NON–DEGENERACY THEOREM

A NOTE FOR GENERIC SMOOTH THEOREM

by

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INVERSION DER MATRIZEN $M(\Delta)$

Sei $\Delta = \{\alpha_1, \ldots\ldots, \alpha_n\}$ ein System einfacher Wurzeln des Wurzelsystems Σ , dann bezeichnen wir mit $M(\Delta)$ die Matrix der Skalarprodukte (α_i, α_j) , $\alpha_i, \alpha_j \in \Delta$ für ein zulässiges Skalarprodukt. Im folgenden werden die $M(\Delta)$ und ihre Inversen für alle irreduziblen Wurzelsysteme angegeben. Ist Σ vom Typ X_y , so bezeichnen wir $M(\Delta)$ mit $M(X_y)$.
Die Nummerierung der einfachen Wurzeln ist vorgenommen wie in Anhang 1 .

$$M(A_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \ldots\ldots & 0 & 0 & 0 \\ & & & & \ldots\ldots\ldots\ldots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & -1 & 2 \end{pmatrix}$$

$$M(B_n) = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & -2 & 0 & \ldots\ldots & 0 & 0 & 0 \\ & & & & \ldots\ldots\ldots\ldots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & -2 & 2 \end{pmatrix}$$

$$M(C_n) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & \ldots\ldots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & \ldots\ldots & 0 & 0 & 0 \\ & & & & \ldots\ldots\ldots\ldots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots\ldots & 0 & -2 & 4 \end{pmatrix}$$

Non–Degeneracy Theorem

A Note for Generic Smoothness Theorem

Lin Weng

Let S be an algebraic surface, H be an ample line bundle on S. We denote by $M_H(r.D.k)$ the moduli space of all isomorphic classes of H–stable rank r vector bundles with fixed determinant bundle D and second Chern class k.

Fixed a line bundle L on S with $h^0(L) \neq 0$, we can introduce a closed subvariety filtration on $M_H(r.D.k)$ as follows:

$$\Sigma_L^1 \subseteq \Sigma_L^2 \subseteq \dots \subseteq \Sigma_L^\infty \subseteq M_H(r.D.k)$$

where $\Sigma_L^i := \{E \in M_H(r.D.k) \,|\, h^0(\mathrm{ad}\, E \otimes L^i) \neq 0\}$ and $\Sigma_i^\infty = \bigcup_i \Sigma_L^i$.

Such kinds of subvarieties are very important in studying vector bundles on surfaces, smooth structures on surfaces, periods theory of surfaces and symplectic structures on surfaces, etc. For examples:

I. If we let $L = K_S$, the canonical line bundle of S, $r = 2$, $D = O_S$, then Donaldson proves the following

<u>Donaldson Lemma</u> [1]. $\dim_{\mathbb{C}} \Sigma_{K_S}^1 \leq 3k + A\sqrt{k} + A$, where A is a positive number which only depends on the geometry of S.

The most important consequence of it is

Corollary [1]. For algebraic surfaces with $p_g > 0$ and sufficiently large k, k–th Donaldson SU(2)–invariants do not vanish. Thus a simple connected algebraic surface with $p_g > 0$ at least has two different smooth structures.

II. If we let $L = K_S$ only, Zuo [5] generalize Donaldson Lemma as follows:

Generic Smoothness Theorem (Zuo [5]).

$$\dim_{\mathbb{C}} \Sigma^1_{Ks} \leq 3k + A\sqrt{k} + A ,$$

where A is a positive number which only depends on the geometry of S and D.

As an immediately consequence, we have

Corollary. For an algebraic surface with $p_g \geq 0$ and sufficiently large k, its k–th Donaldson SO(3)–invariant does not vanish. And $M_H(2.D.k)$ is generic smooth.

III. If we let $L = 2K_S$, and $D = O_S$, we have the following

Analogue of Donaldson Lemma (Tyurin [4]):

$$\dim_{\mathbb{C}} \Sigma^2_{K_S} \leq 3k + A\sqrt{k} + A ,$$

where A only depends on the geometry of S .

Corollary. For algebraic surfaces with $p_g \equiv 1 (\mathrm{mod}\ 2)$, we have a well–defined theory of periods of surfaces, which generalize the theory associated to the pair $(C, J(C))$, where C is an algebraic curve with its Jacobian $J(C)$.

In this paper, we want to generalize Zuo's work on any line bundle L , instead of K_S , i.e. we have the following

Non–Degeneracy Theorem. For any line bundle L on S with $h^0(L) > 0$,

$$\dim_{\mathbb{C}} \Sigma_L^1 \leq 3k + A\sqrt{k} + A ,$$

where A only depends on the geometry of S and L .

Surely, this theorem has I, II, III as special cases. Moreover, we can easily find an application of it in studying symplectic structures on algebraic surfaces. In fact, in this case, we only use Non–Degeneracy Theorem with $L = 2K_S$ [3].

The rough ideal behind the proof of Non–Degeneracy Theorem is from the following analysis:

For any $E \in \Sigma_L^1$, we find that there exists a nonzero section $s \in H^0(\mathrm{ad}\, E \otimes L)$. Consider the degeneracy locus of s. In this way, we can devide Σ_L^1 into two parts

$${}_1\Sigma_L^1 = \{E \in \Sigma_L^1 \,|\, \exists s \neq 0 \in H^0(\text{ad } E \otimes L)\,,\; t \in H^0(L)\,,\; \text{s.t. } \det s + t^2 = 0\}\,;$$

$${}_2\Sigma_L^1 = \Sigma_L^1 \setminus {}_1\Sigma_L^1\,.$$

For ${}_1\Sigma_L^1$, we can find two subline bundles of E, namely, kernel of the nontrivial morphism $E \xrightarrow{s \pm id_E \otimes t} E \otimes L$. Their quotients are rank 1 torsion–free sheaves. But on surfaces, rank 1 torsion–free sheaf is determined by its first Chern class and singularity set (e.g. Kobayashi [2] and Tyurin [4]). We find that E is totally determined by $\ker(s \pm id_E \otimes t)$, $S_1(\text{coker}(s \pm id_z \otimes t))$ and the extension group associated to $s \pm id_E \otimes t$. Now, we can use the fact that E is H–stable to deduce that $c_1(\ker(s \pm id_E \otimes t))$ are only finite many. From here, it is not difficult to prove the following

<u>Lemma</u>: $\dim_{\mathbb{C}}\, {}_1\Sigma_L^1 \leq 3k + A\sqrt{k} + A$.

For ${}_2\Sigma_L^1$, we note that $0 \neq s : E \longrightarrow E \otimes L$ only has degeneracy locus along $(\det s)_0 \in |2L|$, which is an even divisor and is not resulting from $|L|$. By some more or less standard theory, we can find a double covering along with $(\det s)_0$. With a subtle modification, we can prove that E and S do come from a line bundle on the top surface and a morphism from this line bundle to the pull back of E. Just as above, we also can prove that such kinds of line bundles only fit into finite many classes. From here, we can show that ${}_2\Sigma_L^1$ only has a relative low dimension.

As our proof is essentially the same as Zuo's original proof for Generic Smoothness Theorem, which is also pointed out by Zuo himself, here I'd like to prove the lemma only, in order to show how to make a modification. With this, we recommend readers to see [5] for the proof of the rest part.

Proof of the Lemma: Consider the following maps

$$s \pm \mathrm{id}_E \otimes t : E \longrightarrow E \otimes L .$$

As $\det(s \pm \mathrm{id}_E \otimes t) = \det s + t^2 = 0$, we have the following facts:

(1.) $\ker(s \pm \mathrm{id}_E \otimes t)$ are line bundles, denoted them by $L\pm$;

(2.) There exists the following exact sequence

$$0 \longrightarrow O_S(L\pm) \longrightarrow E \longrightarrow O_S(D - L\pm) \otimes J_{\xi\pm} \longrightarrow 0$$

where $J_{\xi\pm}$ are ideal sheaves of clusters $\xi\pm$ on S ;

(3.) $(D - L\pm) \cdot L\pm + |\xi\pm| = k .$ (*)

Furthermore, as

$$(s_{(\pm)}\mathrm{id}_E \otimes t)(s_{(\mp)}\mathrm{id}_E \otimes t) = s^2 - \mathrm{id}_E \otimes t^2 = -\mathrm{id}_E \otimes (\det s + t^2) = 0 ,$$

we have the following non–trivial maps

$$O_S(D - L_{(\pm)}) \otimes J_{\xi_{(\pm)}} \longrightarrow O_S(L + L_{(\mp)}) .$$

Thus

$$D - L\pm \le L + L\mp .$$

Note that H is an ample divisor and E is H–stable, we have

$$(D - L\pm)H \le (L + L\mp)H ,$$

and

$$L\pm \cdot H < D \cdot H/2 .$$

From this, we have

$$-L \cdot H \le \left[-\frac{D}{2} + L\pm\right] \cdot H < 0 . \qquad (**)$$

Next, we want to prove the following

Sublemma. With the same notation as above, we have

1. $\left[-\frac{D}{2} + L\pm\right]^2 \le A$;

2. $\left[-\frac{D}{2} + L\pm\right] \cdot K_S \le A\sqrt{k} + A$;

3. $h^0(K_S + D - 2L\pm) \le A$;

4. $\#\{c_1(L\pm)\} < +\infty$.

Proof: 1. By Hodge Index Theorem,

$$\left(-\frac{D}{2}+L\pm\right)^2 \le \left(\left(-\frac{D}{2}+L\pm\right)H\right)^2 / H^2 \le (LH)^2/H^2 \le A ;$$

2. For any line bundle I, we have a decomposition

$$I = \frac{(I\cdot H)}{H^2}H + I^{\perp H} .$$

Thus

$$\left|\left(-\frac{D}{2}+L\pm\right)\cdot K_S\right| \le \left|\left(-\frac{D}{2}+L\pm\right)^{\perp H}\cdot K_S^{\perp H}\right| + \left|\frac{\left(-\frac{D}{2}+\ L\pm\right)\cdot H}{H^2}\times\frac{K_S\cdot H}{H^2}\times H^2\right| .$$

By Hodge Index Theorem.

$$\text{RHS} \le \sqrt{-\left(\left(-\frac{D}{2}+L\pm\right)^{\perp H}\right)^2}\ \sqrt{-(K^{\perp H})^2} + A \le A\sqrt{-\left(\left(-\frac{D}{2}+L\pm\right)^{\perp H}\right)^2} + A .$$

Note that

$$-\left(\left(-\frac{D}{2}+L\pm\right)^{\perp H}\right)^2 = -\left(-\frac{D}{2}+L\pm\right)^2 + \frac{\left(\left(-\frac{D}{2}\ +L\pm\right)H\right)^2}{(H^2)^2}\,H^2 \overset{(**)}{\le}$$

$$-\left(-\frac{D}{2}+L\pm\right)^2 + \frac{(LH)^2}{H^2} \overset{(*)}{\le} k + A ; \qquad (***)$$

3. It is sufficient to prove that

$$(K_S + D - 2L\pm) \cdot H \le A \, ,$$

which is an easy consequence of (**);

4. It is sufficient to prove that

$$(2H^2)\left[c_1\left[-\frac{D}{2} + L\pm\right]\right] \in H^{1.1}(S) \cap H^2(S,\mathbb{Z})$$

has a bounded norm.

On the other hand, we have

$$2H^2\left[-\frac{D}{2} + L\pm\right] = 2\left[\left[-\frac{D}{2} + L\pm\right] \cdot H\right]H + (2H^2)\left[-\frac{D}{2} + L\pm\right]^{\perp H} ,$$

and $\left[2\left[\left[-\frac{D}{2} + L\pm\right] \cdot H\right]H\right]^2 \overset{(**)}{\le} A \, ,$

$$\left|\left[(2H^2)\left[-\frac{D}{2} + L\pm\right]^{\perp H}\right]^2\right| \overset{(***)}{\le} 4(H^2)^2(k + A) \, . \qquad \text{Q.E.D.}$$

By 4. we can decompose Σ_L^1 into finite many subvarieties

$$\Sigma_L^1 = \dot{\bigcup_i} \; {}^i\Sigma_L^1 \, ,$$

where $E \in {}^i\Sigma_L^1$ comes from the extension 2. with the same $c_1(L\pm)$.

By (2.), it is easy to see that ${}^{i}\Sigma_L^1$ is parameted by $\xi \in \mathrm{Hilb}^{|\xi|}(S)$, $L\pm \in \mathrm{Pic}(S)$ and

$$\mathrm{Ext}^1{}_{O_S}(O_S(D - L_{(\pm)}) \otimes J_{\xi_{(\pm)}},\ O_S(L_{(\pm)})) .$$

Thus

$$\dim_{\mathbb{C}}\Sigma_L^1 \le \max_i\{\dim_{\mathbb{C}}{}^{i}\Sigma_L^1\} \le \max_i\{\dim_{\mathbb{C}}\mathrm{Ext}^1_{O_S}(O_S(D - L\pm) \otimes J_{\xi\pm}, O_S(L_{(\pm)})) +$$

$$2|\xi_{(\pm)}| + q(S)| = \max_i\{h^1(O_S(K_S + D - 2L_{(\pm)}) \otimes J_{\xi_{(\pm)}}) + 2|\xi_{(\pm)}| + q(S)\} .$$

Note that we have the following exact sequence

$$0 \longrightarrow O_S(K_S + D + 2L_{(\pm)}) \otimes J_{\xi_{(\pm)}} \longrightarrow O_S(K_S + D - 2L_{(\pm)}) \longrightarrow$$

$$O_{\xi_{(\pm)}} \otimes O_S(K_S + D - 2L\pm) \longrightarrow 0 ,$$

thus, $h^1(O_S(K_S + D - 2L_{(\pm)}) \otimes J_{\xi_{(\pm)}}) \le |\xi_{(\pm)}| + h^1(O_S(K_S + D - 2L_{(\pm)})) =$

$|\xi_{(\pm)}| - \chi(O_S) + h^0(O_S(K_S + D - 2L_{(\pm)}) + h^2(O_s(K_S + D - 2L_{(\pm)}) -$

$2\left[-\frac{D}{2} + L_{(\pm)}\right]^2 + \left[-\frac{D}{2} + L_{(\pm)}\right] \cdot K_S$

$= |\xi\pm| - \chi_{(O_S)} + h^0(O_S(K_S + D - 2L_{(\pm)}) - 2\left[-\frac{D}{2} + L_{(\pm)}\right]^2 + \left[-\frac{D}{2} + L_{(\pm)}\right] \cdot K_S$

$\overset{2.,3.}{\le} |\xi\pm| + A_1\sqrt{k} + A' - 2\left[-\frac{D}{2} + L_{(\pm)}\right]^2$

$\overset{1}{\le} |\xi_{(\pm)}| + A_1\sqrt{k} + A'' - 3\left[-\frac{D}{2} + L\pm\right]^2$

$\overset{(*)}{=} -2|\xi_{(\pm)}| + 3k + A\sqrt{k} + A .$

Therefore, we have

$$\dim_{\mathbb{C}} {}^1\Sigma_L^1 \le 3k + A\sqrt{k} + A . \qquad \text{Q.E.D.}$$

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