotents: $e_\alpha \cdot e_\beta = \delta_{\alpha\beta} e_\alpha$, resolving the identity operator: $1 = \sum_{\alpha=1}^{s} e_\alpha$. The number s depends on the choice of the set of idempotens in $\mathcal{A}$. There are two important decompositions of the identity operator and correspondingly two sets of the idempotents in $\mathcal{A}$:
(1) *Primitive idempotents.* An idempotent e_α is primitive if it cannot be further resolved into a sum of nontrivial mutually orthogonal idempotents.
(2) *Primitive central idempotents.* An idempotent e'_α is primitive central if it is primitive in the class of central idempotents.

The regular bimodule decomposes into direct sums of irreducible sub-bimodules (two-sided ideals) $\mathcal{A} = \bigoplus_{\alpha=1}^{s'} \mathcal{A}\cdot e'_\alpha = \bigoplus_{\alpha=1}^{s'} e'_\alpha \cdot \mathcal{A}$ with respect to the central idempotents e'_α.

Now we construct idempotents in H_{M+1} which correspond to the symmetrizers and antisymmetrizers. For this purpose we consider two substitutions $\sigma_i \to q\sigma_i$, $\sigma_i \to -q^{-1}\sigma_i$ for the braid group algebra element $\Sigma_{1\to n}$ (4.1.11). As a result, for the algebra H_{M+1} we obtain two sequences of operators $S_{1\to n}$ and $A_{1\to n}$ $(n = 1, \dots M+1)$

$$S_{1\to n} := a^- \,\Sigma_{1\to n}(q\,\sigma_i)\ , \quad A_{1\to n} := a^+ \,\Sigma_{1\to n}(-q^{-1}\,\sigma_i) \tag{4.2.3}$$

$$\left(a^\mp = \frac{q^{\mp\frac{n(n-1)}{2}}}{[n]_q!}\ , \quad [n]_q! := [1]_q\,[2]_q \cdots [n]_q\ , \quad [n]_q = \frac{(q^n - q^{-n})}{(q - q^{-1})} \right),$$

$$\begin{gathered} \sigma_i\, S_{1\to n} = S_{1\to n}\,\sigma_i = q\, S_{1\to n}\ \ (i = 1, \dots n-1)\ , \\ \sigma_i\, A_{1\to n} = A_{1\to n}\,\sigma_i = -\tfrac{1}{q}\, A_{1\to n}\ \ (i = 1, \dots n-1)\ , \end{gathered} \tag{4.2.4}$$

which are symmetrizers and antisymmetrizers, respectively (see [60]). The normalization factors $a^{\mp}$ have been introduced in (4.2.3) in order to obtain the idempotent conditions $S^2_{1\to n} = S_{1\to n}$ and $A^2_{1\to n} = A_{1\to n}$. Here we additionally suppose that $[n]_q \neq 0$, $\forall n = 1, \dots, M+1$. The first two idempotents are

$$S_{12} = \frac{1}{[2]_q}(q^{-1} + \sigma_1) \; , \quad A_{12} = \frac{1}{[2]_q}(q - \sigma_1) \; . \tag{4.2.5}$$

Note that eqs. (4.2.4) immediately follow from the factorization relations (4.1.14), (4.1.19), the form of the first idempotents (4.2.5) and Hecke condition (4.2.1). The projectors $S_{1\to n}$ and $A_{1\to n}$ (4.2.3) correspond to the Young tableaux which have only one row or one column

$$P\left(\boxed{1 \,|\, \dots \,|\, \mathrm{n}}\right) = S_{1\to n} \; , \quad P\left(\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline \mathrm{n} \\ \hline \end{array}\right) = A_{1\to n} \; .$$

It follows directly from (4.2.4) that the idempotents $S_{1\to M+1}$ and $A_{1\to M+1}$ are central in the algebra H_{M+1}.

Consider now the elements y_i $(i = 1, \dots, M)$ (4.1.4) which form a maximal commutative subalgebra Y_{M+1} in H_{M+1}. The elements y_i are called Jucys - Murphy elements and can be easily rewritten in the form (by using Hecke condition (4.2.1) and braid relations (4.1.1))

$$\begin{array}{c} y_i = \sigma_{i-1} \dots \sigma_2 \, \sigma_1^2 \, \sigma_2 \dots \sigma_{i-1} = \\[2ex] = \lambda \sigma_{i-1} \dots \sigma_2 \, \sigma_1 \, \sigma_2 \dots \sigma_{i-1} + \sigma_{i-1} \dots \sigma_3 \, \sigma_2^2 \, \sigma_3 \dots \sigma_{i-1} = \dots = \\[2ex] = \lambda \sum_{k=1}^{i-2} \sigma_{i-1} \dots \sigma_{k+1} \, \sigma_k \, \sigma_{k+1} \dots \sigma_{i-1} + \lambda \sigma_{i-1} + 1 = \\[2ex] = \lambda \sum_{k=1}^{i-2} \sigma_k \dots \sigma_{i-2} \, \sigma_{i-1} \, \sigma_{i-2} \dots \sigma_k + \lambda \sigma_{i-1} + 1 \; . \end{array} \tag{4.2.6}$$

It is interesting that the idempotents which correspond to the symmetrizers and antisymmetrizers (the Young tableaux is only one row or column) can be constructed in the different way using elements y_n.

Proposition 8. *The idempotents $S_{1\to n}$ and $A_{1\to n}$ $(n = 2, \dots M+1)$ (4.2.3) are expressed in term of the Jucys-Murphy elements as*

$$S_{1\to n} = \frac{(y_2 - q^{-2})}{(q^2 - q^{-2})} \frac{(y_3 - q^{-2})}{(q^4 - q^{-2})} \cdots \frac{(y_n - q^{-2})}{(q^{2(n-1)} - q^{-2})} \; , \tag{4.2.7}$$

$$A_{1\to n} = \frac{(y_2 - q^2)}{(q^{-2} - q^2)} \frac{(y_3 - q^2)}{(q^{-4} - q^2)} \cdots \frac{(y_n - q^2)}{(q^{2(1-n)} - q^2)} \; . \tag{4.2.8}$$

Proof. We note that the first two projectors (4.2.5) are

$$S_{12} = \frac{(y_2 - q^{-2})}{(q^2 - q^{-2})} \; , \quad A_{12} = \frac{(y_2 - q^2)}{(q^{-2} - q^2)} \; ,$$

and therefore eqs. (4.2.7) and (4.2.8) are satisfied for $n = 2$. We prove eq. (4.2.7) by induction (the proof of (4.2.8) is analogous). Let eq. (4.2.7) be correct for some $n = k$. We need to prove this equation for $n = k + 1$ or we have to show that

$$S_{1\to k+1} = S_{1\to k} \cdot \frac{(y_{k+1} - q^{-2})}{(q^{2k} - q^{-2})} \ . \tag{4.2.9}$$

Indeed, we substitute here the last expression for Jucys-Murphy elements y_{k+1} (4.2.6) and take into account (4.2.4). As a result we obtain for (4.2.9)

$$\begin{aligned} S_{1\to k+1} &= \tfrac{1}{(q^{2k}-q^{-2})}\, S_{1\to k}\, \Big(\lambda(q^{k-1}\sigma_k \dots \sigma_1 + q^{k-2}\sigma_k \dots \sigma_2 + \dots + \sigma_k) + 1 - q^{-2}\Big) = \\ &= \tfrac{q^k}{[k+1]_q}\, S_{1\to k}\, \Big(q^k\sigma_k \dots \sigma_1 + q^{k-1}\sigma_k \dots \sigma_2 + \dots + q\,\sigma_k + 1\Big) \ , \end{aligned} \tag{4.2.10}$$

which coincides with the definition of symmetrizers (4.1.17), (4.2.3). This ends the proof of the Proposition. □

Since the idempotents $S_{1\to M+1}$ and $A_{1\to M+1}$ are central in the algebra H_{M+1} and represented as the polynomials $\sim (y_2 - t)(y_3 - t)\cdots(y_{M+1} - t)$, (for $t = q^{\mp 2}$; see (4.2.7), (4.2.8)) one can conclude that all symmetric functions in y_i ($i = 2, \dots, M+1$) generate the central subalgebra Z_{M+1} in the Hecke algebra H_{M+1}. To prove this fact one should only check the relations: $[\sigma_k,\, y_n + y_{n+1}] = 0 = [\sigma_k,\, y_n y_{n+1}]$ for all $k < n + 1$.

New identities for the elements y_i follow from the representations (4.2.7) and (4.2.8) (if we use eqs. (4.2.4)):

$$(y_i - q^{2(i-1)})\, S_{1\to n} = 0 \Rightarrow (y_i - q^{2(i-1)})(y_2 - q^{-2})(y_3 - q^{-2})\cdots(y_n - q^{-2}) = 0 \ , \tag{4.2.11}$$

$$(y_i - q^{2(1-i)})\, A_{1\to n} = 0 \Rightarrow (y_i - q^{2(1-i)})(y_2 - q^2)(y_3 - q^2)\cdots(y_n - q^2) = 0 \ , \tag{4.2.12}$$

($i = 2, \dots, n$). Two new types of idempotents (which are primitive orthogonal idempotents for the subalgebra $H_n \in M_{M+1}$) are obtained from these identities:

$$P\left(\begin{array}{|c|c|c|}\hline 1 & \dots & \text{n-1} \\ \hline \text{n} \\ \cline{1-1}\end{array}\right) = \frac{(y_n - q^{2(n-1)})}{(q^{-2} - q^{2(n-1)})} \prod_{k=1}^{n-1} \frac{(y_k - q^{-2})}{(q^{2(k-1)} - q^{-2})} \ , \tag{4.2.13}$$

$$P\left(\begin{array}{|c|c|}\hline 1 & \text{n} \\ \hline \vdots \\ \cline{1-1} \text{n-1} \\ \cline{1-1}\end{array}\right) = \frac{(y_n - q^{2(1-n)})}{(q^2 - q^{2(1-n)})} \prod_{k=1}^{n-1} \frac{(y_k - q^2)}{(q^{2(1-k)} - q^2)} \ . \tag{4.2.14}$$

These idempotents are not central but they are the elements of the commutative subalgebra Y_{M+1}.

Below we describe the construction (see also [119] and references therein) of all primitive orthogonal idempotents $e_\alpha \in H_{M+1}$ which are elements of Y_{M+1} (i.e. functions of the elements y_i) and they are common eigenidempotents of y_i: $y_i e_\alpha = e_\alpha y_i = a_i^{(\alpha)} e_\alpha$ ($i = 1, \dots, M+1$). We denote by $\mathrm{Spec}(y_1, \dots, y_{M+1})$ the set $\{\Lambda(e_\alpha)\}$ $(\forall \alpha)$ of strings of $M+1$ eigenvalues $\Lambda(e_\alpha) := (a_1^{(\alpha)}, \dots, a_{M+1}^{(\alpha)})$. These eigenidempotents define left (and right) submodules $H_{M+1}\, e_\alpha$ ($e_\alpha\, H_{M+1}$) in the regular bimodule of H_{M+1}.

Lemma 1. *The eigenidempotents e and e' with eigenvalues $a_i = a'_i$ ($\forall i = 1, \dots, M$) and $a_{M+1} \neq a'_{M+1}$ define different left (right) submodules in the regular bimodule of H_{M+1}.*

Proof. We proof this Lemma only for the left submodules $H_{M+1}\, e$, $H_{M+1}\, e'$. The case of right submodules can be considered analogously. Consider central element $Z = y_1 y_2 \cdots y_{M+1}$ (symmetric function of y_i). The left action of Z on $H_{M+1}\, e$ and $H_{M+1}\, e'$ gives different constants. Thus, there are no elements $X \in H_{M+1}$ such that $H_{M+1} e' = X\, H_{M+1} e$. □

Now we introduce the important intertwining elements [41] (presented in another form in [124])

$$U_{n+1} = \sigma_n y_n - y_n \sigma_n = \tag{4.2.15}$$

$$= (y_{n+1} - y_n)\sigma_n - \lambda y_{n+1} = \sigma_n (y_n - y_{n+1}) + \lambda y_{n+1} \;\; (1 \leq n \leq M) \; , \tag{4.2.16}$$

subject to relations[11]

$$\begin{gathered} U_{n+1} y_n = y_{n+1} U_{n+1} \; , \quad U_{n+1} y_{n+1} = y_n U_{n+1} \; , \\ [U_{n+1},\, y_k] = 0 \;\; (k \neq n, n+1) \; , \end{gathered} \tag{4.2.17}$$

$$U_n\, U_{n+1}\, U_n = U_{n+1}\, U_n\, U_{n+1} \; , \tag{4.2.18}$$

$$U_{n+1}^2 = (q y_n - q^{-1}\, y_{n+1})\, (q\, y_{n+1} - q^{-1}\, y_n) \; , \tag{4.2.19}$$

(first two relations in (4.2.17) hold for any set of elements $\{y_k\}$ which satisfy $[\sigma_n,\, y_n + y_{n+1}] = 0 = [\sigma_n,\, y_n y_{n+1}]$).

Lemma 2. *The eigenidempotents e and e' with eigenvalues:*

$$\begin{gathered} a_i = a'_i \;\; (\forall i = 1, \dots, M-1) \; , \\ a_M = a'_{M+1} \; , \quad a_{M+1} = a'_M \; , \quad a_M \neq q^{\pm 2} a_{M+1} \; , \end{gathered} \tag{4.2.20}$$

belong to the same irreducible sub-bimodule in the regular bimodule of H_{M+1}.

Proof. Since the algebra Y_{M+1} generated by $\{y_1, \dots, y_{M+1}\}$ is maximal commutative subalgebra in H_{M+1} we have $e' = e''$ if $\Lambda(e') = \Lambda(e'')$. Then, using intertwining element U_{M+1} (4.2.15) we construct the eigenidempotent

$$e'' = \frac{1}{(q^2 a_M - a_{M+1})(a_{M+1} - q^{-2} a_M)}\, U_{M+1} e U_{M+1} \; ,$$

which is well defined in view of the last condition in (4.2.20). The element $U_{M+1} e U_{M+1}$ is not equal to zero, since $U_{M+1}^2 e U_{M+1}^2 = (q^2 a_M - a_{M+1})^2 (a_{M+1} - q^{-2} a_M)^2 e \neq 0$. This inequality follows from the last condition in (4.2.20). For the element $e'' \sim U_{M+1} e U_{M+1}$ we have $\Lambda(e'') = \Lambda(e')$ in view of (4.2.17). Thus, $e'' = e' \Rightarrow e' \sim U_{M+1} e U_{M+1}$ and the eigenidempotents e and e' belong to the same irreducible sub-bimodule in the regular bimodule of H_{M+1}. □

[11]The definition (4.2.15) of intertwining elements is not unique. One can multiply U_{n+1} by a function $f(y_n, y_{n+1})$: $U_{n+1} \to U_{n+1} f(y_n, y_{n+1})$. Then, all eqs. (4.2.17) – (4.2.19) are valid if f satisfies $f(y_n, y_{n+1}) f(y_{n+1}, y_n) = 1$.

Now we associate Young tableaux (related to the primitive orthogonal idempotents) with paths in Young-Ogievetsky graph. By definition Young-Ogievetsky graph is Young graph with vertices which are Young diagrams, edges which indicate inclusions (or branching) of diagrams and indices on the edges corresponding to the eigenvalues of the Jucys-Murphy elements of the same level (O.Ogievetsky was the first who propose to put the eigenvalues of the Jucys-Murphy elements on the edges). Usually the indices on the edges of the graphs of Young type correspond to the multiplicity functions of the branching. In our case the multiplicities are equal to 1 in view of Lemma 1 and maximality of the commutative subalgebra Y_{M+1}. For example, the Young-Ogievetsky graph for H_4 is:

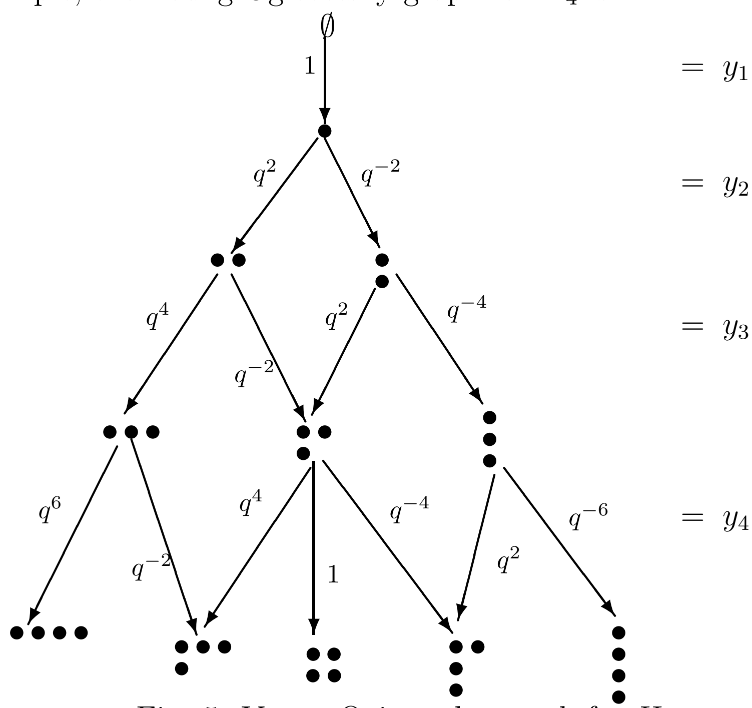


Fig. 5. Young-Ogievetsky graph for H_4.

The paths (associated to Young tableaux) start from the top vertex $\emptyset$ and finish at the vertex labeled by the Young diagram of the same shape as the tableaux. The dimension of the corresponding representation is equal to the number of standard tableaux of this shape or, as we see, the number of paths which lead to this Young diagram from $\emptyset$. For example: the path $\{\emptyset \xrightarrow{1} \bullet \xrightarrow{q^2} \bullet\bullet \xrightarrow{q^{-2}} \begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix} \xrightarrow{1} \begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix}\}$ corresponds to the tableau $\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline\end{array}$, i.e. the shape of the tableau (Young diagram) is given by the shape of the last vertex of the path, while the content of the tableau shows in which sequence the points $\bullet$ appear in the vertices along the path. The edge indices of the path are the eigenvalues of the Jucys-Murphy elements $y_1 = 1$, $y_2 = q^2$, $y_3 = q^{-2}$, $y_4 = 1$ obtained by their action on the idempotent $P\left(\begin{array}{|c|c|}\hline 1&2\\ \hline 3&4\\ \hline\end{array}\right)$. Then, the explicit formula for this idempotent can be constructed by induction. Namely, we take the explicit form of the previous idempotent $P\left(\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\right)$ (related to the previous vertex of the path) and multiply it by the factors $(y_4 - q^4)$ and $(y_4 - q^{-4})$ to forbid the moving from the vertex $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\end{smallmatrix}$ along the edges with labels q^4 and q^{-4} and direct the path along the edge with the index 1 to the vertex $\begin{smallmatrix}\bullet&\bullet\\ \bullet&\bullet\end{smallmatrix}$. As a result, after an obvious renormalization, we obtain

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right) = P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array}\right)\frac{(y_4-q^4)(y_4-q^{-4})}{(1-q^4)(1-q^{-4})}\,. \tag{4.2.21}$$

In the same way one can deduce the chain of identities

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{|c}{}\\\cline{1-1}\end{array}\right) = P\left(\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)\frac{(y_3-q^4)}{(q^{-2}-q^4)} = P\left(\begin{array}{|c|}\hline 1\\\hline\end{array}\right)\frac{(y_2-q^{-2})(y_3-q^4)}{(q^2-q^{-2})(q^{-2}-q^4)}\,, \tag{4.2.22}$$

where we fix $P\left(\begin{array}{|c|}\hline 1\\\hline\end{array}\right)=1$ by definition. Using (4.2.22), the final formula for (4.2.21) can be found

$$P\left(\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\right) = \frac{(y_2-q^{-2})(y_3-q^4)}{(q^2-q^{-2})(q^{-2}-q^4)}\frac{(y_4-q^4)(y_4-q^{-4})}{(1-q^4)(1-q^{-4})}\,. \tag{4.2.23}$$

We note that the described procedure leads automatically to the idempotents which are orthogonal to each other.

This example has demonstrated that all information about primitive orthogonal idempotents for the A-type Hecke algebra is encoded in the Young-Ogievetsky graph given in Fig. 5. Thus, we need to justify this graph and its edge indices. First of all we show that the spectrum of the Jucys-Murphy operators y_j (possible edge indices of the graph) for H_{M+1} is such that

$$\mathrm{Spec}(y_j)\subset\{q^{2\mathbf{Z}_j}\}\quad\forall j=1,2,\ldots,M+1\,, \tag{4.2.24}$$

where $\mathbf{Z}_j$ denotes the set of integer numbers $\{1-j,\ldots,-2,-1,0,1,2,\ldots,j-1\}$. For this purpose we use the important intertwining elements (4.2.15), (4.2.16) and prove (4.2.24) by induction. From Hecke condition (4.2.1) we have

$$(y_2-q^2)(y_2-q^{-2})=0\,. \tag{4.2.25}$$

Thus, $\mathrm{Spec}(y_2)$ satisfies (4.2.24). Assume that the spectrum of y_{j-1} satisfies (4.2.24) for some $j\geq 3$. Consider a characteristic equation for y_{j-1} (cf. (4.2.25)):

$$f(y_{j-1}) := \prod_\alpha (y_{j-1}-a^{(\alpha)}_{j-1}) = 0 \quad (a^{(\alpha)}_{j-1}\in\mathrm{Spec}(y_{j-1}))\,.$$

Using operators U_j and their properties (4.2.17), (4.2.19) we deduce

$$0 = U_j f(y_{j-1})U_j = f(y_j)U_j^2 = f(y_j)(q^2y_{j-1}-y_j)(y_j-q^{-2}y_{j-1})\,. \tag{4.2.26}$$

which means that

$$\mathrm{Spec}(y_j)\subset\Big(\mathrm{Spec}(y_{j-1})\cup q^{\pm 2}\cdot\mathrm{Spec}(y_{j-1})\Big)\,, \tag{4.2.27}$$

and it proves (4.2.24).

Consider a subalgebra $\hat{H}_2^{(i)}$ in H_{M+1} with generators y_i, y_{i+1} and σ_i (for fixed $i \leq M$). We investigate representations of $\hat{H}_2^{(i)}$ such that the elements y_i, y_{i+1} are diagonalizable. Let e be a common eigenidempotent of y_i, y_{i+1}: $y_i e = a_i e$, $y_{i+1} e = a_{i+1} e$. Then, the left action of $\hat{H}_2^{(i)}$ closes on elements $v_1 = e$, $v_2 = \sigma_i e$ and is given by matrices:

$$\sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & -\lambda a_{i+1} \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & \lambda a_{i+1} \\ 0 & a_i \end{pmatrix}, \tag{4.2.28}$$

where we have used the convention $y v_i = v_j \, y_{ji} \;\; \Rightarrow \;\; x\, y\, v_i = v_k \, x_{kj} \, y_{ji}$ to produce matrix representations.

The matrices y_i, y_{i+1} (4.2.28) can be simultaneously diagonalized by the transformation $y \to V^{-1} y V$ where

$$V = \begin{pmatrix} 1 & \frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}, \quad V^{-1} = \begin{pmatrix} 1 & -\frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} \\ 0 & 1 \end{pmatrix}.$$

As a result we obtain the following matrix representation

$$\sigma_i = \begin{pmatrix} -\frac{\lambda\, a_{i+1}}{a_i - a_{i+1}} & 1 - \frac{\lambda^2\, a_i a_{i+1}}{(a_i - a_{i+1})^2} \\ 1 & \frac{\lambda\, a_i}{a_i - a_{i+1}} \end{pmatrix}, \; y_i = \begin{pmatrix} a_i & 0 \\ 0 & a_{i+1} \end{pmatrix}, \; y_{i+1} = \begin{pmatrix} a_{i+1} & 0 \\ 0 & a_i \end{pmatrix}, \tag{4.2.29}$$

where $a_i \neq a_{i+1}$ otherwise y_i, y_{i+1} are not diagonalizable. When $a_{i+1} = q^{\pm 2} a_i$, the 2-dimensional representation (4.2.29) is reduced into the 1-dimensional representation with $\sigma_i = \pm q^{\pm 1}$, respectively. We summarize the above results as:

Proposition 9. (q-Vershik-Okounkov [125], [119]). *Let*

$$\Lambda = (a_1, \dots, a_i, a_{i+1}, \dots, a_n) \in \mathrm{Spec}(y_1, \dots, y_{M+1}) \; ,$$

be a possible spectrum of the commutative set $(y_1, \dots, y_{M+1})$, which corresponds to a primitive eigenidempotent $e_\Lambda \in H_{M+1}$. Then, $a_i = q^{2m_i}$, where $m_i \in \mathbf{Z}_i$ (4.2.24) and

(1) $a_i \neq a_{i+1}$ for all $i < M+1$;

(2) if $a_{i+1} = q^{\pm 2} a_i$, then $\sigma_i \cdot e_\Lambda = \pm q^{\pm 1} e_\Lambda$;

(3) if $a_i \neq q^{\pm 2} a_{i+1}$, then

$$\Lambda' = (a_1, \dots, a_{i+1}, a_i, \dots, a_{M+1}) \in \mathrm{Spec}(y_1, \dots, y_{M+1}) \; ,$$

and the left action of the elements σ_i, y_i, y_{i+1} in the linear span of $v_\Lambda = e_\Lambda$ and

$$v_{\Lambda'} = \sigma_i \, e_\Lambda + \frac{\lambda a_{i+1}}{a_i - a_{i+1}} \, e_\Lambda \; ,$$

is given by (4.2.29).

From this Proposition we conclude that the only admissible subgraphs in the Young-Ogievetsky graph (subgraphs which show all possible two-edges paths with fixed initial and final vertices) are

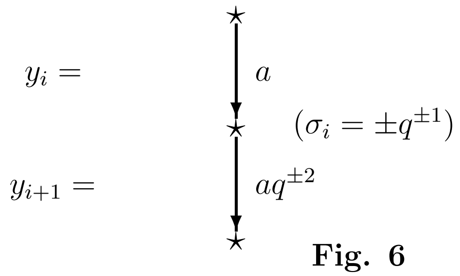


Fig. 6

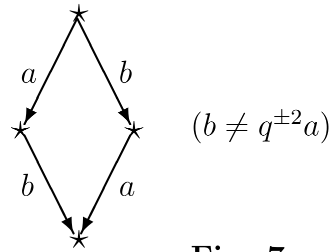


Fig. 7

where stars in the vertices denote Young diagrams. These subgraphs are related to the 1-dimensional and 2-dimensional (it corresponds to the number of paths from the top vertex to the bottom one) representations of the subalgebra generated by $\{y_i, y_{i+1}, \sigma_i\}$. In view of the braid relations $\sigma_i \sigma_{i\pm1} \sigma_i = \sigma_{i\pm1} \sigma_i \sigma_{i\pm1}$ and possible values of σ's presented in Fig. 6 for 1-dimensional (1D) representation subgraphs, we conclude that the chains: $\star \stackrel{a}{\longrightarrow} \star \stackrel{q^{\pm2}a}{\longrightarrow} \star \stackrel{a}{\longrightarrow} \star$ of the 1D representation subgraphs in the Young-Ogievetsky graph in Fig. 5 are forbidden. While admissible chains of 1D representation subgraphs are: $\star \stackrel{a}{\longrightarrow} \star \stackrel{q^{\pm2}a}{\longrightarrow} \star \stackrel{q^{\pm4}a}{\longrightarrow} \star$. These statements and the form of only admissible subgraphs in Figs. 6,7 justify (for the A-type Hecke algebra) the Young-Ogievetsky graph presented in Fig. 5. Indeed, we know the top of the Young-Ogievetsky graph which consists of 3 edges with indices $1, q^2, q^{-2}$ (see Fig. 5). Then, one can explicitly construct "step by step" moving down the whole Young-Ogievetsky graph (with all indices on edges) by using (4.2.27), the form of only admissible subgraphs in Figs. 6,7 and rules for the chains of 1-dim. representations. We stress here two important properties of the Young-Ogievetsky graph:

1.) to each vertex of the graph the number of incoming edges E_{in} less than the number of outgoing edges E_{out} on 1: $E_{out} = E_{in} + 1$;
2.) to each vertex the products of indices a_{in} for incoming and a_{out} for outgoing edges are equal to each other: $\prod_{\alpha=1}^{E_{in}} a_{in}(\alpha) = \prod_{\beta=1}^{E_{out}} a_{out}(\beta)$.

Since the Young-Ogievetsky graph is explicitly known, we can deduce the expressions (in terms of the elements y_k) of all orthogonal primitive idempotents for the Hecke algebra (in the same way as it has been done in (4.2.21) - (4.2.23)). Note that explicit expressions for such primitive orthogonal idempotents are known and have been presented in [114], [115], [119].

Using intertwining elements (4.2.15) one can immediately construct off-diagonal matrix units[12] in a double sided Peirce decomposition of the Hecke algebra $H_{M+1} = \bigoplus_{\alpha,\beta} e_\alpha H_{M+1} e_\beta$. Let $P(X_{\vec{a}})$ be orthogonal primitive idempotent which corresponds to the path $X_{\vec{a}}$ on Young-Ogievetsky graph labeled by the eigenvalues

$$\vec{a} = (1, a_2, \dots, a_{M+1}) \in \mathrm{Spec}(y_1, \dots, y_{M+1}) \; ,$$
$$y_i \, P(X_{\vec{a}}) = P(X_{\vec{a}}) \, y_i = a_i \, P(X_{\vec{a}}) \;\; (\forall i = 1, \dots, M+1) \; .$$

In the case $a_j \neq q^{\pm 2} a_{j+1}$ (see the proof of Lemma 2) we introduce the element $P(X_{s_j \cdot \vec{a}}) \in H_{M+1}$

$$P(X_{s_j \cdot \vec{a}}) := \frac{1}{(q^2 a_j - a_{j+1})(a_{j+1} - q^{-2} a_j)} \, U_{j+1} \, P(X_{\vec{a}}) \, U_{j+1} \quad (\forall j = 1, \dots, M) \tag{4.2.30}$$

[12]Recall that the orthogonal primitive idempotents are diagonal matrix units.

such that

$$P(X_{s_j\cdot\vec a})^2 = P(X_{s_j\cdot\vec a}) \ , \quad \vec y\, P(X_{s_j\cdot\vec a}) = P(X_{s_j\cdot\vec a})\, \vec y = (s_j\cdot\vec a) P(X_{s_j\cdot\vec a}) \ ,$$

$$P(X_{s_j\cdot\vec a}) = \frac{(q^2 y_j - y_{j+1})(y_{j+1} - q^{-2} y_j)}{(q^2 a_j - a_{j+1})(a_{j+1} - q^{-2} a_j)}\, (s_j\cdot P)(X_{\vec a}) \ ,$$

$$P(X_{s_j\cdot\vec a})\, P(X_{\vec a}) = P(X_{\vec a})\, P(X_{s_j\cdot\vec a}) = 0 \ , \tag{4.2.31}$$

where $s_j\cdot\vec a = (a_1,\dots,a_{j+1},a_j,\dots a_{M+1}) \in \mathrm{Spec}(y_1,\dots,y_{M+1})$ is the vector with permuted coordinates a_j and a_{j+1}; $(s_j\cdot P)(X_{\vec a})$ denotes the function $P(X_{\vec a})$ with permuted variables y_i and y_{i+1}. The identity (4.2.31) follows from the fact $P := P(X_{\vec a})\ P(X_{\vec a\,'}) = 0$ for all $\vec a \neq \vec a\,'$ ($\exists j$: $a_j \neq a'_j$) in view of the equations $[P(X_{\vec a}),\, P(X_{\vec a\,'})] = 0$ and $y_j P = a_j P = a'_j P$.

According to (4.2.19) and (4.2.30) we obtain

$$\begin{aligned} &U_{j+1}\, P(X_{\vec a}) = P(X_{s_j\cdot\vec a})\, U_{j+1} =: P(X_{s_j\cdot\vec a}|X_{\vec a}) \ \ (j=1,\dots,M) \ , \\ &P(X_{\vec a})\, U_{j+1} = U_{j+1}\, P(X_{s_j\cdot\vec a}) =: P(X_{\vec a}|X_{s_j\cdot\vec a}) \ \ (j=1,\dots,M) \ , \end{aligned} \tag{4.2.32}$$

In the case $a_j \neq q^{\pm 2} a_{j+1}$, in view of Lemma 2, we have $s_j\cdot\vec a \in \mathrm{Spec}(y_1,\dots,y_{M+1})$ (the path $X_{s_j\cdot\vec a}$ exists in Young-Ogievetsky graph and corresponds to the standard Young tableau). Then, taking into account (4.2.16), we deduce

$$\begin{aligned} &(\sigma_j + \tfrac{\lambda a_{j+1}}{(a_j - a_{j+1})})\, P(X_{\vec a}) = P(X_{s_j\cdot\vec a})\, (\sigma_j - \tfrac{\lambda a_j}{(a_j - a_{j+1})}) = \tfrac{P(X_{s_j\cdot\vec a}|X_{\vec a})}{(a_j - a_{j+1})} \ , \\ &P(X_{\vec a})\, (\sigma_j + \tfrac{\lambda a_{j+1}}{(a_j - a_{j+1})}) = (\sigma_j - \tfrac{\lambda a_j}{(a_j - a_{j+1})})\, P(X_{s_j\cdot\vec a}) = \tfrac{P(X_{\vec a}|X_{s_j\cdot\vec a})}{(a_{j+1} - a_j)} \ . \end{aligned} \tag{4.2.33}$$

The elements $P(X_{s_j\cdot\vec a}|X_{\vec a})$ play the role of the off-diagonal matrix elements in the Peirce decomposition [115]. In the case $a_j = q^{\pm 2} a_{j+1}$ we have

$$U_{j+1}\, P(X_{\vec a}) = 0 = P(X_{\vec a})\, U_{j+1} \ ,$$

and formulas of the type (4.2.33) define the 1-dimensional representation for generator σ_j which corresponds to the Fig. 6.

Another convenient recurrent relations for Hecke symmetrizers and antisymmetrizers (4.2.3), (4.2.7), (4.2.8) are (see e.g. [62], [60], [126]):

$$S_{1\to n} = S_{1\to n-1}\, \frac{\sigma_{n-1}(q^{1-n})}{[n]_q}\, S_{1\to n-1} \ , \quad S_{1\to n} = S_{2\to n}\, \frac{\sigma_1(q^{1-n})}{[n]_q}\, S_{2\to n} \ , \tag{4.2.34}$$

$$A_{1\to n} = A_{1\to n-1}\, \frac{\sigma_{n-1}(q^{n-1})}{[n]_q}\, A_{1\to n-1} \ , \quad A_{1\to n} = A_{2\to n}\, \frac{\sigma_1(q^{n-1})}{[n]_q}\, A_{2\to n} \ , \tag{4.2.35}$$

where

$$\sigma_n(x) := \lambda^{-1}\,(x^{-1}\sigma_n - x\sigma_n^{-1}) \ , \tag{4.2.36}$$

are Baxterized elements (the R-matrix representations of these elements are given in (3.8.5)) for the algebra H_{M+1}. The elements $\sigma_n(x)$ obey the Yang-Baxter equation (the proof of this statement is the same as the analogous proof in (3.8.1) – (3.8.4)):

$$\sigma_n(x)\, \sigma_{n-1}(xy)\, \sigma_n(y) = \sigma_{n-1}(y)\, \sigma_n(xy)\, \sigma_{n-1}(x) \ . \tag{4.2.37}$$

These elements are also normalized by the conditions $\sigma_n(\pm 1) = \pm 1$ and satisfy

$$\sigma_i(x) = \frac{x - x^{-1}}{y - y^{-1}}\, \sigma_i(y) + \frac{yx^{-1} - xy^{-1}}{y - y^{-1}} \quad (\forall x, y \neq \pm 1) \; ,$$

$$\sigma_i(x)\, \sigma_i(y) = \sigma_i(xy) + (x - x^{-1})(y - y^{-1})\lambda^{-2} \; , \tag{4.2.38}$$

and the special case of (4.2.38) is the "unitarity condition"

$$\sigma_i(x)\, \sigma_i(x^{-1}) = \left(1 - \frac{(x - x^{-1})^2}{\lambda^2}\right) \; . \tag{4.2.39}$$

Eqs. (4.2.38), (4.2.39) follow from the Hecke relation (4.2.1). The equivalence of both representations given in (4.2.34) and (4.2.35) is demonstrated by means of Yang-Baxter eq. (4.2.37) (or by means of the obvious automorphism $\sigma_k \to \sigma_{M-k}$ of the Hecke algebra H_M). The equivalence of (4.2.34), (4.2.35) and (4.2.3) can be easily demonstrated if one rewrites first representations of (4.2.34), (4.2.35) in the form

$$\begin{array}{c} S_{1\to n} = \frac{1}{[n]_q!}\, \sigma_1(q^{-1})\, \sigma_2(q^{-2}) \dots \sigma_{n-1}(q^{1-n})\, S_{1\to n-1} \; , \\ A_{1\to n} = \frac{1}{[n]_q!}\, \sigma_1(q)\, \sigma_2(q^2) \dots \sigma_{n-1}(q^{n-1})\, A_{1\to n-1} \; , \end{array} \tag{4.2.40}$$

and, then, use (4.2.4) to compare (4.2.40) with (4.2.3) and (4.1.11). According to (4.2.34), (4.2.35) the first two projectors are (cf. (3.4.19), (4.2.5)):

$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}\right) = S_{12} = \frac{1}{[2]_q}\, \sigma_1(q^{-1}) \; , \quad P\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}\right) = A_{12} = \frac{1}{[2]_q}\, \sigma_1(q) \; ,$$

and their orthogonality readily follows from (4.2.39). One can also express another types of the orthogonal idempotents (not only symmetrizers and antisymmetrizers) in terms of the Baxterized elements:

$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right) = \frac{1}{[3]_q!}\sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1}) \; , \quad P\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = \frac{1}{[3]_q!}\sigma_1(q)\sigma_2(q^{-1})\sigma_1(q) \; ,$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q^{-2})\sigma_1(q^{-1})\sigma_3(q)\sigma_2(q^{-2})\sigma_1(q^{-1}) \; ,$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q)\sigma_2(q^{-1})\sigma_3(q^{-2})\sigma_2(q^{-1})\sigma_1(q) \; ,$$

$$P\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q)\sigma_2(q^{-2})\sigma_1(q^{-1})\sigma_3(q^{-3}) \; ,$$

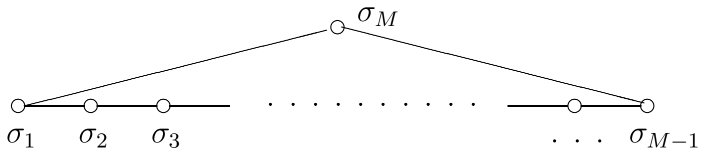
$$P\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}\right) \sim \sigma_1(q^{-1})\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q^3)\sigma_2(q)\sigma_1(q^{-1})\sigma_3(q^{-1}) \; .$$

Consider the quotients of the Hecke algebra H_{M+1} with respect to the additional relations $A_{1\to n} = 0$ ($n \leq M+1$), which are equivalent (see (4.2.35)) to the identities

$$A_{1\to n-1}\, \sigma_{n-1}\, A_{1\to n-1} = \frac{q^{n-1}}{[n-1]_q}\, A_{1\to n-1} \; . \tag{4.2.41}$$

This is the way how the generalized Temperley-Lieb-Martin algebras [117] are defined. As it has been mentioned in [62] the quotient of H_{M+1} with respect to $A_{1\to 3} = 0$ is isomorphic to the Temperley-Lieb algebra.

At the end of this subsection we stress that one can define the infinite dimensional Hecke algebra (4.2.1), (4.2.2) which corresponds to the affine A-type Coxeter graph



We call this algebra a periodic A-type Hecke algebra[13] and denote it as AH_{M+1}. For the algebra AH_{M+1} one can construct the set of $(M-1)$ commuting elements

$$I_k = \sum_{i=1}^{M} \sigma_i \sigma_{i+1} \dots \sigma_{i+k} \quad (k = 0, \dots, M-2) \; , \tag{4.2.42}$$

where we have identified $\sigma_{M+i} = \sigma_i$.

Another infinit dimensional Hecke algebra is affine Hecke algebra $\hat{H}_{M+1}$. We recall that an affine Hecke algebra $\hat{H}_{M+1}$ is defined (see e.g. [118], Chapter 12.3) in terms of the generators σ_i $(i = 1, \dots, M)$ of H_{M+1} and additional generators y_k $(k = 1, \dots, M+1)$ subject relations (cf. (4.1.4), (4.1.5)):

$$y_{k+1} = \sigma_k \, y_k \, \sigma_k \; , \quad y_k \, y_j = y_j \, y_k \; , \quad y_j \, \sigma_i = \sigma_i \, y_j \;\; (j \neq i, i+1) \; , \tag{4.2.43}$$

(the generators $\{y_k\}$ form a commutative subalgebra in $\hat{H}_{M+1}$). Symmetric functions of the elements y_k generate the center of the algebra $\hat{H}_{M+1}$. The interesting property of the algebra $\hat{H}_{M+1}$ is the existence of important intertwining elements [41] (cf. elements ϕ_i in [124], Prop. 3.1): $U_{i+1} = (\sigma_i y_i - y_i \sigma_i) f(y_i, y_{i+1})$ $(1 \leq i \leq M)$, where function f satisfies: $f(y_i, y_{i+1}) f(y_{i+1}, y_i) = 1$. The elements U_i obey the same relations (4.2.17)–(4.2.19) as in the case of the usual A-type Hecke algebra H_{M+1}.

With the help of the algebra $\hat{H}_{M+1}$ one can construct $(M+1)$-dimensional representation for its subalgebra H_{M+1}. Indeed, let a vector v be a 1-dimensional representation for the subalgebra H_{M+1}: $\sigma_i \, v = q \, v$ $(\forall i = 1, \dots M)$. Consider an induced $M+1$-dimensional space with the basis $\{v_1, v_2, \dots v_{M+1}\}$ where $v_k := y_k v$. Then, according to (4.2.43) and Hecke condition (4.2.1) we obtain $M+1$-dimensional representation for generators σ_i:

$$\sigma_i \, v_k = q \, v_k \;\; (k \neq i, i+1) \; , \quad \sigma_i \, v_i = q^{-1} \, v_{i+1} \; , \quad \sigma_i \, v_{i+1} = \lambda \, v_{i+1} + q \, v_i \; ,$$

which is called *Burau representation* of H_{M+1}.

The affine Hecke algebra $\hat{H}_{M+1}$ is intimately related to a reflection equation

$$\sigma_n(x \, z^{-1}) \, K_n(x) \, \sigma_n(x \, z) \, K_n(z) = K_n(z) \, \sigma_n(x \, z) \, K_n(x) \, \sigma_n(x \, z^{-1}) \; , \tag{4.2.44}$$

(here x, z are spectral parameters) which appears, e.g., in the theory of integrable spin chains with boundaries (see below Sec. 5.2). The Baxterized elements $\sigma_n(x)$

[13] As we will see below, in Sec. 5.1, this algebra appears in a formulation of integrable models of periodic spin chains.

are defined in (4.2.36). Taking the reflection operator $K_n(x)$ in the form

$$K_n(x) = \frac{y_n - \xi\, x^2}{y_n - \xi\, x^{-2}} \;, \tag{4.2.45}$$

where ξ is a constant, we find [121] that this $K_n(x)$ is a solution of (4.2.44) if y_n are the affine generators of $\hat{H}_{M+1}$. In particular one can easily reduce (4.2.45) to the solution

$$K_n(x) = y_n + \frac{\beta_0/\xi + \xi + \beta_1 x^2}{x^2 - x^{-2}} \tag{4.2.46}$$

of the reflection equation (4.2.44) if in addition we require that y_n satisfies a quadratic characteristic equation $y_n^2 + \beta_1 y_n + \beta_0 = 0$ $(\forall \beta_0, \beta_1 \in \mathbf{C}\backslash 0)$. The solution (4.2.45) is obviously regular: $K_n(1) = 1$, and obeys a "unitary condition":

$$K_n(x) K_n(x^{-1}) = 1 \;.$$

We stress that the simplest solution (4.2.46) of the reflection equation (4.2.44) has been obtained in [122].

If one finds a solution of eq. (4.2.44) for $n = m$, then, a solution for $n = m+1$ can be constructed by means of the formula:

$$K_{m+1}(x) = (\lambda x)^2 \, \sigma_m(x) K_m(x) \sigma_m(x) \;.$$

In particular one can take $K_{n-1}(x) = 1$ and, using (4.2.37) and (4.2.38), directly check that (cf. (4.2.46))

$$\frac{K_n(x)}{x^2} = \lambda^2 \, \sigma_{n-1}^2(x) = \lambda^2 \left(\sigma_{n-1}(x^2) + \frac{(x - x^{-1})^2}{\lambda^2} \right) = \sigma_{n-1}^2 + \frac{2 - (2 + \lambda^2)\, x^2}{x^2 - x^{-2}} \;,$$

solves eq. (4.2.44).

Remark. Interrelations of periodic AH_M and affine $\hat{H}_M$ Hecke algebras has been discussed in [115]. Here we present more explicit construction [120] of these interrelations which is valid even for the braid group case (when the Hecke condition (4.2.1) is relaxed).

Let $\{\sigma_1, \dots, \sigma_{M-1}\}$ be generators of braid group $\mathcal{B}_M$. We extend the group $\mathcal{B}_M$ by the element X such that

$$X\, \sigma_k \, X^{-1} = \sigma_{k-1} \quad (\forall k = 2, \dots, M-1) \;, \tag{4.2.47}$$

$$X\, \sigma_1 \, X^{-1} = X^{-1} \, \sigma_{M-1} \, X =: \sigma_M \;. \tag{4.2.48}$$

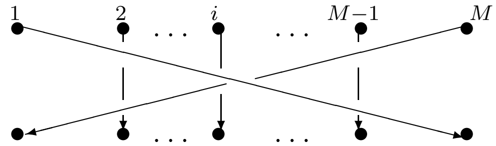
It is not hard to check that the new element σ_M satisfies eqs. (4.1.3) and, therefore, the elements $\{\sigma_1, \dots, \sigma_M\}$ (where σ_M has been defined in (4.2.48)) generate the periodic braid group $\overline{\mathcal{B}}_M$.

Note that X (4.2.47), (4.2.48) can be realized as the inner element of $\mathcal{B}_M$. Indeed, the operator X which solves equations (4.2.47), (4.2.48) can be taken in the form $X = \sigma_{M-1 \leftarrow 1} \in \mathcal{B}_M$, where the notation $\sigma_{M-1 \leftarrow m} := \sigma_{M-1} \dots \sigma_{m+1} \, \sigma_m$ has been used. Then, we define the additional generator σ_M (4.2.48) as:

$$\sigma_M := X \, \sigma_1 \, X^{-1} = \sigma_{M-1 \leftarrow 1} \, \sigma_1 \, \sigma_{M-1 \leftarrow 1}^{-1} = \sigma_{M-1 \leftarrow 1} \, \sigma_{M-1 \leftarrow 2}^{-1} \tag{4.2.49}$$

and its graphical representation is:

$\sigma_M \;=$



(4.2.50)

It is evident that σ_M satisfies (4.1.3) in view of its graphical representation (4.2.50). According to eqs. (4.1.1) and (4.1.3) the elements $\{\sigma_1, \dots, \sigma_M\}$ of the group $\mathcal{B}_M$ (where σ_M has been defined in (4.2.49)) generate the periodic braid group $\overline{\mathcal{B}}_M$ and, therefore, eq. (4.2.49) defines the homomorphism: $\overline{\mathcal{B}}_M \to \mathcal{B}_M$.

Consider the affine braid group $\hat{\mathcal{B}}_M$ generated by the elements $\{\sigma_1, \dots, \sigma_{M-1}, y_1\}$. The generator y_1 satisfies reflection equation and locality conditions

$$\sigma_1\, y_1\, \sigma_1\, y_1 = y_1\, \sigma_1\, y_1\, \sigma_1 \;, \quad [y_1,\, \sigma_k] = 0 \;\; (k = 2, \dots, M-1) \;.$$

Then, the operator

$$X = \sigma_{M-1\leftarrow 1}\, y_1 \in \hat{\mathcal{B}}_M \;,$$

solves eqs. (4.2.47), (4.2.48) and one can introduce new generator $\sigma_M \in \hat{\mathcal{B}}_M$ according to (4.2.48):

$$\sigma_M = \sigma_{M-1\leftarrow 1}\, y_1\, \sigma_1\, y_1^{-1}\, \sigma_{M-1\leftarrow 1}^{-1} \;, \tag{4.2.51}$$

which satisfies (4.1.3). Thus, eq. (4.2.51) defines the homomorphism $\overline{\mathcal{B}}_M \to \hat{\mathcal{B}}_M$.

All this consideration is readily carry over to the Hecke algebra case. Indeed, the definition (4.2.48) of the additional generator σ_M (needed to close the set of the generators σ_k to the periodic chain) looks like the similarity transformation of σ_1. Thus, the characteristic Hecke identity (4.2.1) for the elements σ_1 and σ_M coincides.

4.3 Birman-Murakami-Wenzl algebra

The Birman-Murakami-Wenzl algebra $W\!B_{M+1}(q,\nu)$ is generated by the elements σ_i (4.1.1) and κ_i $(i = 1, \dots, M)$ which satisfy the following relations [123]

$$\kappa_i\, \sigma_i = \sigma_i\, \kappa_i = \nu\, \kappa_i \;, \tag{4.3.1}$$

$$\kappa_i\, \sigma_{i-1}^{\pm 1}\, \kappa_i = \nu^{\mp 1}\, \kappa_i \;, \tag{4.3.2}$$

$$\sigma_i - \sigma_i^{-1} = \lambda\, (1 - \kappa_i) \;, \tag{4.3.3}$$

where $\nu \in \mathbf{C}\backslash\{0, \pm q^{\pm 1}\}$ is an additional parameter of the algebra and $\lambda = q - q^{-1}$. The following relations can be derived from (4.1.1), (4.3.1) - (4.3.3)

$$\kappa_i\, \kappa_i = \mu\, \kappa_i \;, \tag{4.3.4}$$

$$(\; \mu = (\lambda + \nu^{-1} - \nu)/\lambda = -(\nu + q^{-1})(\nu - q)\,(\lambda\nu)^{-1} \;) \;, \tag{4.3.5}$$

$$\kappa_i\, \sigma_{i\pm 1}\, \sigma_i = \sigma_{i\pm 1}\, \sigma_i\, \kappa_{i\pm 1} \;, \tag{4.3.6}$$

$$\kappa_i\, \sigma_{i\pm 1}\, \sigma_i = \kappa_i\, \kappa_{i\pm 1} \;, \tag{4.3.7}$$

$$\kappa_i\, \sigma_{i\pm 1}^{-1}\, \sigma_i^{-1} = \kappa_i\, \kappa_{i\pm 1} \;, \tag{4.3.8}$$

$$\sigma_{i\pm 1}\, \kappa_i\, \sigma_{i\pm 1} = \sigma_i^{-1}\, \kappa_{i\pm 1}\, \sigma_i^{-1} \;, \tag{4.3.9}$$

$$\kappa_i\, \kappa_{i\pm 1}\, \kappa_i = \kappa_i \;, \tag{4.3.10}$$

$$\kappa_{i\pm 1}\, \kappa_i\, (\sigma_{i\pm 1} - \lambda) = \kappa_{i\pm 1}\, (\sigma_i - \lambda) \;, \tag{4.3.11}$$

$$(\sigma_i - \lambda)\, \kappa_{i-1}\, (\sigma_i - \lambda) = (\sigma_{i-1} - \lambda)\, \kappa_i\, (\sigma_{i-1} - \lambda) \;. \tag{4.3.12}$$

Eq. (4.3.4) is deduced by the action of the element κ_i on (4.3.3) and using (4.3.1). Relations (4.3.6) follow from (4.1.1) and (4.3.3). Relations (4.3.7) and (4.3.8) with lower signs are obtained by multiplying (4.3.2) with $\sigma_{i-1}^{\mp 1}\sigma_i^{\mp 1}$ from the right and using (4.3.1) and (4.3.6). Eq. (4.3.9) follows from (4.3.7), (4.3.8). Combining the pair of relations (4.3.2) in the form: $\kappa_i\,(\sigma_{i+1}-\sigma_{i+1}^{-1})\,\kappa_i = (\nu^{-1}-\nu)\,\kappa_i$ and using (4.3.3) and (4.3.4) we derive (4.3.10). Eq. (4.3.11) is proved as following

$$\kappa_{i\pm 1}\,\kappa_i\,(\sigma_{i\pm 1}-\lambda) = \kappa_{i\pm 1}\,\kappa_i\,(\sigma_{i\pm 1}^{-1}-\lambda\kappa_{i\pm 1}) = \kappa_{i\pm 1}\,(\sigma_i-\lambda)\;,$$

where we have used (4.3.3), (4.3.7) and (4.3.10). Eq. (4.3.12) is deduced by means of eq. (4.3.11), its mirror counterpart and eq. (4.3.10). The pairs of eqs. in (4.3.6) - (4.3.10) (with upper and lower signs) are related to each other by the similarity transformations

$$\sigma_{i+1} = V_i\sigma_{i-1}V_i^{-1}\;,\quad \sigma_i = V_i\sigma_i V_i^{-1}$$

where $V_i = \sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1}\sigma_i$ (the braid relations (4.1.1) should be used). We also present the relations

$$\kappa_{i\pm 1}\,\kappa_i\,(\sigma_{i\pm 1}^{-1}+\lambda) = \kappa_{i\pm 1}\,(\sigma_i^{-1}+\lambda)\;, \tag{4.3.13}$$

$$(\sigma_i^{-1}+\lambda)\,\kappa_{i-1}\,(\sigma_i^{-1}+\lambda) = (\sigma_{i-1}^{-1}+\lambda)\,\kappa_i\,(\sigma_{i-1}^{-1}+\lambda)\;, \tag{4.3.14}$$

which are related to (4.3.11), (4.3.12) via the obvious isomorphism $(\sigma_i, q, \nu) \leftrightarrow (\sigma^{-1}, q^{-1}, \nu^{-1})$ of the algebras $W\!B_{M+1}$ (4.1.1), (4.3.1) – (4.3.3).

In fact the pair of relations (4.3.2) (in the definition of the Birman-Murakami-Wenzl algebra) are not independent for the case $\nu \neq \lambda$ [78]. Indeed, using $\kappa_i\,\sigma_{i-1}\,\kappa_i = \nu^{-1}\,\kappa_i$ and (4.1.1) one can deduce $\sigma_{i-1}\sigma_i\kappa_{i-1}\kappa_i = \nu\kappa_i\sigma_{i-1}\kappa_i = \kappa_i$, which is written in the form $\sigma_{i-1}^{-1}\kappa_i = \sigma_i\kappa_{i-1}\kappa_i$. Acting to this relation by $\lambda\kappa_i$ from the left, we deduce

$$\lambda\kappa_i\sigma_{i-1}^{-1}\kappa_i = \lambda\nu\kappa_i\kappa_{i-1}\kappa_i = \nu\kappa_i(\sigma_{i-1}^{-1}-\sigma_{i-1}+\lambda)\kappa_i = \nu\kappa_i\sigma_{i-1}^{-1}\kappa_i + \nu(\lambda\mu-\nu^{-1})\kappa_i\;,$$

which is equivalent to $(\lambda-\nu)(\kappa_i\sigma_{i-1}^{-1}\kappa_i-\nu\kappa_i)=0$ and, thus, to the above statement.

One can construct the analogs of the symmetrizers and antisymmetrizers for the algebra $W\!B_{M+1}$ using the inductive relations similar to that we have considered in the Hecke case (4.2.40):

$$S_{1\to n} = f^{(-)}_{1\to n}\,S_{1\to n-1} = S_{1\to n-1}\,\overline{f}^{(-)}_{1\to n}\;, \tag{4.3.15}$$

$$A_{1\to n} = f^{(+)}_{1\to n}\,A_{1\to n-1} = A_{1\to n-1}\,\overline{f}^{(+)}_{1\to n}\;, \tag{4.3.16}$$

where 1-shuffles are

$$f^{(\pm)}_{1\to n} = \frac{1}{[n]_q!}\,\sigma_1^{(\pm)}(q^{\pm 1})\cdots\sigma_{n-2}^{(\pm)}(q^{\pm(n-2)})\,\sigma_{n-1}^{(\pm)}(q^{\pm(n-1)})\;,$$

$$\overline{f}^{(\pm)}_{1\to n} = \frac{1}{[n]_q!}\,\sigma_{n-1}^{(\pm)}(q^{\pm(n-1)})\,\sigma_{n-2}^{(\pm)}(q^{\pm(n-2)})\cdots\sigma_1^{(\pm)}(q^{\pm 1})\;,$$

and $\sigma_i^{(\pm)}(x)$ are Baxterized elements (cf. (3.11.15)) for the algebra $W\!B_{M+1}$:

$$\sigma^{(\pm)}(x) = \frac{1}{\lambda}(x^{-1}\,\sigma_i - x\,\sigma_i^{-1}) + \frac{(\nu\pm q^{\pm 1})}{(\nu x\pm q^{\pm 1}x^{-1})}\,\kappa_i = \tag{4.3.17}$$

$$= x\left(\mathbf{1} + \frac{1}{\lambda}(x^{-2}-1)\,\sigma_i + \frac{\nu(x^{-2}-1)}{(\nu \pm q^{\pm 1}x^{-2})}\,\kappa_i\right) =$$

$$= x\left(\mathbf{1} + \frac{1}{\lambda}(x^{-2}-1)\,\sigma_i\right)\left(1 + \frac{\lambda(x^{-2}-1)}{(\lambda - \nu(1-x^{-2}))(1 \pm q^{\pm 1}\nu^{-1}x^{-2})}\,\kappa_i\right) .$$

These elements are normalized by the conditions $\sigma^{(\pm)}(\pm 1) = \pm\mathbf{1}$ and satisfy the Yang-Baxter equations

$$\sigma_n^{(\pm)}(x)\,\sigma_{n-1}^{(\pm)}(xy)\,\sigma_n^{(\pm)}(y) = \sigma_{n-1}^{(\pm)}(y)\,\sigma_n^{(\pm)}(xy)\,\sigma_{n-1}^{(\pm)}(x) . \tag{4.3.18}$$

and the "unitarity conditions"

$$\sigma_i^{(\pm)}(x)\,\sigma_i^{(\pm)}(x^{-1}) = \left(1 - \lambda^{-2}\,(x - x^{-1})^2\right) . \tag{4.3.19}$$

Note that the elements $\sigma_i^{(+)}(x)$ and $\sigma_i^{(-)}(x)$ (4.3.17) are related to each other by the transformation $q \leftrightarrow -q^{-1}$, which corresponds to the isomorphism of algebras $W\!B_{M+1}(q,\nu) \simeq W\!B_{M+1}(-q^{-1},\nu)$. To my knowledge the both inequivalent sets $(\pm)$ of the Baxterized elements (4.3.17) have been firstly presented in [127] (see also the original version of this review paper: A. P. Isaev, "Quantum Groups and Yang-Baxter Equations," Sov. J. Part. Nucl. **26** (1995) 501.). The only one set has been found in [128].

It follows from eqs. (4.3.1) - (4.3.3) that the algebra $W\!B_{M+1}$ $(\nu \neq \lambda)$ is a quotient of the braid group algebra (4.1.1) if the additional relations on σ_i are imposed

$$(\sigma_i - q)(\sigma_i + q^{-1})(\sigma_i - \nu) = 0 , \tag{4.3.20}$$

$$(\sigma_i^{-1} + \lambda - \sigma_i)\left(\sigma_{i+1}^{\pm 1}\,(\sigma_i^{-1} + \lambda - \sigma_i) - \lambda\nu^{\mp 1}\right) = 0 .$$

This quotient is finite dimensional and the dimension of $W\!B_{M+1}$ is $(2M+1)!! = 1\cdot 3\cdots(2M+1)$ (this dimension evidently follows from the graphical representation (3.9.29) of the $W\!B_{M+1}$ elements). The whole set of basis elements for the algebra $W\!B_{M+1}$ appear in the expansion of the symmetrizer S_{M+1} (4.3.15) (or antisymmetrizer A_{M+1} (4.3.16)). Note that the quotient of the Birman-Murakami-Wenzl algebra $W\!B_{M+1}$ (4.3.1) - (4.3.3) by an ideal generated by κ_i is isomorphic to the Hecke algebra H_{M+1}.

The first symmetrizer and antisymmetrizer for the algebra $W\!B_{M+1}$ are (cf. eqs. (3.9.2), (3.11.16))

$$\begin{gathered} S_{1\to 2} = \tfrac{1}{[2]_q}\sigma_1^{(-)}(q^{-1}) = \tfrac{(\sigma_1^2 - q^{-2})(\sigma_1^2-\nu^2)}{(q^2-q^{-2})(q^2-\nu^2)} = \\ = \tfrac{1}{[2]_q}(q^{-1} + \sigma_1 + \tfrac{\lambda}{1-q\nu^{-1}}\kappa_1) = \tfrac{1}{q^2-q^{-2}}(\sigma_1^2 - q^{-2})(1 - \mu^{-1}\,\kappa_1) , \end{gathered} \tag{4.3.21}$$

$$\begin{gathered} A_{1\to 2} = \tfrac{1}{[2]_q}\sigma_1^{(+)}(q) = \tfrac{(\sigma_1^2 - q^{2})(\sigma_1^2-\nu^2)}{(q^{-2}-q^{2})(q^{-2}-\nu^2)} = \\ = \tfrac{1}{[2]_q}(q - \sigma_1 - \tfrac{\lambda}{1+q^{-1}\nu^{-1}}\kappa_1) = \tfrac{1}{q^{-2}-q^{2}}(\sigma_1^2 - q^{2})(1 - \mu^{-1}\,\kappa_1) . \end{gathered} \tag{4.3.22}$$

They are obviously orthogonal to each other and to the element κ_1 in view of the characteristic equation (4.3.20). The following eqs. also hold

$$\sigma_1^{(-)}(q)\,S_{1\to 2} = 0 = \kappa_1\,S_{1\to 2} , \quad \sigma_1^{(+)}(q^{-1})\,A_{1\to 2} = 0 = \kappa_1\,A_{1\to 2} ,$$

in view of the unitarity condition (4.3.19). In fact, these eqs. are special cases of the more general relations (for $i = 1, \dots, n-1$)

$$\sigma_i^{(-)}(q)\, S_{1\to n} = S_{1\to n}\, \sigma_i^{(-)}(q) = 0 \; ,$$

$$\sigma_i^{(+)}(q^{-1})\, A_{1\to n} = A_{1\to n}\, \sigma_i^{(+)}(q^{-1}) = 0 \; ,$$

which equivalent to the conditions ($i = 1, \dots n-1$):

$$\begin{array}{c} (\sigma_i - q)\, S_{1\to n} = 0 = S_{1\to n}\, (\sigma_i - q) \; , \quad \kappa_i\, S_{1\to n} = 0 = S_{1\to n}\, \kappa_i \; , \\ (\sigma_i + q^{-1})\, A_{1\to n} = 0 = A_{1\to n}\, (\sigma_i + q^{-1}) \; , \quad \kappa_i\, A_{1\to n} = 0 = A_{1\to n}\, \kappa_i \; , \end{array} \tag{4.3.23}$$

and demonstrate that $S_{1\to M+1}$, $A_{1\to M+1}$ are central idempotents. Eqs. (4.3.23) can be readily proved by means of the analogs of the factorization relations (4.1.14), (4.1.19) or by the induction with using of (4.3.15), (4.3.16) and the Yang-Baxter equations (4.3.18).

We note that the idempotents (4.3.15), (4.3.16) can be easily written in the form [130] (cf. (4.2.34), (4.2.35)):

$$S_{1\to n} = S_{1\to n-1}\, \frac{\sigma_{n-1}^{(-)}(q^{-(n-1)})}{[n]_q}\, S_{1\to n-1} \; , \tag{4.3.24}$$

$$A_{1\to n} = A_{1\to n-1}\, \frac{\sigma_{n-1}^{(+)}(q^{n-1})}{[n]_q}\, A_{1\to n-1} \; . \tag{4.3.25}$$

This inductive definition of the idempotents (4.3.15), (4.3.16) has also been presented in [131] (see Lemma 7.6).

Using the representations (4.3.24), (4.3.25) we prove the analog of Proposition 8 about symmetrizers and antisymmetrizers for the case of the Birman-Murakami-Wenzl algebra.

Proposition 10. *The idempotents $S_{1\to n}$ and $A_{1\to n}$ ($n = 2, \dots M+1$) (4.3.24), (4.3.25) for the Birman-Murakami-Wenzl algebra are expressed in term of the Jucys-Murphy elements y_k ($k = 2, \dots, M$): $y_1 = 1$, $y_{k+1} = \sigma_k y_k \sigma_k$, $[y_k,\, y_m] = 0$ as*

$$S_{1\to n} = \prod_{i=2}^{n} \left(\frac{(y_i - q^{-2})}{(q^{2(i-1)} - q^{-2})}\, \frac{(y_i - \nu^2 q^{-2(i-2)})}{(q^{2(i-1)} - \nu^2 q^{-2(i-2)})} \right) \; , \tag{4.3.26}$$

$$A_{1\to n} = \prod_{i=2}^{n} \left(\frac{(y_i - q^{2})}{(q^{-2(i-1)} - q^{2})}\, \frac{(y_i - \nu^2 q^{2(i-2)})}{(q^{-2(i-1)} - \nu^2 q^{2(i-2)})} \right) \; . \tag{4.3.27}$$

Proof. We deduce the identity (4.3.25) from (4.3.27). The representation (4.3.26) for the symmetrizers (4.3.24) can be justified analogously. The equations (4.3.22) demonstrate that (4.3.27) coincides with (4.3.25) for $n = 2$. Then, we use the induction. Let (4.3.27) coincides with the idempotent (4.3.25) for some fixed $n \geq 2$ and, thus, it is the element which satisfies (4.3.23). We prove that the formulas (4.3.25) and (4.3.27) are equivalent for $A_{1\to n+1}$. In view of the induction conjecture and obvious properties $[A_{1\to n},\, y_{n+1}] = 0$ (since $A_{1\to n}$ is a function of y_i) we obtain from (4.3.27):

$$A_{1\to n+1} = A_{1\to n}\, \frac{(y_{n+1} - q^2)}{(q^{-2n} - q^2)}\, \frac{(y_{n+1} - \nu^2 q^{2(n-1)})}{(q^{-2n} - \nu^2 q^{2(n-1)})}\, A_{1\to n} \; . \tag{4.3.28}$$

We need the identities

$$\sigma_n \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_n \, \kappa_n = \nu^2 \, \sigma_{n-1}^{-1} \dots \sigma_2^{-1} \sigma_1^{-2} \sigma_2^{-1} \dots \sigma_{n-1}^{-1} \, \kappa_n \;\Rightarrow$$
$$y_n \, \sigma_n \, y_n \, \kappa_n = \nu \, \kappa_n \quad \Rightarrow \quad y_{n+1} \, y_n \, \kappa_n = \nu^2 \, \kappa_n \; , \tag{4.3.29}$$

which follow from eq. $\sigma_k \kappa_{k+1} = \sigma_{k+1}^{-1} \kappa_k \kappa_{k+1}$. We also deduce the analogs of the identities (4.2.6) for the Birman-Murakami-Wenzl algebra case:

$$\begin{gathered} y_{n+1} = \sigma_n \dots \sigma_2 \sigma_1^2 \sigma_2 \dots \sigma_n = 1 + \lambda (\textstyle\sum_{i=1}^{n-1} \sigma_i \dots \sigma_{n-1} \sigma_n \sigma_{n-1} \dots \sigma_i + \sigma_n) - \\ - \lambda \nu \, (\textstyle\sum_{i=1}^{n-1} \, \sigma_i^{-1} \dots \sigma_{n-1}^{-1} \kappa_n \sigma_{n-1}^{-1} \dots \sigma_i^{-1} + \kappa_n) \; . \end{gathered} \tag{4.3.30}$$

Using eqs. (4.3.29), (4.3.30) and $A_{1 \to n} y_n = q^{2(1-n)} A_{1 \to n}$ (see eqs. (4.3.23) for $A_{1 \to n}$) we obtain

$$A_{1 \to n} \, y_{n+1} \, A_{1 \to n} = A_{1 \to n} \left(1 + q \, (1 - q^{-2n}) \, \sigma_n + \frac{\nu}{q} \, (1 - q^{2n}) \kappa_n \right) A_{1 \to n} \; , \tag{4.3.31}$$

$$\begin{gathered} A_{1 \to n} \, y_{n+1}^2 \, A_{1 \to n} = A_{1 \to n} y_{n+1} \left(1 + q \, (1 - q^{-2n}) \, \sigma_n + \frac{\nu}{q} \, (1 - q^{2n}) \kappa_n \right) A_{1 \to n} = \\ = A_{1 \to n} [(1 + \lambda q (1 - q^{-2n})) + q (1 - q^{-2n}) (q^2 + q^{-2n}) \sigma_n - \\ + \tfrac{\nu}{q} \, (1 - q^{2n}) (q^2 - q^{-2(n-1)} + q^{-2n} + \nu (\lambda + \nu q^{2(n-1)}) \kappa_n)] A_{1 \to n} \; . \end{gathered} \tag{4.3.32}$$

Then, we substitute (4.3.31) and (4.3.32) into (4.3.28) and finally deduce

$$A_{1 \to n+1} = \frac{q^{-1} \lambda}{(1 - q^{-2(n+1)})} \, A_{1 \to n} \; \left(1 + \frac{(q^{-2n} - 1) \sigma_n}{\lambda} + \frac{\nu (q^{-2n} - 1) \kappa_n}{(q^{-2n+1} + \nu)} \right) A_{1 \to n} \; , \tag{4.3.33}$$

which coincides with (4.3.25). □

One can prove directly the identities (4.3.23) for elements (4.3.26), (4.3.27). We again use the induction. Let (4.3.23) valid for (4.3.27) for some fixed n (it is obvious for $n = 2$). Then, we obtain

$$A_{1 \to n} \, (y_{n+1} - \nu^2 q^{2(n-1)}) \kappa_n = A_{1 \to n} \, (y_{n+1} - \nu^2 \, y_n^{-1}) \kappa_n = 0 \; , \tag{4.3.34}$$

where we have applied identities (4.3.29) and $A_{1 \to n} y_n = A_{1 \to n} q^{-2(n-1)}$. Using eq. (4.3.34) and the relation $[A_{1 \to n}, \, y_{n+1}] = 0$ we prove that $A_{1 \to n+1} \kappa_n = 0$ for (4.3.28). Now consider the following chain of relations

$$\begin{gathered} A_{1 \to n} \, (y_{n+1} - q^2)(\sigma_n + q^{-1}) = A_{1 \to n} \, (\sigma_n y_n \sigma_n - q^2)(\sigma_n + q^{-1}) = \\ = A_{1 \to n} \, (q \, \sigma_n y_n \sigma_n + \sigma_n y_n - \lambda \nu \sigma_n y_n \kappa_n - q^2 \sigma_n - q) = \\ = A_{1 \to n} \, (-\lambda \nu) \, \kappa_n \, (\textstyle\sum_{i=1}^{n-1} (-1)^{n-i} \, q^{i+1-n} \sigma_{n-1}^{-1} \dots \sigma_i^{-1} + q + \\ + \textstyle\sum_{i=1}^{n-2} (-1)^{n-i} \, q^{i-n} \kappa_{n-1} \sigma_{n-2}^{-1} \dots \sigma_i^{-1} + q \, \kappa_{n-1} + \nu q^{2(1-n)}) \; , \end{gathered} \tag{4.3.35}$$

where we have used eqs. (4.3.23), (4.3.29) and (4.3.30). Multiplying eq. (4.3.35) by the factor $(y_{n+1} - \nu^2 q^{2(n-1)})$ from the left and taking into account (4.3.34) we obtain $A_{1\to n+1}(\sigma_n + q^{-1}) = 0$.

Remark 1. The idempotents $S_{1\to n}$ and $A_{1\to n}$ for the Birman-Murakami-Wenzl algebra have been also constructed in another form in [129]. The authors of [129] (as well as the authors of [131]) have not used the Baxterized or Jucys-Murphy elements and, thus, their expressions for $S_{1\to n}$ and $A_{1\to n}$ look rather cumbersome. The idea of using the Baxterized elements (4.3.17) to construct the primitive idempotents $S_{1\to n}$ and $A_{1\to n}$ has been firstly proposed by P.N.Pyatov [130]. After the substitution of (4.3.17) to (4.3.15), (4.3.16) and direct calculations one can derive the formulas for $S_{1\to n}$ and $A_{1\to n}$ presented in [129].

Remark 2. Let the projectors $A_{1\to n+1}$ (or $S_{1\to n+1}$) be zero for some n and $A_{1\to n} \neq 0 \neq S_{1\to n}$. It leads to some constraints on the parameter ν. Indeed, from conditions $\kappa_{n+1} A_{1\to n+1} \kappa_{n+1} = 0$ and $\kappa_{n+1} S_{1\to n+1} \kappa_{n+1} = 0$ we obtain $\kappa_{n+1} \sigma_n^{(+)}(q^n) \kappa_{n+1} = 0$ and $\kappa_{n+1} \sigma_n^{(-)}(q^{-n}) \kappa_{n+1} = 0$, respectively, which equivalent to the equations

$$(\nu + q^{-(2n+1)})(\nu - q^{-(n-1)})(\nu + q^{-(n-1)}) = 0 \; ,$$
$$(\nu - q^{(2n+1)})(\nu - q^{(n-1)})(\nu + q^{(n-1)}) = 0 \; .$$

Thus, all antisymmetrizers $A_{1\to k}$ could be zero for $k > n$ only if ν takes one of the values $\nu = -q^{-(2n+1)}, \pm q^{1-n}$, and, respectively, all symmetrizers $S_{1\to k}$ could be zero for $k > n$ only if $\nu = q^{(2n+1)}, \pm q^{n-1}$.

Remark 3. The homomorphisms of the periodic $\overline{\mathbb{W}B}_M$ algebra to the algebra $\mathbb{W}B_M$ and to the affine algebra $\hat{\mathbb{W}B}_M$ are defined by the same eqs. (4.2.49) and (4.2.51) as in the case of the group algebra of the braid group. Indeed, the characteristic identity for σ_M is the same as for σ_1 while relations

$$\kappa_1 \sigma_M^{\pm 1} \kappa_1 = \nu^{\mp 1} \kappa_1 \; , \quad \kappa_{M-1} \sigma_M^{\pm 1} \kappa_{M-1} = \nu^{\mp 1} \kappa_{M-1} \; ,$$

can be checked directly.

Remark 4. The representation theory for the Birman-Murakami-Wenzl algebra (the Young-Ogievetsky graph, the analog of Proposition 9, the explicit formulas for all primitive idempotents in terms of the Jucys-Murphy elements, intertwiner operators U_k (4.2.17) – (4.2.19), etc.) similar to that presented for the Hecke algebra case in Subsection 4.2 has been developed in [132].

5 APPLICATIONS AND CONCLUSIONS

In the previous sections of the review, we have presented the fundamentals of the theory of quantum groups. We have also considered how it is possible to obtain trigonometric and rational (Yangian) solutions of the Yang-Baxter equation on the basis of the theory of quantum Lie groups. Unfortunately, in the previous sections it was not possible for us to discuss in detail the numerous applications of the theory of quantum groups and the Yang-Baxter equation in both theoretical physics and mathematical physics. In this final section, we shall merely give a brief list of such applications that, in the author's opinion, have definite interest.

Before we do this, we recall that in the physics of condensed matter exactly solvable two-dimensional models are used to describe various layered structures, contact surfaces in electronics, surfaces of superconducting liquids like He II, etc. Two-dimensional integrable field theories are used to describe dynamical effects in one-dimensional spatial systems (such as light tubes, nerve fibers, etc.). In addition, such field theories (and also integrable systems on one-dimensional chains) can also arise on reductions of multidimensional field theories (see, for example, Ref. [133]). Quite recently it has been argued that the one-loop dilatation operator (anomalous dimension operator) of $\mathcal{N}=4$ Super Yang-Mills theory may be identified, in some restricted cases, with the hamiltonians of various integrable quantum (super) spin chains [134]. Similar spin chain models (related to the noncompact Lie groups) had been previously appeared in the QCD context [135].

5.1 Quantum periodic spin chains

We have already mentioned that the quantum inverse scattering method [1] (an introduction to this method that can be readily understood by a wide circle of readers can be found in Refs. [136], [137]) is designed as a constructive procedure for solving quantum two-dimensional integrable systems. In addition, the quantum inverse scattering method makes it possible to construct quantum integrable systems on one-dimensional chains (see, for example, Refs. [86], [97], and [138]). Here we discuss the case of the periodic chains. The generalization to the case of the open chains will be mentioned in the next subsection 5.2. The initial point is the relation (3.8.22) for the L operators, which can be written in the form[14]

$$R_{12}(\theta-\theta')\,L_{K2}(\theta)\,L_{K1}(\theta') = L_{K1}(\theta')\,L_{K2}(\theta)\,R_{12}(\theta-\theta') \; . \tag{5.1.1}$$

Here $L_{Ki}(\theta)$ are $N \times N$ matrices in the auxiliary vector space V_i, with matrix coefficients that are the operators in the space of states of the K-th site of a chain consisting of M sites:

$$L_{Ki}(\theta) = I^{\hat{\otimes}(K-1)}\hat{\otimes}L_i(\theta)\hat{\otimes}I^{\hat{\otimes}(M-K)} \;\rightarrow\; [L_{Ki},\, L_{K'i'}] = 0 \;\; (K \neq K') \; . \tag{5.1.2}$$

In (5.1.2), the symbol $\hat{\otimes}$ denotes the direct product of the operator spaces.

To construct an integrable system, we introduce the monodromy matrix

$$T_i(\theta) = D_i^{(1)}\,L_{1i}(\theta)\,D_i^{(2)}\,L_{2i}\ldots D_i^{(M)}\,L_{Mi}(\theta) \; . \tag{5.1.3}$$

If the matrices $D^{(K)}$ $(1 \leq K \leq M)$ satisfy the relations

$$R_{ij}(\theta)\,D_j^{(K)}\,D_i^{(K)} = D_j^{(K)}\,D_i^{(K)}\,R_{ij}(\theta) \; , \tag{5.1.4}$$

$$[D_i^{(K)},\, D_j^{(J)}] = [D_i^{(J)},\, L_{Kj}] = 0 \; ,$$

then it follows from (5.1.1) that

$$R_{ij}(\theta-\theta')\,T_j(\theta)\,T_i(\theta') = T_i(\theta')\,T_j(\theta)R_{ij}(\theta-\theta') \; . \tag{5.1.5}$$

[14]Usually this equation is written in the form in which the matrix $R_{12}(\theta)$ is substituted by $R_{21}(\theta)$; this is not important, since $R_{21}(\theta)$ satisfies, up to the change of the spectral parameters, the same Yang-Baxter equation (3.8.21) as $R_{12}(\theta)$.

The trace of the monodromy matrix (5.1.3) over the auxiliary space i forms the transfer matrix

$$t(\theta) = Tr_{(i)}\,(T_i(\theta)) \tag{5.1.6}$$

which gives a commuting family of operators: $[t(\theta),\, t(\theta')] = 0$. The commutativity of the transfer matrices follows directly from eq. (5.1.5) if we multiply it by the matrix $(R_{ij}(\theta-\theta'))^{-1}$ from the right and take the trace $Tr_{(i,j)}\,(\ldots)$. Using the family of commuting operators $t(\theta)$ a certain local operator H can be constructed, which is interpreted as the Hamiltonian of the system. Locality of the Hamiltonian is a natural physical requirement and means that H describes the interaction of only nearest-neighbor sites of the chain. The remaining operators in the commuting set $t(\theta)$ give an infinite set of integrals of the motion, indicating the integrability of the constructed system. In many well-known cases, the commuting set is associated with the coefficients in the expansions of $t(\theta)$ over spectral parameter θ. For example, one can consider logarithmic derivatives of the transfer matrix:

$$\mathcal{I}_n = \frac{d^n}{d\theta^n} \ln\left(t(\theta)\, t(0)^{-1}\right)\,|_{\theta=0}\ . \tag{5.1.7}$$

and identify the local Hamiltonian to the first logarithmic derivative of the transfer matrix

$$H \equiv \mathcal{I}_1 = \frac{d}{d\theta} \ln\left(t(\theta)\, t(0)^{-1}\right)\,|_{\theta=0}\ , \tag{5.1.8}$$

where the matrix $t(0)^{-1}$ is introduced in order to obtain the local charges $\mathcal{I}_n$ [139].

Now we consider explicit examples of integrable periodic spin chains. It is clear that from the Yang-Baxter equation (3.8.21) there always follow representations for the L operators (5.1.1) in the form of R matrices:

$$\rho_{V_k}\,(L_{Ki}(\theta)) = R_{ki}(\theta)\ ,\ \ \overline{\rho}_{V_k}\,(L_{Ki}(\theta)) = (R_{ik}(\theta))^{-1}\ . \tag{5.1.9}$$

In this case, the representations of $L_{Ki}(\theta)$ act nontrivially in the space $V_k \otimes V_i$. We choose for L operators the first representation in (5.1.9) and obtain for $T_i(\theta)$ (5.1.3):

$$\begin{gathered} T_i(\theta) = D_i^{(1)}\, R_{1i}(\theta)\, D_i^{(2)}\, R_{2i}(\theta) \ldots D_i^{(M)}\, R_{Mi}(\theta) = \\ = \hat{R}'_{i1}(\theta)\, \hat{R}'_{12}(\theta)\, \hat{R}'_{23}(\theta) \ldots \hat{R}'_{M-1M}(\theta) P_{M-1\,M} \ldots P_{23} P_{12} P_{1i}\ , \end{gathered}$$

where P_{ij} are permutation matrices and $\hat{R}'_{ij}(\theta) = D_i^{(j)}\, \hat{R}_{ij}(\theta)$. Taking the trace $Tr_{(i)}$ we deduce

$$t(\theta) = Tr_{(i)}\left(\hat{R}'_{i1}(\theta)\, \hat{R}'_{12}(\theta)\, \hat{R}'_{23}(\theta) \ldots \hat{R}'_{M-1M}(\theta) P_{Mi}\right) P_{M-1M} \ldots P_{23} P_{12}\ . \tag{5.1.10}$$

We consider rather general case of R matrices which can be normalized such that $\hat{R}_{ij}(\theta) = I + \theta\, h_{ij} + \theta^2 \ldots$, (see, e.g., (3.8.23), (3.8.25), (3.11.19), (3.11.21)). These R-matrices are called *regular* [92]. For the regular R-matrices, using (5.1.10) we obtain

$$t(\theta)\, t(0)^{-1} = I + \theta \left(\sum_{k=1}^{M} h'_{k\,k+1}\right) + \theta^2 \ldots\ \ ,\quad h'_{k\,k+1} := D_k^{(k+1)}\, h_{k\,k+1}\, \left(D_k^{(k+1)}\right)^{-1}\ ,$$

where $D_M^{(M+1)} := D_M^{(1)}$, $h_{M\,M+1} := h_{M\,1}$ and the local Hamiltonian (5.1.8) is

$$H = \sum_{k=1}^{M} h'_{k\,k+1} \; . \tag{5.1.11}$$

If we choose the R-matrix in (5.1.10) in the form of the trigonometric solution (3.11.19), then we obtain

$$h_{j\,j+1} = \frac{1}{2}\left(\hat{R}_{j\,j+1} + \hat{R}^{-1}_{j\,j+1} - \lambda\,\beta_{\pm}\,\mathbf{K}_{j\,j+1}\right) \; , \tag{5.1.12}$$

where $\beta_{\pm} = \frac{\alpha_{\pm}-1}{\alpha_{\pm}+1}$, $\alpha_{\pm} = \pm q^{\pm 1}\,\nu^{-1}$ and the parameter ν is fixed for different quantum (super)groups in (3.11.12). We note that the Hamiltonians (5.1.11) with the densities (5.1.12) (and $D^{(k)} = 1$) are the R matrix representations of the operators:

$$H_{\pm} = \frac{1}{2}\sum_{j=1}^{M}\left(\sigma_j + \sigma_j^{-1} + \lambda\,\frac{\nu \mp q^{\pm 1}}{\nu \pm q^{\pm 1}}\,\kappa_j\right) \; ,$$

where σ_i, κ_i ($i = 1, \dots, M$) obey (4.1.1), (4.3.1) – (4.3.3) with periodic identifications: $\sigma_{M+i} = \sigma_i$, $\kappa_{M+i} = \kappa_i$. It is natural to call the algebra with such generators as a periodic Birman-Murakami-Wenzl algebra. The case $\kappa_i = 0$ corresponds to the periodic system with the Hamiltonian: $H = \left(\sum_{j=1}^{M}\sigma_j - \frac{\lambda M}{2}\right)$, where σ_i are generators of the periodic A-type Hecke algebra AH_{M+1} (see Sec. 4.2). In the R matrix representation: $\sigma_i \to \hat{R}_i$, $\sigma_M \to \hat{R}_{M\,1}$, where $\hat{R}$ is $GL_q(2)$ matrix (3.4.8), this Hamiltonian describes the periodic XXZ Heisenberg model.

For the Yangian R matrices (3.11.21) we obtain $SO(N)$ ($\epsilon = +1$) and $Sp(N)$ ($N = 2n$, $\epsilon = -1$) invariant spin chain models with local Hamiltonian densities (see, e.g. [86]): $h_{l\,l+1} = \left(P_{l\,l+1} + \frac{2}{2\epsilon - N}K^{(0)}_{l\,l+1}\right)$ (for closed chains we imply $O_{M\,M+1} = O_{M\,1}$), where as usual $P_{l\,l+1}$ are the transposition matrices and the matrices $K^{(0)}_{l\,l+1}$ have been defined in (3.9.6). The $Osp(N|2m)$ invariant spin chain model corresponds to the densities $h_{l,l+1} = \left(\mathcal{P}_{l\,l+1} + \frac{2}{2+2m-N}\,\mathcal{K}^{(0)}_{l\,l+1}\right)$ which are deduced from (3.11.22). These Yangian models are generalizations of the XXX Heisenberg models of magnets. We recall that XXX model can be obtained if we take the special limit $q \to 1$ in the XXZ model or choose the $gl(2)$ Yangian R matrix (3.8.25) as the representation of L operators in (5.1.9).

Using in (5.1.9), (5.1.10) the elliptic solution (3.13.3), (3.13.8) of the Yang-Baxter equation, we recover, for $N = 2$, the XYZ spin chain model [110] while for $N > 2$ we obtain its integrable generalizations.

At the end of this subsection we stress that using the transfer matrix (5.1.6) one can construct an integrable 2-dimensional statistical model on the $(M \times L)$ lattice with periodic boundary conditions. Namely, one should consider the partition function

$$Z = Tr_{(1\dots M)}(\,\underbrace{t(\theta_0)\cdots t(\theta_0)}_{L}\,) = Tr_{(1\dots M)}\left(\prod_{i=1}^{L} Tr_{(i)}(D_i^{(1)} L_{1i}(\theta_0)\dots D_i^{(M)} L_{Mi}(\theta_0))\right) \; ,$$

where the combination $D_i^{(K)} L_{Ki}(\theta_0)$ (for a special value of the spectral parameter $\theta = \theta_0$) defines a weight of the statistical system in the site (K, i) and $Tr_{(1\dots M)}$ are the traces over the operator spaces.

5.2 Factorizable scattering: S-matrix and boundary K matrix

The Yang-Baxter equation (3.8.20):

$$S_{23}(\theta-\theta')\, S_{13}(\theta)\, S_{12}(\theta') = S_{12}(\theta')\, S_{13}(\theta)\, S_{23}(\theta-\theta') \tag{5.2.1}$$

together with the subsidiary relations of unitarity and crossing symmetry

$$S_{12}(\theta)\, S_{21}(-\theta) = I_{12} \; , \quad S_{12}(\theta) = (S_{21}(i\pi-\theta))^{t_1} \; , \tag{5.2.2}$$

uniquely determine factorizable S matrices (with a minimal set of poles) describing the scattering of particle-like excitations in (1+1)-dimensional integrable relativistic models [3]. Eqs. (5.2.2) guarantee that the S matrix $S_{12}(\theta)$ is invertible and skew-invertible (see (3.1.9)). The matrix $S^{i_1 i_2}_{j_1 j_2}(\theta)$ is interpreted as the S matrix for the scattering of two neutral particles with isotopic spins i_1 and i_2 to two particles with spins j_1 and j_2, and the spectral parameter θ is none other than the difference of the rapidities of these particles. For charged particles the crossing symmetry relation (5.2.2) should be written in the form $S_{12}(\theta) = (S_{2\bar{1}}(i\pi-\theta))^{t_1}$, where S-matrix $S_{2\bar{1}} = S^{i_2 \bar{i}_1}_{k_2 \bar{k}_1}$ describes particle-antiparticle scattering. The many-particle S matrices decompose into products of two-particle matrices (factorization). In this sense, the Yang-Baxter equation (5.2.1) is the condition of uniqueness of the determination of the many-particle S matrices.

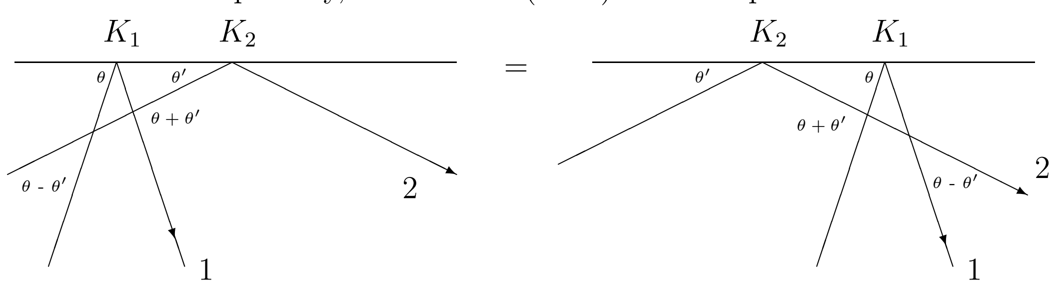
The reflection equation [140] – [145] which depends on the spectral parameters,

$$S_{12}(\theta-\theta')\, K_1(\theta)\, S_{21}(\theta+\theta')\, K_2(\theta') = K_2(\theta')\, S_{12}(\theta+\theta')\, K_1(\theta)\, S_{21}(\theta-\theta') \; , \tag{5.2.3}$$

determines together with unitarity condition

$$K^i_j(\theta)\, K^j_m(-\theta) = \delta^i_m \; , \tag{5.2.4}$$

and the relations (5.2.1) and (5.2.2) factorizable scattering of particles (solitons) on a half-line (see, e.g., [140], [141], [146], [147]). In this case, the operator matrix $K^i_j(\theta)$ describes reflection of a particle with rapidity θ at a boundary point of the half-line. Graphically, the relation (5.2.3) can be represented in the form



We recall [141] that factorizable scattering on a half-line can be described by a Zamolodchikov algebra with generators $\{A^i(\theta)\}$ and boundary operator B which satisfy the defining relations

$$\begin{gathered} A^i(\theta)\, A^j(\theta') = S^{ij}_{kl}(\theta-\theta')\, A^l(\theta')\, A^k(\theta) \; , \quad A^i(\theta)\, B = K^i_j(\theta)\, A^j(-\theta)\, B \;\; \Rightarrow \\ A_1(\theta)\, A_2(\theta') = S_{12}(\theta-\theta')\, A_2(\theta')\, A_1(\theta) \; , \quad A_1(\theta)\, B = K_1(\theta)\, A_1(-\theta)\, B \; . \end{gathered} \tag{5.2.5}$$

The consistency conditions for this algebra give rise to the Yang-Baxter equation (5.2.1), to the unitarity conditions (5.2.2), (5.2.4) and the reflection equation (5.2.3) for matrices S and K.

The reflection equation (5.2.3) can be used [142] – [145], [148] for the construction of the quantum group invariant integrable spin systems (see e.g. [5]) on the chains with nonperiodic boundary conditions. Indeed, if $T(\theta)$ satisfies (5.1.5) for $R_{ij}(\theta) \equiv S_{ji}(\theta)$ and $K(\theta)$ satisfies (5.2.3), then $\mathcal{T}(\theta) = T(\theta)\, K(\theta)\, [T(-\theta)]^{-1}$ is also a solution of (5.2.3). It follows from the transformation

$$A(\theta) \;\to\; [T(\theta)]^{-1}\, A(\theta) \; , \quad B \;\to\; B \; ,$$

of the algebra (5.2.5) (it is covariance transformation of the first relation in (5.2.5)). The matrix $\mathcal{T}(\theta)$ is called *Sklyanin monodromy matrix*. It relates to the partition function of an integrable lattice model with nontrivial boundary conditions which defined by the reflection matrix $K(\theta)$. The set of commuting integrals (including the Hamiltonian of the model) is given by the transfer matrix $t(\theta)$ which is constructed as a special trace

$$t(\theta) = Tr\left(\mathcal{T}(\theta)\, \overline{K}(\theta)\right) = Tr\left(T(\theta)\, K(\theta)\, [T(-\theta)]^{-1}\, \overline{K}(\theta)\right) \; , \tag{5.2.6}$$

where $\overline{K}(\theta)$ is any solution of a conjugated reflection equation [142] – [145], [148]:

$$S^t_{12}(\theta-\theta')\, \overline{K}^t_2(\theta')\, \Psi^t_{12}(\theta+\theta')\, \overline{K}^t_1(\theta) = \overline{K}^t_1(\theta)\, \Psi^t_{21}(\theta+\theta')\, \overline{K}^t_2(\theta')\, S^t_{21}(\theta-\theta') \; , \tag{5.2.7}$$

such that $[\overline{K}^i_j(\theta),\, K^m_n(\theta')] = 0 = [\overline{K}^i_j(\theta),\, T^m_n(\theta')]$. In (5.2.7) the matrix Ψ_{12} is the skew-inverse matrix for S_{12} (see (3.1.9)):

$$\Psi^{t_1}_{12}(\theta)\, S^{t_1}_{12}(\theta) = I_{12} = S^{t_1}_{12}(\theta)\, \Psi^{t_1}_{12}(\theta) \; ,$$

and we have used the following notation $S^t_{12} := S^{t_1 t_2}_{12}$. The proof of the identity $[t(\theta),\, t(\theta')] = 0$ for the transfer matrix $t(\theta)$ (5.2.6) is the following [142]. We need only to prove this identity for the case $T(\theta) = 1$. Then, we have ($\theta^\pm = \theta \pm \theta'$)

$$t(\theta')\, t(\theta) = Tr_{12}\left(K_2(\theta')\, K^t_1(\theta)\, \overline{K}_2(\theta')\, \overline{K}^t_1(\theta)\right) =$$

$$= Tr_{12}\left(K_2(\theta')\, K^t_1(\theta)\, S^{t_1}_{12}(\theta^+)\, \Psi^{t_1}_{12}(\theta^+)\, \overline{K}_2(\theta')\, \overline{K}^t_1(\theta)\right) =$$

$$= Tr_{12}\left(\left(K_2(\theta')\, S_{12}(\theta^+)\, K_1(\theta)\right)^{t_1} \left(\overline{K}^t_2(\theta')\, \Psi^t_{12}(\theta^+)\, \overline{K}^t_1(\theta)\right)^{t_2}\right) =$$

using eq. (5.2.3) we deduce

$$= Tr_{12}\left(K_1(\theta)\, S_{21}(\theta^+)\, K_2(\theta')\, S^{-1}_{21}(\theta^-) \left(\overline{K}^t_2(\theta')\, \Psi^t_{12}(\theta^+)\, \overline{K}^t_1(\theta)\right)^t\, S_{12}(\theta^-)\right) =$$

and applying here the conjugated reflection equation (5.2.7) and transpositions we finally obtain

$$= Tr_{12}\left(\left(K_1(\theta)\, S_{21}(\theta^+)\, K_2(\theta')\right)^{t_2} \left(\overline{K}^t_1(\theta)\, \Psi^t_{21}(\theta^+)\, \overline{K}^t_2(\theta')\right)^{t_1}\right) =$$

$$= Tr_{12}\left(K_1(\theta)\, K^t_2(\theta')\, S^{t_2}_{21}(\theta^+)\, \Psi^{t_2}_{21}(\theta^+)\, \overline{K}_1(\theta)\, \overline{K}^t_2(\theta')\right) = t(\theta)\, t(\theta') \; .$$

Now we take in (5.2.3) the limit $\theta,\ \theta' \to \pm\infty$ in such a way that $\theta - \theta' \to \pm\infty$, and at the same time we set

$$K(\theta)|_{\theta\to\infty} = L\ , \quad S_{12}(\theta)|_{\theta\to\infty} = R_{12}\ .$$

$$K(\theta)|_{\theta\to-\infty} = L^{-1}\ , \quad S_{12}(\theta)|_{\theta\to-\infty} = (R_{21})^{-1}\ .$$

Then (5.2.3) goes over into (3.2.30), and this is the reason why all algebras with defining relations of the type (3.2.30) are called the reflection equation algebras [143] – [145].

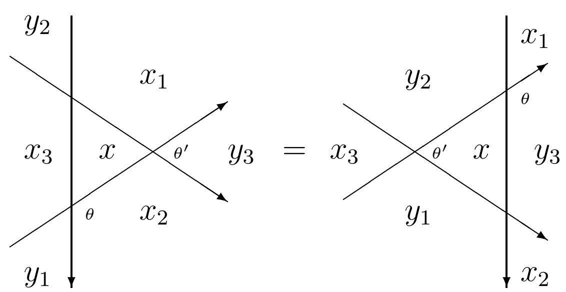
Note that every solution of the Yang-Baxter equation (5.2.1) with the conditions (5.2.2) determines an equivalence class of quantum integrable systems with the given factorizable S matrix. Thus, every classification of solutions of the Yang-Baxter equation is, to some extent, a classification of the integrable systems with the properties indicated above.

The $3D$ analog of the Yang-Baxter (triangle) equation (3.8.24), (5.2.1) is called *tetrahedron equation* [4], [154] and defines a consistence condition for the $3D$ factorizable scattering of strings. A 3-dimensional model of such factorizable scattering has firstly been proposed by A.Zamolodchikov in [4]. Then, this $3D$ model has been generalized in [149]. A 3-dimensional version of the $2D$ reflection equation (5.2.3) (the tetrahedron reflection equation) has been constructed in [150].

From the mathematical point of view the higher dimensional generalizations of the Yang-Baxter equation are related to Manin-Schechtman higher braid groups [151] and n-categories [152].

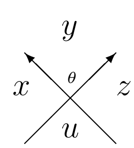
5.3 Yang-Baxter equations in calculations of multiloop Feynman diagrams

We mention the application of the Yang-Baxter equation in multiloop calculations in quantum field theory. There is a form of the Yang-Baxter equation (see Refs. [2], [153], and [154]) that can also be represented in the form of the triangle equation (3.8.24), but the indices x, x_i, y_i are ascribed, not to the "lines," but to the "faces":



(5.3.0)

where θ, θ' are angles (spectral parameters), a summation is over the index x and

$R^{xy}_{uz}(\theta) =$ 

The analytical form of (5.3.0) is

$$\sum_x R^{x_1\,x}_{y_2\,x_3}(\theta-\theta')\, R^{x\,x_2}_{x_3\,y_1}(\theta)\, R^{x_1 y_3}_{x\,x_2}(\theta') = \sum_x R^{y_2\,x}_{x_3\,y_1}(\theta')\, R^{x_1\,y_3}_{y_2\,x}(\theta)\, R^{y_3\,x_2}_{x\,y_1}(\theta-\theta')\ . \quad (5.3.1)$$

We have already considered a solution of this equation in Sec. 3.12. Indeed, one can show that eq. (3.13.10) is equivalent to eq. (5.3.1) if we put (for the notation see Sect. 3.12):

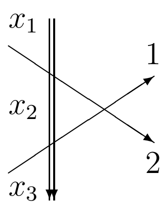
$$R^{x_1\, x}_{x_2\, x_3}(\theta) = \omega^{\frac{1}{2}<x-x_2\,,\,x_1+x_3>+<x_2,x>}\, W_{x+x_2-x_1-x_3}(\theta)\ , \qquad (5.3.2)$$

where the indices x, x_i are 2D vectors, e.g. $x = (\alpha_1, \alpha_2)$ and $\alpha_{1,2} \in \mathbf{Z}_N$. Thus, (5.3.2), (3.13.8) and (3.13.9) solve the face type Yang-Baxter equation (5.3.1).

There is a transformation from the vertex type Yang-Baxter equation (3.8.21) to the face type (5.3.1) using intertwining vectors $\psi_i^{x_1x_2}$ (see e.g. [145] and Refs. therein), where i is a vertex index, while x_1, x_2 are face indices. The vectors $\psi_i^{x_1x_2}$ satisfy intertwining relations

$$\psi^{x_2x_1}_{\langle 2}(\theta-\theta')\,\psi^{x_3x_2}_{\langle 1}(\theta)\,R_{12}(\theta') = \sum_x R^{x_1\,x}_{x_2x_3}(\theta')\,\psi^{xx_1}_{\langle 1}(\theta)\,\psi^{x_3x}_{\langle 2}(\theta-\theta') \qquad (5.3.3)$$

which are represented graphically in the form (here the angles are the same as in (5.3.0))



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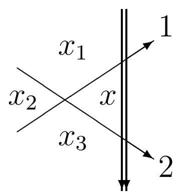
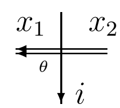


, $\quad \psi_i^{x_1x_2}(\theta) :=$


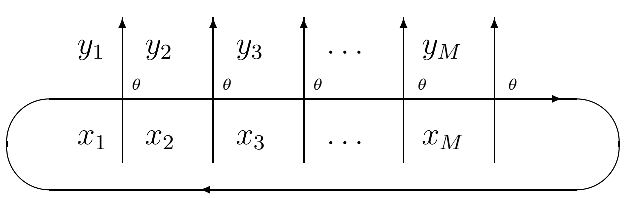
.

Then, the face type Yang-Baxter equation (5.3.1) is obtained from the vertex equation (3.8.21) if we act on it by $(\psi^{y_2x_1}_{\langle 3}\,\psi^{x_3y_2}_{\langle 2}\,\psi^{y_1x_3}_{\langle 1})$ from the left.

The relations (5.3.0), (5.3.1), like (3.8.24), give the conditions of integrability of two-dimensional lattice statistical systems (interaction round face models) with weights determined by the R matrices $R^{xy}_{uz}(\theta)$. In this case the transfer matrix has the form

$$t^{y_1y_2\ldots y_M}_{x_1x_2\ldots x_M}(\theta) = R^{y_1y_2}_{x_1x_2}(\theta)\,R^{y_2y_3}_{x_2x_3}(\theta)\,R^{y_3y_4}_{x_3x_4}(\theta)\ldots R^{y_My_1}_{x_Mx_1}(\theta)\ ,$$

and its graphical representation is



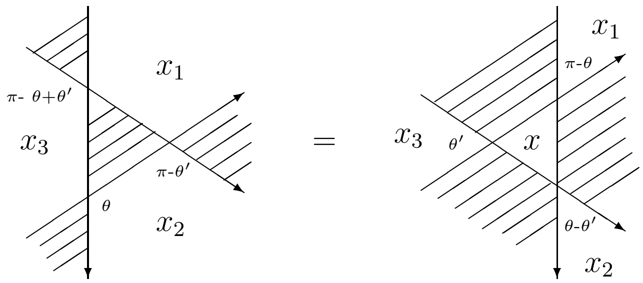
,

while a partition function for a periodic system on $(M \times K)$ lattice is given by the standard formula: $Z = Tr_{1\ldots M}(t(\theta))^K$.

We now note that the Yang-Baxter equation (5.3.0), (5.3.1) has a solution in the form $R^{xy}_{uz}(\theta) = G^y_u(\theta)\,\bar{G}^x_z(\pi-\theta)$, where the matrices G^y_u, $\bar{G}^x_z = \bar{G}^z_x$ satisfy the star triangle relation (see, for example, Refs. [2] and [153])

$$f(\theta,\theta')\bar{G}^{x_1}_{x_3}(\pi-\theta+\theta')\,G^{x_2}_{x_3}(\theta)\,\bar{G}^{x_1}_{x_2}(\pi-\theta') = \sum_x G^x_{x_3}(\theta')\,\bar{G}^{x_1}_x(\pi-\theta)\,G^{x_2}_x(\theta-\theta')\ , \quad (5.3.4)$$

and $f(.,.)$ is an arbitrary function such that $f(\theta,\theta') = f(\theta,\theta-\theta')$. The relations (5.3.4) for $f=1$ can be represented graphically in the form



The Feynman diagrams which will be considered here are graphs with vertices connected by lines labeled by numbers (indices). With each vertex we associate the point in the D-dimensional space R^D while the lines of the graph (with index α) are associated with the massless Feynman propagator

$$x \overset{\alpha}{\text{———}} x' \qquad = \qquad \frac{\Gamma(\alpha)}{(x-x')^{2\alpha}}$$

(which is a function of two points x, x' in D dimensional space-time):

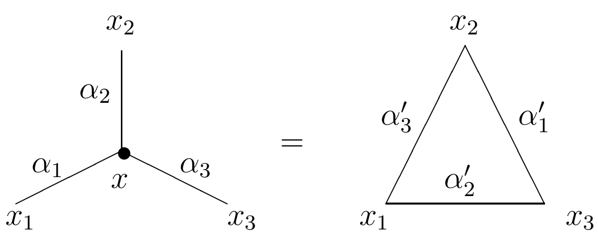
$$G_D(x - x'|\alpha) = \frac{\Gamma(\alpha)}{(x-x')^{2\alpha}} = \frac{\Gamma(\alpha)}{(\sum_\mu (x-x')_\mu (x-x')^\mu)^\alpha} \;, \qquad (5.3.5)$$

where $\Gamma(\alpha)$ is the Euler gamma-function, $D = 4 - 2\epsilon$ is the dimension of space-time, $(x)_\mu$ $(\mu = 1, 2, \ldots, D)$ are its coordinates, $\alpha = D/2 - 1 + \epsilon'$, and ϵ and ϵ' are, respectively, the parameters of the dimensional and analytic regularizations. The boldface vertices $\bullet$ denote that the corresponding points x are integrated over R^D: $\frac{1}{\pi^{D/2}} \int d^D x$. These diagrams are called the Feynman diagrams in the configuration space.

The propagator (5.3.5) satisfies the relation

$$\begin{gathered} \tfrac{1}{\pi^{D/2}} \int d^D x \prod_{i=1}^3 G_D(x - x_i|\alpha_i) \overset{\sum \alpha_i = D}{=} \\ G_D(x_1 - x_2|\alpha'_3)\, G_D(x_2 - x_3|\alpha'_1)\, G_D(x_3 - x_1|\alpha'_2) \;, \end{gathered} \qquad (5.3.6)$$

which is represented as the star-triangle identity for Feynman diagrams:



where $\alpha_1 + \alpha_2 + \alpha_3 = D$ and $\alpha'_i := D/2 - \alpha_i$. Eq. (5.3.6) can be readily obtained if we put $(x_3)^\mu = 0$ $\forall \mu$ and in the right-hand side of (5.3.6) make a simultaneous inversion transformation of the variables of integration, $(x)_\mu \to (x)_\mu / x^2$, and of the coordinates $(x_{1,2})^\mu$. The relations (5.3.4) and (5.3.6) are equivalent if we set

$$G^x_{x'}(\theta) = \bar{G}^x_{x'}(\theta) = G_D(x - x'|\frac{D}{2}(1 - \frac{\theta}{\pi})) \;, \;\; f(\theta, \theta') = 1 \;, \;\; \sum_x = \int \frac{d^D x}{\pi^{D/2}} \;. \quad (5.3.7)$$

Thus, the analytically and dimensionally regularized massless propagator (5.3.5) satisfies the infinite-dimensional star-triangle relation (5.3.4) and accordingly, on the basis of (5.3.5) and (5.3.7), we can construct solutions of the Yang-Baxter equation (5.3.0), (5.3.1). This remark was made in Ref. [153], in which calculations were made of vacuum diagrams with an infinite number of vertices corresponding to a planar square lattice (ϕ^4 theory, $D=4$), a planar triangular lattice (ϕ^6 theory, $D=3$), and a honeycomb lattice (ϕ^3 theory, $D=6$). The star-triangle relation (5.3.6) (known also as *the uniqueness relation*) was used in addition for analytic calculation of the diagrams that contribute to the 5-loop β- function of the $\phi^4_{D=4}$ theory [155] and of massless ladder diagrams [156], [157]. By means of identity (5.3.6) the symmetry groups of dimensionally and analytically regularized massless diagrams have been investigated [157], [158], [159][15]. We emphasize that an extremely interesting problem is that of massive deformation of the propagator function (5.3.5) and the corresponding deformation of the star-triangle relation (5.3.6).

There is an elegant operator interpretation [157] of the star-triangle identity (5.3.6). Indeed, consider a D-dimensional Heisenberg algebra with generators $\hat{q}_\mu = \hat{q}_\mu^\dagger$ and $\hat{p}_\mu = \hat{p}_\mu^\dagger$ ($\mu = 1, \ldots, D$) subject to defining relations

$$[\hat{q}_\mu, \, \hat{p}_\nu] = \mathbf{i} \, \delta_{\mu\nu} \; , \quad (\mu, \nu = 1, 2, \ldots, D) \; , \tag{5.3.8}$$

where $\hat{q}_\mu$ and $\hat{p}_\mu$ are operators of coordinate and momentum, respectively. Consider a representation of the algebra (5.3.8) in the linear vector space of complex functions $\psi(x_\mu)$ on $\mathbf{R}^D$:

$$\hat{q}_\mu \, \psi(x) = x_\mu \, \psi(x) \; , \quad \hat{p}_\mu \, \psi(x) = -\mathbf{i} \, \partial_\mu \, \psi(x) \; .$$

It will be convenient to realize the action of elements $\hat{A}$ of Heisenberg algebra as the action of integral operators: $\hat{A} \, \psi(x) = \int d^D y \, \langle x|\hat{A}|y\rangle \, \psi(y)$. The integral kernels $\langle x|\hat{A}|y\rangle$ can be considered as the matrix elements of $\hat{A}$ on the states $|x\rangle := |\{x_\mu\}\rangle$ and $\langle x| = |x\rangle^\dagger$ such that

$$\langle y|x\rangle = \delta^D(y-x) \; , \quad \hat{q}_\mu |x\rangle = x_\mu \, |x\rangle \; , \quad \int d^D x \, |x\rangle \, \langle x| = \hat{1} \; . \tag{5.3.9}$$

We extend the Heisenberg algebra by the elements $\hat{q}^{2\alpha} := (\hat{q}^\mu \hat{q}_\mu)^\alpha$ and pseudo-differential operators $\hat{p}^{-2\beta} := (\hat{p}^\mu \hat{p}_\mu)^{-\beta}$ ($\forall \alpha, \beta \in \mathbf{C}$). The corresponding integral kernels are:

$$\langle x|\hat{q}^{2\alpha}|y\rangle = x^{2\alpha} \, \delta^D(x-y) \; , \quad \langle x|\frac{1}{\hat{p}^{2\beta}}|y\rangle = a(\beta) \, \frac{1}{(x-y)^{2\beta'}} \; , \tag{5.3.10}$$

where $a(\beta) = \frac{\Gamma(\beta')}{\pi^{D/2} \, 2^{2\beta} \, \Gamma(\beta)}$, $\beta' = D/2 - \beta$ and $\beta' \neq 0, -1, -2, \ldots$.

For the extended Heisenberg algebra one can prove [157] that the set of operators: $H_\alpha := \hat{p}^{2\alpha} \, \hat{q}^{2\alpha}$ ($\forall \alpha \in \mathbf{C}$) is commutative. The commutativity condition $[H_\alpha, \, H_{-\beta}] = 0$ is represented in the form

$$\hat{p}^{2\alpha} \hat{q}^{2\gamma} \hat{p}^{2\beta} = \hat{q}^{2\beta} \hat{p}^{2\gamma} \hat{q}^{2\alpha} \; , \quad (\gamma = \alpha + \beta) \; . \tag{5.3.11}$$

Then, it is not hard to see that this identity, written for integral kernels by means of (5.3.10), is equivalent to the star-triangle relation (5.3.6). One should act on

[15] Here the symmetry of diagrams means the symmetry of corresponding perturbative integrals.

(5.3.11) by vectors $\langle x_1 - x_3|$ and $|x_2 - x_3\rangle$ from the left and right, respectively, and insert the unit $\hat{1}$ (5.3.9):

$$\langle x_1 - x_3|\hat{p}^{2\alpha} \Big(\int d^D x \, |x\rangle \, \langle x| \Big) \, \hat{q}^{2\gamma} \hat{p}^{2\beta} |x_2 - x_3\rangle = \langle x_1 - x_3|\hat{q}^{2\beta} \hat{p}^{2\gamma} \hat{q}^{2\alpha} |x_2 - x_3\rangle \;\Rightarrow$$

$$\int d^D x \, \langle x_1 - x_3|\hat{p}^{2\alpha}|x\rangle \, x^{2\gamma} \, \langle x|\hat{p}^{2\beta}|x_2 - x_3\rangle = (x_1 - x_3)^{2\beta} \langle x_1 - x_3|\hat{p}^{2\gamma}|x_2 - x_3\rangle (x_2 - x_3)^{2\alpha} \, .$$

Applying here the second eq. in (5.3.10) we obtain (5.3.6) for $\alpha = -\alpha_1'$, $\beta = -\alpha_2'$ and $\gamma = -\alpha_3$.

An important (from the point of view of the physical applications) problem in the analytical evaluations of massless multi-loop Feynman integrals is the representation of the D-dimensional integral (one of the way to define this divergent integral is given in [158], [157]):

$$Tr(\hat{q}^{2\alpha_1} \hat{p}^{2\beta_1} \hat{q}^{2\alpha_2} \hat{p}^{2\beta_2} \dots \hat{q}^{2\alpha_n} \hat{p}^{2\beta_n}) := \int d^D x \, \langle x|\hat{q}^{2\alpha_1} \hat{p}^{2\beta_1} \dots \hat{q}^{2\alpha_n} \hat{p}^{2\beta_n}|x\rangle \, ,$$

as a complex function of the indices α_i, β_i and parameter $D = 4 - 2\epsilon$ (the dimension of the space-time). The well known conjecture is that the coefficients in the expansion of this integral over ϵ (around physical value $D = 4$ of the space-time dimension) are expressed only in terms of the multiple zeta values and polylogarithms.

One can deduce another star-triangle relation [160] ($x_i \in \mathbf{R}^D$)

$$\Big(\frac{2\bar{\alpha}_1 \bar{\alpha}_3}{\alpha_2} \Big)^{D/2} \, W(x_3 - x_1|\alpha_1) \, W(x_1 - x_2|\alpha_2) \, W(x_2 - x_3|\alpha_3) =$$

$$= \int \frac{d^D x}{\pi^{D/2}} \, W(x_1 - x|\bar{\alpha}_3) \, W(x_3 - x|\bar{\alpha}_2) \, W(x_2 - x|\bar{\alpha}_1) \, , \qquad (5.3.12)$$

where $W(x|\alpha) = \exp{(-x^2/(2\alpha))}$ and the map

$$\bar{\alpha}_i = \frac{\alpha_1 \alpha_2 \alpha_3}{(\alpha_1 + \alpha_2 + \alpha_3)} \, \frac{1}{\alpha_i} \, , \quad \alpha_i = \frac{\bar{\alpha}_1 \bar{\alpha}_3 + \bar{\alpha}_2 \bar{\alpha}_3 + \bar{\alpha}_1 \bar{\alpha}_2}{\bar{\alpha}_i} \, , \qquad (5.3.13)$$

is the well known star-triangle transformation for resistances in electric networks. The identity (5.3.12) is related to the local Yang-Baxter equation [161] and is also rewritten in the operator form [160]

$$W(\hat{q} \,|\, \alpha_1) \, W(\hat{p} \,|\, \alpha_2^{-1}) \, W(\hat{q} \,|\, \alpha_3) = W(\hat{p} \,|\, \bar{\alpha}_3^{-1}) \, W(\hat{q} \,|\, \bar{\alpha}_2) \, W(\hat{p} \,|\, \bar{\alpha}_1^{-1}) \, . \qquad (5.3.14)$$

To obtain (5.3.12) from (5.3.14) we have used the representations

$$\langle x|e^{\frac{1}{\alpha}\hat{q}^2}|y\rangle = e^{\frac{1}{\alpha}(x)^2} \, \delta^D(x - y) \, , \quad \langle x|e^{-\frac{1}{2}\alpha\hat{p}^2}|y\rangle = (2\pi\alpha)^{-D/2} \, e^{-\frac{1}{2\alpha}(x-y)^2} \, .$$

It is tempting to apply identities (5.3.12) – (5.3.14) for the investigations of symmetries and analytical calculations of massive perturabative multi-loop integrals written in the α- representation. Besides, we hope that the local star-triangle relations (5.3.12), (5.3.14) will help in the construction of the massive deformation of the star-triangle relation (5.3.6).

• Note that we have not considered at all the numerous applications of quantum Lie groups and algebras with deformation parameters q satisfying the conditions

$q^N = 1$, i.e., when the parameters q are equal to the roots of unity. These applications (see, for example, Ref. [162]) appear mostly in the context of the topological and 2D conformal field theories and are associated with the specific theory of representations of such quantum groups that, generally speaking, can no longer be regarded as the deformation of the classical Lie groups and algebras.

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