SPECTRAL RIGIDITY OF GROUP ACTIONS: APPLICATIONS TO THE CASE $\mathbf{gr}\langle t, s;\ ts = st^2\rangle$

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Abstract. We apply a technique to study the notion of spectral rigidity of group actions to a group $\mathrm{gr}\langle t, s;\ ts = st^2\rangle$. As an application, we prove that there exist rank one weakly mixing transformations conjugate its square, thereby giving a positive answer to a well-known question.

Introduction

In [1] we introduced a notion of spectral rigidity of group actions. This notion is essential only for noncommutative groups (see Remark 8 in [1]). We say that a property holds for a *typical* element from a topological space D (or a typical element of a topological space D has some property) if the set of elements from D with this property contains a dense G_δ subset of D. Following [1], we say that an element h of a group H has spectral rigidity if for a typical H-action T the set of essential values of the spectral multiplicity function of T_h, named $M(T_h)$, is constant. If every element of H has this property, we say that the group H has spectral rigidity. It is easy to see that the notion of spectral rigidity can be considered as one of invariants (in general, not complete) with respect to group, metrical, or unitary isomorphisms.

It is well known that a typical transformation has rank one, and is weakly mixing. However, it is not conjugate to its square (see [9]). More information about these notations can be found in [5]. The main goal of this paper is to show how the technique to study the spectral rigidity works in the case of actions of a group $G = \mathrm{gr}\langle t, s;\ ts = st^2\rangle$. Let us recall that even in this simple case we have no useful spectral theorem, because G is wild (see [10]).

1. Preliminaries and the basic result

Let T be a transformation defined on a non-atomic standard Borel probability space $(X, \mathcal{F}, \mu)$. The set of all transformations (automorphisms of the σ- algebra of measurable sets) of (X, μ) is a Polish (complete metrizable separable) topological group, noted $\mathbf{Aut}(\mu)$, with respect to the weak (or coarse) topology (see [7]) defined by

$$T_n \to T \Leftrightarrow \mu(T_n^{-1} A \Delta T^{-1} A) \to 0 \text{ for each measurable } A$$

(we identify transformations if they are coincide up to a set of measure zero).

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The spectral properties of T are those of the induced unitary operator on $L_2(\mu)$ defined by

$$\widehat{T} : L_2(\mu) \to L_2(\mu); \quad \widehat{T} f(x) = f(Tx).$$

This is Koopman's representation of dynamical systems, therefore if $TS = ST^2$, then $\widehat{S}\widehat{T} = \widehat{T^2}\widehat{S}$. Let us recall that a transformation T has *rank* n if n is the smallest number such that, for any k, there exist integers $h_{k,i}$ and towers (columns)

$$A_{k,i}, TA_{k,i}, \ldots, T^{h_{k,i}-1}A_{k,i}, \quad i = 1, \ldots, n$$

such that all levels $T^j A_{k,i}$, $0 \le j < h_{k,i}$, and the remaining set form a measurable partition of X, say ξ_k, and $\xi_k \to \varepsilon$, i.e. for any measurable set A there are ξ_k-measurable sets A_k such that $\mu(A \Delta A_k) \to 0$ as $k \to \infty$. A transformation has infinite rank if there is no such number n.

In this paper we study metrical properties of some elements of typical G-actions, in particular, we answer a well-known question (see [4],[5],[6]) by proving the following theorem:

Theorem 1.1. *There exists a weakly mixing rank one transformation conjugate to its square.*

For the proof of Theorem 1.1, we apply a group action approach. Given a discrete group G, a G-action T_g is any homomorphism $T : G \to \mathbf{Aut}(\mu)$. The G-action T is called *free* if $\mu\{x \in X : (\exists g \ne 1)\ T_g x = x\} = 0$. The set of all G-actions is equipped with the weak topology defined by

$$T(n) \to T \Leftrightarrow (\forall g \in G)\ T_g(n) \to T_g.$$

When G is countable, the weak topology makes the action space, noted Ω_G, a Polish space.

Theorem 1.1 is a natural corollary of the following main theorem:

Theorem 1.2. *For a typical G-action T, T_t is a weakly mixing rank one transformation.*

2. Natural properties and basic notation for G

Denote by Q_2 the subgroup of G generated by all conjugations of t, i.e. $g^{-1}tg$, $g \in G$. Following [11], a subset of G, say F, tiles G if there is a set of centers $C \subseteq G$ such that $\{Fc : c \in C\}$ is a partition of G.

Proposition 2.1.
(1) *Q_2 is an abelian group consisting of elements $s^{-m}t^n s^m$, $m, n \in \mathbf{Z}$.*
(2) *Q_2 is isomorphic to a subgroup of $\mathbf{R}$ via an isomorphism ϕ defined by $\phi(s^{-m}t^n s^m) = n2^m$, $m, n \in \mathbf{Z}$.*
(3) *$G' = Q_2$.*
(4) *$G = \sqcup_{i \in \mathbf{Z}} s^i G'$. Therefore $G/G' \cong \mathbf{Z}$.*
(5) *There exists a Følner sequence of sets, say F_k, such that every F_k tiles G.*

Proof. The proof is almost obvious. Indeed, part 1 is a consequence of the following equality:

$$s^m t^p s^{-m} s^q t^r s^{-q} = s^q t^{r+p2^{q-m}} s^{-q} = s^q t^r s^{-q} s^m t^p s^{-m}, \tag{2.1}$$

where $q \ge m \ge 0$.

It is easy to see that the map ϕ is well defined. Using (2.1) once more, we prove part 2. The statement

$$(\forall g \in Q_2)\ (\exists g_1, g_2 \in Q_2)\ [gs^{\pm 1} = s^{\pm 1} g_1\ \&\ gt^{\pm 1} = g_2]$$

is trivial. This implies that $G = \sqcup_{i \in \mathbf{Z}} s^i Q_2$, because $s^i \in Q_2$ iff $i = 0$.

It is easy to check that $[g_1, g_2] \in Q_2$ for $g_j = s^{i_j} q_j$, $q_j \in Q_2$. Thus $G' \subseteq Q_2$.

To prove both parts 3 and 4, it remains to mention that $Q_2 \subseteq G'$, because $t = [s^{-1}t, s] \in G'$.

Part 5, were we replaced G by Q_2, holds for some sequence of subsets , say $F\prime_k$, of Q_2. Therefore $\cup_{0 \le i < k} s^i F\prime_k$ is the sequence we need in part 5. Proposition 2.1 is proved.

□

3. Approximations

3.1. **The density of conjugations to any fixed free action in Ω_G.** For any countable group G, the set of free actions is dense in Ω_G (see [3]). Thus we only need to approximate a free action of G, say $T\prime$, by conjugations of a fixed free action T of G. Given a finite set $F \subseteq G$, and a measurable set $A \subseteq X$ with $\mu(A) > 0$, $F \times A$ is said to form a (Rokhlin) tower if the sets $T_g A$, $g \in F$ are mutually disjoint. We say that such a tower is an (ε)-tower if $\mu(\bigcup_{g \in F} T_g A) > \varepsilon$. It said that the Rokhlin lemma is valid for F, and a free action T if for all $\varepsilon > 0$ there is a measurable set A such that $F \times A$ forms a $(1-\varepsilon)$-tower of T. Consider the same sets F_k as in Proposition 2.1. The Rokhlin lemma is valid for every F_k and every free G-action, because F_k tiles G, and is finite (see [11], and [12] for the proofs of theorems announced in [11]). Therefore we can choose a measurable set A ($A\prime$) such that $F_k \times A$ ($F_k \times A\prime$) forms a $(1 - 1/k)$-tower for T ($T\prime$, respectively). Obviously, A and $A\prime$ can be chosen such that $\mu(A) = \mu(A\prime)$. Therefore there exists a metrical isomorphism $S_k : A\prime \to A$, i.e. a measure-preserving map (onto) which is one-to-one up to zero measure subsets of $A\prime$ and A. It is clear that S_k can be extended to a metrical isomorphism of (X, μ) and (X, μ) (hence it is an automorphism) such that

$$S_k T\prime_g x = T_g S_k x \text{ for any } \quad g \in F_k, x \in A\prime.$$

It is easy to see that for any $g \in G$

$$S_k^{-1} T_g S_k x = T\prime_g x \quad \text{ if } x \in T\prime_{g_1} A\prime \text{ for some } \quad g_1 \in F_k \cap g^{-1} F_k.$$

Therefore,

$$(\forall g \in G)\ [\mu\{x \in X : S_k^{-1} T_g S_k x \ne T\prime_g x\} < \frac{1}{k} + \frac{|F_k \backslash g^{-1} F_k|}{|F_k|} \to 0 \quad \text{as } k \to \infty].$$

This implies that

$$S_k^{-1} T S_k \to T\prime \quad \text{as } k \to \infty.$$

3.2. **An example of a free action of G such that T_t is weakly mixing.** Let W be a measurable flow on X (i.e. some continuous homomorphism $W : \mathbf{R} \to \mathbf{Aut}(X, \mu)$) such that the transformation W^1 is weakly mixing. Define a transformation $T_0 : \Omega \to \Omega$ by

$$T_0 = \times_{i=-\infty}^{\infty} W^{2^i},$$

where $\Omega = \prod_{i=-\infty}^{\infty} X$. Also consider the right shift map, say S, acting on Ω by

$$(Sx)_i = x_{i-1},\ i \in \mathbf{Z},$$

where y_i is the i^{th} coordinate of $y \in \Omega$. Clearly, maps T_0, S are weakly mixing transformations of the space (Ω, ν), where $\nu = \times_{i=-\infty}^{\infty}\mu$, and the map

$$t \to T_0, \quad s \to S$$

can be naturally extended to a homomorphism from G to a group generated by T_0 and S, so a pair (T_0, S) defines an action of G, say T. Using Proposition 2.1, to prove the freeness of T, we only need to show that

$$|k| + |l| \neq 0 \Rightarrow \nu\{x \in \Omega : T_0^k x = S^l x\} = 0.$$

It is trivial in the case $l = 0$, because W is the aperiodic transformation. To have the same in the case $l \neq 0$, it is remain to mention that for any transformation R of a non-atomic standard Borel probability space $(X, \mathcal{F}, \mu)$ we have $\mu \times \mu\{(x, Rx) : x \in X\} = 0$. This implies that T is a free action.

3.3. **An example of a free action of G such that T_t has rank 1.** Let Λ be a countable discrete subgroup of $\mathbf{T}$ such that the restriction of the map $\phi : \Lambda \to \Lambda, \phi(\lambda) = \lambda^2$, to this subgroup is a group automorphism, and Λ has no roots of unity. By the theorem of Pontryagin, the group of characters of Λ, noted $\widehat{\Lambda}$, is an abelian compact group. It is clear that ϕ^* is an automorphism of $\widehat{\Lambda}$, where ϕ^* is defined by $(\phi^*\chi)(\lambda) = \chi(\phi(\lambda))$ for any character $\chi(\lambda)$. Consider a rotation $S : \widehat{\Lambda} \to \widehat{\Lambda}, S(\chi) = \chi_0\chi$, where $\chi_0(\lambda) = \lambda$ is a character of Λ. Maps ϕ^*, S are transformations of a Lebesgue space $(\widehat{\Lambda}, \mu)$, where μ is a normalized Haar measure on $\widehat{\Lambda}$. Clearly, $\phi^*(\chi) = \chi^2$, and $\phi^* \circ S = S^2 \circ \phi^*$. All characters of $\widehat{\Lambda}$ form an orthonormal basis of $L_2(\widehat{\Lambda}, \mu)$. By the theorem of Pontryagin, every character of $\widehat{\Lambda}$ has the following form

$$\lambda(\chi) = \chi(\lambda)$$

for some $\lambda \in \Lambda$. Clearly,

$$\widehat{S}\lambda(\chi) = \lambda(S\chi) = \lambda(\chi_0)\lambda(\chi) = \lambda\lambda(\chi),$$
$$\widehat{\phi^*}\lambda(\chi) = \lambda(\chi^2) = \lambda^2(\chi).$$

Therefore S is an ergodic transformation having discrete spectrum (moreover, $\Lambda_d(S) = \Lambda$, where $\Lambda_d(S)$ is a set of all eigenvalues of $\widehat{S}$). This implies that S has rank 1 (see [8]).

It is clear that a pair (ϕ^*, S) defines a G-action T via an identification $T_s = {\phi^*}^{-1}$, $T_t = S$. To prove a freeness of T, it is enough to show that

$$(\forall f \in L_2)((\forall k, l \in \mathbf{Z})\ [[\widehat{\phi^*}^k f = \widehat{S}^l f] \Rightarrow [\text{either } k = l = 0 \text{ or } f = \text{const}]]. \tag{3.1}$$

Moreover, we can assume that $k \geq 0$. The transformation S is totally ergodic, because Λ has no roots of unity. Therefore (3.1) is true in the case $k = 0$. Let $k > 0$. Using a uniqueness of a representation of f in the form $\sum_\lambda c_\lambda\lambda(\chi)$, where $\sum_\lambda |c_\lambda|^2 < \infty$, we have

$$[\widehat{\phi^*}^k f = \widehat{S}^l f] \Leftrightarrow [\sum_\lambda c_\lambda\lambda^{2^k}(\chi) = \sum_\lambda c_\lambda\lambda^l\lambda(\chi)],$$

and $|c_\lambda| = |c_{\lambda^{2^k}}| = |c_{\lambda^{2^{k+1}}}| = \ldots.$

If $\lambda \in \Lambda\backslash\{1\}$, then $\lambda, \lambda^{2^k}, \lambda^{2^{k+1}}, \ldots$ are pairwise different. Thus $c_\lambda = 0$ if $\lambda \neq 1$. Therefore $f \equiv c_1$. It follows both (3.1) and the freeness of the G-action T.

We need the following modification of Theorem 5 in [1]

Theorem 3.1. *Suppose D is a dense subset of Ω_G, and $g \in G$; then*

(1) *If T_g is weakly mixing for any $T \in D$, then T_g is weakly mixing for a typical T from Ω_G.*
(2) *If T_g has rank 1 for any $T \in D$, then T_g has rank 1 for a typical T from Ω_G.*

Remark 3.2. Let us mention that Theorem 3.1 is also true in the case of any discrete group G such that Ω_G is a Polish space with respect to the weak topology. Moreover part 2 is true if we replace the property to have rank 1 by the property to have at most rank k. The proof of this remark is a certain modification of the proofs of Theorem 3.1 and Theorem 5 in [1].

Proof. Choose a countable subset of D, say $\{T(k) \in D : k \in \mathbf{N}\}$, which is dense in Ω_G. Let $\varepsilon_i \to 0$ as $i \to \infty$ for some sequence ε_i. Consider

$$B(\varepsilon_1, \varepsilon_2, \dots) = \bigcap_k \bigcup_{i \geq k} U_{\varepsilon_i}(T(i)),$$

where $U_\varepsilon(T) = \{S \in \Omega_G : \rho(T, S) < \varepsilon\}$, ρ is a metric of the space Ω_G. It is easy to check that Ω_G has no isolated points. This implies that $\bigcup_{i \geq k} T(i)$ is also dense for any k. Therefore $B(\varepsilon_1, \varepsilon_2, \dots)$ is a dense G_δ set.

Obviously, to prove Theorem 3.1, it is enough to find a sequence ε_i such that $B(\varepsilon_1, \varepsilon_2, \dots)$ consists of group actions having properties as required in parts 1,2.

Fix a dense sequence of elements in $L_2(X, \mu)$, f_i, $i = 1, 2, \dots$.

To prove part 1, we consider the weak operator topology. Next, we use the well-known fact that a transformation S is weakly mixing if and only if $\widehat{S}^{n_i} \to \int$ for some sequence n_i, and $\int$ is the operator of orthogonal projection onto the space of constants. Let $0 < \delta_n \to 0$ as $n \to \infty$. Choose a sequence $k_i(n)$ such that $\widehat{T}_g^{k_i(n)}(n) \to \int$ as $i \to \infty$. Fix a number $m_n = k_{i(n)}(n)$ such that

$$(\forall p, q \leq n)\ |\langle \widehat{T}_g^{m_n}(n) f_p, f_q\rangle - \langle \int f_p d\mu, f_q\rangle| < \frac{\delta_n}{2}. \tag{3.2}$$

Obviously, for any $k \in \mathbf{Z}$ and $h_1, h_2 \in L_2(X, \mu)$, the map $\phi : \Omega_G \to \mathbf{R}$ defined by $\phi(T) = \langle \widehat{T}_g^k h_1, h_2\rangle$ is continuous.

Thus there exists $\varepsilon_n > 0$ such that

$$\rho(T(n), T) < \varepsilon_n \Rightarrow$$

$$(\forall p, q \leq n)\ |\langle \widehat{T}_g^{m_n}(n) f_p, f_q\rangle - \langle \widehat{T}_g^{m_n} f_p, f_q\rangle| < \frac{\delta_n}{2}. \tag{3.3}$$

Let $T \in B(\varepsilon_1, \varepsilon_2, \dots)$, where ε_i were chosen above. Then, for some $n_j \to \infty$, we have $T \in U_{\varepsilon_{n_j}}(T(n_j))$. Using (3.2) and (3.3), we obtain that

$$|\langle \widehat{T}_g^{m_{n_j}} f_p, f_q\rangle - \langle \int f_p d\mu, f_q\rangle| < \delta_{n_j} \text{ for } p, q = 1, \dots, n_j.$$

Hence

$$(\forall p, q)\ \lim_j \langle \widehat{T}_g^{m_{n_j}} f_p, f_q\rangle = \langle \int f_p d\mu, f_q\rangle.$$

Part 1 is proved.

Let us prove part 2. Let $0 < \delta_n \to 0$ as $n \to \infty$. Since $T_g(n)$ has rank 1, there exist integers $k_i(n)$, towers

$$A_i(n), T_g(n)A_i(n), \dots, T_g^{k_i(n)-1}(n)A_i(n)$$

and corresponding partitions $\xi_i(n)$ of X such that $\xi_i(n) \to \varepsilon$ as $i \to \infty$. Choose a sequence $i(n) > n$ such that $\xi_{i(n)}(n) \to \varepsilon$ as $n \to \infty$. Obviously, for any $k \in \mathbf{Z}$ the map $\phi : \Omega_G \to \mathbf{Aut}(\mu)$ defined by $\phi(T) = \widehat{T}_g^k$ is continuous. Thus there exists $\varepsilon_n > 0$ such that

$$\rho(T(n), T) < \varepsilon_n \Rightarrow$$

$$\max_{0<j<k_{i(n)}(n)} \mu(T_g^j(n)A_{i(n)}(n) \triangle T_g^j A_{i(n)}(n)) < \frac{\delta_n}{k_{i(n)}(n)} \mu(A_{i(n)}(n)). \tag{3.4}$$

Let $T \in B(\varepsilon_1, \varepsilon_2, \ldots)$, where ε_i were chosen above. Then, for some $n_j \to \infty$, we have $T \in U_{\varepsilon_{n_j}}(T(n_j))$.

Consider the following set

$$A\prime_j = A_{i(n_j)}(n_j) \cap T_g^{-1} T_g(n_j) A_{i(n_j)}(n_j) \cap \ldots$$
$$\cap T_g^{1-k_{i(n_j)}(n_j)} T_g^{k_{i(n_j)}(n_j)-1}(n_j) A_{i(n_j)}(n_j).$$

Obviously,

$$T_g^l A\prime_j \subseteq T_g^l(n_j) A_{i(n_j)}(n_j), \quad l = 0, \ldots, k_{i(n_j)}(n_j) - 1.$$

Therefore the sets $T_g^l A\prime_j$ ($l = 0, \ldots, k_{i(n_j)}(n_j) - 1$) and the remaining part of X form a partition of X, say η_j. Using (3.4), we obtain that

$$\frac{\mu(A\prime_j)}{\mu(A_{i(n_j)}(n_j))} > 1 - \frac{\delta_{n_j}}{2} \to 1 \text{ as } j \to \infty.$$

This implies that $\eta_j \to \varepsilon$ as $j \to \infty$, because $\xi_{i(n_j)}(n_j) \to \varepsilon$ as $j \to \infty$. Thus T_g has rank 1. Theorem 3.1 is proved. □

4. The proof of Theorem 1.2

Proof. To prove Theorem 1.2, it is enough to collect all remarks of Section 3. Namely, denote by A the property of a G-action T to have a weakly mixing T_t. Properties we consider are invariant with respect to any metrical isomorphism (in particular, any conjugation). Therefore the set of actions having the property A is dense. Finally, Theorem 3.1 implies that A holds for a typical G-action. It is remain to repeat the same arguments for the set of G-actions T satisfying the rank $T_t = 1$ condition. Theorem 1.2 is proved. □

5. Closing remarks

There are many observations about metrical properties of dynamical systems conjugated to their composition squares (see [5], [6]). We restrict our attention to the "typical" case, i.e. we consider metrical properties of T_t for a typical G-action T. It is not difficult to prove that in our case T_t is rigid, i.e. $T^{k_i} \to E$ for some sequence k_i, where E means the identity transformation. Therefore the typical transformation is not mixing. It is also easy to see that the typical transformation T_t has a simple singular spectrum. Moreover, the maximal spectral type of $\widehat{T}_t$ is invariant and ergodic with respect to a map $\phi : \mathbf{T} \to \mathbf{T}$, $\phi(\lambda) = \lambda^2$, where the ergodicity means that every invariant Borel set has zero measure or is the torus up to a set of zero measure.

All statements of this paper can be naturally extended to the cases both maps T_t^q, where $q = k2^l$, $k \neq 0$, $l \in \mathbf{Z}$, and $G_n = \mathrm{gr}\langle t, s;\ ts = st^n \rangle$, where $n \neq 0$.

However let us mention that the case $n = -1$ is essentially different, because (see [1], Theorem 7) for a typical G_{-1}-action T, transformation T_t has a homogeneous spectrum of multiplicity 2 in the orthogonal complement of the constant functions.

References

1. O. Ageev, *The homogeneous spectrum problem in ergodic theory*, to appear in Invent. Math.
2. I.P. Cornfel'd , S.V. Fomin, Ya. G. Sinai, *Ergodic Theory* (Springer-Verlag, 1980).
3. E. Glasner, J.L. King, A zero-one law for dynamical properties, In Topol. Dyn. and Appl. **215** (1998), 231–242
4. G. Goodson, *A survey of recent results in the spectral theory of ergodic dynamical systems*, J. Dynam. Control Systems **5** (1999), 173–226.
5. G. Goodson, *Ergodic dynamical systems conjugate to their composition squares*, Acta Math. Univ. Comenian. (N.S.) **71** (2002), 201–210.
6. G. Goodson, *Spectral properties of ergodic dynamical systems conjugate to their composition squares*, preprint.
7. P.R. Halmos, *Lectures on ergodic theory* (N.Y.: Chelsea Publ. Comp., 1956).
8. A. del Junco, *Transformations with discrete spectrum are stacking transformations*, Canad. J. Math. **24** (1976), 836–839.
9. A. del Junco, *Disjointness of measure-preserving transformations, self-joinings and category*, Prog. Math. **10** (1981), 81–89.
10. A.A. Kirillov, *Elements of the theory of representations* (Springer-Verlag, 1976).
11. D. Ornstein, B. Weiss, *Ergodic theory of amenable group actions. I. The Rohlin lemma*, Bull. Amer. Math. Soc. (N.S.) **2** (1980), 161–164.
12. D. Ornstein, B. Weiss, *Entropy and isomorphism theorems for actions of amenable groups*, J. Analyse Math. **48** (1987), 1–141.

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