Framework and Results of Stochastic Spectral Analysis

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Abstract

The framework of stochastic spectral analysis is explained. The central and initial magnitude is the transition density function in a Hausdorff space. Free and perturbed Feller operators are introduced. Spectral theoretical results can be obtained by compactness, continuity in Kato-Feller norms, semiclassical and large coupling estimates. A collection of results illustrates each possibility.

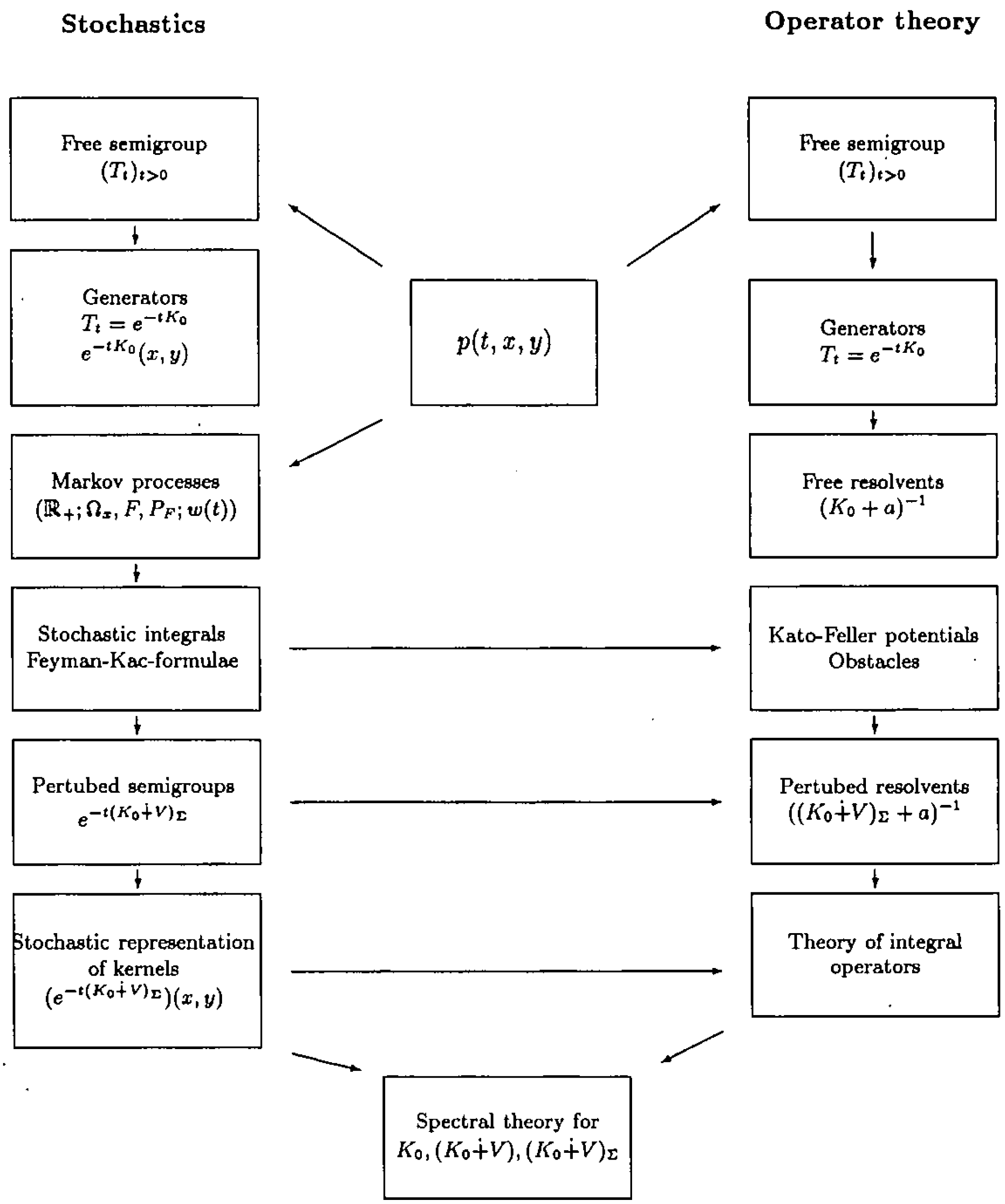
1 The framework of stochastic spectral analysis

The centre of this theory is a function

$$p : (0, \infty) \times E \times E \to [0, \infty)$$

(E - second countable Hausdorff space). This function has different names depending on the field of mathematics which is studied. In stochastic analysis it is a transition density function of a Markov process, in the theory of partial differential equations it is called fundamental solution. In operator theory it is an integral kernel of a semigroup. The following scheme shows that $p(t, x, y)$, $t \in (0, \infty), x, y \in E$, is the main link between operator theory and stochastic analysis. The consequence is that one can use the theory of stochastic processes to study the spectral behaviour of large classes of operators. On the other hand it directs the interest in the theory of Markov processes to spectral analytic properties of their generators.

Framework



Stochastics
Operator theory
Free semigroup $(T_t)_{t>0}$
Free semigroup $(T_t)_{t>0}$
Generators $T_t = e^{-tK_0}$ $e^{-tK_0}(x,y)$
$p(t,x,y)$
Generators $T_t = e^{-tK_0}$
Markov processes $(\mathbb{R}_+;\Omega_x,F,P_F;w(t))$
Free resolvents $(K_0+a)^{-1}$
Stochastic integrals Feyman-Kac-formulae
Kato-Feller potentials Obstacles
Pertubed semigroups $e^{-t(K_0\dot{+}V)_\Sigma}$
Pertubed resolvents $((K_0\dot{+}V)_\Sigma + a)^{-1}$
Stochastic representation of kernels $(e^{-t(K_0\dot{+}V)_\Sigma})(x,y)$
Theory of integral operators
Spectral theory for $K_0, (K_0\dot{+}V), (K_0\dot{+}V)_\Sigma$

Of course the whole theory is only interesting if $p(t,x,y)$ can not be estimated by the Wiener density. On the other hand the assumptions on $p(t,x,y)$ have to admit the use of stochastic analysis. For that we (Demuth, van Casteren, 1989 and 1992) established the following Basic Assumptions on Stochastic Spectral Analysis, shortly denoted as BASSA:

BASSA

1.Existence and Symmetry

Let $(E,\mathcal{E})$ be a second countable locally compact Hausdorff space with Borel field $\mathcal{E}$. A non-negative Radon measure is assumed on E and denoted by dx. Let p be a continuous function mapping $(0,\infty)\times E\times E\to[0,\infty)$ with

$$\int_A p(t,x,y)\,dy\le 1\;,$$

$t>0, x\in E, A\subset\mathcal{E}$ and

$$\int_E p(s,x,u)\,p(t,u,y)\,du=p(s+t,x,y)$$

Moreover p is assumed to be symmetric, i.e.

$$p(t,x,y)=p(t,y,x)$$

for all t>0 and all x, $y\in E$.

2.Continuity

Let C_∞ be the set of continuous functions vanishing at infinity. For any $f\in C_\infty$ and any $x\in E$ we assume

$$\lim_{t\to 0}\int f(y)\,p(t,x,y)\,dy=f(x)$$

3.Feller property

For any $f\in C_\infty$ we assume that the function

$$x\to\int_E f(y)\,p(t,x,y)\,dy\;\in\;C_\infty(E)$$

Under these assumptions exists a strong Markov process $(\mathbb{R}_+,\Omega_x,\mathcal{F},P_{\mathcal{F}},w(t))$ with the following properties:

The one-dimensional distribution is

$$P_F(w(t)\in B)=\int_B p(t,x,y)\,dy$$

$t > 0$, B Borel subset of E. Its sample paths are P_F-almost surely right continuous and possess P_F-almost sure left hand limits in E, and they start in $w(0) = x$. The free Feller operator K_0 is then the L^2-generator of the Feller semigroup determined by $p(t, x, y)$, i.e.

$$(e^{-tK_0} f)(x) = E_x\{f(w(t))\} = \int p(t, x, y) f(y)\, dy$$

and the free resolvents are given by

$$[(K_0 + a1)^{-1} f](x) = \int_0^\infty e^{-as} E_x\{f(w(s))\}\, ds$$

where a is strictly positive. The class of free Feller operators contains a variety of operators: second order elliptic differential operators with variable unbounded coefficients, Laplace Beltrami operators on locally finite Riemannian manifolds, pseudo-differential operators, relativistic Hamiltonians of quantum mechanics. Feller operators are free Feller operators together with a regular or singular perturbation. They can be introduced naturally by studying the properties of

$$E_x\{e^{-\int_0^t V(w(s))\, ds} f(w(t))\} =: (P_V(t) f)(x)$$

where V is a real-valued function on E. $P_V(t)$ is a strongly continuous, quasi-bounded semigroup on $L^2(E)$ with the selfadjoint generator $K_0 \dot{+} V$ if V is a Kato-Feller potential, i.e. if $V = V_+ - V_-$ satisfies

$$\lim_{\tau \to 0} \sup_x \int_0^\tau ds\, E_x\{V_-(w(s)) + \chi_B(w(s) V_+(w(s)))\} = 0$$

where B is a compact subset of E. Moreover, $P_V(t)$ is an integral operator and its kernel has the explicit representation

$$(e^{-t(K_0 \dot{+} V)})(x, y) = E_x^{y,t}\{e^{-\int_0^t V(w(s))\, ds}\}$$

where $E_x^{y,t}\{\,\}$ is the conditional Feller expectation. Instead of finite V_+ one can also include infinitely high parts of V_+. Let $V_+(x) = 1_\Gamma(x)\beta$ where Γ is some closed subset of E; β is a positive parameter tending to infinity. Let $S_\Gamma = S_\Gamma(w)$ be the penetration time of w in Γ, i.e.

$$S_\Gamma := inf\{\tau > 0 : \int_0^\tau 1_\Gamma(w(s))\, ds > 0\}$$

Then

$$E_x\{e^{-\int_0^t V_-(w(s))\, ds} \chi\{w : S_\Gamma > t\} f(w(t))\}$$

restricted to $L^2(\Sigma)$, $\Sigma = E \setminus \Gamma$, is a Feller semigroup. Its generator is denoted by $(K_0 \dot{+} V_-)_\Sigma$. Alltogether we have the following integral kernel of regularly and singularly perturbed Feller semigroups:

$$(e^{-t(K_0 \dot{+} V_-)_\Sigma})(x,y) \;=\; E_x^{y,t}\{e^{-\int_0^t V_-(w(s))\,ds}, S_\Gamma \,>\, t\}$$

and

$$s - \lim_{\beta \to \infty} e^{-t(K_0 \dot{+} V_- + \beta 1_\Gamma)} f \;=\; e^{-t(K_0 \dot{+} V_-)_\Sigma} f$$

$f \in L^2(\Sigma)$. Coming back to our framework $p(x,t,y)$, given by BASSA, determines the free Feller semigroup, the class of free Feller operators, the corresponding Markov process. The expectation of the process provides perturbations of K_0. In all the cases the semigroups and resolvents are integral operators. Their kernels have explicit representations in terms of conditional Feller measures.

2 Principle spectral theoretical results

Assume always BASSA, two Kato-Feller potentials V and W, and the singularity region Γ as described above. Then there are several possibilities to study the spectral data of the Feller operators determined by the investigation of resolvent or semigroup differences.

Compactness: It is possible to find conditions on p, V, W or Γ such that the differences

$$e^{-t(K_0 \dot{+} V)} - e^{-t(K_0 \dot{+} W)},$$

$$(K_0 \dot{+} V + a)^{-1} - (K_0 \dot{+} W + a)^{-1},$$

$$J e^{-t(K_0 \dot{+} V)} - e^{-t(K_0 \dot{+} V)_\Sigma} J,$$

$$J\,(K_0 \dot{+} V + a)^{-1} - (K_0 \dot{+} W + a)_\Sigma^{-1} J$$

(where J is defined by $Jf := f \upharpoonright_\Sigma$, $\Sigma = E \backslash \Gamma$, Γ singularity region) are trace class, Hilbert-Schmidt, or compact operators. The conditions link always the density function $p(t,x,y)$ with V, W, or Γ. In order to verify these conditions one needs more information on p.Very often it is sufficient to have the L^1-L^∞ smoothing, i.e.

$$\sup_{x,y} p(t,x,y) \;<\; \infty\,.$$

Moreover it is often very useful that the perturbed kernels satisfy

$$e^{-t(K_0 \dot{+} V)}(x,y) \;\le\; c\, e^{ct}\, p^{1/2}(t,x,y) \sup_{x,y\in E} p^{1/2}(t,x,y)$$

Examples of results are given in the next section.

Continuity in V: For any Kato-Feller potential the Kato-Feller norm

$$\|V\|_{KF} \;=\; \sup_x \int_0^1 ds\, E_x\{V(w(s))\}$$

exists. Then the resolvent difference for regular resolvent values a, a large enough, can be estimated by this Kato-Feller-norm

$$\|(K_0 \dot{+} V + a)^{-1} \;-\; (K_0 \dot{+} W + a)^{-1}\| \;\le\; c\|V \;-\; W\|_{KF}$$

For applications it is important that we treat here the operator norm. That can be used to study also the behaviour of these resolvents in the limiting absorption case. Let φ be a nonvanishing continuous function mapping E into $\mathbb{R}_+$ with $\varphi^{-1} \le 1$. For special real positive values λ it turns out that

$$\sup_{\varepsilon\in[0,1]} \|\,\varphi^{-1}[(K_0 \dot{+} V - \lambda + i\varepsilon)^{-1} \;-\; (K_0 \dot{+} W - \lambda + i\varepsilon)^{-1}]\varphi^{-1}\,\| \;\le\; c\,\|\,(V - W)\varphi^2\|_{KF}\,.$$

Again the operator norm (in weighted L^2-spaces) is studied. That implies consequences for any spectral property depending on the resolvents near the real axis.

Semiclassical limits: As explained in section 1 one has explicit representations for the kernels of the semigroups $e^{-t(K_0 \dot{+} V)}$. That remains true if we introduce a parameter $\hbar^2$, i.e. if we study generators of the form $\hbar^2 K_0 \dot{+} V$. For certain potentials the behaviour of

$$e^{-t(\hbar^2 K_0 \dot{+} V)} \;-\; e^{-t(\hbar^2 K_0 \dot{+} W)}$$

for small $\hbar^2$ can be studied.

Large coupling behaviour: The singularly perturbed semigroup $e^{-t(K_0 \dot{+} V_-)_\Sigma}$ was obtained by limits of semigroups the generators of which have finite potential heights

$$e^{-t(K_0 \dot{+} V_- + \beta 1_\Gamma)}\,.$$

The operator resolvent norm is

$$\|\,J(K_0 \dot{+} V_- \;+\; \beta\, 1_\Gamma)^{-1} \;-\; ((K_0 \dot{+} V)_\Sigma \;+ a)^{-1}\,J\| \;=:\; f(\beta)$$

$f(\beta)$ is mainly determined by

$$\sup_{x\in\Sigma}\left[E_x\left\{e^{-\beta\int_0^1 1_\Gamma(w(s))\,ds}\,, S_\Gamma \;<\; 1\right\}\right]$$

where S_Γ is the penetration time of Γ. For certain boundaries $\delta\Gamma$ the last term can be estimated uniformly in x.

3 Collection of results

In order to illustrate the kind of conditions typical in stochastic spectral analysis we collect some results concerning the principles mentioned in the preceding section. We always assume BASSA, Kato-Feller potentials and closed singularity regions Γ. Proofs are omitted. They are given in the articles referred. Hints are not given because it seems to be unmodest to mention always our names.

Compactness

Proposition 1 : *The semigroup difference*

$$e^{-t(K_0 \dot{+} V)} - e^{-tK_0}$$

is a Hilbert-Schmidt-operator if

$$\sup_{x,y} p(t,x,y) < \infty$$

and if

$$\int_0^{2t} d\lambda \, \lambda \int dx \int dy \, |V(x)| \, |V(y)| \, p(\lambda,x,y) < \infty$$

Proposition 2 : *The resolvent difference*

$$(K_0 \dot{+} V + a)^{-1} - (K_0 + a)^{-1}$$

is a trace class operator if

$$\int_0^{\infty} d\lambda \, \lambda \, e^{-a\lambda} \int dy \, E_y^{y,\lambda}\{e^{-\int_0^\lambda V(w(s))\,ds}\} |V(y)| < \infty$$

Proposition 3 : *For singular pertubations the difference*

$$e^{-t(K_0)_{B\setminus\Gamma}} - e^{-tK_0}$$

is Hilbert-Schmidt if

$$\sup_{x,y} p(x,t,y) < \infty$$

and if

$$\int dx \, [P_F\{S_\Gamma < t; \, w(0) = x\}]^2 < \infty$$

The singular semigroup difference is a trace class operator if

$$\int dx \, [P_F\{S_\Gamma < t \, , w(0) = x\}]^{1/2} < \infty$$

(see also Stollmann 1992).

Continuity in V

Proposition 4 : *$K_0 \dot{+} V$ and $K_0 \dot{+} W$ are selfadjoint operators in $L^2(E)$. Let $E_V(.)$, $E_W(.)$ denote its spectral measures. Let y be a non-vanishing Borel-function (typically $y(x) = (1+|x|^2)^\alpha$, $\alpha > 0$). Let*

$$[(K_0 + a)^{-1}|\varphi|^2](x) \ \leq \ c|\varphi(x)|^2$$

for all $x \in E$. For one of the potentials, take V, we assume

$$\sup_{\lambda\in\Delta} \|\varphi^{-1}(K_0 \dot{+} V + \lambda + i0)^{-1}\varphi^{-1}\| < \infty$$

where $\Delta = (\alpha, \beta)$ is an interval in $\mathbb{R}_+$, α, β no eigenvalues of $K_0 \dot{+} V$ or $K_0 \dot{+} W$. Let $\|(V-W)\varphi^2\|_{KF}$ be sufficiently small, such that the last estimate holds also for $K_0 \dot{+} W$. Then for $\lambda_0 \in \Delta$ we get

$$\left\| \varphi^{-1} \left[\frac{dE_V(\lambda)}{d\lambda} - \frac{dE_W(\lambda)}{d\lambda} \right] \varphi^{-1} \right\|_{\lambda=\lambda_0} \leq c_{(\lambda_0, a)} \left\| (V - W)\varphi^2 \right\|_{KF}$$

The constant $c_{(\lambda_0,a)}$ can be estimated quantitatively.

Proposition 5 : *Let V and W be Kato-Feller potentials in $L^1(E)$. Assume*

$$\int_0^\infty d\lambda \, \lambda e^{-a\lambda} \sup_x p(\lambda, x, x) < \infty.$$

Then the wave operators

$$\Omega_\pm(K_0 \dot{+} V, K_0) := s - \lim_{t\to\pm\infty} e^{it(K_0 \dot{+} V)} e^{-itK_0} P_{ac}(K_0)$$

and $\Omega_\pm(K_0 \dot{+} W, K_0)$ exist. ($P_{ac}(K_0)$-projection operator onto the absolutely continuous subspace of K_0). Define the scattering operator by

$$S_V := \Omega_+^*(K_0 \dot{+} V, K_0)\Omega_-(K_0 \dot{+} V, K_0)$$

Both S_V and S_W commutes with K_0, providing that the corresponding scattering matrices $S_V(\lambda)$, $S_W(\lambda)$ are well defined. Assume that for some λ_0

$$\|V\varphi^{-1}(K_0 \dot{+} W - \lambda_0 - i0)^{-1}\varphi^{-1}\| < 1$$

Let $\sup_x |\varphi(x)V(x)| < \infty$ and $\sup_x |\varphi(x)W(x)| < \infty$. The operator norm of the scattering matrices is a norm in the fiber of the spectral resolution of the absolutely continuous subspace of K_0. This norm can be estimated as

$$\|S_V(\lambda_0) - S_W(\lambda_0)\| :\leq \ c(\lambda_0) \ \left\| (V - W)\varphi^2 \right\|_{KF}$$

Semiclassical limits

Proposition 6 : *Let B be a compact set in E and $(h^2K_0+V)_B$ the Feller operator with Dirichlet boundary conditions on δB. Assume positive V such that*

$$V = V1_{B\setminus\Gamma} + V1_\Gamma$$

$$V1_\Gamma \geq \gamma 1_\Gamma$$

$\Gamma \subset B$, *i.e. V is larger than a constant γ on Γ. Let ψ_{h^2} be the ground state of $(h^2K_0+V)_B$, i.e.*

$$((h^2K_0+V)_B\,\psi_{h^2})(x) = E_{h^2}\psi_{h^2}(x)$$

Let $\sup_{x,y} p(t,x,y) < \infty$ *and* $E_{h^2} = h^2E$. *Then*

$$|\psi_{h^2}(x)| \leq e^E\, E_x\{e^{-h^{-2}\gamma T_{1,\Gamma}(w)}\}$$

where $T_{t,\Gamma}(w) := meas\{s, s \leq t, w(s) \in \Gamma\}$ is the spending time of the trajectory w in Γ. If we consider x in a subset $\tilde{\Gamma} \subset \Gamma$ with $dist(\tilde{\Gamma}, B\setminus\Gamma) \geq r$, a uniform estimate is possible:

$$|\psi_{h^2}(x)| \leq e^E\, E_0\{e^{-h^{-2}\gamma T_{1,B(r)}(w)}\}$$

where $B(r)$ is a ball of radius r with centre in the origin. The right hand side tends to zero as $h \to 0$. A rate of convergence can be given for special K_0.

Large coupling limits

Proposition 7 : *Let $V \equiv 0$ and compare $K_\beta = K_0 + \beta 1_\Gamma$ with $(K_0)_\Sigma$, $\Sigma = E\setminus\Gamma$ for large parameters β. Denote again $Jf = f\upharpoonright_\Sigma$. Then*

$$\|Je^{-tK_\beta} - e^{-t(K_0)_\Sigma}J\| \leq \sup_{x\in\Sigma} E_x\{e^{-\beta T_{t,\Gamma}}, T_{t,\Gamma} > 0\}$$

($T_{t,\Gamma}$ is the spending time defined in Proposition 6)

Remark: To estimate the Laplace transform of the spending time (or occupation time) is a difficult problem. If $K_0 = -\Delta$ in $L^2(\mathbb{R}^n)$ it is done recently by Demuth, Kirsch, Mc Gillivray (1993)and explained in another contribution of these proceedings.

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