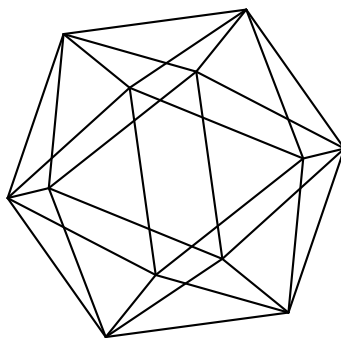# Max-Planck-Institut für Mathematik
Bonn

(Super)-character and (super)-denominator formulae for
finite dimensional and affine Lie superalgebras with
non-degenerate, supersymmetric, invariant bilinear
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by

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(Super)-character and (super)-denominator formulae for finite dimensional and affine Lie superalgebras with non-degenerate, supersymmetric, invariant bilinear forms

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Abstract: In this paper we give a proof for the Weyl character and super-character formulae for all irreducible highest weight integrable modules of finite dimensional and affine Lie superalgebras with non-degenerate supersymmetric, invariant bilinear forms, except for modules of level 0 of affine Lie superalgebras with zero dual Coxeter number. We deduce a denominator and super-denominator formulae for these Lie superalgebras except for affine Lie superalgebras with zero dual Coxeter number.

1 Finite dimensional Lie superalgebras

1.1 Introduction

Unless otherwise stated, a Lie superalgebra will mean a simple finite dimensional with symmetrizable Cartan matrix. A finite dimensional Lie superalgebra is of this type if and only if it is either a simple Lie algebra or one of the following ones:

$$A(m,n), B(m,n), C(n), D(m,n), F_4, G_3, D(2,1;\alpha).$$

The study of their representations was started in [K3]. More precisely a character formula was computed for what the author called their typical representations. An irreducible representation of highest weight Λ with respect to a base Π is typical if and only if there are no isotropic odd roots α such that $(\Lambda + \rho, \alpha) = 0$, where ρ is the Weyl vector with respect to the base Π. In this case, its character formula is of the same form as the one in the context of finite dimensional simple Lie algebras. The problem in the arbitrary case is the existence of both positive and negative non-diagonal entries in the Cartan matrix or in other words the existence of real roots both of positive and negative norms. Indeed this makes it impossible to apply the standard proof of the character formula used in the

Borcherds-Kac-Moody setup [B, K4] as it relies heavily on inequalities. The level of complications depends on the maximal possible dimension of an isotropic subspace of the set of roots, otherwise known as the defect (as defined in [KW1]). In particular, if a Lie superalgebra has defect 0, or equivalently has no isotropic odd simple roots, which is the case when the odd part of the Lie superalgebra is trivial or G is of type $B(0, n)$, there are no problems. When the defect is equal to 1 – i.e. for the exceptional Lie superalgebras of type F_4, G_3, $D(2, 1; \alpha)$, and those of type $A(0, n)$, $A(m, 0)$, $B(1, n)$, $B(m, 1)$, $C(n)$, $D(m, 1)$ –, the character formula for all irreducible integrable highest weight representations was computed in [KW1]. When the defect is equal to or less than 2, there is a base Π with respect to which the Cartan matrix is of Borcherds-Kac-Moody type, i.e. its non-diagonal entries are all non-positive and hence the proof of the character formula given in [Ray] for irreducible integrable highest weight modules for which $(\Lambda, \alpha_i) \geq 0$ for all simple roots $\alpha_i \in \Pi$ of Borcherds-Kac-Moody superalgebras applies. In particular, it gives the denominator formula for the Lie superalgebras of defect at most 2.

There have been several attempts to derive the (super)-character formula for atypical modules. Apart from the papers cited above the following is a far from exhaustive list of articles dealing with several special cases: [BL], [F], [J], [JHKT-M], [S1] and [S2]. In [KW1], the authors conjectured a denominator formula for all finite dimensional cases. This was proved recently using combinatorial methods recently in [G].

In this second part, we give a proof of the character formula for all the irreducible integrable highest weight modules, equivalently finite dimensional irreducible. From the (super)-character formula, we deduce a (super)-denominator formula. Our proof is not combinatorial and applies to all the finite dimensional Lie superalgebras with symmetrizable Cartan matrix. The main idea behind the proof is independent of special features of the different types of finite dimensional Lie superalgebras and is heavily dependent on the concept of odd reflections constructed by V. Serganova in [LSS].

1.2   Notation and some fundamental properties

In this section, we fix the notation that will be used throughout the first part of this paper and give basic properties of the Lie superalgebras in question.

1. $\mathbb{Z}_+$ will stand for the set of all non-negative integers and $\mathbb{N}$ for the set of positive integers.

2. Let G be a finite dimensional Lie superalgebra with a non degenerate, super-symmetric, invariant bilinear form $(.,.)$.

3. Let H be a Cartan subalgebra of G and $\Delta \leq H^*$, $\Delta_0 \leq H^*$, $\Delta_1 \leq H^*$ be respectively the set of roots, even root, and odd roots with respect to the Cartan subalgebra H. Set $\breve{\Delta}_0 = \{\alpha \in \Delta_0 : \frac{1}{2}\alpha \notin \Delta\}$.

4. Let $\Pi = \{\alpha_1, \cdots, \alpha_N\}$ be a base of the root system Δ. Let $E = \mathbb{R}\Pi$ be the real vector space spanned by the base Π. The base Π is a basis of the dual space H^*.

   Convention used in this paper: By Cartan matrix we mean the matrix of the restriction of the bilinear form to the Cartan subalgebra H. In the usual sense, a Cartan matrix has a different meaning: its diagonal entries are equal to 2 or 0 only. It may be symmetrizable but not necessarily symmetric. When it is symmetrizable, its symmetric version (i.e. its product with an appropriate diagonal matrix) can have entries both non-positive and non-negative and not always equal to ± 2 is what we call Cartan matrix in this paper. However, the Killing form may be trivial and hence the usual Cartan matrix trivial, but there may be other non-degenerate bilinear supersymmetric invariant forms as is the case for $A(n,n)$.

   Let $h_i \in H$, $1 \leq i \leq N$, be such that $(h_i, h_j) = (\alpha_i, \alpha_j)$.

   Let Δ^+ be the set of positive roots with respect to the base Π. Set
   $$\Delta_0^+ = \Delta^+ \cap \Delta_0, \quad \Delta_1^+ = \Delta^+ \cap \Delta_1;$$
   and Π_0 to be the base of the Lie algebra with positive root system Δ_0^+. For Lie algebras or Lie superalgebras of type $A(m,n)$, $C(n)$, $B(0,n)$, $\alpha \in \Pi_0$ if and only if $\alpha \in \Pi$ or $\frac{1}{2}\alpha \in \Pi$.

   For Lie superalgebras of type $B(m,n)$, $m > 0$, $D(m,n)$ or exceptional, we define θ to be the positive root with the following property:
   $$\theta \in \Pi_0, \quad \theta, \frac{1}{2}\theta \notin \Pi.$$
   Indeed in these cases, $\alpha \in \Pi_0$ if and only if $\alpha \in \Pi$ or $\frac{1}{2}\alpha \in \Pi$ or $\alpha = \theta$.

5. Let
   $$G_\alpha = \{x \in G : [h,x] = \alpha(h)x, h \in H\}.$$
   As $\dim G < \infty$, $\dim G_\alpha = 1$ for all $\alpha \in \Delta$.

Set $e_i \in G_{\alpha_i}$, $f_i \in G_{-\alpha_i}$ to be the generators of the superalgebra G, where $[e_i, f_i] = h_i$ and $[h, e_i] = (h, h_i)e_i$, $[h, f_i] = -(h, h_i)f_i$ for all $h \in H$. Note that our generators are multiples of the usual generators taken when the Cartan matrix is assumed to be symmetrizable but not necessarily symmetric.

6. For a root $\alpha \in \Delta$, write $\mathrm{supp}(\alpha)$ for the support of α. By abuse of language we use this terminology for both the simple roots and their indices.

7. As in [KW1],

$$e(\lambda) : E = \mathbb{R}\Delta \to \mathbb{R}$$
$$\mu \mapsto e^{(\lambda, \mu)}$$

is an exponential function and the expressions $\frac{\sum_{\lambda \in H^*} a_\lambda e(\lambda)}{\sum_{\lambda \in H^*} b_\lambda e(\lambda)}$ are rational exponential functions, namely the numerator and denominator are finite linear combinations of exponentials $e(\lambda)$. Set

$$R = \frac{\prod_{\alpha \in \Delta_0^+}(1 - e(-\alpha))}{\prod_{\alpha \in \Delta_1^+}(1 + e(-\alpha))}$$

and

$$\check{R} = \frac{\prod_{\alpha \in \Delta_0^+}(1 - e(-\alpha))}{\prod_{\alpha \in \Delta_1^+}(1 - e(-\alpha))}$$

to be respectively the Weyl denominator and superdenominator with respect to the base Π.

8. Let ρ be the Weyl vector with respect to the base Π, i.e.

$$(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) \quad \forall\, 1 \leq i \leq N.$$

Let ρ_0 be the Weyl vector of the Lie algebra G_0 with respect to the base Π_0. Set $\rho_1 = \rho_0 - \rho$. As $\dim G < \infty$,

$$\rho_0 = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha, \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

Because the bilinear form on G is non-degenerate, there is a unique Weyl vector in E.

9. Let $V = V(\Lambda)$ be the finite dimensional G-module of highest weight $\Lambda \in H^*$ with respect to the base Π, i.e. for all

$$\alpha \in \Delta^+ \text{ such that } |\alpha|^2 \neq 0,\ \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_+.$$

4

The vector $v_\Lambda \in V$ will denote a highest weight vector of the G-module $V(\Lambda)$. Set $P(\Lambda)$ to be the set of weights of the module $V(\Lambda)$. For $\mu \in H^*$, we write $V_\mu = \{v \in V : h.v = \mu(h)v, \forall\, h \in H\}$. Then, $\dim V_\mu < \infty$ for all $\mu \in H^*$ and following [K4]

$$\operatorname{ch} V = \sum_{\mu \leq \Lambda} (\dim V_\mu) e(\mu) \in \mathcal{E},$$

where $\mathcal{E}$ is the algebra over $\mathbb{C}$ of formal series of the form

$$\sum_{\mu \in H^*} c_\mu e(\mu)$$

where $c_\mu \in \mathbb{C}$ and $c_\mu = 0$ for μ outside the union of a finite number of sets of the type $D(\lambda) = \{\mu \in H^* : \mu \leq \lambda\}$.

10. Let W be the Weyl group of the Lie superalgebra G. For $w \in W$, let $l(w)$ be the number of simple reflections needed to write w as a word and $\breve{l}(w)$ the number of simple reflections corresponding to the set $\breve{\Delta}_0^+$ needed to write w. Set

$$\epsilon(w) = (-1)^{l(w)} \quad \text{and} \quad \breve{\epsilon}(w) = (-1)^{\breve{l}(w)}.$$

We define the following subgroups of the Weyl group:

$$W_+ = \langle r_\alpha : |\alpha|^2 > 0 \rangle$$
$$W_- = \langle r_\alpha : |\alpha|^2 < 0 \rangle,$$

i.e.

$$W = W_- \times W_+.$$

11. $G_0 = (G_0)_+ \oplus (G_0)_-$, where $(G_0)_+ = \langle G_\alpha : \alpha \in \Delta_0, |\alpha|^> 0 \rangle$ and $(G_0)_- = \langle G_\alpha : \alpha \in \Delta_0, |\alpha|^< 0 \rangle$. To express the (super)-denominator formula in a simple manner, we need to choose one of these summands. We do this as follows (see [KW1]). Setting θ' to be the maximal root in Δ^+, define the dual Coxeter number

$$h^\vee = \frac{1}{2}(\theta', \theta' + 2\rho)$$

to be the half of the eigenvalue of the action of the Casimir operator on the Lie superalgebra G. $h^\vee = 0$ if and only if G is of type $A(n,n)$, $D(n+1,n)$, or $D(2,1;a)$ (see [KW1]). Set

$$\Delta_0^\sharp = \begin{cases} \{\alpha \in \Delta_0^+ : |\alpha|^2 h^\vee > 0\} & \text{if } h^\vee \neq 0 \\ \{\alpha \in \Delta_0^+ : |\alpha|^2 > 0\} & \text{if } G \text{ is of type } A(n,n) \\ \{\alpha \in \Delta_0^+ : |\alpha|^2 |\theta|^2 < 0\} & \text{if } G \text{ is of type } D(n+1,n) \text{ or } D(2,1;a) \end{cases}$$

and
$$W^{\sharp} = \langle r_{\alpha} : \alpha \in \Delta^{\sharp} \rangle.$$

12. In the appendix of [LSS], V. Serganova introduced the concept of odd reflections for finite dimensional simple Lie superalgebras with Cartan matrix. As the proof is heavily based on the use of these reflections, we remind the reader of their definition. For $1 \le i \le N$, if $|\alpha_i|^2 = 0$, the linear map s_{α_i} on H^* given by

$$s_{\alpha_i}(\alpha_j) = \begin{cases} \alpha_j & \text{if } j \ne i \text{ and } (\alpha_j, \alpha_i) = 0 \\ \alpha_j + \alpha_i & \text{if } (\alpha_j, \alpha_i) \ne 0 \\ -\alpha_i & \text{if } j = i. \end{cases}$$

is the corresponding odd reflection.

Write $\Pi_i := s_{\alpha_i}(\Pi)$. This is another base of the set of roots Δ whose simple roots are:

$$\beta_j^i = \begin{cases} \alpha_j & \text{if } i \ne j \text{ and } (\alpha_i, \alpha_j) = 0 \\ \alpha_i + \alpha_j & \text{if } (\alpha_i, \alpha_j) \ne 0 \\ -\alpha_i & \text{if } i = j. \end{cases}$$

Define
$$\Delta_{i0}^+ := \Delta_0 \cap \mathbb{Z}_+ \Pi_i, \quad \Delta_{i1}^+ := \Delta_1 \cap \mathbb{Z}_+ \Pi_i.$$

The notation
$$\lambda \le_j \mu$$
will mean $\mu - \lambda \in \mathbb{Z}_+ \Pi_j$. More generally if Π' is a base then
$$\lambda \le_{\prime} \mu$$
will mean $\mu - \lambda \in \mathbb{Z}_+ \Pi'$. From now on Π' will denote a base obtainable from the base Π by successive applications of odd reflections. A consequence of the definition of odd reflections is the following:

Corollary 1. *Let $\alpha_i \in \Pi$ be such that $|\alpha_i|^2 = 0$. Then the set of positive even (resp. odd) roots with respect to the base Π_i is*

$$\Delta_0^+ \quad \text{(resp.} \quad (\Delta_1^+ - \{\alpha_i\}) \cup \{-\alpha_i\})$$

and
$$\rho + \alpha_i$$
is a Weyl vector with respect to the base Π_i.

Proof. Let

$$\alpha = \sum_{i=1}^{n} k_i \alpha_i \in \Delta^+.$$

Suppose that $\alpha \neq \alpha_i$. We show that α remains a positive root with respect to the base Π_i. We know from Proposition 1 that α is a root and hence we only need to check positivity.

Suppose first that there is an index $j \in \mathrm{supp}(\alpha)$ such that $j \neq i$ and $(\alpha_i, \alpha_j) = 0$. As $\alpha_j \in \Pi_i$ and $k_j > 0$, it then follows that $\alpha \in \mathbb{Z}_+ \Pi_i)$.

Hence we can suppose that for all $j \in \mathrm{supp}(\alpha)$ such that $j \neq i$, $(\alpha_i, \alpha_j) \neq 0$. Then, for $j \neq i$,

$$\alpha_j = (\alpha_i + \alpha_j) + (-\alpha_i).$$

Since $\alpha_i + \alpha_j, -\alpha_i \in \Pi_i$, $\alpha_j \in \mathbb{Z}_+(\Pi_i)$. Hence as $k_j > 0$ it again follows that $\alpha \in \mathbb{Z}_+(\Pi_i)$. The first two statements follow.

It remains to check that $\rho_i = \rho + \alpha_i$ is the Weyl vector with respect to the base Π_i. Now

$$(\rho + \alpha_i, \alpha_j) = (\rho, \alpha_j)$$

for all $(\alpha_i, \alpha_j) = 0$. In particular,

$$(\rho + \alpha_i, -\alpha_i) = 0.$$

If $(\alpha_i, \alpha_j) \neq 0$, then

$$(\rho + \alpha_i, \alpha_i + \alpha_j) = \frac{1}{2}(\alpha_j, \alpha_j) + (\alpha_i, \alpha_j)$$
$$= \frac{1}{2}(\alpha_i + \alpha_j, \alpha_i + \alpha_j)$$

Therefore $\rho + \alpha_i$ is a Weyl vector with respect to the base Π_i. $\qquad\square$

Set

$$\rho_i = \rho + \alpha_i.$$

We will write Λ^i for the highest weight of the finite dimensional simple G-module V with respect to the base Π_i. More generally, for the base Π', we will write ρ' and Λ'.

1.3 (Super)-character and (super)-denominator formulae

1.3.1 Statement of main results

Let $V = V(\Lambda)$ be an integrable highest-weight irreducible G-module with highest weight Λ with respect to the base Π. Set $S_{\Lambda+\rho}$ to be a maximal subset of Δ^+ consisting of isotropic roots, mutually orthogonal and orthogonal to the weight $\Lambda + \rho$. The cardinality of the set $S_{\Lambda+\rho}$ is called the degree of atypicality of the module V. It is independent of the base. The cardinality of the maximal isotropic subset of the set of roots is the defect of the Lie superalgebra and clearly the atypicality of a module is at most equal to the defect of the Lie superalgebra. If $S_{\Lambda+\rho} = \emptyset$, then the module V is typical. In this case, the character formula was proved in [K3]. Our proof for the general case includes the typical case and is different from the one in [K3]. Based on the notation in [KW1], for a subset U of the Weyl group W, set

$$\mathcal{T}_{\Lambda+\rho} = \cup_{T \leq S_{\Lambda+\rho}} \{\sum_{\alpha \in T} \alpha\}$$

to be the subset of H^* of all possible sums of roots in $S_{\Lambda+\rho}$ and

$$\mathcal{S}_{\Lambda+\rho} = S_{\Lambda+\rho} \cup \{\beta \in \mathcal{T}_{\Lambda+\rho} : \beta \notin S_{\Lambda+\rho}, \beta = \sum_{\gamma \in \Delta_1^+, \gamma \notin S_{\Lambda+\rho}} \gamma\}$$

to be the subset of $\mathcal{T}_{\Lambda+\rho}$ consisting of roots in $S_{\Lambda+\rho}$ and of sums of at least two roots in $S_{\Lambda+\rho}$ that can be written as sums of odd roots not in $S_{\Lambda+\rho}$

Proposition 1. *1. If the atypicality of the Lie superalgebra G is equal to 1, then $\mathcal{S}_{\Lambda+\rho} = S_{\Lambda+\rho}$;*

2. if the base Π contains a unique isotropic root then $\mathcal{S}_{\Lambda+\rho} = \mathcal{T}_{\Lambda+\rho}$;

3. if $S_{\Lambda+\rho} \leq \Pi$, then $\mathcal{S}_{\Lambda+\rho} = S_{\Lambda+\rho}$.

$$\Gamma_{\Lambda, S_{\Lambda+\rho}, \Pi, U} = \sum_{w \in U} \epsilon(w) w \left(\frac{e(\Lambda + \rho)}{\prod_{\alpha \in \mathcal{S}_{\Lambda+\rho}} (1 + e(-\alpha))} \right)$$

and

$$\breve{\Gamma}_{\Lambda, S_{\Lambda+\rho}, \Pi, U} = \sum_{w \in U} \breve{\epsilon}(w) w \left(\frac{e(\Lambda + \rho)}{\prod_{\alpha \in \mathcal{S}_{\Lambda+\rho}} (1 - e(-\alpha))} \right)$$

Define

$$\dot{\jmath}_{\Lambda, S_{\Lambda+\rho}, \Pi, U}$$

(resp.
$$\breve{j}_{\Lambda,S_{\Lambda+\rho},\Pi,U})$$
to be the the cardinality of the set

$$\{w \in W : \frac{e(w(\Lambda + \rho))}{\prod_{\alpha \in S_{\Lambda+\rho}}(1 + e(-w(\alpha)))} = \frac{e(\Lambda + \rho)}{\prod_{\alpha \in S_{\Lambda+\rho}}(1 + e(-\alpha))}\}$$

(resp.

$$\{w \in W : \frac{e(w(\Lambda + \rho))}{\prod_{\alpha \in S_{\Lambda+\rho}}(1 - e(-w(\alpha)))} = \frac{e(\Lambda + \rho)}{\prod_{\alpha \in S_{\Lambda+\rho}}(1 - e(-\alpha))}\})$$

Define

$$\text{ch}_{\Lambda,S_{\Lambda+\rho},\Pi,U} = j_{\Lambda,S_{\Lambda+\rho},\Pi}^{-1} e(-\rho) R^{-1} \Gamma_{\Lambda,S_{\Lambda+\rho},\Pi,U}$$

and

$$\text{sch}_{\Lambda,S_{\Lambda+\rho},\Pi,U} = j_{\Lambda,S_{\Lambda+\rho},\Pi}^{-1} e(-\rho) \breve{R}^{-1} \breve{\Gamma}_{\Lambda,S_{\Lambda+\rho},\Pi,U}$$

We generalize the notion of a tame module given in [KW1]:

Definition 1. *The G-module $V = V(\Lambda)$ is said to be generalized tame (with respect to the base Π) if*

$$\text{ch} V = \text{ch}_{\Lambda,S_{\Lambda+\rho},\Pi,W}.$$

When the degree of atypicality of the module V is at most 1, the G-module V is said to be tame.

We need to distinguish between two types of isotropic roots.

Definition 2. *The isotropic root $\alpha \in \Delta^+$ is said to be of the first kind if there is a base $\Pi' = s_p \cdots s_1(\Pi)$, where the s_i are odd reflections, such that $\alpha \in \Pi'$. Otherwise the isotropic root α is said to be of the second kind.*

Proposition 2. *When the Lie algebra G is of type $A(m,n)$, $C(n)$ or is exceptional, all isotropic roots are of the first kind. This is not the case when the Lie algebra G is of type $B(m,n)$, $m > 0$, or $D(m,n)$.*

Theorem 1. *Let G is either a Lie algebra or a Lie superalgebra of type $A(m,n)$, $C(n)$, or $B(0,n)$. Then all finite dimensional irreducible G-modules are generalized tame with respect to all bases.*

When the Lie superalgebra G is of type $B(m,n)$, $m > 0$, $D(m,n)$ or is exceptional, set

$$\tilde{W} = \langle r_\alpha : \alpha \in \Pi_0 - \{\theta\}\rangle$$

and

$$W' = \langle r_\theta, r_\alpha : \alpha \in \Pi_0 : |\alpha||\theta| < 0\rangle.$$

Theorem 2. *Let the Lie superalgebra G be of type $B(m,n)$, $m > 0$, or $D(m,n)$. Set α to be the maximal isotropic root of the second kind such that $i \in \mathrm{supp}(\alpha)$ implies that $|\alpha_i||\theta| \leq 0$.*

1. *If*
$$\frac{2(r_\theta(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} \leq 0$$

   (equivalently the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is infinite dimensional), then the G-module V is tame and

$$S_{\Lambda + \rho} = \{\alpha\} \ \text{or} \ \emptyset;$$

2. *if the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is finite dimensional then the module V is not generalized tame and*

$$\mathrm{ch}\, V = \begin{cases} \mathrm{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, \tilde{W}} & \text{if } \frac{2(\Lambda+\rho, \theta)}{(\theta, \theta)} \leq 0 \\ \mathrm{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, W} - \mathrm{ch}_{r_\theta(\Lambda+\rho)-\rho, S_{\Lambda+\rho}, \Pi, W'} & \text{otherwise} \end{cases}$$

   Moreover, if $\frac{2(\Lambda+\rho, \theta)}{(\theta, \theta)} \leq 0$, then all isotropic roots in the set $S_{\Lambda+\rho}$ are of the first kind and otherwise $S_{\Lambda+\rho} = \{\alpha\}$.

Theorem 3. *Let G be an exceptional Lie superalgebra. Then,*
$$W = \langle r_\theta, r_\alpha : |\alpha||\theta| < 0 \}.$$

1. *If*
$$\frac{2(r_\theta(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} \leq 0$$

   (equivalently the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is infinite dimensional), then the G-module V is tame;

2. *if the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is finite dimensional then it not generalized tame and*

$$\mathrm{ch}\, V = \begin{cases} \mathrm{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, \tilde{W}} & \text{if } \frac{2(\Lambda+\rho, \theta)}{(\theta, \theta)} \leq 0 \\ \mathrm{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, W} - \mathrm{ch}_{r_\theta(\Lambda+\rho)-\rho, S_{\Lambda+\rho}, \Pi, W} & \text{otherwise} \end{cases}$$

As a consequence, the denominator formula can be nicely expressed with respect to a base with special properties.

Theorem 4. *Let G be a finite dimensional Lie superalgebra with a symmetrizable Cartan matrix. Let Π be a base containing a maximal isotropic subset S of the set of roots Δ. Then the denominator formula is:*
$$e(\rho)R = \Gamma_{0, S, \Pi, W^\sharp}.$$

In particular, when the Lie superalgebra is of type $A(m,n)$ or $C(n)$ or $B(0,n)$, the trivial module is generalized tame and when the Lie superalgebra is of type $B(m,n)$, $m > 0$, $D(m,n)$ or is exceptional, it is not generalized tame.

Theorem 5. *For a Lie superalgebra G with symmetrizable Cartan matrix, replacing* ch *with* sch *and Γ with $\check{\Gamma}$ in Theorems 1, 2, 3 and 4 gives the super-character for the module V and the super-denominator formula.*

1.3.2   Proof: Part I

In the rest of this section we prove these theorems. As mentioned in section 2, we fix an arbitrary base Π. The arguments in [K3] or [K4] lead to the next equality:

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \mathrm{ch}\, V$$

$$= \sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)), \qquad (1)$$

where $c_\Lambda = 1, c_\lambda \in \mathbb{Z}$.

In the first part, we aim to prove the following result:

Proposition 3. *Let Π be a base containing at most one isotropic simple root and all of whose non-isotropic roots are even. Let $\lambda \in H^*$ be such that $c_\lambda \neq 0$.*

1. When G is a Lie superalgebra of type $A(m,n)$ or $C(n)$,

$$\lambda + \rho = w(\Lambda + \rho) - \sum_{\alpha \in S_{w(\Lambda+\rho)}} k_\alpha \alpha$$

where $w \in W$, for all α, $k_\alpha \in \mathbb{Z}_+$ and $k_\alpha \neq 0$ implies that the root α is a positive isotropic root. Moreover, when $k_\beta, k_\alpha \neq 0$, $(\alpha, \beta) = 0$.

2. Suppose that G is a Lie superalgebra of type $B(m,n)$, $m > 0$ or $D(m,n)$.

(a) If $r_\theta(\Lambda + \rho) \not\leq \Lambda + \rho$, then $S_{\Lambda+\rho} \neq \emptyset$ and

$$\lambda + \rho = w(\Lambda + \rho) - \sum_{\alpha \in S_{w(\Lambda+\rho)}} k_\alpha \alpha$$

for some $w \in \tilde{W}$, $k \in \mathbb{N}$ and mutually orthogonal isotropic roots α of the first kind;

(b) otherwise either $S_{\Lambda+\rho} = \emptyset$ in which case

$$\lambda + \rho = w(\Lambda + \rho)$$

or $S_{\Lambda+\rho} = \{\alpha\}$, where α is the maximal isotropic positive root of the second kind containing only simple roots of non-negative norm in its support and

$$\lambda + \rho = w(\Lambda + \rho - k\alpha).$$

3. *When G is an exceptional Lie superalgebra, $S_{\Lambda+\rho} = \emptyset$, in which case*

$$\lambda + \rho = w(\Lambda + \rho)$$

or $S_{\Lambda+\rho} = \{\alpha\}$, where α is an isotropic root of the first kind and

$$\lambda + \rho = w(\Lambda + \rho) - kw(\alpha)$$

for some $w \in \tilde{W}_+$ if $(\Lambda + \rho, \theta) > 0$ and for some $w \in W$ otherwise.

We do not as yet restrict ourselves to a base with the properties described in Proposition 2. We assume that G is not a Lie algebra or of type $B(0,n)$ as in these cases there are no isotropic roots and hence the usual proof of the character formula applies. This will avoid mentioning these cases throughout the proof.

Note that $c_\lambda \neq 0$ implies that $\lambda \leq \Lambda$. The point is that there may be other sums for which this is not the case. In other words, with respect to a base $\Pi' = \{\beta_1, \cdots, \beta_N\}$ with positive even (resp. odd) root set Δ'_0 (resp. Δ'_1), we may have:

$$e(\rho) \prod_{\alpha \in \Delta_0'^+} (1 - e(-\alpha)) \mathrm{ch}\, V$$
$$= \sum_{|\lambda+\rho|^2 = |\Lambda+\rho|^2} d_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_1'^+} (1 + e(-\alpha)), \qquad (2)$$

with some weight $\lambda \in H^*$ such that $d_\lambda \neq 0$ but $\lambda \nleq_{\prime} \Lambda$.

Suppose that $1 \leq j \leq n$, $\alpha_j \in \Delta_1$ and $|\alpha_j|^2 = 0$. We show that this situation arises when we consider equality (1) with respect to the base Π_j. Corollary 1 tells us that equality (1) can be rewritten as follows:

$$e(\rho_j - \alpha_j) \prod_{\alpha \in \Delta_{j0}^+} (1 - e(-\alpha)) \text{ch} \, V$$

$$= \sum_{\lambda \leq \Lambda, \, |\lambda + \rho|^2 = |\Lambda + \rho|^2} c_j e(\lambda + \rho) e(-\alpha_j) \prod_{\alpha \in \Delta_{j1}^+} (1 + e(-\alpha)) \qquad (3)$$

Let us find the highest weight Λ^j of the G-module V with respect to the base Π_j.

Lemma 1. *With respect to the base Π_j, the highest root of the G-module V is*

$$\Lambda^j = \begin{cases} \Lambda - \alpha_j & \text{if } (\Lambda, \alpha_j) \neq 0 \\ \Lambda & \text{if } (\Lambda, \alpha_j) = 0 \end{cases}.$$

Proof. $f_j v_\Lambda = 0$ if and only if $(\Lambda, \alpha_j) = 0$ and hence the result follows since $f_j^2 v_\Lambda = 0$, the simple root α_j being isotropic. $\qquad\square$

As a result:

Corollary 2. $|\Lambda + \rho|^2 = |\Lambda^j + \rho_j|^2$.

Proof. This is a direct consequence of Lemma 1 when $(\Lambda, \alpha_j) \neq 0$ and it follows when $(\Lambda, \alpha_j) = 0$ since $(\rho, \alpha_j) = 0$. $\qquad\square$

Simplifying equality (3), from Lemma 1 and Corollary 2, we get:

$$e(\rho_j) \prod_{\alpha \in \Delta_{j0}^+} (1 - e(-\alpha)) \text{ch} \, V$$

$$= \sum_{\lambda \leq \Lambda, \, |\lambda + \rho|^2 = |\Lambda^j + \rho_j|^2} c_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_{j1}^+} (1 + e(-\alpha)) \qquad (4)$$

We will say that equality (4) is equality (1) with respect to the base Π_j. Note that this is not the same as substituting the base Π_j for the base Π in equality (1): indeed, $\lambda + \rho = (\lambda - \alpha_j) + \rho_j$ and $c_\lambda \neq 0$ implies $\lambda - \alpha_j \leq \Lambda - \alpha_j \leq \Lambda^j$ but $\lambda - \alpha_j \leq_j \Lambda^j$ may not hold. More generally we will use the phrase "equality (1) with respect to the base Π'" for bases Π' obtainable from the base Π by successive applications of odd reflections. Writing

$$\lambda = \Lambda - \sum_i k_i \alpha_i \in H^*,$$

$$\lambda + \rho =$$
$$\begin{cases} \Lambda^j + \rho_j - \sum_{i \neq j} k_i \beta_i^j - (\sum_{i,(\alpha_i,\alpha_j) \neq 0} k_i - k_j)\beta_j^j & \text{if } (\alpha_j, \Lambda) \neq 0 \\ \Lambda^j + \rho_j - \sum_{i \neq j} k_i \beta_i^j - (\sum_{i,(\alpha_i,\alpha_j) \neq 0} k_i - k_j - 1)\beta_j^j & \text{otherwise} \end{cases}$$
$$(5)$$

In other words, with respect to the base Π_j, the coefficient of the base element β_j^j may be positive. The proof of the theorems is based on the observation that with respect to the base Π_j, for each weight λ appearing on the right hand side of equality (4), i.e. with $c_\lambda \neq 0$, at most one summand, namely that corresponding to the simple root β_j^j, may be positive.

Lemma 2. *Let*
$$\lambda = \Lambda - \sum_{i \neq j} k_i \beta_i + k_j \beta_j \in H^*$$

be a weight for which $d_\lambda \neq 0$ in equality (2) and such that for all i, $k_i \in \mathbb{Z}_+$ and $k_j > 0$ for some index $1 \leq j \leq n$. Set
$$\mu = \Lambda - \sum_{i \neq j} k_i \beta_i.$$

Then
$$|\beta_j| = 0$$
and for all integers $s \geq 1$,
$$d_{\mu+s\beta_j} = (-1)^{s-1} d_{\mu+\beta_j}.$$
In particular, $d_{\mu+\beta_j} \neq 0$,
$$(\lambda, \beta_j) = 0 = (\mu, \beta_j)$$
and for all integers $s \geq 0$,
$$|\mu + s\beta_j + \rho|^2 = |\Lambda + \rho|^2.$$

Moreover if the term $e(\mu + \rho)$ does not appear on the left hand side of equality (2), then $d_\mu \neq 0$ and in particular, $d_{\mu+s\beta_j} = (-1)^s d_\mu$ for all $s \in \mathbb{N}$.

Proof. Let the weight $\lambda = \Lambda - \sum_{i \neq j} k_i \beta_i + k_j \beta_j$ be such that $d_\lambda \neq 0$, $k_i \geq 0$ for all $1 \leq i \leq N$ and $k_j > 0$. Since no terms on the L.H.S. of equality (2) equals $e(\lambda)$,
$$d_\lambda + \sum_\gamma d_{\lambda+\gamma} = 0,$$

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where the sum is taken over sums γ of distinct odd roots. We prove the result by induction on $k = \sum_{i \neq j} k_i$. Let k be minimal. Then, $d_{\lambda + \beta_j} \neq 0$ and $d_{\lambda + \gamma} = 0$ for all $\gamma \neq \beta_j$. So

$$d_\lambda + d_{\lambda + \beta_j} = 0.$$

Next replacing λ by $\lambda + \beta_j$ we get

$$d_{\lambda + \beta_j} + d_{\lambda + 2\beta_j} = 0.$$

More generally, for all integers $s \geq 0$,

$$d_{\lambda + s\beta_j} = (-1)^s d_\lambda.$$

In particular

$$(\lambda + \rho + s\beta_j, \lambda + \rho + s\beta_j) = (\lambda + \rho, \lambda + \rho)$$

for all integers $s \geq 0$ and so

$$(\beta_j, \beta_j) = 0$$

and

$$(\lambda, \beta_j) = 0$$

since $(\rho, \beta_j) = 0$.

Suppose that $k_j > 1$. If $d_{\lambda - \beta_j} = 0$, then as $\beta_j \in \Delta_1^+$, there is a term on the right hand side of equality (2) equal to $e(\lambda - \beta_j)$. However this is false if $k_j - 1 > 0$. In this case, the above arguments imply that $d_{\lambda - \beta_j} = -d_\lambda$. Continuing in this manner we can deduce that if

$$\mu = \Lambda - \sum_{i \neq j} k_i \beta_i,$$

then for all integers $s \geq 1$,

$$d_{\mu + s\beta_j} = (-1)^{s-1} d_{\mu + \beta_j}.$$

Moreover if the term $e(\mu + \rho)$ does not appear on the left hand side of equality (2), then $d_\mu \neq 0$ and in particular, $d_{\mu + s\beta_j} = (-1)^s d_\mu$ for all $s \in \mathbb{N}$.

Now

$$d_{\mu + \beta_j} (\sum_{s \geq 0} (-1)^s e(s\beta_j)) \prod_{\alpha \in \Delta_1'} (1 + e(-\alpha))$$

$$= d_{\mu + \beta_j} (1 + e(b_j))^{-1} \prod_{\alpha \in \Delta_1'} (1 + e(-\alpha))$$

$$= d_{\mu + \beta_j} e(-\beta_j) \prod_{\alpha \in \Delta_1' - \{\beta_j\}} (1 + e(-\alpha))$$

Therefore the result follows by induction on k. $\qquad \square$

As we have seen previously, we can apply Lemma 2 to equality (4) with respect to the base Π_j. Since $\beta_j^j = -\alpha_j$, this gives:

Corollary 3. *Suppose that $1 \leq j \leq r$ with $|\alpha_j|^2 = 0$. Let $\lambda \in H^*$ be such that $c_\lambda \neq 0$ and*

$$\lambda + \rho = \Lambda^j + \rho_j - \sum_{i=1}^{n} l_i \beta_i^j.$$

Suppose that $l_j < 0$. Set $\mu = \lambda - (l_j + 1)\alpha_j$. Then for all integers $s \geq 1$,

$$c_{\mu - s\alpha_j} = (-1)^s c_\mu$$

and

$$(\mu, \alpha_j) = 0.$$

In particular, $c_\lambda = (-1)^{(l_j+1)} c_\mu$ and $c_\mu \neq 0$. In particular, $\mu \leq \Lambda$ and $\mu \leq_j L_j$. Furthermore, the term $e(\mu + \rho)$ appears in the left hand side of equality (1) or $c_{\mu + \alpha_j} \neq -c_\mu$.

Proof. We only prove the last inequality and statement. $\mu + \rho = (\mu + \beta_j^j) + \rho_j$ and $\mu + \beta_j^j \leq_j \Lambda^j$. In particular, $\mu \leq_j \Lambda^j$. From Lemma 2 we know that either the term $e(\mu + \alpha_j + \rho)$ appears in the left hand side of equality (4) or $c_{\mu + \alpha_j} = -c_\mu$. Since $\rho_j = \rho + \alpha_j$, the former case is equivalent to the term $e(\mu + \rho)$ appearing of the left hand side of equality (1). $\square$

Since at each base change by an odd permutation, precisely one base vector changes sign, the following is a consequence of the fact that $e(\mu + \rho)$ appears in the left hand side of equality (1).

Lemma 3. *Let $\lambda, \mu \in H^*$ and the simple root α_j be as described in Corollary 4. Suppose that the term $e(\mu + \rho)$ appears in the left hand side of equality (1). Let β_k^j be an isotropic simple root of the base Π_j and $\Pi_{j,k} = s_k(\Pi_j)$. Then, writing $\mu + \rho$ as a linear combination of the basis vectors in $\Pi_{j,k}$, the coefficient of $s_k(\beta_k^j)$ is at most 1.*

Note that there are two types of isotropic roots. Let $\alpha \in \Delta_1^+$ be such that $|\alpha|^2 = 0$. Then, either there is a base Π' (as described in section 1, obtainable from Π after successive applications of odd reflections) such that $\alpha \in \Pi'$ or for any such base Π', $\alpha \notin \Pi'$. We will say that the root $\pm \alpha$ is either of the first or second kind. If there are isotropic roots, then all of them are of the first kind if and only if the Lie superalgebra G is of type $A(m, n)$ or $C(n)$.

As a consequence of Lemmas 2, 3 and Corollary 3, applying successive odd reflections, the following can be deduced:

Corollary 4. *Let $\lambda \in H^*$ be such that $c_\lambda \neq 0$. Then there is a weight $\mu \in H^*$ for which the height $\Lambda - \mu$ is such that $c_\mu \neq 0$ such that*
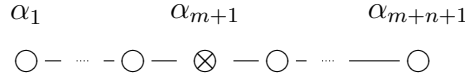
$$\lambda + \rho = \mu + \rho - \sum_{\alpha \in \Delta_1^+} k_\alpha \alpha,$$
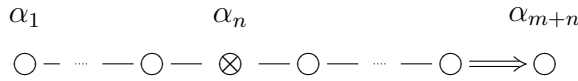
where $k_\alpha \in \mathbb{Z}_+$

1. *$k_\alpha \neq 0$ implies that $\alpha \in \Delta_1^+$, $(\alpha, \mu) = 0 = (\alpha, \beta)$ if $k_\beta \neq 0$; in particular the root α is isotropic of the first kind; and*

2. *either $\mu + \rho \leq_, \Lambda' + \rho'$ for all bases Π' and $\mu + \rho + \alpha$ $\Lambda' + \rho'$, for bases Π' with respect to which the root α is simple; or the term $e(\mu + \rho)$ appears in the left hand side of equality (1). If the latter holds then there is a base $\Pi' = \{\beta_1, \cdots, \beta_N\}$ of the above described type and an isotropic simple root $\beta_k \in \Pi'$ such that $\mu + \rho = \Lambda' + \rho' + \beta_k - \sum_{i \neq k} l_i \beta_i$, where for all $i \neq k$, $l_i \in \mathbb{Z}_+$. Furthermore if β_k for $k \neq i$ is an isotropic simple root, then writing $\mu + \rho$ as a linear combination of simple roots in $s_{\beta_k}(\Pi')$, the coefficient corresponding to $s_{\beta_k}(\beta_k)$ is at most 1.*

For reasons of simplicity which will become obvious later, from now on we assume that the base Π contains one isotropic simple root and all other simple roots are even. This is always possible (see [K1]):
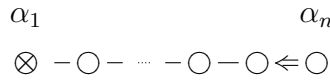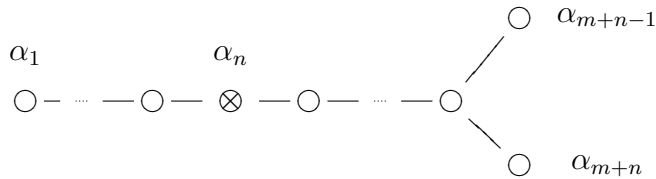
$A(m, n)$:

$$\overset{\alpha_1}{\bigcirc} - \cdots - \bigcirc - \overset{\alpha_{m+1}}{\otimes} - \bigcirc - \cdots - \overset{\alpha_{m+n+1}}{\bigcirc}$$

$B(m, n)$ $(m \neq 0)$:

$$\overset{\alpha_1}{\bigcirc} - \cdots - \bigcirc - \overset{\alpha_n}{\otimes} - \bigcirc - \cdots - \overset{\alpha_{m+n}}{\bigcirc} \Longrightarrow \bigcirc$$

$C(n)$:

$$\overset{\alpha_1}{\otimes} - \bigcirc - \cdots - \bigcirc - \overset{\alpha_n}{\bigcirc} \Leftarrow \bigcirc$$

$D(m, n)$:

$$\overset{\alpha_1}{\bigcirc} - \cdots - \bigcirc - \overset{\alpha_n}{\otimes} - \bigcirc - \cdots - \bigcirc \overset{\displaystyle \nearrow \bigcirc\ \alpha_{m+n-1}}{\underset{\displaystyle \searrow \bigcirc\ \alpha_{m+n}}{}}$$

$D(2,1;a)$:

$$\alpha_1 \quad \alpha_2 \quad \alpha_3$$

$$\bigcirc - \otimes \underset{a}{-} \bigcirc$$

$F(4)$:

$$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4$$

$$\otimes - \bigcirc \Leftarrow \bigcirc - \bigcirc$$

$G(3)$:

$$\alpha_1 \quad \alpha_2 \quad \alpha_3$$

$$\otimes - \bigcirc \Leftarrow \bigcirc$$

We set $\alpha_l \in \Pi$ to be the unique simple isotropic root. Define $\tilde{W}$ to be the following subgroup of the Weyl group W:

$$\tilde{W} = \langle r_i : |\alpha_i|^2 \neq 0 \rangle.$$

Without loss of generality, we assume that when $\tilde{W} \neq W$, $|\theta|^2 < 0$.

We make the following observations which can be easily checked:

Lemma 4. *If G is a Lie algebra or a Lie superalgebra of type $A(m,n)$ or $C(n)$, then $\tilde{W} = W$. Otherwise $W = \langle \tilde{W}, r_\theta \rangle$. When the Lie superalgebra G has defect 1, the root θ is the unique even positive root of negative norm. When the Lie superalgebra G is of type $B(m,n)$ or $D(m,n)$ with $m > 0$, if $\alpha, \beta \in \Delta_0^+$ such that $r_\alpha \in \tilde{W}$ and $r_\beta \notin \tilde{W}$, then $|\alpha|^2 \neq |\beta|^2$.*

We next study the action of the group $\tilde{W}$ on the weights $\lambda \in H^*$ such that $c_\lambda \neq 0$.

Lemma 5. *For all weights $\lambda \in H^*$ such that $c_\lambda \neq 0$,*

$$w(\lambda + \rho) \leq \Lambda + \rho$$

for all $w \in \tilde{W}$.

Proof. Since $\tilde{W}$ is the Weyl group of the Lie algebra with simple root system $\{\alpha_i \in \Pi\} \cup \{2\alpha_i : \alpha_i \in \Pi\}$, it suffices to prove the result for $w = r_{\alpha_i}$, where $|\alpha_i|^2 \neq 0$ (see [K4], 3.12). Without loss of generality, we may assume that $|\alpha_i|^2 > 0$. Applying the reflection r_i to both sides of equality (1) we get:

$$- e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{ch} V$$

$$= \sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(r_i(\lambda + \rho)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)) \qquad (6)$$

If $\lambda \in H^*$ is such that $c_\lambda \neq 0$, then

$$\lambda + \rho = \Lambda + \rho - \sum_{i=1}^{n} k_i \alpha_i,$$

where $k_i \in \mathbb{Z}_+$ for all $1 \leq i \leq N$. Set

$$\tau = r_i(\lambda + \rho) - \rho.$$

Then,

$$\tau + \rho = \Lambda + \rho - \sum_{i=1}^{n} a_i \alpha_i,$$

where the coefficients $a_i \in \mathbb{Z}$ are integers but are not necessarily non-negative. For all $j \neq i$, $a_j = k_j \geq 0$. If $a_i < 0$, applying Lemma 2 leads to a contradiction since $|\alpha_i|^2 \neq 0$. Therefore for all $1 \leq j \leq n$, $a_j \geq 0$ and so

$$\tau + \rho \leq \Lambda + \rho.$$

Given the definition of the group $\tilde{W}$, the result now follows.

$\square$

A consequence of Lemma 5 is the following:

Lemma 6. *Let $\lambda \in H^*$ be a weight such that $c_\lambda \neq 0$. Then for all $w \in \tilde{W}$, $c_{w(\lambda+\rho)-\rho} = \epsilon(w)c_\lambda$.*

Proof. Suppose first that $w = r_i$ for some simple root $\alpha_i \in \Pi$ such that $|\alpha_i|^2 \neq 0$. By Lemma 5, for any $\lambda \in H^*$ such that $c_\lambda \neq 0$, there is a weight $\mu \in H^*$ such that $\lambda + \rho = r_i(\mu + \rho)$, $\mu \leq \Lambda$ and $|\mu + \rho| = |\Lambda + \rho|$. Hence, we may write

$$\sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} d_\lambda e(\lambda + \rho) = \sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(r_i(\lambda + \rho)).$$

On the other hand, applying r_i to to equality (1), we can deduce that

$$\sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} (d_\lambda + c_\lambda)e(\lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)) = 0. \qquad (i)$$

Suppose there exists a weight $\lambda \in T$ for which $d_\lambda + c_\lambda \neq 0$. We may take λ to be such that the height $\Lambda - \lambda$ is minimal with this property. Then, equation (i) gives

$$d_\lambda + c_\lambda + \sum_{\mu} d_{\lambda+\mu} + c_{\lambda+\mu} = 0,$$

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the sum being taken over all distinct sums μ of positive odd roots. This proves the result for $w = r_i$.

The result for arbitrary $w \in \tilde{W}$ follows by induction on the number of generating reflections of the above type needed in any given expression for w. □

Lemmas 5 and 6 and Corollary 5 immediately imply the next result.

Corollary 5. *Let G be a Lie superalgebra of type $A(m,n)$ or $C(n)$. Then for all weights $\lambda \in H^*$ such that $c_\lambda \neq 0$,*

$$w(\lambda + \rho) \leq \Lambda + \rho$$

and

$$c_{w(\lambda+\rho)-\rho} = \epsilon(w)c_\lambda$$

for all $w \in W$.

We first concentrate on the Lie superalgebras G for which $\tilde{W} = W$. Let us consider the vector space V as a G_0-module. The G_0-module is the direct sum of finitely many finite dimensional irreducible G_0-modules $V^1, \cdots, V^m$ of highest weight Γ^i.

If $\lambda + \rho_0$ is a weight such that the term $e(\lambda + \rho_0)$ appears in the left hand side of

$$e(\rho_0) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \mathrm{ch}\, V$$

$$= \sum_{\lambda \leq \Lambda,\, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(\lambda + \rho_0) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)), \qquad (8)$$

then it appears in the expression

$$e(\rho_0)\Pi_{\alpha \in \Delta_0^+}(1 - e(-\alpha)) \sum_{i=1}^m \mathrm{ch}\, V^i = \sum_{i=1}^m \sum_{w \in W} (-1)^w e(w(\Gamma^i + \rho_0))$$

Lemma 7. *For all $1 \leq i \leq m$, if $\Lambda \neq \Gamma^i$, then $\Lambda - \Gamma^i$ is the sum of distinct odd positive roots.*

Proof. Since

$$V = U(N_0^-)U(N_1^-)v_\Lambda,$$
$$v = v_1 + v_2,$$

where

$$v_1 \in U(N_1^-)v_\Lambda \quad \text{and} \quad v_2 \in U(N_0^-)N_0^- U(N_1^-)v_\Lambda.$$

Then $v_1 \neq 0$ as v is a highest vector. We can therefore deduce that $\Lambda - \lambda$ is a sum of odd positive roots. Since $v_1 \in U(N_1^-)v_\Lambda$, there are no repeats in this sum. □

Lemma 8. *Let μ be a weight satisfying the conditions of Corollary 4. Then, for all $w \in \tilde{W}$, $w(\mu + \rho) \leq_\prime \Lambda + \rho$ for all bases Π'.*

Proof. We first prove the result for $w = 1$. Suppose first that $\mu + \rho \leq_\prime \Lambda' + \rho'$. From Corollary 1 and Lemma 1 it follows that $\Lambda' + \rho' \leq_\prime \Lambda + \rho$. Hence the result follows in this case. Remember from Corollary 1 that the set Δ_0^+ remains the set of positive even roots with respect to the base Π', and so $\rho'_0 = \rho_0$. Otherwise, by definition of μ and from above discussion, there is an irreducible G_0-submodule $\check{V}$ of V of highest weight Γ (with respect to the base Π') such that for some $w \in W$,

$$\mu + \rho = w(\Gamma + \rho_0) - \rho'_1.$$

Hence

$$\mu + \rho = \Gamma + \rho_0 - \beta - \rho'_1,$$

where $0 \leq_\prime \beta$. Equivalently,

$$\mu = \Gamma - \beta + \rho' - \rho.$$

Now, $\Gamma \leq_\prime \Lambda'$. Corollary 1 tells us that $\rho' - \rho$ is a sum of positive (with respect to the base Π') isotropic roots and by induction of the number of odd reflections needed to arrive at the base Π' from the base Π, Lemma 1 that this is not only also the case of $\Lambda' - \Lambda$ but that $\Lambda' - \Lambda \leq \rho' - \rho$. Therefore, $\mu \leq_\prime \Lambda$.

We next consider the more general case $\tau + \rho = w(\mu + \rho)$, where $w \in \tilde{W}$. For $w \in \tilde{W}$, Lemma 6 tells us that $c_{w(\mu+\rho)-\rho} \neq 0$. Moreover for any isotropic positive root of the first kind α, $w(\alpha) > 0$. Hence it follows from Corollary 2 that the weight τ satisfies the same conditions (stated in Corollary 4) of the root μ. Hence the above arguments applied to the weight τ instead of the weight μ imply that $\tau \leq_\prime \Lambda$. $\qquad\square$

For each base Π', set

$$W_{\Pi'}$$

to be the subgroup of the group W generated by reflections r_α such that $\alpha \in \Pi'$ or its support (with respect to the base Π') contains two distinct isotropic roots and set

$$\tilde{W}_{\Pi'} := W_{\Pi'} \cap \tilde{W}.$$

So $\tilde{W} = \tilde{W}_\Pi$.

Lemma 9. *Let $\mu \in H^*$ be a weight satisfying the conditions expressed in Corollary 4 such that $\frac{(\mu+\rho,\alpha_i)}{(\alpha_i,\alpha_i)} > 0$ for $\alpha_i \in \Pi$ such that*

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$|\alpha_i|^2 \neq 0$. *Then, for all $w \in \tilde{W}_{\Pi'}$,*

$$w(\mu) \leq_, w(\Lambda)$$

for all bases Π'. In particular when G is either a Lie algebra or of type $A(m, n)$ or $C(n)$, for all $w \in W$,

$$w(\tau) \leq_, w(\Lambda)$$

for all bases Π'.

Proof. We first prove the result for simple reflections and start by considering the base Π. Set $r \in \tilde{W}$.

Case 1: $r = r_{\alpha_i}$, where $|\alpha_i|^2 \neq 0$

Case (a): the term $e(r(\mu + \rho))$ appears in the left hand side of equality (1).

Since the reflection r multiplies equality (1) by -1, and hence the term $e(\mu + \rho)$ appears in the left hand side of equality (1) Then,

$$\mu + \rho = w(\Gamma + \rho_0) - \rho_1$$

for some $w \in W$ and some highest weight Γ of a G_0-submodule of V. Hence,

$$\mu + \rho_0 = w(\Gamma + \rho_0).$$

Since $\frac{(\mu+\rho,\alpha_i)}{(\alpha_i,\alpha_i)} \geq 0$ for all non-isotropic root α_i; this also holds for $\mu + \rho_0$. Therefore, $\frac{(w(\Gamma+\rho_0),\alpha)}{(\alpha,\alpha)} \geq 0$ for all positive roots $\alpha \in \langle \alpha_i : |\alpha_i|^2 \neq 0 \rangle$. Note that as α is an even root it is positive with respect to any base Π'. As a consequence if $\tilde{W} = W$, then $w = 1$. Otherwise by assumption the root θ has negative norm and hence as $W = W_- \times W_+$, either $w = 1$ or $w = r_\theta w_1$, where w_1 is a product of reflections r_{α_i}, where $|\alpha_i|^2 < 0$ and of the reflection r_θ. Now from the above, $w_1 = r_{\alpha_i} w_2$ for some root α_i of negative norm for which $(\theta, \alpha_i) > 0$. There are no such roots and so $w = 1$ or $w = r_\theta$. In all cases

$$\mu = \Gamma - t\theta,$$

where t is a non-negative integer. Then

$$\mu = \Lambda' - t\theta - \gamma,$$

where γ is sum of distinct odd positive (with respect to the base Π') roots. Now $\Lambda' = \Lambda$ or $\Lambda + \rho - \rho'$. Since β_i is a sum of positive (with respect to the base Π) roots all of the same norm, $r(\rho) = \rho - s\beta_i$, where $s \geq 1$. As $\beta_i \in \Pi'$, $r(\rho') = \rho' - \beta_i$. As a consequence,

$$r(\mu) \leq' r(\Lambda)$$

since the root θ being non-isotropic remains positive with respect to all bases Π'.

Case (b): The term $e(r(\mu + \rho))$ does not appear on the left hand side of equality (1).

We apply the reflection r to both sides of equality (1). Since the term $e(r(\mu+\rho))$ does not appear in the left hand side of equality (1), if

$$r(\mu + \rho) = r(\Lambda + \rho) + s\alpha_i - \sum_{j \neq i} k_j \alpha_j$$

with $s \in \mathbb{N}$ and $k_j \in \mathbb{Z}_+$, then the arguments of Lemma 2 imply that $|\alpha_i|^2 = 0$, contradicting assumptions.

This proves the result for simple reflections and the base Π.

Next consider a set $\Pi' = \{\beta_1, \cdots, \beta_N\}$.

Case 2: $r = r_{\beta_i}$, where $|\beta_i|^2 \neq 0$

$$r_{\beta_i}(\mu) \not\leq_{\prime} r_{\beta_i}(\Lambda). \qquad (i)$$

Since $|\beta_i|^2 \neq 0$, $\beta_i \in \Delta_0^+$ and in particular is positive with respect to all bases Π'. Hence β_i is a positive sum of the simple roots in Π_0, the base of the positive root system Δ_0^+. Since β_i must be even (by definition of the base Π), this forces β_i to be a simple root in the base Π. Therefore, by Case 1, $r_{\beta_i}(\mu) \leq r_{\beta_i}(\mu)$. As a consequence, assumption (i) implies that

$$r(\mu) = r(\Lambda) - \sum_{j \neq i} a_j \beta_j + s\beta_i,$$

where $s \in \mathbb{N}$ and $a_j \in \mathbb{Z}_+$. However by Lemma 8,

$$\mu = \Lambda - \sum_{j=1}^{n} k_j \beta_j,$$

where $k_j \in \mathbb{Z}_+$. Hence applying the reflection r to both sides of this equality contradicts the fact that $s > 0$.

Therefore, the result holds for all reflections r with respect to any base Π'.

Case 3: Suppose $r = r_\alpha \in \tilde{W}$ is such that the support of the root α contains two distinct simple (with respect to the base Π') isotropic roots.

Let the root $\alpha \in \Delta_0^+$ be such that its support with respect to the base Π' contains two distinct isotropic roots. Then,

$$\alpha = \beta_{j_1} + \cdots + \beta_{j_s},$$

where $|\beta_{j_1}|^2 = 0 = |\beta_{j_s}|^2$; for $1 < i, k < s$, $|\beta_{j_i}|^2 = |\beta_{j_k}|^2 \neq 0$ and $|\alpha|^2 |\beta_{j_k}|^2 < 0$. Since the reflection r_α is a product of reflections r_{α_i}, it follows that the bases Π'_{j_1} and Π'_{j_s} are obtainable from the base Π by successive applications of odd reflections in such a way that at each stage the isotropic simple root chosen is positive with respect to the base Π. Moreover with respect to these bases, the height of the root α is strictly smaller to its height with respect to the base Π'. Therefore by Case 1 and induction,

$$r_\alpha(\tau) \leq_{\prime,j_1} r_\alpha(\Lambda)$$

and

$$r_\alpha(\tau) \leq_{\prime,j_s} r_\alpha(\Lambda).$$

This forces

$$r_\alpha(\tau) \leq_\prime r_\alpha(\Lambda).$$

Next suppose that the result holds for all element $w \in \tilde{W}$ of length less than l (as a product of reflections of the above types) and consider the element $r_\alpha w$, where $\alpha \in \Pi'$ or is as in Case 2. Then $\frac{(w(\mu),\alpha)}{(\alpha,\alpha)} < 0$ implies that the length of $r_\alpha w$ is strictly less than that of w (see 3.11 in [K4]) and hence the result holds by induction. Otherwise we can apply the above arguments to the weight $w(\mu+\rho)$ (when this is equal to $w'(\Gamma + \rho_0) - \rho_1$, the element $w\prime$ is such that $w'(\Gamma + \rho_0) = \Gamma + \rho_0 - \gamma$, where $r_\alpha(\gamma) > 0$) and the result follows for $r_\alpha w$. Hence the result holds for all $w \in \tilde{W}$. $\qquad \square$

We are now ready to prove the main result about weight μ for the Lie superalgebras G for which $W = \tilde{W}$. Note that if μ satisfies the conditions of Corollary 4, then so do $w(\mu + \rho) - \rho$ for all $w \in \tilde{W}$.

Corollary 6. *Let G be a Lie superalgebra of type $A(m,n)$ or $C(n)$. Let $\mu \in H^*$ be a weight as described in Corollary 4 such that for all simple non-isotropic roots $\alpha_i \in \Pi$, $\frac{(\mu+\rho,\alpha_i)}{(\alpha_i,\alpha_i)} \geq 0$. Then,*

$$\mu = \Lambda.$$

Proof. Since $W = \tilde{W}$, by Lemma 9, $w(\mu) \leq_\prime w(\Lambda)$ for all $w \in W$ and all bases Π'. We show that $\mu = \Lambda$. To do this we choose a base $\Pi' = \{\beta_1, \cdots, \beta_N\}$ such that for all $1 \leq i \leq r$, $|\beta_i|^2 = 0$ and for $i > r$, $|\beta_i|^2 > 0$.

Case 1. $r = 2p + 1$.

Set $\mu = \Lambda - \sum_{i=1}^N k_i \beta_i$. Since $\mu \leq_{\prime,1} \Lambda$, $k_2 \geq k_1$. Then, consider $r = r_{\beta_2 + \cdots + \beta_N}$. Since $r(\mu) \leq_\prime \Lambda$, considering the coefficient

of β_N in $r(\mu - \Lambda)$, we get $k_1 \geq k_2$. Therefore, setting $\gamma_1 = \beta_1 + \beta_2$ and considering the Lie superalgebra with base $\gamma_1, \beta_3, \cdots, \beta_N$, we get

$$k_1 = k_2 = k_3.$$

Therefore by induction,

$$k_1 = \cdots = k_N.$$

Therefore

$$\mu = \Lambda - k_1(\beta_1 + \cdots + \beta_N).$$

Applying successive odd reflections to the base Π', we get a base

$$\tilde{\Pi} = s_{\beta_r + \cdots + \beta_N} \cdots s_{\beta_r + \beta_{r+1}} s_{\beta_r} \cdots s_{\beta_3} s_{\beta_1} \Pi'$$

and

$$\mu = \Lambda - k_1 \gamma_i$$

for some $\gamma_i \in \tilde{\Pi}$. Then, applying Lemma 8 to the base $\tilde{\Pi}$ and $s_{\gamma_i}(\tilde{\Pi})$, we have $\mu \leq_{\gamma_i} \Lambda$. Therefore, $k_1 = 0$ and so

$$\mu = \Lambda.$$

Case 2. $r = 2p$

Consider the reflection $r = r_{\beta_1 + \cdots + \beta_N}$. Then,

$$r(\mu - \Lambda) = k_1 \beta_N + k_N \beta_1 - \sum_{j \neq 1, N} l_j \beta_j,$$

where $l_j \geq 0$ for all $1 < j < N$. Therefore,

$$k_1 = 0 = k_N.$$

Hence the support of the weight $\mu - \Lambda$ generates a Lie superalgebra with base $\beta_2, \cdots, \beta_{N-1}$ of the type of Π' but with $r - 1$ isotropic roots. Therefore applying Case 1 we once again get

$$\mu = \Lambda.$$

$\square$

We next consider the cases when $W \neq \tilde{W}$. Note that the exceptional Lie superalgebras do not have isotropic roots of the second kind. Remember that we have assumed $|\theta|^2 < 0$. We first show a technical property which will be useful both now and later on. For the proof the assumption that the base Π contains a unique odd root and which moreover is isotropic is needed. Hence our taking Π to be such a base.

Lemma 10. *Let G be a Lie superalgebra of type $B(m,n)$, $m > 0$, $D(m,n)$ or exceptional. If $\Gamma_i \neq \Lambda$, then $|\Gamma_i + \rho|^2 \neq |\Lambda + \rho|^2$*

Proof. Suppose that for some $\Gamma = \Gamma_i \neq \Lambda$,

$$|\Gamma_i + \rho|^2 = |\Lambda + \rho|^2 \qquad (i).$$

Let v be a highest weight vector of the G_0-submodule of V of highest weight Γ. Then

$$e_{\alpha_i} v = 0 \quad \forall 1 \leq i \leq N \text{ such that } |\alpha_i|^2 \neq 0. \qquad (ii)$$

Hence, applying the Casimir operator on the vector v we can deduce that

$$\sum_{\mu \in \Delta_1^+} f_\mu e_\mu v = 0, \qquad (iii)$$

where $(e_\mu, f_\mu) = 1$. By assumption there is a unique simple isotropic root in Π', say β_j. Therefore $\mu \in \Delta_1$ implies that

$$\mu = \alpha_j + \mu_+ + \mu_-, \qquad (iv)$$

where $\mu_\pm \in \Delta_0^+$ or $\mu_\pm = 0$, $|\mu_+|^2 > 0$ and $|\mu_-|^2 < 0$; or

$$\mu + \alpha_j \in \Delta_0^+. \qquad (v)$$

Note that if the odd root μ is of type (iv) then it is not of type (v) and conversely. When $\mu_\pm \neq 0$, let $e_{\mu_\pm} \in G_{\mu_\pm}$ be such that

$$e_{\alpha_j + \mu_\pm} = [e_{\alpha_j}, e_{\mu_\pm}].$$

When $\mu + \alpha_j \in \Delta_0^+$, set $f_{\mu+\alpha_j} \in G_{\mu+\alpha_j}$ to be such that

$$f_\mu = [f_{\mu+\alpha_j}, e_{\alpha_j}].$$

Next, apply the operator e_{α_j} to both sides of equality (iii). If $\mu + \alpha_j \in \Delta$, then

$$[e_{\alpha_j}, f_\mu] = 0,$$
$$[e_{\alpha_j}, e_\mu] \neq 0,$$

and

$$([e_{\alpha_j}, e_\mu], f_{\mu+\alpha_j}) = (e_\mu, f_\mu) = 1$$

If $\mu + \alpha_j \notin \Delta$, then

$$[e_{\alpha_j}, e_\mu] = 0,$$
$$[e_{\alpha_j}, f_\mu] = 0,$$

unless $\mu_- = 0$ or $\mu_+ = 0$, and

$$(e_{\mu_\pm}, [e_{\alpha_j}, f_{\alpha_j+\mu_\pm}]) = ([e_{\alpha_j}, e_{\mu_\pm}], f_{\alpha_j+\mu_\pm}) = 1.$$

Therefore, using condition (ii), we get

$$(\Gamma, \alpha_j)e_j v + \sum_{\gamma \in \Delta_+} f_\gamma e_\gamma e_j v = 0.$$

Writing Ω for the Casimir operator, this is equivalent to

$$(\Gamma, \alpha_j)e_j v + \Omega(e_j v) - |\Gamma + \alpha_j + \rho|^2 e_j v = 0.$$

Since the G-module V is highest weight of highest weight Λ,

$$\Omega(e_j v) = |\Lambda + \rho|^2 e_j v$$

(see [K2], [K4] or [Ray]). If $e_j v = 0$ then v is a highest weight vector of the G-module V and so $\Gamma = \Lambda$. Hence as $\Gamma \neq \Lambda$,

$$(\Gamma, \alpha_j) - |\Gamma + \alpha_j + \rho|^2 = -|\Lambda + \rho|^2.$$

This together with assumption (i) forces

$$(\Gamma, \alpha_j) = 0$$

since $|\alpha_j|^2 = 0$.

Since $(\Gamma, \gamma)(\gamma, \gamma) \geq 0$ for all non-isotropic positive roots γ, this implies that $(\Gamma, \alpha_i) = 0$ for all $\alpha_i \in \Pi$ of positive norm. Therefore by Lemma 7 and considering the action of G on the vector v, it follows that

$$\Lambda = \gamma + \Gamma,$$

where γ is an isotropic positive root such that

$$S(\gamma) = \{\alpha_j, \alpha_i : \alpha_i \in \Pi, |\alpha_i|^2 < 0\}.$$

Hence

$$(\Lambda, \theta) = 0$$

and

$$(\Lambda, \alpha_j) > 0.$$

Considering the support of the root θ, it follows that there is a simple root α_i such that $|\alpha_i|^2 > 0$ but $(\Lambda, \alpha_i) < 0$, contradicting the integrability of the module V. This proves that $\Gamma = \Lambda$. $\square$

We are now ready to prove our main result about weights μ satisfying Corollary 4 in the case when $W \neq \tilde{W}$. We do this in two parts.

Lemma 11. *Let G be a Lie superalgebra of type $B(m,n)$, $m > 0$, $D(m,n)$, or exceptional and $\mu \in H^*$ a weight as described in Corollary 4 such that for all simple non-isotropic roots $\alpha_i \in \Pi$, $\frac{(\mu+\rho, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$. Then, either*

1. $(\Lambda + \rho, \alpha) = 0$ *for some isotropic root α of the first kind; in which case $\mu = \Lambda$ and $r_\theta(\Lambda + \rho) \not\leq \Lambda + \rho$; or*

2. $(\Lambda + \rho, \alpha) \neq 0$ *for all isotropic root α of the first kind; in this case $w(\mu + \rho) \leq \Lambda + \rho$ and $w(\Lambda + \rho) \leq \Lambda + \rho$ for all $w \in W$.*

Proof. Let $\alpha_l \in \Pi$ be the unique isotropic simple root in the base Π and Π_0 be the base of the positive even root system Δ_0^+. Then, $\Pi_0 = \{\alpha_i : i \neq l\} \cup \{\theta\}$. We deal with two cases.

Case 1: $(\Lambda + \rho, \alpha) \neq 0$ for all isotropic root α of the first kind

By 3.12 in [K4] and Lemma 5, it suffices to prove properties (2) for $w = r_\theta$. Let $\mu \in H^*$ be a weight as described above.

There is a base Π' with respect to which the root θ is simple. We choose Π' so that the number p of odd reflections $s_1,..., s_p$ such that $\Pi' = s_1 \cdots s_p(\Pi)$ is minimal. Consider first the base $\tilde{\Pi} = s_1(\Pi')$. We write $\Pi' = \{\gamma_1, \cdots, \gamma_n\}$ and $\tilde{\Pi} = \{\beta_1, \cdots, \beta_N\}$. We use the notation $\tilde{\leq}$ for the order relation, $\tilde{\Lambda}$ and $\tilde{\rho}$ for the highest weight of V and the Weyl vector with respect to the base $\tilde{\Pi}$.

$$w'(\mu + \rho) \tilde{\leq} \mu + \rho$$

for all $w' \in \tilde{W}$ since $\frac{(\mu + \rho, \alpha)}{(\alpha, \alpha)} \geq 0$ for all $\alpha \in \langle \alpha_i : i \neq l \rangle$ and the set of positive even roots with respect to any base obtainable from the base Π by applications of odd reflections remains invariant. We show that

$$r_\theta(\mu + \rho) \tilde{\leq} \Lambda + \rho. \tag{i}$$

Hence suppose that this inequality does not hold. Let $\beta_l \in \tilde{\Pi}$ be the isotropic simple root such that the odd reflection $s_1 = s_{\beta_l}$. Set

$$\mu + \rho = \tilde{\Lambda} + \tilde{\rho} - \sum_i k_i \beta_i$$

and

$$r_\theta(\mu + \rho) = \Lambda + \rho - \sum_i a_i \beta_i. \tag{ii}$$

If $i \neq l$ then $(\theta, \beta_i) \geq 0$.

Claim: $(\tilde{\Lambda} + \tilde{\rho}, \theta) \leq 0$.

$(\Lambda' + \rho', \theta) \leq 0$ since the module V is integrable and by definition of the Weyl vector $(\rho', \theta) = \frac{1}{2}(\theta, \theta)$. The assumption of Case 1 and Lemma 1 imply that

$$\Lambda' + \rho' = \tilde{\Lambda} + \tilde{\rho}, \tag{iii}$$

and hence our claim follows.

Let $\Pi^j = s_j \cdots s_p(\Pi)$ and Λ^j, ρ^j the highest weight of the module V and the Weyl vector with respect to this base and $\leq^j$ is the associated order relation.

Our assumption implies that $\Lambda + \rho = \Lambda^j + \rho^j$ for all j and hence by Lemma 8,

$$\mu + \rho \leq^j \Lambda^j + \rho^j$$

for all $j = 1, \cdots, p$.

Let γ_q be the isotropic root in the base Π' which belongs to the support of the root θ. Moreover, if $\mu + \rho + \beta_l \nleq \Lambda' + \rho'$, then clearly $r_\theta(\mu + \rho) \tilde{\leq} \tilde{L} + \tilde{\rho}$ and hence $r_\theta(\mu + \rho) \leq_I \Lambda' + \rho'$.

Applying the reflection r_θ to equality (1) with respect to the base Π', since $\theta \in \Pi'$, as in Lemmas 5 and 10, $r_\theta(\mu + \rho) \leq_I \Lambda' + \rho'$. Therefore equality (iii) and above claim imply that in expression (ii), for all $i \neq l$, $b_i \in \mathbb{Z}_+$ and $b_q \in \mathbb{Z}_-$. Applying r_θ to equality (1) with respect to the base $\tilde{\Pi}$ we get

$$-e(\tilde{\rho}) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \mathrm{ch}\, V = \sum_\lambda c_\lambda e(r_\theta(\lambda + \rho)) \prod_{\alpha \in \tilde{\Delta}_1^+} (1 + e(-\alpha)).$$

Indeed, as $\theta \in \Pi_0$, the only positive root $\alpha \in \Delta_0^+$ such that $r_\theta(\alpha) < 0$ is β, whereas $\alpha \in \Delta_1^+$ and $r_\theta(\alpha) < 0$ if and only if $\alpha = \beta_i + \gamma$ for a root of positive norm γ such that $\gamma \leq \theta$.

In consequence, by Lemma 2,

$$r_\theta(\mu + \rho) = \Lambda + \rho - \sum_{i \neq l} a_i \beta_i + \beta_q$$

and

$$(r_\theta(\mu + \rho), \beta_q) = 0.$$

Equivalently,

$$(\mu + \rho, \alpha) = 0, \qquad\qquad (iv)$$

where $\alpha = -r_\theta(\beta_l)$ is the highest isotropic root whose support does not contain negative norm simple roots. However $\alpha = \frac{1}{2}\theta + \gamma$, where $\gamma \in \Delta_0^+$ is a root of positive norm. By assumption $(\mu + \rho, \theta) > 0$. Therefore, property (iv) cannot hold. This proves the result in this case for the base $\tilde{\Pi}$.

We next consider the base $s_2(\tilde{\Pi})$. The above argument now applies to this base and thus tells us that $\mu + \rho - r_\beta(\mu + \rho)$ is a sum of positive roots with respect to this base. Hence by induction on the number of reflections p, the result follows for the weight μ.

Case 2: $(\Lambda + \rho, \alpha) = 0$ for some isotropic root α of the first kind.

We keep the notation of Case 1.

Case (a): there is a maximal index j such that $\mu + \rho \leq^j \Lambda^j + \rho^j$.

By abuse of notation, for simplicity's sake, write

$$\Pi^j = \{\alpha_1, \cdots, \alpha_N\}.$$

Let α_p be the isotropic simple root such that $\alpha_p \in \mathrm{supp}(\theta)$ (with respect to this base). This means the odd reflection $s_{j-1} = s_{\alpha_p}$. Therefore from Lemma 2 we know that $(\mu + \rho, \alpha_p) = 0$. The definition of the index j forces

$$(\Lambda^j + \rho^j, \alpha_p) = 0.$$

Set $\mu + \rho = \Lambda + \rho - \sum_{i=1}^{N} k_i \alpha_i$. By definition of the weight τ and by Lemmas 8 and 9, whether the simple root α_1 is isotropic or not, $k_1 \leq k_2$. Next consider the simple root α_2, we get $k_2 \leq k_3$. Continuing in this manner, we get $k_1 \leq k_2 \leq \cdots \leq k_N$ in the $B(m, n)$ and exceptional cases. Therefore,

$$\mu + \rho = \Lambda + \rho - l_1(\alpha_1 + \cdots + \alpha_N) - l_2(\alpha_2 + \cdots + \alpha_N) - \cdots - l_N \alpha_N, \quad (v)$$

However, this contradicts the fact that $\mu + \rho \not\leq \Lambda^{j+1} + \rho^{j+1}$. In the $D(m, n)$ case, $k_1 \leq k_2 \leq \cdots \leq k_{N-2} \leq k_{N-1} + k_N$. Hence in this case,

$$\mu + \rho = l_1(\alpha_1 + \cdots + \alpha_{N-1}) - \cdots - l_{N-1}\alpha_{N-1} - l_N(\alpha_1 + \cdots + \alpha_{N-2} + \alpha_N) \quad (vi)$$

So the previous arguments once again imply a contradiction. Hence in all cases, $k_i = 0$ for all i and

$$\mu = \Lambda.$$

Case (b): $\mu + \rho \leq^j \Lambda^j + \rho^j$ for all j

The assumption of Case 2 implies that $(\Lambda + \rho, \theta) > 0$. Considering equalities (v) and (vi) with respect to the base Π, let $\theta' = 2(\alpha_i + \cdots + \alpha_{l-1}) + \theta$, where i is minimal such that $k_i \neq 0$. It follows that

$$r_{\theta'}(\mu + \rho) \not\leq \Lambda + \rho.$$

As the set $\Pi_0 = \{\alpha_i, \theta : i \neq l\}$, this forces

$$r_\theta(\mu + \rho) \not\leq \Lambda + \rho.$$

Hence

$$r_\theta(\mu + \rho) \not\lesssim \tilde{\Lambda} + \tilde{\rho}.$$

As θ is a simple root in the base Π^1 and the reflections r_{α_i} for $i \leq l - 1$ belong to $W_{\Pi'}$, the arguments of Case 1

$$r_\theta(\mu + \rho) = \tilde{\Lambda} + \tilde{\rho} - \sum_{i \neq q} b_i \beta_i + \beta_q$$

with respect to the base $\tilde{\Pi}$. If the term $e(r_\theta(\mu + \rho) - \beta_q)$ appears in the left hand side of equality (iv) then by the discussion preceding Lemma 7,

$$r_\theta(\mu + \rho) - \beta_q = w(\Gamma + \tilde{\rho}_0) - \tilde{\rho}_1$$

for some highest weight Γ of a G_0-submodule of the module V and some element $w \in W$. As the expression $\tilde{\Lambda} + \tilde{\rho} - r_\theta(\mu + \rho) - \beta_q$ does not contain the simple isotropic root in the support of the root θ, $w \in \tilde{W}$ and so Lemma 10 tells us that

$$r_\theta(\mu + \rho) - \beta_q = w(\Lambda + \rho).$$

Equivalently,

$$\mu + \rho = r_\theta w(\Lambda + \rho) - \beta,$$

where $\beta = -r_\theta(\beta_q)$ is the highest isotropic roots whose support does not contain any simple roots of negative norm. However, considering the coefficient of β_q in the expression $\Lambda + \rho - r_\theta w(\Lambda + \rho)$, the assumption of Case 2 and the fact that $\mu \leq \Lambda$, forces

$$r_\theta w(\Lambda + \rho) \leq w(\Lambda + \rho) + \frac{1}{2}\theta.$$

Since $(\mu + \rho, \gamma) > 0$ and $(-\beta, \gamma) \leq 0$ for all positive roots of positive norm, it follows that

$$w \in W_- \cap \tilde{W}.$$

As a result, since $\mu + \rho \tilde{\leq} \tilde{\Lambda} + \tilde{\rho}$,

$$\mu + \rho = w(\tilde{\Lambda} + \tilde{\rho}).$$

Hence by assumption on μ, $w = 1$. So,

$$\mu = \tilde{\Lambda} + \tilde{\rho} - \rho.$$

Let j be the maximal index satisfying $\tilde{\Lambda} + \tilde{\rho} = \Lambda^j + \rho^j$. If $j \neq p$, then $\Lambda^j = \Lambda^{j-1}$. Therefore,

$$\mu = \Lambda^{j-1} + \rho^{j-1} - \gamma - \rho,$$

where $\gamma \in \Pi^{j-1}$ is the simple root in the support of the root θ. As there is a unique isotropic root γ of the first kind such that $(\Lambda + \rho, \gamma) = 0$, it follows that $\Lambda^{j-1} + \rho^{j-1} = \Lambda + \rho$. As a result,

$\mu = \Lambda - \gamma$. However since $\gamma = \alpha_l + \cdots + \alpha_k$, where $k < n$, this contradicts (v) or (vi). Therefore $j = 1$ and so

$$\mu = \Lambda.$$

The arguments of Lemma 2 applied to equality (iv) tell us that

$$c_{\mu+\rho+\beta} \neq 0.$$

If

$$\mu + \rho + \beta \not\leq \Lambda^j + \rho^j \qquad (vii)$$

for some index $1 \leq j \leq p$, then Case (a) tells us that

$$\mu + \rho + \beta = w(\Lambda + \rho)$$

for some $w \in \tilde{W}$. Equivalently,

$$\mu + \rho = w(\Lambda + \rho - w^{-1}(\beta)),$$

where $w^{-1}(\beta) > 0$ since $w\tilde{W}$ and $w^{-1}(\beta)$ is an isotropic root of the first kind as the weight $\Lambda + \rho$ cannot be orthogonal to isotropic roots of the first and of the second kind. Moreover, considering equality (iv) and (v), calculations contradict condition (vii). Therefore,

$$\mu + \rho + \beta \leq \Lambda^j + \rho^j$$

for all indices $1 \leq j \leq p$. So the weight $\mu + \beta$ satisfies the conditions of Corollary 4. Hence from what precedes, we can deduce that there is an element $w \in \tilde{W}$ such that

$$w(\mu + \rho) + \beta \leq \Lambda^j + \rho^j$$

for all indices $1 \leq j \leq p$. Since $w(\beta) > 0$, this forces

$$\mu + \rho + \sum \alpha \leq \Lambda + \rho,$$

where α is as a big a sum as we wish of positive roots. This is clearly false. Hence Case (b) does not occur and this proves the result. $\quad\square$

Remark It is important to note that for any arbitrary element $r \in W - \tilde{W}$, it is not always necessarily true that $c_{r(\lambda+\rho)-\rho} \neq 0$ for a weight $\lambda \in H^*$ such that $c_\lambda \neq 0$ and $w(\lambda + \rho) \leq \Lambda + \rho$ for all $w \in W$. This is why not all finite dimensional irreducible G-modules are tame when the Lie superalgebra G is of type $B(m,n)$, $m > 0$, $D(m,n)$, or exceptional. We will prove this later.

Let μ be a weight satisfying the conditions of Corollary 4 and set

$$\tau_1 + \rho = w(\mu + \rho)$$

be such that the height $\Lambda + \rho - w(\mu + \rho)$ is minimal. Set

$$\tau = \begin{cases} \tau_1 & \text{if } W = \tilde{W} \text{ or } (\tau_1 + \rho, \theta) \leq 0 \text{ when } |\theta|^2 < 0 \\ r_\theta(\tau + \rho) - \rho & \text{otherwise} \end{cases}$$

In the first case, the height of $\Lambda + \rho - (\tau + \rho)$ is clearly minimal among the elements of the set $\{\Lambda + \rho - w(\tau + \rho) : w \in W\}$. In the latter case, this remains true because $r_{\alpha_i} r_\theta(\tau_1 + \rho) \leq r_\theta(\tau_1 + \rho)$ for all $|\alpha_i|^2 \neq 0$. Moreover, applying the reflection r_θ to both sides of equality (1), we get

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{ch} V = \sum_\lambda c_\lambda e(r_\theta(\lambda + \rho)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha)).$$

Since $c_{\lambda + \rho + kr_\theta(\alpha)} = 0$ for $k >> 0$ (otherwise, $\lambda + \rho + kr_\theta(\alpha) \leq \Lambda + \rho$), it follows that the weight $r_\theta(\tau_1 + \rho)$ satisfies the conditions of Corollary 4. In summary,

1. $c_{\tau_1} \neq 0$ (note that we do not know $c_\tau \neq 0$ in the second case); and

2. $\frac{(\tau, \alpha)}{(\alpha, \alpha)} \geq 0$ for all non-isotropic roots $\alpha \in \Delta_0^+$ conjugate under the action of the group $\tilde{W}$ to a simple root in Π; and

3. $\tau \leq_, \Lambda'$ for all bases Π' obtainable from the base Π by successive applications of odd reflections; and

4. $\tau + \rho \leq_, \Lambda' + \rho'$ for all bases Π', or the term $\tau + \rho$ appears in the left hand side of equality (1) with respect to the base Π'.

5. for all $w \in W$, $w(\tau + \rho) \leq \Lambda + \rho$ and for all $w \in \tilde{W}$, $c_{w(\tau+\rho)-\rho} = \epsilon(w) c_{(\tau)}$; and

6. $w(\tau) \leq_, w(\Lambda)$ for all $w \in W_{\Pi'}$.

When the base Π' is such that its maximal root has negative norm, $W_{\Pi'} = \tilde{W}$ and the last property is simply Lemma 11. Otherwise there is a base $\tilde{\Pi}$ obtainable from the base by applications of odd reflections with respect to which $W_- \leq W_{\tilde{\Pi}}$ and the subgroup $W_{\Pi'}$ is generated by simple root reflections. Therefore when the maximal root of a base Π' has negative norm the arguments of Lemma 11 apply.

Corollary 7. *Suppose that $(\Lambda + \rho, \alpha) \neq 0$ for all isotropic roots of the first kind.*

1. *When G is a Lie superalgebra of type $B(m,n)$, $m > 0$ or $D(m,n)$, let α be the isotropic positive root of the second kind of maximal height whose support only contains roots of non-negative norm. Then,*

$$\tau = \Lambda - k\alpha,$$

for some integer $k \geq 0$;

2. *When the Lie superalgebra G is exceptional, $\tau = \Lambda$.*

Proof. In each case there is a non-isotropic root containing one distinct isotropic simple root in its support. If Π' is a base with respect to which the root θ still contains only one distinct simple root in its support, then $\tilde{W}_{\Pi'} = \tilde{W}$. Let $\tilde{\Pi}$ be a base with this property for which there exists some odd reflection s such that with respect to the base $s(\tilde{\Pi})$, the root θ is either simple or only contains non-isotropic simple roots in its support. Set $\Pi' = s(\tilde{\Pi})$.

Case 1: the Lie superalgebra G is exceptional or of type $B(m,n)$.

Part of the Dynkin diagram corresponding to the base $\tilde{\Pi}$ in the $B(m,n)$ case is as follows:

$$\cdots\!\!-\!\!\otimes\!\!\Longrightarrow\!\!\bigcirc$$

For simplicity of notation, we keep the notation $\alpha_1, \cdots, \alpha_N$ for the simple roots in $\tilde{\Pi}$ and the the notation Λ, ρ for the highest weight of the module V and the Weyl vector with respect to the base $\tilde{\Pi}$. So the odd reflection is: $s = s_{N-1}$, where $\alpha_{N-1} \in \mathrm{supp}(\theta)$ (with respect to the base $\tilde{\Pi}$) and $|\alpha_{N-1}|^2 = 0$.

From the proof of Lemma 11 we know that

$$\tau = \Lambda - \sum_{i=1}^{N} \sum_{i=1}^{n} l_i(\alpha_i + \cdots + \alpha_N), \qquad (i)$$

where for each i, $l_i \in \mathbb{Z}_+$. Hence with respect to the base Π',

$$\tau = \Lambda - \sum_{i=1}^{N} \sum_{i=1}^{n-2} l_i(\beta_1 + \cdots + \beta_N) - l_2\beta_N - l_1(\beta_{N-1} + \beta_N). \qquad (ii)$$

We consider in turn the roots $\beta_i + \cdots + \beta_N$. Depending on whether they have positive or negative norm we use equality 1 or 2 and apply

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property 6 (stated before the Lemma) satisfied by the weight τ and the reflection $r_{\beta_i+\cdots+\beta_N}$. This gives

$$l_1 = \cdots = l_{N-2} = 0.$$

Next considering the reflections r_{β_N} and r_{α_N} we get $l_N = l_{N-1}$ and so

$$\tau = \Lambda - l_N\alpha, \qquad (ii)$$

where the isotropic root α is as described in the Lemma in the $B(m,n)$ and $D(m,n)$ cases. In the exceptional cases, $l_N = 0 = l_{N-1}$ by definition of the weight τ as all isotropic roots are of the first kind.

Case 2: the Lie superalgebra G is of type $D(m,n)$.

In this case, the root θ is the sum of two distinct simple isotropic roots (with respect to the base $s(\tilde{\Pi})$. There are two odd reflection $s_{j_1} \neq s_{j_2}$ such that $\beta \in s_{j_i}s(\tilde{\Pi})$ for $i = 1, 2$ and we set $\Pi'^i = s_{j_i}s(\tilde{\Pi})$. Set

$$\Pi' = s(\tilde{\Pi}) = \{\beta_1, \cdots, \beta_N\}.$$

Part of the Dynkin diagram corresponding to the base $\tilde{\Pi}$ is as follows:
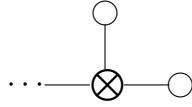
Therefore once again, using equality (vi) from the proof of Lemma 11 and similar arguments to the previous case, we get

$$\tau = \Lambda - l\alpha, \qquad (ii)$$

for some integer $l \geq 0$, where the isotropic root α is as described in the Lemma. $\qquad \square$

For any weight $\lambda \in H^*$, fix a maximal isotropic set $S_\lambda \leq \Pi$ of roots orthogonal to λ, i.e.

$$S_\lambda = \{\alpha \in \Pi : (\lambda, \alpha) = 0 = (\alpha, \beta)\,\forall \beta \in S_\lambda\}$$

and S_λ is maximal with this property. Proposition 2 is now an immediate consequence of the definition of the weight τ, Corollary 4 and Lemmas 9 and 11.

1.3.3 Proof: part II

Now that we know what conditions the weight $\lambda \in H^*$ must satisfy if $c_\lambda \neq 0$ in equality (1), we need to prove the converse or more precisely compute the coefficients c_λ. We first need to observe the following:

Lemma 12. *Suppose that $|S_{\Lambda+\rho}| > 1$. Then the set $S_{\Lambda+\rho}$ contains only isotropic roots of the first kind and the Lie superalgebra G has defect at least 2. Moreover, for any subset $S \leq S_{\Lambda+\rho}$ of cardinality at least 2, there is a unique set S' of positive odd roots satisfying the following:*

1. $S' \cap S_{\Lambda+\rho} = \emptyset$; and

2. $\sum_{\gamma \in S} \gamma = \sum_{\beta \in S'} \beta$; and

3. for any proper subsets $T < S$, $T' < S'$, $\sum_{\gamma \in T} \gamma \neq \sum_{\beta \in T'} \beta$.

Proof. As shown in [K1] the atypicality of the module is equal at most to the defect of the Lie superalgebra. Hence the first part follows. If there is a root of the second kind in $S_{\Lambda+\rho}$ then by Lemma 11, $|S_{\Lambda+\rho}| = 1$. Hence for the second part, without loss of generality, we may assume that $G = A(m, n)$. Note that there are two possibilities: either there is a root $\alpha \in S_{\Lambda+\rho}$ such that all simple roots in $\text{supp}(\alpha)$ apart from that of norm 0 have norms of the same sign and there is a simple root of non-zero norm; or there is no such root in the set $S_{\Lambda+\rho}$. We first assume that the latter holds. Then, keeping the notation used in the statement of the Lemma, for $S \leq S_{\Lambda+\rho}$, if j is maximal such that $j \in \text{supp}(\alpha)$, $\alpha \in S$, then $\alpha_l + \cdots + \alpha_j \in S'$ (where α_l is the unique isotropic simple root in Π). $S' \cap S_{\Lambda+\rho} = \emptyset$ for otherwise $(\Lambda + \rho, \alpha_i) = 0$ for some $i \neq l$, which cannot happen since the module V being integrable, $\frac{(\Lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \geq 0$ and $\frac{(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} > 0$. The other case follows by symmetry. $\square$

Lemma 12 proves statement 2 of Proposition 1. We set

$$T_{\Lambda+\rho} = S_{\Lambda+\rho}$$

if the atypicality of the module V is 1 and

$$T_{\Lambda+\rho} = S_{\Lambda+\rho} \cup (\cup_{S \leq S_{\Lambda+\rho}; |S|>1} \{\sum_{\gamma \in S} \gamma\}).$$

Lemma 13.

$$\sum_{k_\alpha, \alpha \in S_{\Lambda+\rho}} (c_{\Lambda+\rho-\sum_{\alpha \in S_{\Lambda+\rho}} k_\alpha \alpha}) e(\Lambda + \rho - \sum_{\alpha \in S_{\Lambda+\rho}} k_\alpha \alpha) =$$

$$\frac{e(\Lambda + \rho)}{\prod_{\alpha \in T_{\Lambda+\rho}}(1 + e(-\alpha))}$$

unless the Lie superalgebra G is exceptional or of type $B(m,n)$, or $D(m,n)$, $S_{\Lambda+\rho} = \{\alpha\}$, where α is a root of the second kind, and $\frac{2(r_\theta(\Lambda+\rho)-\rho,\theta)}{(\theta,\theta)} \geq 0$. In this case

$$c_\Lambda = 1 \quad and \quad c_{\Lambda-k\alpha} = 0 \quad \forall k \geq 1.$$

Proof. Let

$$\lambda + \rho = \Lambda + \rho - \sum_{\alpha \in S_{\Lambda+\rho}} k_\alpha \alpha,$$

where $k_\alpha \in \mathbb{Z}_+$. Considering equality (1) and since $c_\lambda \neq 0$ implies that $\lambda \leq \Lambda$,

$$c_{\Lambda+\rho} = 1.$$

From Corollary 8 there are two cases to consider.

Case 1: All roots in the set $S_{\Lambda+\rho}$ are of the first kind

Let $\alpha \in S_{\Lambda+\rho}$ and Π' be a base for which $\alpha \in \Pi'$ such that the number of odd reflections needed to arrive at the base Π' from the base Π is minimal. Then, $\Lambda + \rho = \Lambda' + \rho'$. Consider equality (1) with respect to the base Π^p (namely equality (4)). The terms $e(\Lambda + \rho - k\alpha)$ for $k > 0$ do not appear on the left hand side of equality (4). Also, $w(\Lambda + \rho) - \gamma \neq \Lambda + \rho - k\alpha$, where $1 \neq w \in \tilde{W}$ such that $w(\Lambda + \rho) \leq \Lambda + \rho$ and γ is a sum of distinct positive odd roots. Therefore, as the root α is simple with respect to the base Π', we must have $c_{\Lambda+\rho-k\alpha} \neq 0$. Hence from Lemma 2 it follows that

$$c_{\Lambda+\rho-k\alpha} = (-1)^k.$$

Now,

$$\sum_{k \geq 0}(-1)^k e(\Lambda+\rho-k\alpha) \prod_{\beta \in \Delta_1^+}(1+e(-\beta)) = e(\Lambda+\rho) \prod_{\beta \in \Delta_1^+\{\alpha\}}(1+e(-\beta)).$$

$$(i)$$

Let $\alpha' \in S_{\Lambda+\rho} - \{\alpha\}$ be the of minimal height. So choosing α to be of minimal height in $S_{\Lambda+\rho}$ we set $\Pi^1 = \Pi'$. Replace the base Π by the base Π^1. Consider the base Π^2 for which $\alpha' \in \Pi^2$ such that the number of odd reflections needed to arrive at the base Π^2 from the base Π^1 is minimal. Considering equality (1) with respect to

the base Π^2 and taking account of equality (i), it follows that the terms $e(\Lambda + \rho - k\alpha - k'\alpha)$ do not appear on the left hand side of this equality for $k + k' > 0$. Note that

$$\Lambda + \rho - k\alpha - k'\alpha' = \Lambda + \rho - (k-l)\alpha - (k'-l)\alpha' - l(\alpha + \alpha').$$

Considering Lemma 12, we get by induction on $k + k'$ that

$$c_{\Lambda+\rho-k\alpha-k'\alpha'}$$
$$= -c_{\Lambda+\rho-(k-1)\alpha-k'\alpha'} - c_{\Lambda+\rho-k\alpha-(k'-1)\alpha'} - c_{\Lambda+\rho-(k-1)\alpha-(k'-1)\alpha'}$$
$$= \sum_{l=0}^{\min(k,k')} (-1)^{k+k'+l}.$$

Therefore in this case the result follows by induction on the number of roots in $S_{\Lambda+\rho}$ or equivalently the degree of atypicality .

Case 2: $\alpha \in S_{\Lambda+\rho}$ is of the second kind

Claim: for all $k \geq 1$, the terms $e(\Lambda + \rho - k\alpha)$ do not appear on the left hand side of equality (1)

Suppose that the term $e(\Lambda+\rho-k\alpha)$ appears in the left hand side of equality (1). Then, from the discussion preceding Lemma 7,

$$\Lambda + \rho_0 - k\alpha = w(\Gamma + \rho_0)$$

for some $w \in W$ and some highest weight Γ of an irreducible component of the G_0-module V. Suppose that $k >> 0$ so that

$$(\Lambda + \rho_0 - k\alpha, \theta) > 0. \qquad (ii)$$

Therefore, by 3.11 in [K4], $w = r_\theta w_1$ for some $w_1 \in W$ such that $l(w_1) < l(w)$. This in turn gives $r_\theta(\Lambda + \rho_0) + k\alpha_l$, where α_l is the unique isotropic root in the base Π. Since $(\Lambda + \rho_0, \theta) > 0$, the same type of arguments imply that $w_1 = r_\gamma$, where $r_\gamma \in W_+$ and $r_\gamma(\alpha_l) = \alpha$. As a consequence

$$r_\gamma r_\theta(\Lambda + \rho_0) + k\alpha = \Gamma + \rho_0.$$

Since $\Gamma \leq \Lambda$, $\frac{2(\Lambda+\rho_0)}{(\theta,\theta)} \geq k$. However this contradicts assumption (ii). Therefore, if (ii) holds then the term $e(\Lambda+\rho-k\alpha)$ does not appear on the left hand side of equality (1). Next suppose that

$$(\Lambda + \rho_0 - k\alpha, \theta) \leq 0.$$

Similar arguments as before lead to

$$\Lambda + \rho_0 - k\alpha = w(\Gamma + \rho_0),$$

for some $w \in W_+ \leq \tilde{W}$. This gives

$$\Lambda + \rho - k\alpha = w(\Gamma + \rho),$$

and so by Lemma 10, $\Gamma = \Lambda$. Considering the coefficient of α_l in the expression $w(\Lambda + \rho) + k\alpha$, we get a contradiction. This proves our Claim.

As a consequence for all $k \geq 1$, since $k\alpha$ is not the sum of odd roots distinct from the root α,

$$c_{\Lambda+\rho-k\alpha} = (-1)^k$$

unless there is some $1 \neq w \in \tilde{W}$ and a positive sum of distinct odd roots γ such that $w(\Lambda+\rho)-\gamma = \Lambda+\rho-\alpha$. This happens if and only if $(r_\theta(\Lambda + \rho) - \rho, \theta) \leq 0$. In this case, $c_{\Lambda+\rho-\alpha} = 0$ and so $c_{\Lambda+\rho-k\alpha} = 0$ for all $k \geq 1$. This proves the result in Case 2. $\qquad\square$

By assumption on the base Π, $\tilde{W}$ is generated by simple roots. Hence for any isotropic root $\alpha \in \Delta^+$, $w(\alpha) > 0$ for all $w \in \tilde{W}$. Therefore

$$w(\Lambda + \rho) - \sum_{\alpha \in S_{w(\Lambda+\rho)}} k_\alpha \alpha = w(\Lambda + \rho - \sum_{w^{-1}(\alpha) \in S_{\Lambda+\rho}} k_\alpha w^{-1}(\alpha)).$$

In particular, setting $\lambda+\rho = \Lambda+\rho-\sum_{\alpha \in S_{\Lambda+\rho}} k_\alpha \alpha$, since $\lambda+\rho \leq \Lambda+\rho$, by Corollary 4 and Lemmas 6 and 13, for all $w \in \tilde{W}$,

$$c_{w(\lambda+\rho)-\rho} = \epsilon(w)c_\lambda.$$

Note that in the case of the base Π,

$$j_{\Lambda,\Pi} = 1$$

since the Weyl W is generated by reflections corresponding to roots containing at most one isotropic simple root in their support.

Case I: The Lie superalgebra G is of type $A(m,n)$ or $C(n)$

Since $\tilde{W} = W$, Theorem 1 follows with respect to the base Π in these cases, namely all the modules V are generalized tame.

Case II: The Lie superalgebra G is of type $B(m,n)$ or $D(m,n)$

Case 1: all the roots in the set $S_{\Lambda+\rho}$ are of the first kind

Then Corollary 8 and Lemma 13 tell us that the module is not generalized tame. More precisely, the character is as stated in Theorem 2.

Case 2: $S_{\Lambda+\rho} = \{\alpha\}$, where α is a root of the second kind.

We know from Corollary 8 that it is the highest isotropic root all of whose support consists of non-negative norm simple roots and that $(\Lambda + \rho, \theta) \leq 0$.

Case (a): $(r_\theta(\Lambda + \rho) - \rho, \theta) > 0$

As a consequence, from Corollary 5, letting $w_1, \cdots, w_s$ be a set of representatives of right cosets of the subgroup $\tilde{W}$ in the Weyl group W, equality (1) becomes

$$
e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{ch} V
$$

$$
= \sum_{w \in \tilde{W}} \left(\sum_{w_i : w_i(\alpha) > 0} \epsilon(w) c_{w_i(\Lambda+\rho)} e(w w_i(\Lambda + \rho)) \prod_{\alpha \in \Pi - \{w w_i(\alpha)\}} (1 - e(-\alpha)) \right.
$$

$$
+ \sum_{w_i : w_i(\alpha) < 0} \epsilon(w) c_{w_i(\Lambda+\rho)} e(w w_i(\Lambda + \rho - \alpha)) \prod_{\alpha \in \Pi - \{w w_i(\alpha)\}} (1 - e(-\alpha))
$$

$$
\text{(i)}
$$

We know that $\Pi_0 = \{\alpha_i, \theta : i \neq l\}$ and that for $w = r_\beta$, $\beta \in \Pi$,

$$
\epsilon(w) e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \operatorname{ch} V = \sum_\lambda c_\lambda e(w(\Lambda + \rho)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha))
$$

(for $\beta = \theta$, see proof of Lemma 11). Therefore for all $w \in W$,

$$
\epsilon(w) e(w(\rho)) \prod_{\alpha \in \Delta_0^+} (1 - e(-w(\alpha))) \operatorname{ch} V = \sum_\lambda c_\lambda e(\Lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 + e(-w(\alpha)))
$$

$$
\text{(ii)}
$$

We set $w_1 = 1$, $w_2 = r_\theta$ and without loss of generality, we may assume that for all i, $w_i \in \{r_j, r_\theta : |\alpha_j| < 0\}$. Moreover, by 3.12 in [K4], for each i, there is an element $u_i \in \tilde{W}$ such that

$$
\frac{(u_i w_i(\Lambda + \rho), \alpha_j)}{(\alpha_j, \alpha_j)} > 0
$$

for all $j \neq l$. Without loss of generality, we consider w_i to the coset representative for which these inequalities hold. Note that the elements $r_{\gamma_{i,j}}$, where

$$
\gamma_{i,j} = \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_m) + \theta, \quad i > 0
$$

$$
\gamma_{0,0} = \theta, \quad \gamma_{0,j} = 2(\alpha_j + \cdots + \alpha_m) + \theta,
$$

together with the identity element are a set of right coset representatives. We order the elements w_i so that if $w_i r_{\gamma_{i_1,j_1}} \in \tilde{W}$, and

$w_k r_{\gamma_{i_2,j_2}} \in \tilde{W}$, then $i > k$ if the $\mathrm{ht}(\gamma_{i_1,j_1}) > \mathrm{ht}(\gamma_{i_1,j_1})$. From equalities (i) and (ii), it follows that for w_i $c_{w_i(\Lambda+\rho)} = \epsilon(w_i)$ if for all w_j such that $j < i$,

$$rw_j(\Lambda + \rho) - \gamma \neq w(\Lambda + \rho)$$

for all $r \in \tilde{W}$ and all sums γ of positive distinct odd roots.

So we start with $r = r_\theta$. Suppose that

$$r_\theta(\Lambda + \rho) = w(\Lambda + \rho) - \gamma, \qquad (iii)$$

where $w \in \tilde{W}$ and γ is a sum of distinct positive odd roots. From (i) it follows that none of the odd roots appearing in the sum are equal to $w(\alpha)$. Let $t = \frac{2(\Lambda+\rho,\theta)}{(\theta,\theta)}$. Then,

$$\Lambda + \rho - t\theta = w(\Lambda + \rho) - \gamma.$$

Hence, considering the support of θ, $w \in W_+$ and all simple roots in the support of γ have non-negative norm.

Suppose that $w(\Lambda + \rho, \alpha_i) < 0$ for some $|\alpha_i|^2 > 0$. Then $(\gamma, \alpha_i) < 0$. However there is a unique odd root μ with only non-negative norm simple roots in its support such that $(\mu, \alpha_i) < 0$ (namely $\alpha_l + \cdots + \alpha_i$). Considering the left hand side of equality (iii), we get $(\Lambda + \rho, \alpha_i) = 0$ which cannot hold. So $w = 1$. In other words,

$$r_\theta(\Lambda + \rho) = \Lambda + \rho - \gamma. \qquad (iv)$$

Now the sum of all positive odd roots is $2\rho_1 = p\theta + \mu$, where $(\mu, \theta) = 0$. Since $(w(\alpha), \theta) = (\alpha, w^{-1}(\theta)) = (\alpha, \theta) > 0$, it follows that

$$t \leq p - 1$$

Moreover, $\frac{2(\rho_1,\theta)}{(\theta,\theta)} = p$, $\rho_1 = \rho_0 - \rho$ and $(\rho_0, \theta) = \frac{1}{2}(\theta, \theta)$ since $\theta \in \Pi_0$. Therefore our assumption implies that

$$c_{r_\theta(\Lambda+\rho)} = -1.$$

As a consequence in equality (i),

$$\sum_{w\in\tilde{W}} \sum_{k\geq 0} \epsilon(w) c_{wr_\theta(\Lambda+\rho-\alpha)} e(wr_\theta(\Lambda + \rho - k\alpha)) \prod_{\alpha\in\Delta_1^+} (1 + e(-\alpha))$$
$$= \sum_{w\in\tilde{W}} \epsilon(w) e(wr_\theta(\Lambda + \rho - \alpha)) \prod_{\alpha\in\Delta_1^+ - \{w(\alpha_l)\}} (1 + e(-\alpha))$$

More generally we next consider w_i. Suppose that

$$w_i(\Lambda + \rho) = w(\Lambda + \rho) - \gamma, \qquad (v)$$

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where $w \in W$ and w is not in the same coset as w_i. Therefore,

$$w^{-1}w_i(\Lambda + \rho) = \Lambda + \rho - w^{-1}(\gamma).$$

So $w^{-1}w_i = rw_j$ for some $r \in \tilde{W}$. Since $(\theta, \alpha_j) \geq 0$ and $(\Lambda + \rho, \alpha_j) \leq 0$ for all $j \leq m$, it follows from the conditions satisfied by the element w_j that if k is the minimal index in the support of a reflection appearing in a minimal product for w_j then

$$w_j^{-1}(\theta) = r_{2(\alpha_k + \cdots + \alpha_m) + \theta}(\theta).$$

As a consequence, from our assumption

$$\frac{2(w_j(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} < 0.$$

Therefore equality (v) cannot hold. In conclusion in Case (a), the module V is tame and part of Theorem 2 with respect to the base Π follows.

Case (b): $\frac{2(r_\theta(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} \geq 0$

Then, Lemma 13 implies that

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha)) \mathrm{ch}\, V =$$

$$\sum_{w \in \tilde{W}} e(w(\Lambda + \rho)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha))$$

$$+ \sum_{w \notin \tilde{W}} c_{w(\Lambda + \rho - k\alpha)} e(w(\Lambda + \rho - k\alpha)) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha))$$

The assumption of case (b) implies that the term $e(r_\theta(\Lambda + \rho))$ does not appear on the left hand side of this equality for otherwise, $r_\theta(\Lambda + \rho) = \Gamma + \rho$ for some highest weight Γ of a G_0-component of the module V, contradicting Lemma 10. However the above shows that there is a sum γ of positive distinct odd roots such that $r_\theta(\Lambda + \rho) = \Lambda + \rho - \gamma$. Hence as $c(\Lambda) = 1$, we must have

$$c_{r_\theta(\Lambda + \rho)} = -1.$$

Similarly the terms $e(r_\theta(\Lambda + \rho - k\alpha)$ for $k \geq 1$ do not appear on the left hand side of equality (1), but neither do they cancel out on the right hand side and so

$$c_{r_\theta(\Lambda + \rho - k\alpha)} = 0 \quad \forall\, k \geq 1.$$

In this case, calculations show that we always have

$$\frac{(r_{2\alpha_j + \cdots + \theta}(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} < 0.$$

As a result, with similar arguments as above for all

$$w \in W = \langle r_\theta, r_i : |\alpha_i|^2 > 0 \rangle,$$

$$c_{w(\Lambda + \rho - k\alpha)} = (-1)^k \epsilon(w).$$

This proves Theorem 2 with respect to the base Π.

Case III: The Lie superalgebra G is exceptional

There is a unique positive root of negative norm θ and $\tilde{W} = W_+$. Hence the previous case simplifies and we get: $(\Lambda + \rho, \theta) > 0$, then

$$\text{ch}\, V = R^{-1} \text{ch}\,_{\Lambda, S_{\Lambda + \rho}, \Pi, W_+}.$$

If $(\Lambda + \rho, \theta) \leq 0$ and the assumption of Case II.2.(a) holds then the module V is tame. If $(\Lambda + \rho, \theta) \leq 0$ and assumption of Case II.2.(b) holds then the character is as given in Theorem 3.

1.3.4 Proof: Part III

We now consider an arbitrary base.

Case 1: $(\Lambda, \alpha_l) \neq 0$

As $\Lambda + \rho = \Lambda_l + \rho_l$, from equalities (1) and (4), the results follow with respect to the base Π_m when $S_{\Lambda + \rho}$ only consists of roots of the first kind. Otherwise, this is clearly true for tame modules. So suppose that $(\Lambda + \rho, \alpha) = 0$ where α is an isotropic root of the second kind and that the module V is not tame. If Π' is the base obtainable from the base Π by applications of odd reflections containing a unique isotropic root and such that W_- is generated by simple root reflections, then calculations show that

$$(\text{ch}\,_{\Lambda, \Pi, W} - \text{ch}\,_{\Lambda, \Pi, \langle W_+, r_\theta \rangle}) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha)) =$$

$$(\text{ch}\,_{\Lambda', \Pi', W} - (\text{ch}\,_{\Lambda, \Pi', \langle W_-, r_{\theta'} \rangle}) \prod_{\alpha \in (\Delta_1')^+} (1 - e(-\alpha))$$

Therefore the result follows for all bases obtainable from the base Π by applications of odd reflections in this case.

Case 2: $(\Lambda, \alpha_l) = 0$.

Corollary 8 tells us that in this case all the roots in the set $S_{\Lambda+\rho}$ are of the first type. Then,

$$\frac{e(-\alpha_l)}{(1 + e(-\alpha_l))} = \frac{1}{(1 + e(\alpha_l))}$$

and $-\alpha_l \in \Pi_l$. All other isotropic roots in the set $S_{\Lambda+\rho}$ positive with respect to the base Π remain positive with respect to the base Π_l. Moreover, note that in this case the set

$$S_{\Lambda_l+\rho_l} = (S_{\Lambda+\rho} - \{\alpha_l\}) \cup \{\beta_l\}$$

is a maximal isotropic set orthogonal to the weight $\Lambda_l + \rho_l$ in the set Δ_l^+, the set of positive roots generated by the base Π_l. Therefore the results again holds with respect to the base $\Pi_l = \{\beta_1, \cdots, \beta_N\}$ for the set $S_{\Lambda_l+\rho_l}$. If $S_{\Lambda+\rho} = \{\alpha_l\}$, then we are done. Suppose that $|S_{\Lambda+\rho}| > 1$. Then $S_{\Lambda_l+\rho_l}$ may not be the unique maximal isotropic subset orthogonal to the weight $\Lambda_l + \rho_l$ in Δ_l^+. Indeed, we know from the above that in this case, there is a root $\alpha \in S_{\Lambda+\rho}$ of the type

$$\alpha = \alpha_i + \cdots + \alpha_l + \cdots + \alpha_j,$$

where without loss of generality $|\alpha_i|^2 = \cdots = |\alpha_{l-1}|^2 = -2$ and $|\alpha_{l+1}|^2 = \cdots = |\alpha_j|^2 = 2$. With respect to the base Π_l this becomes

$$\alpha = \beta_i + \cdots + \beta_l + \cdots + \beta_j.$$

Therefore

$$(\Lambda_l + \rho_l, \beta_i + \cdots + \beta_{l-1} + \beta_{l+1} + \cdots + \beta_j) = 0.$$

However as $|\beta_{l-1}|^2 = 0 = |\beta_{l+1}|^2$ and $|\beta_i|^2 \neq 0$ for $i < l - 1$ and $i > l + 1$, the roots $\beta_i + \cdots + \beta_{l-1}$ and $\beta_{l+1} + \cdots + \beta_j$ have norm 0 (whereas the roots $\alpha_i + \cdots + \alpha_{l-1}$ and $\alpha_{l+1} + \cdots + \alpha_j$ do not). Hence we may have

$$(\Lambda_l + \rho_l, \beta_i + \cdots + \beta_{l-1}) = 0,$$

in which case we also have

$$(\Lambda_l + \rho_l, \beta_{l+1} + \cdots + \beta_j) = 0.$$

This is equivalent to

$$(\Lambda + \rho, \alpha_i + \cdots + \alpha_{l-1}) = -1 \quad \text{and} \quad (\Lambda + \rho, \alpha_{l+1} + \cdots + \alpha_j) = 1.$$

Therefore this may hold only if $j = l + 1$ and $i = l - 1$. In this case the set

$$S'_{\Lambda_l+\rho_l} = \{\beta_{l-1}, \beta_{l+1}\} \cup (S_{\Lambda+\rho} - \{\alpha_{l-1}, \alpha\})$$

together with the set $S_{\Lambda_l+\rho_l}$ are the two possible maximal isotropic subsets orthogonal to the weight $\Lambda_l + \rho_l$ in Δ_l^+. Therefore we need to show that the result holds for the set $S'_{\Lambda_l+\rho_l}$. Without loss of generality, we need only consider the case $A(1,1)$ and the trivial module V with highest weight $\Lambda = 0$. Simple calculations then show that

$$\text{ch}_{0,S_{\rho_2},\Pi_2,W} =$$

$$e(\rho)\Big(\frac{1}{(1+e(-\beta_2))(1+e(-\beta_1-\beta_2-\beta_3))(1+e(-\beta_1-2\beta_2-\beta_3))}$$

$$-\frac{1}{(1+e(-\beta_1))(1+e(\beta_3))(1+e(-\beta_1-\beta_3))}$$

$$-\frac{1}{(1+e(\beta_1))(1+e(-\beta_3))(1+e(-\beta_1-\beta_3))}$$

$$+\frac{1}{(1+e(\beta_1+\beta_2+\beta_3))(1+e(\beta_2))(1+e(-\beta_1-\beta_3))}\Big)$$

$$= e(\rho_l)\Big(\frac{1}{(1+e(-\beta_1))(1+e(-\beta_3))}$$

$$-\frac{1}{(1+e(\beta_2))(1+e(-\beta_1-\beta_2-\beta_3))}$$

$$= \frac{1}{2}\text{ch}_{0,S_{\rho_2},\Pi_2,W} = \text{ch}_{0,\{\beta_1,\beta_3\},\Pi_2,W_+}$$

As this calculations shows there may be bases Π' for which there exists more than one set $S_{\Lambda'+\rho'}$. We need to consider this possibility. Indeed in the case at hand, $(\Lambda_l + \rho_l, \beta_{l-1}) = 0$, in which case $(\Lambda_l + \rho_l, \beta_{l+1}) = 0$ and conversely. Therefore by induction it follows that statement 3 of Proposition 1 and Theorems 1 and 2 hold for all bases obtainable from the base Π by applications of odd reflections. It follows that by induction on the number of odd reflections needed to arrive at the base Π' from the base Π the result holds for all bases Π'. Since clearly the Theorems hold for the base $w(\Pi)$ for all $w \in W$, it follows that they hold for all bases.

1.3.5   Proof: Part IV

We next consider the denominator formula. Its clearest expression is with respect to a base containing a maximal isotropic subset of Δ. To see this, we first need a technical result about sets. When the Lie superalgebra G if of type $B(m,n)$ or $D(m,n)$ we choose a base with respect to which $W^\sharp$ is generated by simple root reflections. For simplicity of notation, we rename this base $\Pi = \{\alpha_1, \cdots, \alpha_N\}$ if necessary.

Lemma 14. *Let S and T be maximal isotropic subsets of Δ. Then, there is a unique element $w \in W^\sharp$ such that for all $\alpha \in T$, $w_+(\alpha) \in S$ or $-w_+(\alpha) \in S$.*

Proof. Without loss of generality, we may assume that $S, T \leq \Delta^+$. Without loss of generality, assume that $W^\sharp = W_+$ (note that this assumption may not be consistent with our assumption above that $|\theta|^2 < 0$ when $r_\theta \notin \tilde{W}$, but this does not matter since this latter fact will not be used in this proof). Let $\alpha_l \in \Pi$ be the isotropic simple root.

Case 1: $|S| = 1$

We may in this case assume without loss of generality that $S = \{\alpha_l\}$. Then, $\tilde{W} = W^\sharp$. Then for all i, $|\alpha_i|^2 \geq 0$. Let α be an isotropic root in Δ^+. For all $w \in W^\sharp$, $w(\alpha) > 0$. Let $w \in \tilde{W}$ be such that $\mathrm{ht}(w(\alpha))$ is minimal. For simplicity of notation, we may without loss of generality write $w = 1$. Hence,

$$(\alpha, \alpha_i) \leq 0 \quad \text{and} \quad (\alpha_l, \alpha_i) \leq 0$$

for all $i \neq l$. Since $|\alpha|^2 = 0$,

$$\alpha_l \in \mathrm{supp}(\alpha).$$

So

$$(\alpha, \alpha_l) \leq 0.$$

Let

$$\alpha = \sum_i k_i \alpha_i.$$

Then,

$$0 = (\alpha, \alpha) = \sum_i k_i(\alpha, \alpha_i)$$

forces $k_i = 0$ for all i such that $(\alpha, \alpha_i) \neq 0$. Therefore,

$$0 = (\alpha, \alpha_l) = \sum_i k_i(\alpha_i, \alpha_l),$$

which in turn gives $k_i = 0$ for all i for which $(\alpha_i, \alpha_l) \neq 0$. It follows that $(\alpha_i, \alpha_l) = 0$ for all $i \in \mathrm{supp}(\alpha)$. As the support of the root α cannot be disconnected, we get $\alpha = \alpha_l$, proving the result in this case.

Case 2: $|S| > 1$

In this case all the roots in S are of the first kind (as can be easily seen from the structure of the Lie superalgebras $B(m, n)$ and $D(m, n)$) and we may take

$$S = \{\alpha_l, \alpha_{l-1} + \alpha_l + \alpha_{l+2}, \alpha_{l-2} + \cdots + \alpha_{l+2}, \cdots\}.$$

It is not hard to see that for all $r_i \in \tilde{W}_-$, there are roots $\alpha, \beta \in S$ such that $r_i(\alpha) = \alpha - \alpha_i$ and $r_i(\beta) = \beta + \alpha_i$. Moreover as $(r_i(\alpha), r_i(\beta)) = 0$, these two roots are unique. Conversely, given any root $\alpha \in S$, there is a unique reflection $r_i \in \tilde{W}_-$ (respectively, $r_j \in \tilde{W}_-$) such that $r_i(\alpha) = \alpha - \alpha_i$ (respectively, $r_j(\alpha) = \alpha + \alpha_j$) unless $\alpha = \alpha_l$ (respectively, $\alpha = \alpha_1 + \cdots + \alpha_{2l+1}$), in which case there are none. Therefore, for all $w \in \tilde{W}_-$

$$w(\sum_{\alpha \in S} \alpha) = \sum_{\alpha \in S} \alpha.$$

Now, let $w \in \tilde{W}$ be such that $w(T) \cap S$ is maximal. Suppose that $w(T) \neq S$. We may then also assume that among all such w, the height of $\sum_{\alpha \in w(T) - (w(T) \cap S)} \alpha$ is minimal. Without loss of generality, we write $w = 1$. Suppose that $\alpha \in T - S$ such that $\alpha + \alpha_i \in \Delta$ for some simple root α_i of negative norm. From the above, there is a unique root $\beta \in \Delta^+$ such that $\beta - \alpha_i \in \Delta$. If $\beta \in S$ then the above implies that $\alpha \in S$, contradicting the definition of the root α. A similar argument holds if $\alpha - \alpha_i \in \Delta$. Therefore for all $\alpha \in T - S$, if $r_i \in \tilde{W}_-$ and $r_i(\alpha) \neq \alpha$, then $r_i(\beta) \neq \beta$ forces $\beta \notin S$. In other words, for all $\gamma \in T \cap S$, $(\gamma, \alpha_i) = 0$. By the above assumption for all $\alpha \in T - S$, it follows that for all $\alpha \in T - S$, $\alpha - \alpha_i \notin T - S$. If $\alpha \neq \alpha_l$, then this forces $|r_i(T) \cap S| > |T \cap S|$. Hence, $T = (r_i(T) \cap S) \cup \{\alpha_l\}$. However $\alpha_l \in S$, which gives a contradiction. Hence $T = S$. As $w_-(S) = S$ for all $w_- \in \tilde{W}_-$ and for all $w_+ \in \tilde{W}_+$, $w_- w_+ = w_+ w_-$, the result follows in this case. $\qquad\square$

Considering the set S described in the proof of Lemma 15 and the calculations preceding Lemma 15, it follows that there is a base containing a maximal isotropic subset (though it may not be the set S). Lemma 15 also tells us that for all $w \in \tilde{W}_-$, $w(T) = T$ for all maximal isotropic subsets in Δ and more particularly that if $\alpha \in T$ such that $r_i(\alpha) \neq \alpha$, then there is a root $\beta \in T$ such that $r_i(\beta) \neq \beta$ and $r_i(\alpha) < \alpha$ if and only if $r_i(\beta) > \beta$. Let $\tilde{\Pi} = \{\beta_1, \cdots, \beta_N\}$ be any base containing a maximal isotropic subset S. Suppose that $\beta_i \in S$. What precedes implies that there is a reflection $r_\gamma \in \tilde{W}_-$ such that $r_\gamma(\beta_i) < 0$. So

$$\gamma = \beta_i + \cdots + \beta_j,$$

where $|\beta_i|^2 = 0 = |\beta_j|^2$ and if $k \in \text{supp}(\beta) - \{i, j\}$ (here the support is taken with respect to the base $\tilde{\Pi}$), then $|\beta_k|^2 > 0$. Moreover the above also show that $\beta_j \in S$. Also the roots β_k are short roots in Δ and so $r_{\beta_k} \in \tilde{W}$. Set $w = r_\gamma r_{\gamma - \beta_i - \beta_j}$. Note that $|\gamma|^2 |\gamma - \beta_i - \beta_j|^2 < 0$.

As a consequence,

$$\frac{e(w(\rho))}{(1 + e(-w(\beta_i)))(1 + e(-w(\beta_j)))}$$
$$= \frac{e(\rho + \beta_i + \beta_j)}{(1 + e(\beta_i))(1 + e(\beta_j))}$$
$$= \frac{e(\rho)}{(1 + e(-\beta_i))(1 + e(-\beta_j))}$$

Hence by induction on $|S|$, for the trivial irreducible module V we get

$$\mathrm{ch}\, V = R^{-1}\mathrm{ch}_{0,S,\tilde{\Pi},W^\sharp}.$$

Note that the trivial module for the Lie superalgebras of type $B(m,n)$, $m > 0$, $D(m,n)$ and exceptional is never tame since $(\rho, \theta) > 0$ in these cases.

1.3.6 Proof: Part V

The super-character and super-denominator formulae follow in exactly similar fashion. We only need to notice the following changes. This time equality (1) is replaced by:

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha))\mathrm{sch}\,(V) =$$

$$\sum_{\substack{\lambda \leq \Lambda \\ |\lambda + \rho|^2 = |\Lambda + \rho|^2}} \check{c}_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))$$

Note that $\check{\epsilon}(w) = \epsilon(w)$ for all $w \in \tilde{W}$ for the base Π since the non-isotropic simple roots in this base are all even. Moreover if the Lie superalgebra G is not of type $B(m,n)$, then $\check{\Delta}_0 = \Delta_0$ since the only odd roots are isotropic. Hence in these cases, $\check{\epsilon}(w) = \epsilon(w)$ for all $w \in W$. We have to be careful that $\check{c}_{\Lambda - k\alpha} = 1$ for all $k \geq 0$ and $\alpha \in S_{\Lambda + \rho}$ since the product on the right hand side of the above equation contains minus signs and not plus signs as is the case for Weyl denominator. Moreover $\check{\epsilon}(w)$ is needed rather than $\epsilon(w)$ for Lie superalgebras of type $B(m,n)$. Indeed if we apply the reflection r_θ to the above equality, letting β be the highest isotropic root with only non-negative norm simple roots in its support, and as $|\alpha_l| = 0$,

we then get:

$$e(\rho + \frac{1}{2}\theta)(1 + e(\theta)) \prod_{\alpha \in (\Delta_0^+ - \{\theta\})} (1 - e(-\alpha))\mathrm{sch}\,(V) =$$

$$\sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} \check{c}_\lambda e(r_\theta(\lambda + \rho))(1 - e(\alpha_l))(1 - e(\beta))(1 - e(\frac{1}{2}\theta))$$

$$\times \prod_{\alpha \in \Delta_1^+ - \{\beta, \alpha_l, \frac{1}{2}\theta\}} (1 - e(-\alpha))$$

Equivalently,

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha))\mathrm{sch}\,(V) =$$

$$\sum_{\substack{\lambda \leq \Lambda \\ |\lambda+\rho|^2 = |\Lambda+\rho|^2}} \check{c}_\lambda e(r_\theta(\lambda + \rho)) \prod_{\alpha \in \Delta_1^+} (1 - e(-\alpha))$$

In other words, contrary to the case of equality (1), which gets multiplied by -1, this equality does not.

2 Affine Lie superalgebras

2.1 Introduction

Unless otherwise stated, in this second part we will simply say a Lie superalgebra for an affine one with non degenerate, supersymmetric, invariant bilinear form. By this we mean an affinization of any of the following finite dimensional simple Lie superalgebras:

$$A(m,n), B(m,n), C(n), D(m,n), F_4, G_3, D(2,1;\alpha),$$

as classified in [Y], hence including both twisted and untwisted ones. Affine Kac-Moody superalegbras, namely the restricted subclass of affine Lie superalgebras without isotropic simple roots, were studied and a character formula computed for all their irreducible integrable highest weight modules in [K2]. The technical problems arising in the calculation of the character formula in the affine case is the same as in the finite dimensional one. This is the reason why standard methods do not work.

There have been some attempts to derive this formula in particular cases. In [KW1], the authors consider the $A(1,2)^{(1)}$ case

and conjecture a denominator formula for untwisted affine cases. This is studied in greater detail in [KW2]. Based on her computations in the finite dimensional context in [G1], the author gives a proof of the denominator formula for the affine Lie superalgebras with non-zero dual Coxeter number in [G2]. The zero Coxeter case has recently been proved in [GR].

In this second part, we give a proof of the (super)-character formula for all the irreducible partially integrable highest weight modules except when the Lie superalgebra is affine with dual Coxeter number equal to zero and the module is of level 0. The meaning of integrability needs to be clarified. When G is a finite dimensional Lie superalgebra, integrability means the module is finite dimensional. Equivalently all the subalgebras of G isomorphic to $sl(2)$ generated by non-isotropic roots act (locally) finitely. However as was pointed out in [KW1] and [Rao], if we restrict ourselves to this condition for affine Lie superalgebras, then for several of them only the trivial module satisfies this condition. Hence in the affine case, we will take the definition given in [KW1] (see section 2). This in effect amounts to partial integrability (as defined in [DP])or weak integrability (as defined in [RF]) rather than full integrability. Our proof of the character formula relies heavily on the proof of the character formula in the finite dimensional case given in the first part. We deduce a denominator formula for affine Lie superalgebras with non-trivial dual Coxeter number; in this case, there is a technical problem in our method, which we do not address in this paper. The main idea behind the proof is independent of special features of the different types of affine Lie superalgebras with symmetrizable Cartan matrix, though final calculations have to take these into account.

2.2   Notation and some fundamental properties

In this section, we fix the notation that will be used throughout the second part and give basic properties of the Lie superalgebras in question. We do not restate any notation, results or explanations already given in Part 1.

1. Let $\hat{G}$ be an affine Lie superalgebra with a non degenerate, supersymmetric, invariant bilinear form $(.,.)..$ If $\hat{G}$ is an untwisted affine Lie superalgebra, then set $\hat{G} = G^{(1)}$. If $\hat{G}$ is twisted then set $\hat{G} = L^{(i)}$, where L is a finite dimensional simple Lie superalgebra. So there is a diagram automorphism of L of order $i \geq 2$ and we set the finite dimensional Lie superalgebra G to be the 1-eigenspace of this autormorphism.

2. Let H be a Cartan subalgebra of G and $\Delta \leq H^*$, $\Delta_0 \leq H^*$, $\Delta_1 \leq H^*$ be respectively the set of roots, even root, and odd roots with respect to the Cartan subalgebra H. Set

$$\hat{\Delta}_0 = \{\alpha \in \Delta_0 : \frac{1}{2}\alpha \notin \Delta\}.$$

3. Let $\hat{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_N\}$ be a base of the root system Δ. Let $\hat{E} = \mathbb{R}\hat{\Pi}$ be the real vector space spanned by the base $\hat{\Pi}$.

Let $h_i \in H$, $0 \leq i \leq n$, be such that $(h_i, h_j) = (\alpha_i, \alpha_j)$.

Let $\hat{\Delta}^+$ be the set of positive roots with respect to the base $\hat{\Pi}$. Set
$$\hat{\Delta}_0^+ = \hat{\Delta}^+ \cap \hat{\Delta}_0, \quad \hat{\Delta}_1^+ = \hat{\Delta}^+ \cap \hat{\Delta}_1;$$

$\hat{\Pi}_0$ to be the base of the Lie algebra with positive root system $\hat{\Delta}_0^+$; Set

$$\delta = \sum_{i=1}^{n} a_i \alpha_i$$

to be the isotropic imaginary root in $\hat{\Delta}^+$ such that all imaginary roots are integral multiples $m\delta$ with $m \in \mathbb{Z} - \{0\}$. As finite dimensional Lie algebras do not contain imaginary roots, $a_i > 0$ for all $0 \leq i \leq n$. Moreover, $(\delta, \alpha) = 0$ for all $\alpha \in \hat{\Delta}$ (see [KW1]). Define the weight $\Phi_0 \in H^*$ as follows:

$$(\Phi_0, \alpha_i) = 0 \text{ if } i \neq 0 \text{ and } (\Phi_0, \delta) = a_0^{-1}$$

Then, $\hat{\Pi} \cup \{\Phi_0\}$ is a basis of the dual space $\hat{H}^*$.

4. Let
$$\hat{G}_\alpha = \{x \in \hat{G} : [h, x] = \alpha(h)x, h \in \hat{H}\}$$

and
$$\mathrm{mult}(\alpha) = \dim \hat{G}_\alpha.$$

Then, $\mathrm{mult}(\alpha) = 1$ if $\alpha \in \hat{\Delta} - \{n\delta : \pm n \in \mathbb{N}\}$ and

$$\mathrm{mult}(n\delta) = \mathrm{mult}(\delta)$$

for all $n \in \mathbb{N}$.

Set $e_i \in G_{\alpha_i}$, $f_i \in G_{-\alpha_i}$ to be the generators of the derived sub-superalgebra $[\hat{G}, \hat{G}]$, where $[e_i, f_i] = h_i$ and $[h, e_i] = (h, h_i)e_i$, $[h, f_i] = -(h, h_i)f_i$ for all $h \in \hat{H}$.

5. As in [KW1], let $Y = \{\mu \in \hat{H}^* : Re(\delta, \mu) > 0\}$. The expressions $\frac{\sum_{\lambda \in H^*} a_\lambda e(\lambda)}{\sum_{\lambda \in H^*} b_\lambda e(\lambda)}$ are meromorphic functions on the domain Y. Set

$$\hat{R} = \frac{\prod_{\alpha \in \Delta_0^+}(1 - e(-\alpha))^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_1^+}(1 + e(-\alpha))^{\text{mult}(\alpha)}}$$

and

$$\overset{\circ}{\hat{R}} = \frac{\prod_{\alpha \in \Delta_0^+}(1 - e(-\alpha))^{\text{mult}(\alpha)}}{\prod_{\alpha \in \Delta_1^+}(1 - e(-\alpha))^{\text{mult}(\alpha)}}$$

are respectively the Weyl denominator and super-denominator with respect to the base Π.

6. Let $\hat{\rho}$ be the Weyl vector with respect to the base $\hat{\Pi}$, i.e.

$$(\hat{\rho}, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i) \quad \forall 0 \leq i \leq n,$$

$\hat{\rho}_0$ be the Weyl vector of the Lie algebra $\hat{G}_0$ with respect to a base $\hat{\Pi}_0$. Set $\hat{\rho}_1 = \hat{\rho}_0 - \hat{\rho}$.

By the definition of the weight Φ_0,

$$\hat{\rho} = \rho + a_0(\frac{1}{2}|\alpha_0|^2 - (\rho, \alpha_0))\Phi_0$$

As in supp6.4 in [K4], $\theta' = \rho - a_0\alpha_0$ is the maximal root in Δ^+. Hence

$$(\hat{\rho}, \delta) = (\rho, \theta') + \frac{1}{2}a_0^{-1}|\theta'|^2.$$

If $a_0 = 1$ then (ρ, δ) is the dual Coxeter number $h^\vee$.

Because the bilinear form is of corank 1 on the space E, the Weyl vector is only unique modulo $\mathbb{R}\delta$.

7. In [Y], H. Yamane extended the concept of odd reflections to affine Lie superalgebras. The action of these linear maps on the space $\hat{E}$ is the natural extension of the definition given in section 1.2. For $|\alpha_i|^2 = 0$, write $\hat{\Pi}_i := s_{\alpha_i}(\hat{\Pi})$ and define

$$\hat{\Delta}_{i0}^+ := \hat{\Delta}_0 \cap \mathbb{Z}_+\Pi_i, \quad \hat{\Delta}_{i1}^+ := \hat{\Delta}_1 \cap \mathbb{Z}_+\hat{\Pi}_i.$$

The notation

$$\lambda \leq_j \mu$$

will mean $\mu - \lambda \in \mathbb{Z}_+\hat{\Pi}_j$. More generally if $\hat{\Pi}'$ is a base obtainable from the base Π by successive applications of odd reflections, then

$$\lambda \leq_I \mu$$

will mean $\mu - \lambda \in \mathbb{Z}_+\hat{\Pi}'$. As a consequence of the definition,

Corollary 8. *Let* $1 \leq i \leq N$ *be such that* $\alpha_i \in \hat{\Delta}_1$ *and* $|\alpha_i|^2 = 0$. *Then, with respect to the base* $\hat{\Pi}_i$, *the root* δ *remains the positive imaginary root such that if* $\alpha \in \hat{\Delta}$ *is a positive imaginary root with respect to the base* $\hat{\Pi}_i$, *then* $\alpha = l\delta$ *for some integer* $l > 0$.

Proof. Let δ_i be the positive imaginary root such that if $\alpha \in \Delta$ is a positive imaginary root with respect to the base Π_i, then $\alpha = s\delta$ for some integer $s > 0$. Since δ is an imaginary root, $\delta = l\delta_i$ for some integer l and symmetrically, $\delta_i = l_i\delta$ for some integer l_i. Therefore, $\delta = \pm\delta_i$. If $|\hat{\Pi}| = 1$ then G is a Lie algebra and there is nothing to prove. Otherwise, there is an index $j \in \operatorname{supp}(\delta)$ such that $j \neq i$ and so δ remains positive with respect to the base $\hat{\Pi}_i$. Equivalently, $\delta = \delta_i$. $\qquad\square$

For the proof of the following result, see Corollary 1.

Corollary 9. *Let* $\alpha_i \in \hat{\Pi}$ *be such that* $|\alpha_i|^2 = 0$. *Then the set of positive even (resp. odd) roots with respect to the base* $\hat{\Pi}_i$ *is*

$$\hat{\Delta}_0^+ \quad \text{(resp.} \quad (\hat{\Delta}_1^+ - \{\alpha_i\}) \cup \{-\alpha_i\})$$

and

$$\hat{\rho} + \alpha_i$$

is a Weyl vector with respect to the base Π_i.

Set

$$\hat{\rho}_i = \hat{\rho} + \alpha_i.$$

Note also that $(\hat{\rho}, \delta)$ is invariant of the base chosen. Indeed this is clear if we consider the base $w(\Pi)$, where $w \in W$. By Corollaries 1 and 2, $(\hat{\rho} + \alpha_i, \delta) = (\hat{\rho}, \delta)$ since $(\delta, \alpha_i) = 0$ (as $(\delta, \alpha) = 0$ for all $\alpha \in \Delta$).

As a consequence:

Lemma 15. *If* $a_0 \neq 1$, *then*

$$(\hat{\rho}, \delta)h^\vee \geq 0$$

and $(\hat{\rho}, \delta) \neq 0$.

Proof. By assumption, $a_0 > 1$.

Claim: $(\rho, \theta')((\rho, \theta') + \frac{1}{2}|\theta'|^2) \geq 0$ Since all expressions are independent of the base chosen, without loss of generality we assume the base Π to contain at most one isotropic simple root.

Hence if the finite dimensional Lie superalgebra G is of type $A(m,n)$ or $C(n)$ then $|\theta'|^2 = 0$ so that our claim holds. If G is of type $B(0,n)$ then we may assume all simple roots to have positive norm and hence our claim holds. When G is of type $B(m,n)$, $m > 0$ or $D(m,n)$ or is exceptional, without loss of generality, assume that $|\theta'|^2 < 0$. If $(\rho, \theta') < 0$, then our claim again holds. So suppose that $(\rho, \theta') > 0$. In this case, $\theta' = \theta$ and a case by case study shows that our claim holds unless G is of type $B(1,1)$. However from p.19 in [Y] we know that $B(1,1)$ only has untwisted affinizations, in which case $a_0 = 1$, contradicting assumption.

Without loss of generality, assume that $(\rho, \theta') > 0$. If $(\theta', \theta') \geq 0$. Then the result follows immediately. Otherwise, $(\hat{\rho}, \delta) > 0$ also follows from the above claim and the fact that $a_0^{-1} < 1$. $\quad\square$

8. A highest weight $\hat{G}$-module $V = V(\Lambda)$ of highest weight $\Lambda \in \hat{H}^*$ with respect to the base Π, is integrable if for all

$$\alpha \in \Delta^+ \text{ such that } |\alpha|^2 \neq 0, \; \frac{2(\Lambda, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_+. \qquad (int)$$

Few representations satisfy condition (int). Indeed suppose that the level of the irreducible highest weight representation is non-trivial, i.e. $(\Lambda, \delta) \neq 0$. Without loss of generality, assume that

$$(\Lambda, \delta) > 0.$$

Now, $\hat{G}_0 = (\hat{G}_0)_+ \oplus (\hat{G}_0)_-$, where $(\hat{G}_0)_+ = \langle \hat{G}_\alpha : \alpha \in \hat{\Delta}_0, |\alpha|^{>}0 \rangle$ and $(\hat{G}_0)_- = \langle \hat{G}_\alpha : \alpha \in \hat{\Delta}_0, |\alpha|^{<}0 \rangle$. If $\hat{G}_0^- \neq 0$, let $\alpha \in \Delta_0^+$ be a root of negative norm. Then, for all $n \in \mathbb{Z}_+$, $\alpha + n\delta \in \hat{\Delta}^+$ is a root of negative norm. However, for $n >> 0$,

$$\frac{(\Lambda, n\delta + \alpha)}{(\alpha, \alpha)} < 0.$$

Let $V = V(\Lambda)$ be an irreducible highest weight G-module with highest weight $\Lambda \in \hat{H}^*$ and let v_Λ be a highest weight vector of V. For reasons of simplicity, by abuse of notation, we will also write Λ for its restriction to H.

We need to choose one of the components of $\hat{G}_0$. We do this as follows. From Lemma 1, we know that if $(\hat{\rho}, \delta) = 0$ then the Lie superlagebra G is untwisted. From [KW1] it therefore follows that $(\hat{\rho}, \delta) = 0$ if and only if the finite dimensional Lie

superalgebra G is of type $A(n,n)$, $D(n+1,n)$, or $D(2,1;a)$ and the affine Lie superalgebra G is untwisted. Set

$$(\hat{\Delta})_0^\sharp = \begin{cases} \{\alpha \in (\hat{\Delta}_0^+ : |\alpha|^2(\hat{\rho},\delta) > 0\} & \text{if } (\hat{\rho},\delta) \neq 0 \\ \{\alpha \in (\hat{\Delta})_0^+ : |\alpha|^2 > 0\} & \text{if } G \text{ is of type } A(n,n) \\ \{\alpha \in (\hat{\Delta})_0^+ : |\alpha|^2|\theta|^2 < 0\} & \text{if } G \text{ is of type } D(n+1,n) \text{ or } D(2,1;a). \end{cases}$$

Definition 3. *The highest module G-$V = V(\Lambda)$ of highest weight Λ is said to be partially integrable if the following hold:*

(a) *the G-module generated by the highest weight vector v_Λ is finite dimensional; and*

(b) *for all roots $\alpha \in \hat{\Delta}_0^\sharp$, the elements $x \in G_{\pm\alpha}$ act locally finitely on the module V.*

In other words, if a module is partially integrable, it satisfies condition (int) for at least one component of the even part of the Lie superalgebra. Note that in [KW1] these modules are called integrable.

From now on the G-module $V = V(\Lambda)$ will denote a module satisfying Definition 1. Set $P(\Lambda)$ to be the set of weights of the module $V(\Lambda)$. For $\mu \in \hat{H}^*$, we write

$$V_\mu = \{v \in V : h.v = \mu(h)v, \forall h \in \hat{H}\}.$$

Then, $\dim V_\mu < \infty$ for all $\mu \in \hat{H}^*$ and following [K4]

$$\operatorname{ch} V = \sum_{\mu \leq \Lambda} (\dim V_\mu)e(\mu) \in \mathcal{E},$$

where $\mathcal{E}$ is now the algebra over $\mathbb{C}$ of formal series of the form

$$\sum_{\mu \in \hat{H}^*} c_\mu e(\mu)$$

where $c_\mu \in \mathbb{C}$ and $c_\mu = 0$ for μ outside the union of a finite number of sets of the type $D(\lambda) = \{\mu \in \hat{H}^* : \mu \leq \lambda\}$.

We assume that if the dual Coxeter number is zero then the G-module V has non-trivial level, i.e. $(\Lambda,\delta) \neq 0$.

9. Let $\hat{W}$ be the Weyl group of the Lie superalgebra G. For $w \in \hat{W}$, let $l(w)$ be the number of simple reflections needed to write w as a word and $\hat{l}(w)$ the number of simple reflections corresponding to the set $\hat{\Delta}_0^+$ needed to write w. Set

$$\epsilon(w) = (-1)^{l(w)} \quad \text{and} \quad \hat{\epsilon}(w) = (-1)^{\hat{l}(w)}.$$

We define the following subgroups of the Weyl group:

$$\hat{W}_+ = \langle r_\alpha : |\alpha|^2 > 0 \rangle$$

$$\hat{W}_- = \langle r_\alpha : |\alpha|^2 < 0 \rangle,$$

i.e.

$$\hat{W} = \hat{W}_- \times \hat{W}_+.$$

Set

$$\hat{W}^\sharp = \langle W, r_\alpha : \alpha \in \hat{\Delta}^\sharp \rangle.$$

Equivalently,

$$\hat{W}^\sharp = T^\sharp \ltimes W,$$

where $T^\sharp$ is the group of translations [K4] induced by the lattice $\mathbb{Z}\hat{\Delta}_0^\sharp$.

2.3 (Super)-character and (super)-denominator formulae

2.3.1 Statement of main results

Let $V = V(\Lambda)$ be a partially integrable highest-weight irreducible $\hat{G}$-module with highest weight Λ with respect to the base Π. The reason why the character formula of partially integrable modules of level 0 of affine Lie superalgebras for which $(\rho, \delta) = 0$ does not follow from the method used and would need extra arguments is given in the following result.

Proposition 4. *Suppose that* $(\Lambda, \delta) \neq 0$ *or* $(\rho, \delta) \neq 0$. *Then,* $|S_{\Lambda+\rho}| < \infty$.

Proof. Suppose that the proposition is false. Since $|\Delta_f| < \infty$, in this case there is a root $\alpha \in \Delta_f^+$ such that for infinitely many integers n, $(\Lambda + \rho, \alpha + n\delta) = 0$. Therefore, $(\Lambda + \rho, \delta) = 0$. However by definition of a partially integrable module, this forces $(\Lambda, \delta) = 0 = (\rho, \delta)$, contradicting assumption. $\square$

We restate here the list of affine Lie superalgebras for which $(\rho, \delta) = 0$ (see section 2.1 for the proof).

Proposition 5. *For the affine Lie superalgebra* G, $(\rho, \delta) = 0$ *if and only* G *is untwisted of type* $A(n, n)^{(1)}$, $D(n + 1, n)^{(1)}$ *or* $D(2, 1; a)$. *In particular,* (ρ, δ) *is then the dual Coxeter number.*

We generalize the notion of a tame module of an affine Lie superalgebra given in [KW1]:

Definition 4. *The $\hat{G}$-module $V = V(\Lambda)$ is said to be generalized tame (with respect to the base Π) if*

$$\operatorname{ch} V = \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, W^\sharp}.$$

When $|S_{\Lambda+\rho}| = 1$, the $\hat{G}$-module V is said to be tame.

We need to distinguish between two types of isotropic roots.

Definition 5. *The isotropic root $\alpha \in \Delta^+$ is said to be of the first kind if there is a base $\Pi' = s_p \cdots s_1(\Pi)$, where the s_i are odd reflections, such that $\alpha \in \Pi'$. Otherwise the isotropic root α is said to be of the second kind.*

Theorem 6. *Let G be either a finite dimensional Lie algebra or a Lie superalgebra of type $A(m,n)$, $C(n)$, or $B(0,n)$. Assume that if Lie superalgebra $\hat{G}$ is of type $A(n,n)^{(1)}$, then the module V is not of level 0, i.e. $(\Lambda, \rho) \neq 0$. Then the $\hat{G}$-module V is generalized tame with respect to all bases.*

Theorem 7. *Let the Lie superalgebra G be of type $B(m,n)$, $m > 0$, or $D(m,n)$. Assume that the Lie superalgebra G is not of type $D(m,n)^{(2)}$ or $A(m, m+2n-1)^{(4)}$ and when it is of type $D(n+1,n)^{(1)}$, assume that the module V is not of level 0. Set α to be the maximal isotropic root of the second kind in Δ_f^+ such that if $i \in \operatorname{supp}(\alpha)$, then $|\alpha_i||\theta| \leq 0$.*

1. *If the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is infinite dimensional, then the $\hat{G}$-module V is generalized tame.*

2. *if the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is finite dimensional then the $\hat{G}$-module V is not generalized tame and*

$$\operatorname{ch} V = \begin{cases} \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes \tilde{W}} & \text{if } \frac{2(\Lambda+\rho, \theta)}{(\theta,\theta)} \leq 0 \\ \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes W} - \operatorname{ch}_{r_\theta(\Lambda+\rho)-\rho, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes W'} & \text{otherwise} \end{cases}.$$

   Moreover, if $\frac{2(\Lambda+\rho, \theta)}{(\theta,\theta)} \leq 0$, then $S_{\Lambda+\rho} \leq \Delta^+$ contains only isotropic roots of the first kind and otherwise $S_{\Lambda+\rho} = \{n\delta \pm \alpha\}$.

Theorem 8. *Let the Lie superalgebra G be exceptional. When it is of type $D(2,1;a)^{(1)}$, assume that the module V is not of level 0.*

1. *If the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is infinite dimensional, then the $\hat{G}$-module V is tame;*

2. *if the G-module with highest weight $r_\theta(\Lambda + \rho) - \rho$ is finite dimensional then the module V is not generalized tame and*

$$\operatorname{ch} V = \begin{cases} \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes \tilde{W}_f} & \text{if } \frac{2(\Lambda+\rho, \theta)}{(\theta, \theta)} \leq 0, \\ \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes W_f} - \operatorname{ch}_{r_\theta(\Lambda+\rho)-\rho, S_{\Lambda+\rho}, \Pi, T^\sharp \ltimes W_f} & \text{otherwise.} \end{cases}$$

Let the Lie superalgebra $\hat{G}$ be of type $D(m, n)^{(2)}$ or $A(m, m+ 2n-1)^{(4)}$. Then G is of type $B(m, n-1)$. Let M be the finite dimensional Lie superalgebra with base $\Pi_M = \{\alpha_0, \cdots, \alpha_{N-1}\}$ and root system Δ_M. Let $(\Pi_M)_0$ be the base of $(\Delta_M) \cap \Delta_0^+$. Let $\theta_L \in (\Pi_M)_0$ be the root such that $\theta_L, \frac{1}{2}\theta_L \notin \Pi_M$. Consider the subgroup of translations $\tilde{T} \leq T^\sharp$ induced by the lattice generated by simple roots whose norm is of the same sign as the (ρ, δ) when $\hat{G}$ is not of type $D(2, 1; a)^{(1)}$ and by simple roots of positive norm when $\hat{G}$ is of type $D(2, 1; a)^{(1)}$ (in which case as mentioned earlier, $(\rho, \delta) = 0$).

Theorem 9. *Let the Lie superalgebra $\hat{G}$ be of type $D(m, n)^{(2)}$ or $A(m, m + 2n - 1)^{(4)}$. The module V is never generalized tame and*

$$\operatorname{ch} V = \begin{cases} \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, \tilde{T} \ltimes \tilde{W}} & \text{if } \frac{2(\Lambda+\rho, \theta)}{(\theta, \theta)} \leq 0 \\ \operatorname{ch}_{\Lambda, S_{\Lambda+\rho}, \Pi, \tilde{T} \ltimes W} - \operatorname{ch}_{r_\theta(\Lambda+\rho)-\rho, S_{\Lambda+\rho}, \Pi, \tilde{T} \ltimes W'} & \text{otherwise} \end{cases}.$$

As a consequence, the denominator formula can be nicely expressed with respect to a base with special properties.

Theorem 10. *Assume that the Lie superalgebra $\hat{G}$ is not of type $A(n, n)^{(1)}$, $D(n + 1, n)^{(1)}$ or $D(2, 1; \alpha)^{(1)}$. Let Π be a base containing a maximal isotropic subset S of the set of roots Δ. Then the denominator formula is:*

$$e(\rho)R = \Gamma_{0, S, \Pi, T^\sharp \ltimes W^\sharp}$$

unless $\hat{G}$ is of type $D(m, n)^{(2)}$ or $A(m, m+2n-1)^{(4)}$; in which cases,

$$e(\rho)R = \Gamma_{0, S, \Pi, \tilde{T} \ltimes W^\sharp}$$

For the definition of W' and $\tilde{W}$ see discussion preceding Theorem 2 and for $W^\sharp$ see section 1.2.

Theorem 11. *Replacing* ch *with* sch *in Theorems 1, 2, 3 and 4 gives the super-character for the $\hat{G}$-module V and the super-denominator formula.*

Note that even for the cases $A(m, n)^{(1)}$ in general our result is not equivalent to the conjecture in [KW1] since the roots in $S_{\Lambda+\rho}$ may not be contained in the finite root system Δ of the finite dimensional Lie superalgebra $A(m, n)$.

2.3.2 The proof

In the rest of this section we prove these theorems. The main parts of the proof of the finite dimensional case given in the first part apply in its exact form to the affine Lie superalgebra $\hat{G}$. Hence we do not repeat these arguments. We once again consider the equality (1):

$$e(\rho) \prod_{\alpha \in \Delta_0^+} (1 - e(-\alpha))^{\mathrm{mult}(\alpha)} \mathrm{ch}\, V$$

$$= \sum_{\lambda \leq \Lambda, |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(\lambda + \rho) \prod_{\alpha \in \Delta_1^+} (1 + e(-\alpha))^{\mathrm{mult}(\alpha)}, \qquad (1)$$

Clearly Corollary 4 folds for any base $\hat{\Pi}$. We need to note that as in the finite dimensional case, if we consider the module V as a G_0-module, then from Lemma 9.6 in [K4] that for all weights $\lambda \in H^*$, there are finitely many submodules $U^1, \cdots, U^m$ and a subset $J \subset \{1, \cdots, m\}$ such that

$$0 = U^0 \subset U^1 \subset \cdots \subset U^m = V$$

and if $i \in J$, U^i/U^{i-1} is isomorphic to an irreducible G_0-module V^i with highest weight some $\Lambda^i \geq \lambda$; if $i \notin J$, then $(U^i/U^{i-1})_\tau = 0$ for every weight $\tau \geq \lambda$. As a consequence, if $\lambda + \rho_0$ is a weight such that the term $e(\lambda + \rho_0)$ appears in the expression $e(\rho_0)\Pi_{\alpha \in \Delta_0^+}(1 - e(-\alpha))$, then there are finitely many irreducible G_0-submodules $V^1, \cdots, V^m$ of highest weight Γ^i such that it appears in the expression

$$e(\rho_0)\Pi_{\alpha \in \Delta_0^+}(1 - e(-\alpha)) \sum_{i=1}^m \mathrm{ch}\, V^i$$

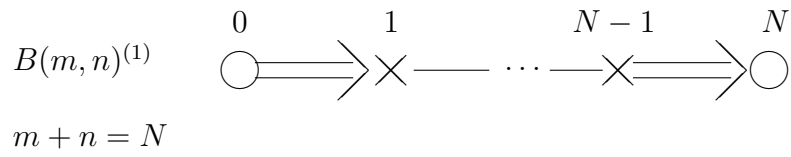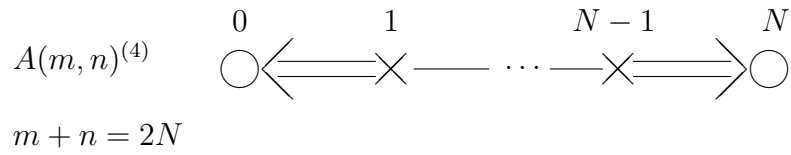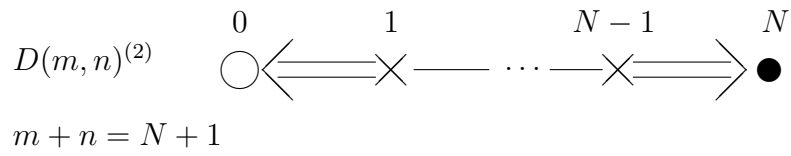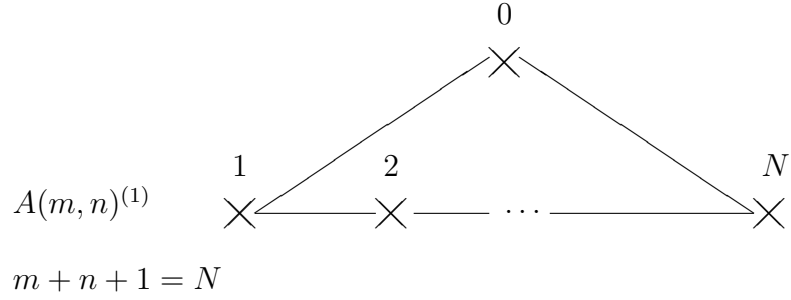$$= \sum_{i=1}^m \sum_{w \in W_+} (-1)^w e(w(\Gamma^i + \rho_0))$$

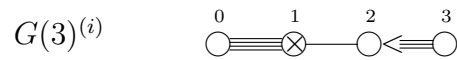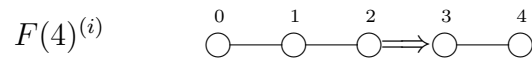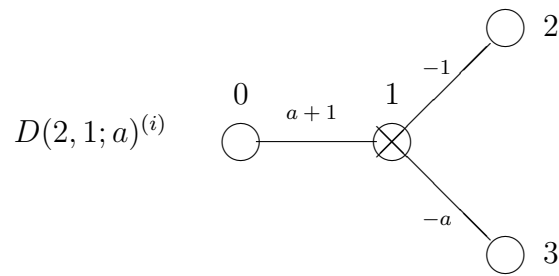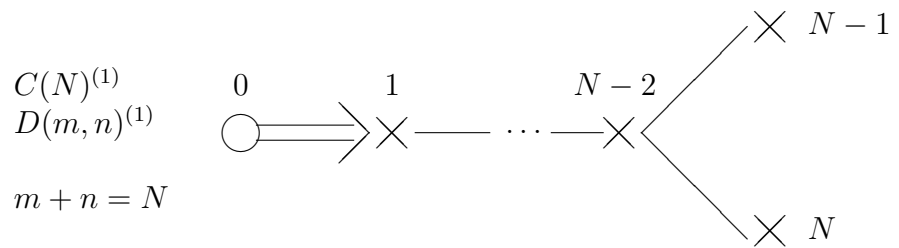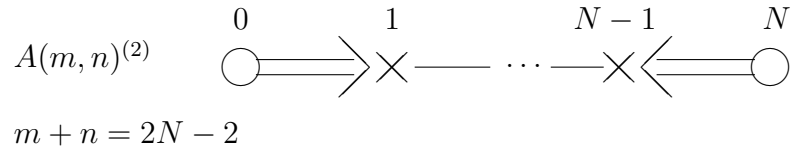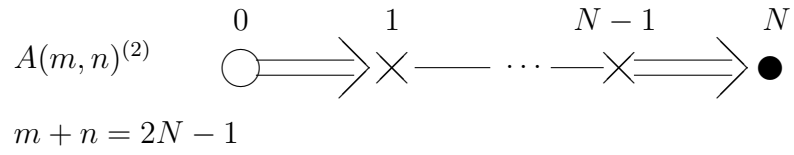Without loss of generality, we will assume that $(\rho, \delta) \geq 0$ and hence

$$W^\sharp = \begin{cases} \langle W_+, W_f \rangle & \text{if } (\rho, \delta) > 0 \\ W & \text{otherwise} \end{cases}.$$

Moreover we assume that either $(\rho, \delta) \neq 0$ or $(\Lambda, \delta) \neq 0$ or equivalently that

$$(\Lambda + \rho, \delta) \neq 0.$$

We first consider bases $\hat{\Pi}$ for which the finite type base Π contains a unique isotropic root and is such that $\hat{\Delta}_{0}^{\sharp} \cap \Delta$ is generated by simple roots in Π. For the Dynkin diagrams, considering the list given in [Y], we take:

$$A(m,n)^{(1)}$$
$$m + n + 1 = N$$

$$D(m,n)^{(2)}$$
$$m + n = N + 1$$

$$A(m,n)^{(4)}$$
$$m + n = 2N$$

$$B(m,n)^{(1)}$$
$$m + n = N$$

$A(m,n)^{(2)}$
$$0 \quad 1 \quad N-1 \quad N$$

$m+n = 2N-1$

$A(m,n)^{(2)}$
$$0 \quad 1 \quad N-1 \quad N$$

$m+n = 2N-2$

$C(N)^{(1)}$
$D(m,n)^{(1)}$
$$0 \quad 1 \quad N-2 \qquad N-1$$
$$N$$

$m+n = N$

$D(2,1;a)^{(i)}$
$$0 \quad \overset{a+1}{\quad} \quad 1 \qquad \overset{-1}{\quad} 2$$
$$\overset{-a}{\quad} 3$$

$F(4)^{(i)}$
$$0 \quad 1 \quad 2 \quad 3 \quad 4$$

$G(3)^{(i)}$
$$0 \quad 1 \quad 2 \quad 3$$

A case by case study shows that in all cases, except for $D(m,n)^{(2)}$ and $A(m, m+2n-1)^{(4)}$, any root of positive norm has either no isotropic simple root in its support or has an even num-

ber of distinct isotropic simple roots in its support. We therefore separate the proof into two cases.

Case 1: the Lie superalgebra G is neither of type $D(m,n)^{(2)}$ nor of type $A(m, m + 2n - 1)^{(4)}$

Hence arguments in Lemma 5 and 9 imply the following

Lemma 16. *For all weights $\lambda \in H^*$ such that $c_\lambda \neq 0$,*

$$w(\lambda + \rho) \leq \Lambda + \rho$$

for all $w \in \hat{W}_+$.

As a consequence, Lemma 6 also holds:

Lemma 17. *Let $\lambda \in H^*$ be a weight such that $c_\lambda \neq 0$. Then for all $w \in W_+$, $c_{w(\lambda+\rho)-\rho} = \epsilon(w)c_\lambda$.*

Set $\tilde{\hat{W}}_{\Pi'}$ to be subgroup of the Weyl group $\hat{W}$ generated by reflections $r_\alpha \in T^\sharp \ltimes \tilde{W}$ such that the support of the root α with respect to the base Π' contains either no isotropic simple roots or an even number of distinct ones. Lemma 8 holds since all previous results hold:

Lemma 18. *Let $\mu \in H^*$ be a weight satisfying the conditions expressed in Corollary 4 such that $\frac{(\mu+\rho,\alpha)}{(\alpha,\alpha)} > 0$ for $\alpha \in \hat{\Delta}^\sharp \cup \langle \alpha_i \in \Pi : |\alpha_i|^2 < 0 \rangle$. Then, for all $w \in \tilde{\hat{W}}_{\Pi'}$,*

$$w(\mu) \leq_, w(\Lambda)$$

for all bases Π'. In particular when G_f is either a Lie algebra or of type $A(m,n)$ or $C(n)$, for all $w \in W^\sharp$,

$$w(\tau) \leq_, w(\Lambda)$$

for all bases Π'.

Corollary 10. *Let $\mu \in H^*$ be a weight satisfying the conditions in Lemma 18. Then, $\mu = \Lambda$ or $\mu = \Lambda - k\alpha$, where α is an isotropic root of the second kind in the root system generated by a finite type sub-base of the base Π.*

Proof. Set

$$\mu + \rho = \Lambda + \rho - \sum_{i=1}^{N} k_i \alpha_i,$$

where $k_i \in \mathbb{Z}_+$. This may be written as follows:

$$\mu = \Lambda - l\delta - \sum_{i \neq j} l_i \alpha_i,$$

where $l \leq l_i \in \mathbb{Z}_+$.

Let G^j be the finite dimensional Lie superalgebra with base $\Pi^j = \Pi - \{\alpha_j\}$ and consider the irreducible highest weight G^j-module U_j of highest weight Λ (though it may not be finite dimensional). The same arguments in the proof of Lemmas 9, 11 and 12 force

$$\mu = \Lambda - l\delta - \sum_{\alpha} k_\alpha \alpha,$$

where $k_\alpha \neq 0, k_\beta \neq 0$ implies that the roots α and β are mutually orthogonal isotropic positive isotropic in the root system Δ^j generated by the finite type base Π^j or

$$\mu + \rho = \Lambda + \rho - l\delta - k\alpha,$$

where the root $\alpha \in \Delta^j$ has positive norm.

In the latter case, since $(\Lambda + \rho, \delta) > 0$ and

$$|\mu + \rho|^2 = |\Lambda + \rho|^2, \qquad (i)$$

we get

$$k = l = 0.$$

In the former case, from Proposition 2, either all the roots α are of the first kind or the sum contains a unique root α of the second kind with the properties described in Corollary 7. If all the roots α are of the first kind, since $l \leq l_i$ for all i, by considering adequate bases Π', $\mu \leq_\prime \Lambda$ forces $k_\alpha = 0$ for all α. Since $(\Lambda + \rho, \delta) \neq 0$, considering (i) we then get $l = 0$.

Finally suppose that

$$\mu = \Lambda - l\delta - k\alpha,$$

where $\alpha \in \Delta^j$ is of the second kind. In this case we know from Lemmas 11 and 12 that $\frac{(\Lambda+\rho,\theta^j)}{(\theta^j,\theta^j)} > 0$, where θ^j is the maximal root in Δ^j. Equality (i) implies that $(\Lambda + \rho, \alpha) \leq 0$ and $(\Lambda + \rho, \alpha) < 0$ if $l \neq 0$. In the latter case however, what precedes forces $|\theta^j|^2 < 0$ and so $(\Lambda + \rho, \delta) < 0$, contradicting assumption, and proving the result. $\qquad \square$

Lemmas 12 and 13 hold. All positive isotropic roots in Δ^+ can be written as $n\delta + \alpha$, where α is a positive isotropic root in some

finite type sub-system of Δ. Note that, as shown in the previous proof, $(\Lambda + \rho, k\delta + \alpha) = 0$, where α is a root of the second kind in some finite type sub-system of Δ, then $k = 0$. As a consequence, Theorems 6, 7, and 8 follow for the base $\hat{\Pi}$ as in part 1 because of Lemma 17.

Case 2: the Lie superalgebra G is either of type $D(m,n)^{(2)}$ or of type $A(m, m + 2n - 1)^{(4)}$

Let M be the Lie superalgebra with base $\alpha_0, \cdots, \alpha_{N-1}$ and root system Δ_M and the root θ_M be in Δ_M what the root θ is in Δ. Then $|\theta|^2 |\theta|_M^2 < 0$ and

$$\frac{2(r_\theta(\Lambda + \rho) - \rho, \theta)}{(\theta, \theta)} \leq 0$$

if and only if

$$\frac{2(r_{\theta_M}(\Lambda + \rho) - \rho, \theta)}{(\theta_M, \theta_M)} \geq 0.$$

Without loss of generality, assume that $|\theta|^2 < 0$. Then, the arguments used in the Case 1 together with the previous observation give Theorem 9 for the base Π.

The arguments in the first part give these theorems with respect to an arbitrary base, as well as the super-character formula and the (super)-denominator formula. Note that with respect to the base Π, the coefficients $j_{\Lambda, S_{\Lambda+\rho}, \Pi, U}$ and $\hat{j}_{\Lambda, S_{\Lambda+\rho}, \Pi, U}$ in the formulae is always 1 and hence as there are only finitely many bases up to conjugacy by the Weyl group, this coefficient is finite with respect to an arbitrary base Π.

The problem with the case $(\Lambda + \rho, \delta) = 0$ comes from Proposition 4. Since $(\Lambda + \rho, \delta) = 0$ if and only if $(\Lambda, \delta) = 0$ and $(\rho, \delta) = 0$, it only corresponds to the (super)-denominator formula for untwisted affine Lie superalgebras with 0 Coxeter number as for twisted affine Lie superalgebras, $(\rho, \delta) \neq 0$ as has been shown previously.

3 References

[BL]. J. Bernstein, D. Leites, A formula for the characters of the irreducible finite dimensional representations of lie superalgebras of series gl and sl, *Acad. Bulgare Sci.* **33** (1980), 1049–1051

[B]. R.E. Borcherds, Generalized Kac-Moody algebras, *J. Algebra* **115 (2)** (1988), 501–512

[CCLL]. D. Chapovalov, M. Chapovalov, A. Lebedev, D. Leites, The classification of almost affine (hyperbolic) Lie superalgebras, *Preprint arXiv:0906.1860v1*

[DP]. I. Dimitrov, I. Penkov, Partially and fully integrable modules over Lie superalgebras, *Studies in Advanced Mathematics* (series editor, S.-T. Yau, **4**, AMS and Internatl. Press 1997, 46–67

[F]. A. Frumkin, The irreducible characters of the Lie superalgebras of type $A(n, m)$ and filtrations of their Kac modules, *Israel J. Math.* **96** (1996), 267–279

[G1]. M. Gorelik, Weyl denominator identity for finite-dimensional Lie superalgebras, *Preprint arXiv*

[G2]. M. Gorelik, Weyl denominator identity for affine Lie superalgebras with non-zero dual coxeter number, *Preprint arXiv: 0911.5594v1*, 30 Nov 2009

[GR]. M. Gorelik, S. Reif, Weyl denominator identity for affine Lie superalgebras with zero Coxeter number, *private communication*

[J]. J. Van der Jeugt, Character formulae for the Lie superalgebra $C(n)$, *Comm. Algebra* **19** (1991), no. 1, 199-222

[JHKT-M]. J. Van der Jeugt, J.W.B. Hughes, R.C. King, J. Thierry-Mieg, A character formula for the singly atypical modules of the Lie superalgebra $sl(m/n)$, *Comm. Algebra* **18** (1990), no. 10, 3453–3480

[K1]. V. G. Kac, Lie Superalgebras, *Advances in Mathematics* **26** (1977), 8–96

[K2]. V. G. Kac, Infinite-Dimensional Algebras, Dedekind's η-Function, Classical Möbius Function and the Very Strange Formula, *Advances in Mathematics* **30** (1978), 85–136

[K3]. V. G. Kac, Characters of typical representations of classical Lie superalgebras, *Comm. Algebra* **5** (1977), no. 8, 889–897

[K4]. V. G. Kac, "Infinite dimensional Lie algebras", third ed., Cambridge University Press, 1990

[KW1]. V.G. Kac, W. Wakimoto, Integrable highest weight modules over affine lie superalgebras and number theory, *Progress in Math.* **123** (1994), 415–456

[KW2]. V.G. Kac, W. Wakimoto, Integrable highest weight modules over affine superalgebras and Appell's function, *Comm. Math. Phys.* **215** (2001), 631–682

[LSS]. D. Leites, M. Saveliev, V. Serganova, Embeddings of $osp(n/2)$ and the associated nonlinear supersymmetic equations, *Group theoretical Methods in Physics* (Utrecht), **1**, Yurmala, 1985, VNU Sci. Press, 1986, 255-297

[Rao]. S. Eswara Rao, Complete reducibility of integrable modules for affine Lie (super)algebras, *J. Algebra* **264** (2003), no. 1, 269–278

[RF]. S. Eswara Rao, V. Futorny, Integrable modules for affine Lie superalgebras, *Trans. Amer. Math. Soc.* **361** (2009), 5435–5455

[Ray]. U. Ray, A character formula for generalized Kac-Moody superalgebras, *J. Alg.* **177** no.1 (1995), 154–163

[S1]. V. Serganova, Characters of simple Lie superalgebras, *Proc. ICM* (1998), 583–594

[S2]. V. Serganova, Kac-Moody superalgebras and integrability, *Preprint*

[Y]. H. Yamane, On the defining relations of the affine Lie superalgebras and affine quantized universal enveloping superalgebras, *Publ. Res. Inst. Math. Sci.* **35** (1999), 321–390; (Errata), *Publ. Res. Inst. Math. Sci.* **37** (2001), 615–619

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