Cayley Surfaces in Affine Differential Geometry

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A Cayley surface in affine space $\mathbf{R}^3$ is given as the graph of a cubic polynomial, say, $z = xy + y^3/6$. This ruled surface is an improper affine sphere which is also one of the homogeneous nondegenerate affine surfaces (see [1], p.243, also [2], Chapter 12).

One of the further properties of the surface is that its (nonzero) cubic form is parallel relative to the induced affine connection. The purpose of this paper is to show that this property alone characterizes the Cayley surface up to an equiaffine transformation in $\mathbf{R}^3$. Namely, we prove the following

Theorem. _Let_ M^2 _be a nondegenerate surface in_ $\mathbf{R}^3$. _Let_ ∇ _be the induced affine connection and let_ h _be the fundamental form (affine metric). If_ $\nabla^2 h = 0$ _but_ $\nabla h \neq 0$, _then_ M^2 _is congruent to (an open subset of) the Cayley surface by an equiaffine transformation of_ $\mathbf{R}^3$.

We shall follow the terminology and notation in [3] and [4], which provide a modern introduction to affine differential geometry. A quick review of the basic notions and facts is provided in Section 1. In Section 2, we study the behavior of the cubic form for dimension 2. In Section 3 we show that the assumption $\nabla^2 h = 0$ but $\nabla h \neq 0$ implies that the induced connection is flat and, consequently, the surface is the graph of a certain function $z = F(x,y)$ such that the Hessian determinant is ± 1. In Section 4 we discuss the reduction of the Hessian matrix to a simple form by an equiaffine change of the coordinates system x,y. This argument makes use of an inner product of signature $(-,+,+)$ in the space of symmetric 2×2 matrices. Once we obtain the function F from the reduced Hessian matrix, our surface is shown to be equiaffinely congruent to the standard Cayley surface.

1. Affine surfaces.

Let f be an immersion of an n-dimensional differentiable manifold M^n into an (n+1)-dimensional affine space $\mathbf{R}^{n+1}$ with a fixed parallel volume element ω. Choose any transversal vector field ξ on M. For vector fields X and Y, we may write

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$$
$$D_X \xi = - f_*(SX) + \tau(X)\xi,$$

where ∇ is the induced affine connection on M^n, the bilinear symmetric tensor h the fundamental form, the (1,1) tensor S the shape operator, and τ the transversal connection form. We also introduce a volume element θ on M^n by setting

$$\theta(X_1,\ldots,X_n) = \omega(X_1,\ldots,X_n,\xi)$$

for any tangent vectors $X_1,\ldots,X_n$.

Whether h is degenerate or nondegenerate is independent of the choice of ξ. When h is nondegenerate, we say that the hypersurface M^n is nondegenerate. It is a fundamental fact in classical affine differential geometry that if M^n is nondegenerate, then we can choose ξ uniquely such that

1) $\tau = 0$, which implies that θ is parallel relative to ∇;

2) the volume element for h coincides with θ.

The uniquely determined ξ is called the affine normal and the corresponding h the affine metric. The induced connection ∇ and the volume element θ together define an equiaffine structure on M^n.

The covariant differential $C = \nabla h$ is called the cubic form of M^n. It is related to the difference tensor K between the induced connection ∇ and the Levi-Civita connection $\hat{\nabla}$ for the affine metric. If $K_X Y = \nabla_X Y - \hat{\nabla}_X Y$, then we have

$$h(K_X Y, Z) = -\tfrac{1}{2} C(X,Y,Z) \quad \text{for any tangent vectors } X,Y,Z.$$

Thus $C = 0$ if and only if ∇ and $\hat{\nabla}$ coincide.

Because of condition 2) above, we have apolarity: trace $K_X = 0$. If we express h and C by their components relative to any basis in the tangent space or any local coordinate system, apolarity can be expressed by

$$\sum h^{ij} C_{ijk} = 0, \quad \text{where} \quad [h^{ij}] = [h_{ij}]^{-1}.$$

It is a classical theorem due to Pick and Berwald that a nondegenerate hypersurface with vanishing cubic form is a quadric. This result has been extended. See [5] for the proof including the classical case.

2. Cubic form on an affine surface.

We now consider exclusively nondegenerate affine surfaces M in $\mathbf{R}^3$. We wish to study the behavior of the cubic form in more detail. Some of the information given below appears in [6].

Let V be a 2-dimensional real vector space with a nondegenerate inner product h. Let C be a nonzero cubic form, namely, a 3-linear symmetric function $V \times V \times V$, which satisfies the apolarity condition relative to h. By a null direction of C, we mean a direction of a vector $X \neq 0$ such that $C(X,X,X) = 0$.

Lemma 1. If h is elliptic (that is, positive-definite), then C has three distinct null directions.

Proof. Take a basis $\{e_1, e_2\}$ such that $h_{11} = h_{22} = 1$, $h_{12} = h_{21} = 0$. By apolarity we have $C_{111} + C_{221} = 0$ and $C_{112} + C_{222} = 0$. Setting $a = C_{111}$ and $b = C_{112}$, we have for $x = x^1 e_1 + x^2 e_2$

$$C(x,x,x) = a\,(x^1)^3 + 3b\,(x^1)^2 x^2 - 3a\,x^1 (x^2)^2 - b\,(x^2)^3.$$

Case where $b = 0$. Then

$$C(x,x,x) = a\,x^1\,[(x^1)^2 - 3\,(x^2)^2]$$

so that $(0,1)$, $(\sqrt{3}, 1)$, and $(\sqrt{3}, -1)$ give three distinct null directions.

Case where $b \neq 0$. Writing $t = x^2/x^1$ and $c = a/b$, solving the equation $C(x,x,x) = 0$ is reduced to solving

$$f(t) \equiv t^3 + 3\,ct^2 - 3t - c = 0.$$

One can show that this equation has three distinct roots by checking the values of f at two critical points:

$$f(-c - (c^2+1)^{\frac{1}{2}}) > 0 \quad \text{and} \quad f(-c + (c^2+1)^{\frac{1}{2}}) < 0. \qquad \square$$

Lemma 2. If h is hyperbolic (that is, indefinite), then C has either

a) one null direction of multiplicity 1 and C can be written in the form

$C(x,x,x) = \mu(x)\, g(x,x)$, where μ is a 1-form and g is a definite inner product on V, each unique up to a scalar; or

b) a null direction of multiplicity 3, in which case there is a nonzero x in V such that $h(x,x) = 0$ and $C(x,y,z) = 0$ for all y,z in V. The direction of x is uniquely determined.

We have case b) if and only if Pick's invariant $h(C,C)$ is 0.

Proof. We use a null basis $\{e_1, e_2\}$ so that $h_{11} = h_{22} = 0$ and $h_{12} = 1$. From apolarity we get $C_{112} = C_{122} = 0$. Thus

$$C(x,x,x) = C_{111}(x^1)^3 + C_{222}(x^2)^3.$$

case a). If $C_{111} \neq 0$ and $C_{222} \neq 0$, let α and β be their real cubic roots. Then

$$C(x,x,x) = (\alpha x^1 + \beta x^2)(\alpha^2 (x^1)^2 - \alpha\beta x^1 x^2 + \beta^2 (x^2)^2).$$

We may define a 1-form μ by $\mu(x) = \alpha x^1 + \beta x^2$ and an inner product g by $g(x,y) = \alpha^2 x^1 y^1 - \frac{1}{2}\alpha\beta(x^1y^2 + x^2y^1) + \beta^2 x^2 y^2$. Clearly, g is positive definite and C has only one null direction. The uniqueness assertion is also obvious.

case b). If $C_{111} = 0$, then $X = (1,0)$ is a null direction of multiplicity 3. Since $C(X, e_i, e_j) = C_{1ij} = 0$ for all i,j, we see that X is in the kernel of C. Obviously, $h(X,X) = 0$. If $C_{222} = 0$, then $X = (0,1)$ is the vector. The uniqueness is easy to see.

The additional statement in Lemma 2 follows from

$$h(C,C) = \sum h^{ip} h^{jq} h^{kr} C_{ijk} C_{pqr} = 2\, C_{111} C_{222}$$

in terms of the same null basis $\{e_1, e_2\}$. □

Remark. Each of the two cases in Lemma 2 is actually possible at a point of an affine surface. For example, for the graph of $z = xy + (x^3 + y^3)/6$ at the point $(0,0,0)$ the vector $\partial/\partial x - \partial/\partial y$ gives the only null direction of the cubic form. On the other hand, for the Cayley surface, namely, for the graph of $z = xy + y^3/6$, the cubic form has a null direction of multiplicity 3 at every point.

3. Consequence of $\nabla C = 0$, $C \neq 0$.

We prove

Lemma 3. *Let M be a nondegenerate surface in* $\mathbf{R}^3$ *such that the cubic form is parallel but not 0. Then* ∇ *is flat and M is the graph of a function* $z = F(x,y)$ *defined on a certain domain D of the (x,y)-plane and the Hessian determinant of F is* ± 1.

Proof. First assume that h is elliptic. Let p be a point of M and consider the three distinct null directions at p that exist by Lemma 1. We may assume that they are given by two linearly independent tangent vectors e_1, e_2 and the vector $e_1 + e_2$. Since C is parallel, parallel displacement of these tangent vectors along any curve from p gives rise to three distinct null directions at each point. This means that each linear transformation φ belonging to the linear holonomy group of ∇ based at p leaves the directions of e_1, e_2, and $e_1 + e_2$ invariant. Thus φ must be a scalar multiple of the identity transformation. But since there is a parallel volume element θ, the determinant of φ is 1, which means that φ is the identity. This shows that the holonomy group consists of the identity transformation and ∇ is flat, that is, the curvature tensor R is 0.

Now for an affine surface (or hypersurface), it is known that $R = 0$ implies that the shape operator S is 0. Indeed, this follows easily from the Gauss equation: $R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY$, see [5]. From the second basic equation in Section 1 the affine normal ξ is parallel in $\mathbf{R}^3$. It follows that M is affinely equivalent to the graph of a certain function $z = F(x,y)$ on a domain D of the (x,y)-plane. Since the affine normal ξ is thus identified with the vector (0,0,1) in the (x,y,z)-space, the Hessian matrix of F expresses the fundamental form h relative to $\partial/\partial x, \partial/\partial y$. The condition that θ coincides with the volume element of h is equivalent to the fact that the Hessian determinant has absolute value 1. (For the detail, see the remark following (7) in [3].) The components of $C = \nabla h$ are the third partial derivatives of F and those of $\nabla C = \nabla^2 h$ are the fourth partial derivatives of F. Hence $\nabla C = \nabla^2 h = 0$ means that each second partial derivative of F is an affine function of the form $ax + by + c$.

Now consider the case where h is hyperbolic. Again, we show that ∇ is flat and hence M is the graph in the manner stated just above.

Since C is parallel, the behavior of C as Lemma 2 remains the same for all points. Namely, we have either case a) at every points or case b) at every point. In the first case, we have a positive definite inner product g at each point. Since C is parallel, the holonomy group of ∇ at a point p leaves g_p invariant up to a scalar. Since each element φ has determinant 1, it must leave g_p invariant, that is, it is a rotation. On the other hand, φ leaves the only null direction invariant and cannot be a proper rotation. Thus the holonomy group consists of the identity transformation and h is flat.

We now deal with the second case so we have at each point a vector X, unique up to a scalar, such that $h(X,X) = 0$ and $C(X,U,V) = 0$ for all U and V. We may choose locally two vector fields X and Y such that

1) $h(X,X) = 0$; 2) $h(X,Y) = 1$; 3) $h(Y,Y) = 0$;

4) $C(X,U,V) = 0$ for all vectors U, V;

5) $C(Y,Y,Y) = 1$.

In the following we use the fact that $(\nabla_X h)(U,V) = C(X,U,V) = 0$ for all U and V and $(\nabla_Y h)(X,U) = C(Y,X,U) = 0$ for all U. Now taking ∇_X of 1) we obtain $h(\nabla_X X,X) = 0$, which implies $\nabla_X X = \lambda X$ for some function λ. From 5) we get $C(\nabla_X Y,Y,Y) = 0$, which implies that $\nabla_X Y = \nu X$. (We shall see in a moment that $\lambda = \nu = 0$.)

From 2) we get $h(\nabla_X X,Y) + h(X,\nabla_X Y) = 0$. Since $h(\nabla_X X,Y) = \lambda$ and $h(X,\nabla_X Y) = h(X,\nu X) = 0$, we get $\lambda = 0$. From 3) we get $h(\nabla_X Y,Y) = 0$, which implies $\nu = 0$. We have thus far shown $\nabla_X X = 0$ and $\nabla_X Y = 0$.

From 1) we get $h(\nabla_Y X,X) = 0$, which implies $\nabla_Y X = \mu X$ for some function μ. From 5) we get $C(\nabla_Y Y,Y,Y) = 0$, which implies $\nabla_Y Y = \tau X$.

From 2) we get $h(\nabla_Y X,Y) + h(X,\nabla_Y Y) = 0$, which implies $\mu = 0$, that is, $\nabla_Y X = 0$. Finally, from 3) we get $(\nabla_Y h)(Y,Y) + 2\,h(\nabla_Y Y,Y) = 0$. By 5) and $h(\nabla_Y Y,Y) = h(\tau X,Y) = \tau$, we find $\tau = -\frac{1}{2}$, namely, $\nabla_Y Y = -\frac{1}{2}X$.

To summarize: $\nabla_X X = \nabla_X Y = \nabla_Y X = 0$ and $\nabla_Y Y = -\frac{1}{2}X$. We get $[X,Y] = 0$. Also we have $R(X,Y)Y = R(X,Y)X = 0$, that is, $R = 0$. Again, we have M as the graph of a function $z = F(x,y)$ as before. □

4. Reduction of the Hessian matrix.

We consider a differentiable function $x^3 = F(x^1,x^2)$ defined on a domain

D of the (x^1,x^2)-plane. We may assume that D contains (0,0). Denote the Hessian matrix by $H = [F_{ij}]$, where $F_{ij} = \partial^2 F/\partial x^i \partial x^j$.

We assume that

(I) $\det [F_{ij}] = \pm 1$ at all points;

(II) each F_{ij} is an affine function of x^1 and x^2. Not all F_{ij} are constant functions (corresponding to the condition $C \neq 0$).

We shall show that actually $\det [F_{ij}] = 1$ at all points and find an equiaffine change of the coordinates (x^1,x^2) to (x,y) which reduces the Hessian matrix to the form

$$\begin{bmatrix} 0 & 1 \\ 1 & \beta y \end{bmatrix}, \quad \beta \neq 0.$$

We begin by stating without proof

Lemma 4. Consider a coordinate change of the form

$$\overline{x}^1 = p^1{}_1 x^1 + p^1{}_2 x^2$$

$$\overline{x}^2 = p^2{}_1 x^2 + p^2{}_2 x^2$$

and think of the function $F(x^1,x^2)$ as a function $\overline{F}(\overline{x}^1,\overline{x}^2)$. Then the Hessian matrix $\overline{H} = [\overline{F}_{ij}]$, where $\overline{F}_{ij} = \partial^2\overline{F}/\partial\overline{x}^i\partial\overline{x}^j$, is related to the original Hessian matrix $H = [F_{ij}]$ by $H = {}^tP\,\overline{H}\,P$, where P is the matrix whose (i,j)-component is $p^i{}_j$.

Next, we consider, in the vector space $\mathbf{gl}(2,\mathbf{R})$, the inner product with signature (-,-,+,+) given by

$$\langle A,A'\rangle = -ad' - a'd + bc' + b'c.$$

The corresponding quadratic form is simply $\langle A,A\rangle = -\det A$. Let $\mathbf{s}(2)$ denote the subspace of all symmetric matrices in $\mathbf{gl}(2,\mathbf{R})$. The restriction of the inner product to $\mathbf{s}(2)$ has signature (-,+,+), that is, it is a Lorentzian inner product.

Now for any $P \in SL(2,\mathbf{R})$, the mapping

$$X \in \mathbf{s}(2) \rightarrow {}^tP\,X\,P \in \mathbf{s}(2)$$

preserves the inner product and hence is a linear isometry. We may easily verify that $SL(2,\mathbf{R})/\{\pm I_2\}$ is isomorphic to the rotation group of $\mathbf{s}(2)$. In other words, for any linear isometry of $\mathbf{s}(2)$ there is a suitable

P which induces it in the manner above.

We now consider the affine mapping given by the Hessian matrix

$(x^1, x^2) \mapsto H(x^1, x^2) \in \mathbf{s}(2)$,

which we may write in the form

$$H(x^1, x^2) = x^1 A + x^2 B + C, \text{ with constant } A, B, C \text{ in } \mathbf{s}(2).$$

If we set $x^1 = 0$, the determinant of $x^2B + C$ must be ± 1. Thus

$$-\det(x^2B + C) = \langle x^2B + C, x^2B + C\rangle = \langle B, B\rangle (x^2)^2 + 2\langle B, C\rangle x^2 + \langle C, C\rangle = \pm 1.$$

So we must have $\langle B, B\rangle = \langle B, C\rangle = 0$ and $\langle C, C\rangle = \pm 1$. Now we can eliminate the case $\langle C, C\rangle = -1$, because then the restriction of the inner product to $\{X \in \mathbf{s}(2); \langle X, C\rangle = 0\}$ is positive definite and cannot contain a null vector B, unless $B = 0$. If $B = 0$, consideration of the line $x^1A + C$ will lead to $\langle A, A\rangle = \langle A, C\rangle = 0$ and thus $A = 0$ by the same argument. Thus $\langle C, C\rangle = -1$ leads to constancy of $H(x^1, x^2)$, contrary to the assumption. Hence $\langle C, C\rangle = 1$.

Since A and B are two null vectors in $\{X \in \mathbf{s}(2); \langle X, C\rangle = 0\}$ whose dimension is 2, they are linearly dependent, say, $A = kB$. Thus $H(x^1, x^2) = (kx^1 + x^2)B + C$. Now since $\langle B, B\rangle = \langle B, C\rangle = 0$ and $\langle C, C\rangle = 1$, we can find an isometry $X \mapsto {}^tP\,X\,P$ of $\mathbf{s}(2)$, with $P \in SL(2, \mathbf{R})$ which takes B into B_1 and C into C_1, where

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } C_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By using this matrix P, we consider an equiaffine change of the coordinate system from (x^1, x^2) to, say, (x, y). By Lemma 4 we see that the Hessian of F relative to (x, y) is of the form $(\alpha x + \beta y)B_1 + C_1$, i.e.

$$\begin{bmatrix} 0 & 1 \\ 1 & \alpha x + \beta y \end{bmatrix}.$$

So the original function as a function of x, y is such that

$$F_{xx} = 0, \quad F_{xy} = F_{yx} = 1, \quad F_{yy} = \alpha x + \beta y.$$

Then $F_{yyx} = \alpha$. On the other hand, $F_{yxy} = 0$. Hence $\alpha = 0$ and $F_{yy} = \beta y$. We have thus proved the assertion in the beginning of this section. Incidentally, we should remark that the affine metric of our surface turns out to be hyperbolic.

From the Hessian matrix in our hand we find

$$F(x,y) = \beta y^3/6 + xy + ax + by + c,$$

where a, b, c are certain constants. By changing the coordinates from (x,y,z) to $(\bar{x},\bar{y},\bar{z})$, where $\bar{x}=x$, $\bar{y}=y$, $\bar{z}=z-(ax+by+c)$, we can assume $F(x,y) = \beta y^3/6 + xy$. Now change (x,y) to $(\beta^{1/3}x,\ y/\beta^{1/3})$. We finally get the form $z = xy + y^3/6$. Thus our surface is equiaffinely congruent to the graph of this function.

References

[1] W. Blaschke, Vorlesungen über Differentialgeometrie, II, Springer, Berlin, 1923

[2] H. Guggenheimer, Differential Geometry, McGraw-Hill, New York, 1963

[3] K. Nomizu, On completenes in affine differential geometry, Geometriae Dedicata 20 (1986), 43-49

[4] K. Nomizu and U. Pinkall, On the geometry of affine immersions, Math. Z. 195(1987), 165-178

[5] K. Nomizu and U. Pinkall, Cubic form theorem for affine immersions, Proceedings Oberwolfach Conference on Affine Differential Geometry, November 1986, to appear in Resultate der Mathematik.

[6] P.A. Schirokow and A.P. Schirokow, Affine Differentialgeometrie, Teubner, Leipzig, 1962

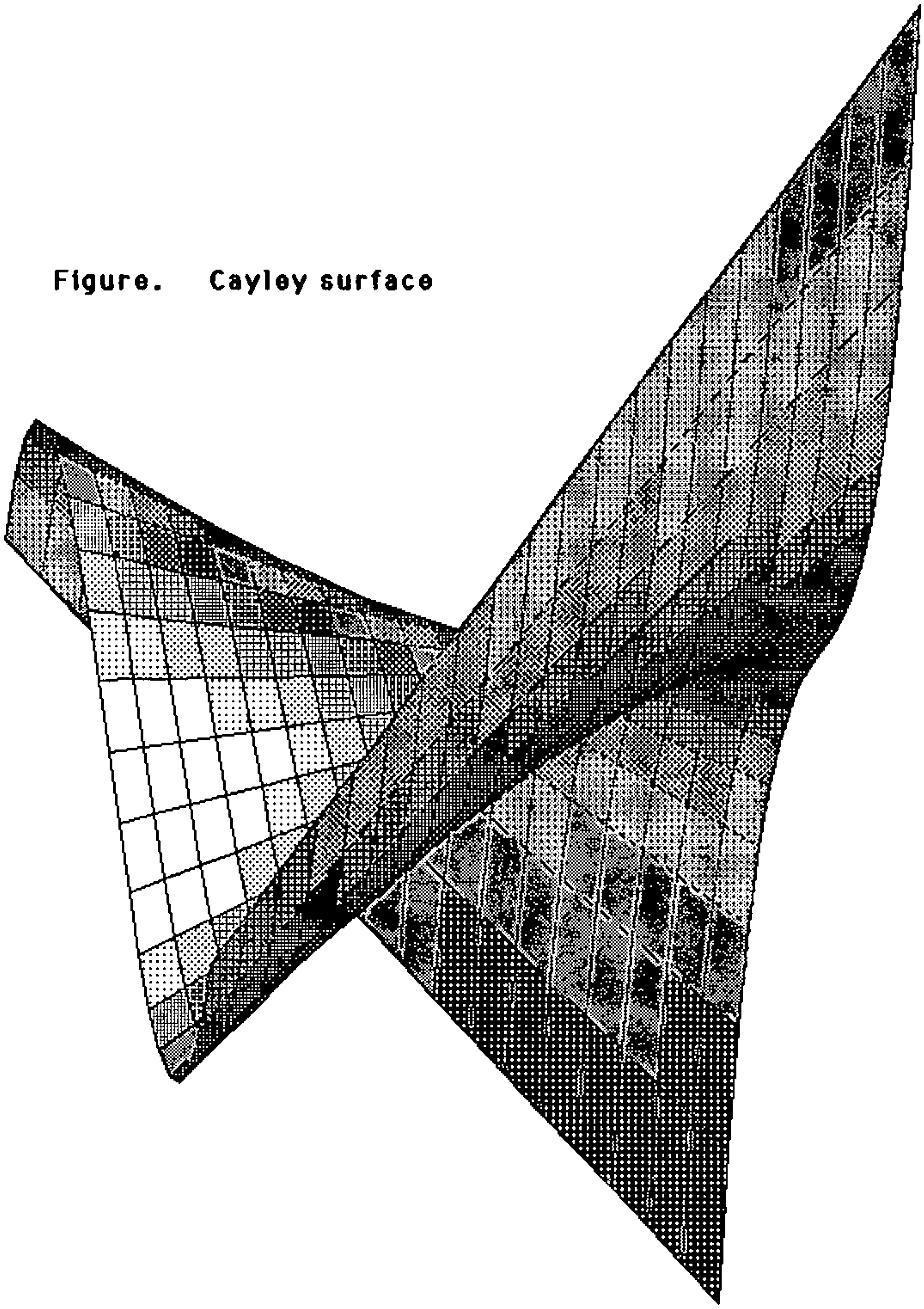

Figure. Cayley surface