ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$

The detailed version, **Part 3**

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In memory of Professor Alexander Buchstab

Through the shade of sleep,
Through the cloudy strife
Of Death and of Life
P.B. Shelley.

Content

§0. Foreword

At the end of the second part, we have come to the necessity of studying the zeros of the polynomial

$$D^\wedge(z,\eta)=(\eta+1)^3(\eta+\gamma_1)^3-2^3(1+\gamma_1)^3z\eta^3. \tag{3.0.1,}$$

where $\gamma_1=\frac{1-\delta_0}{1+\delta_0}$ for $\delta_0=\frac{1}{\Delta}$ and $\Delta\in\mathbb{N}+1$. I do this here in the paragraph 1. Then I give, as compensation for the heavy paragraph 1, an easy paragraph 2.

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§1. "Exercise."

Let $z = -r^3 \exp(3i\phi)$, where $1 \le r$, $-\frac{\pi}{3} < \phi \le \frac{\pi}{3}$. Let further $\psi = \phi + \frac{2k\pi}{3}$, where $k = -1, 0, 1$. Then

$$D^{\wedge}(z,\eta) = \prod_{k=-1}^{1} \left((\eta+1)(\eta+\gamma_1) + 2r\exp\left(i\phi + \frac{2ik\pi}{3}\right)(1+\gamma_1)\eta \right) \tag{3.1.0}$$

So, we have to study the zeros of the polynomial

$$D^{\vee}(z,\eta) = (\eta+1)(\eta+\gamma_1) + 2r\exp(i\psi)(1+\gamma_1)\eta \tag{3.1.1}$$

for $1 \le r$ and $-\pi < \psi \le \pi$. We shall study the zeros of the polynomial (3.1.1) under more general conditions

$$-1 < \delta_0 < 1,\ 1 \le r,\ \gamma_1 = (1-\delta_0)/(1+\delta_0),\ -\pi \le \psi \le \pi. \tag{3.1.2}$$

Putting $\eta = \theta(1+\gamma_1), \theta^{\vee} = -\exp(-i\psi)\theta$, we get the equalities

$$D^{\vee}(z,\eta) = (1+\gamma_1)^2 D_1^{\vee}(z,\theta), \tag{3.1.3}$$

where

$$D_1^{\vee}(z,\theta) = \theta^2 + 2\left(\frac{1}{2} + r\exp(i\psi)\right)\theta + \frac{\gamma_1}{(1+\gamma_1)^2} = \exp(2i\psi)D_2^{\vee}(z,\theta^{\vee}) \tag{3.1.4}$$

with

$$D_2^{\vee}(z,\theta^{\vee}) = (\theta^{\vee})^2 - 2\left(\frac{\exp(-i\psi)}{2} + r\right)\theta^{\vee} + \frac{\gamma_1\exp(-\ 2i\psi)}{(1+\gamma_1)^2}. \tag{3.1.5}$$

Let

$$D_0(r,\psi,\delta_0) = r^2 + r\exp(-i\psi) + \left(\frac{\delta_0}{2}\right)^2\exp(-\ 2i\psi). \tag{3.1.6}$$

If we put $t = \cos(\psi)$, then

$$Im(D_0(r,\psi,\delta_0)) = -\left(\frac{(\delta_0)^2}{2}t + r\right)\sin(\psi), \tag{3.1.7}$$

$$\operatorname{Re}(D_0(r,\psi,\delta_0)) = \frac{(\delta_0)^2t^2}{2} + rt + \ r^2 - \left(\frac{\delta_0}{2}\right)^2. \tag{3.1.8}$$

Since $r \geq 1 \geq (\delta_0)^2$, it follows that

$$\min_{-\pi \leq \psi \leq \pi} \mathrm{Re}\left(D_0(r,\psi,\delta_0)\right) = r^2 - r + \left(\frac{\delta_0}{2}\right)^2 \geq 0; \tag{3.1.9}$$

and the equality $Re(D_0(r,\psi,\gamma)) = 0$ is reached only if $r = 1, \delta_0 = 0, \psi = \pm\pi$; this is the only case when the equality $D(r,\psi,\delta_0) = 0$ holds. Clearly, for $t = \cos(\psi)$, the following equality will hold:

$$|D_0(r,\psi,\delta_0)|^2 = r^4 + r^2 + \left(\frac{\delta_0}{2}\right)^4 + \tag{3.1.10}$$

$$2r^2\left(\frac{\delta_0}{2}\right)^2(2t^2-1) + 2r(r^2 + \left(\frac{\delta_0}{2}\right)^2)t =$$

$$r^2 + \left(r^2 - \left(\frac{\delta_0}{2}\right)^2\right)^2 + 2r\left(r^2 + \left(\frac{\delta_0}{2}\right)^2\right)t + r^2(\delta_0)^2t^2.$$

Below, in order to extract the square root w from the complex number $z = x + iy$, we use the well known formulas

$$w = \varepsilon_1\left(\sqrt{\frac{|z|+x}{2}} + i\,\mathrm{sign}\;(y)\sqrt{\frac{|z|-x}{2}}\right), \tag{3.1.11}$$

where $\varepsilon_1^2 = 1, z \notin (-\infty, 0)$, and

$$w = \varepsilon_1\varepsilon_2 i\sqrt{|z|}, \tag{3.1.12}$$

where $\varepsilon_1^2 = 1,\ \varepsilon_2^2 = 1,\ z \in (-\infty, 0)$. By fixed ε_1 and ε_2, formulas (3.1.11) - (3.1.12) determine the defined on $\mathbb{C}$ branch $w_0(\varepsilon_1,\varepsilon_2,z)$ of multivalued function $z^{1/2}$. This branch is regular in the domain $\mathbb{C}\backslash(-\infty,0]$ and has the continuous restrictions on each of the sets $\mathbb{C}\backslash(-\infty,0)$ and $\{z \in \mathbb{C}\colon \varepsilon_2 Im(z) \geq 0\}$. In view of (3.1.11),

$$Re(w_0(1,\varepsilon_2,z)) \geq 0, \tag{3.1.13}$$

where $z \in \mathbb{C}\backslash(-\infty,0)$, and the equality in (3.1.13) for this case is fulfilled only if $z = 0$. According to the Riemann - Schwarz principle of symmetry, since $w_0(\varepsilon_1,\varepsilon_2,z) \geq 0$ for $z > 0$, it follows that

$$w(\varepsilon,1,z) = \overline{w(\varepsilon,1,\bar{z})}, \tag{3.1.14}$$

where $\varepsilon^2 = 1,\ z \in \mathbb{C}\backslash(-\infty, 0)$. We are interested now in the behavior of $D_0(r, \psi, \delta_0)$ when ψ changes in $[-\pi, \pi]$. According to (3.1.9), when ψ changes in $[-\pi, \pi]$, the value of

$$D_0(r, \psi, \delta_0)$$

remains in the set $\mathbb{C}\backslash(-\infty, 0)$, where the function

$$w(1, 1, z) = w(1, -1, z)$$

is continuous. Therefore, the value

$$w_0(1, 1, D_0(r, \psi, \delta_0)) = w_0(1, -1, D_0(r, \psi, \delta_0)) \tag{3.1.15}$$

in this case is continuous on the set

$$-\pi \leq \psi \leq \pi, 1 \leq r. \tag{3.1.16}$$

It will further be denoted by $R_0(r, \psi, \delta_0)$. Since

$$R_0(r, -\pi, \delta_0) = R_0(r, \pi, \delta_0),$$

it follows that the function $R_0(r, \psi, \delta_0)$, as function from ψ, admits $2\pi-$periodical continuation on $\mathbb{R}$. Since

$$D_0(r, 0, \delta_0) = r^2 + r + (\delta_0/2)^2 > 0 \tag{3.1.17}$$

and

$$D_0(r, \psi, \delta_0) = \overline{D_0(r, -\psi, \delta_0)},$$

it follows from (3.1.14) that

$$R_0(r, \psi, \delta_0) = \overline{R_0(r, -\psi, \delta_0)}. \tag{3.1.18}$$

In (3.1.6), we considered the value $D_0(r, \psi, \delta_0)$ only for $|\delta_0| < 1$, but it may also be considered for $\delta_0 = 1$, and in this case, according to (3.1.6),

$$D_0(r, \psi, 1) = \left(r + \frac{\exp(-i\psi)}{2}\right)^2;$$

the inequality (3.1.9) is also fulfilled for $\delta_0 = 1$, and the value

$$w_0(1, 1, D_0(r, \psi, 1)) = w_0(1, -1, D_0(r, \psi, 1)) \tag{3.1.19}$$

in this case is continuous on the set (3.1.16). It will further be denoted by $R_0(r,\psi,1)$. Since

$$R_0(r,-\pi,1)=R_0(r,\pi,1), \tag{3.1.20}$$

it follows that the function $R_0(r,\psi,1)$, as function from ψ, admits $2\pi-$periodical continuation on $\mathbb{R}$ and (3.1.18) also holds for $\delta_0=1$. Since

$$\mathrm{Re}\left(r+\frac{\exp(-i\psi)}{2}\right)=r+\frac{cos(\psi)}{2}>0, \tag{3.1.21}$$

if $r\geq 1$, in view of (3.1.13), it follows that

$$R_0(r,\psi,1)=r+\frac{\exp(-i\psi)}{2}. \tag{3.1.22}$$

According to (3.1.22), for $r\geq 1$, $|\delta_0|<1$ and $k=0,1$, the values

$$\theta_k^{\vee}(r,\psi,\delta_0)=R_0(r,\psi,1)+(-1)^kR_0(r,\psi,\delta_0) \tag{3.1.23}$$

compose the set of all the roots of the polynomial (3.1.5), the values

$$\theta_k^{\wedge}(r,\psi,\delta_0)=-e^{i\psi}(R_0(r,\psi,1)+(-1)^kR_0(r,\psi,\delta_0)) \tag{3.1.24}$$

compose the set of all the roots of the polynomial (3.1.4) and, according to (3.1.3), the values

$$\eta_k^{\wedge}(r,\psi,\delta_0)=-(1+\gamma_1)e^{i\psi}(R_0(r,\psi,1)+(-1)^kR_0(r,\psi,\delta_0)) \tag{3.1.25}$$

compose the set of all the roots of the polynomial (3.1.1). It is clear that the values (3.1.25), as functions from ψ, are $2\pi-$periodical. Clearly, the six numbers

$$\eta_{k,l}^{*}(\delta_0)=\eta_k^{\wedge}(1,(2l-1)\pi/3,\delta_0), \tag{3.1.26}$$

where $k=0,1$ and $l=0,1,2$, compose the set of all the roots of the polynomial (3.1.1) for $z=1$. It is clear that

$$\eta_{k,2}^{*}(\delta_0)\in\mathbb{R},\eta_{k,1}^{*}(\delta_0)=\overline{\eta_{k,0}^{*}(\delta_0)}. \tag{3.1.27}$$

Let, as in [8] and [9], (2.4.55),

$$h^{\sim}(\eta)=(\eta^2-1)^3(1-\delta_0)^{-3d_1}2^{-6}\eta^{3d_1}. \tag{3.1.28}$$

Let further

$$F_0^{\vee}(\eta)=\prod_{l=0}^{2}\prod_{k=0}^{1}(\eta-h^{\sim}(\eta^*_{k,l}))$$

and $\mathfrak{D}_0^{\vee}$ be the discriminant relatively to η of the polynomial $F_0^{\vee}(\eta)$. The goal of this paragraph is the proof of the following

Theorem 3.1.1. *If $\Delta = 13$, then the following inequality holds:*

$$\mathfrak{D}_0 \neq 0. \tag{3.1.29}$$

To prove this theorem we shall first prove the following

Theorem 3.1.2. *If $\Delta = 13$, then the following inequalities hold:*

$$|h^{\sim}(\eta^*_{0,l})| > |h^{\sim}(\eta^*_{1,l})|,$$

where $l = 0, 1$;

$$0 = |h^{\sim}(\eta^*_{0,2})| < |h^{\sim}(\eta^*_{1,2})|.$$

Then we shall prove

Theorem 3.1.3. *If $\Delta = 13$, then $h^{\sim}(\eta^*_{k,l}) \notin \mathbb{R}$, where $k = 0, 1$ and $l = 0, 1$. Theorem 3.1.1 follows from (3.1.27), Theorem 3.1.3 and Theorem 3.1.2.*

Lemma 3.1.1. *The value $|D_0(r,\psi,\delta_0)|$ is increasing and concave (convex upwards) function from $t = \cos(\psi)$.*

Proof. According to (3.1.10),

$$|D_0(r,\psi,\delta_0)| = \sqrt{at^2+bt+c},$$

where

$$a = r^2(\delta_0)^2, b = 2r\left(r^2+\left(\frac{\delta_0}{2}\right)^2\right), c = r^2+\left(r^2-\left(\frac{\delta_0}{2}\right)^2\right)^2.$$

In view of

$$\frac{\partial}{\partial t}\sqrt{at^2+bt+c} = \frac{2at+b}{2\sqrt{at^2+bt+c}}$$

$$\left(\frac{\partial}{\partial t}\right)^2\sqrt{at^2+bt+c} = \frac{-(2at+b)^2+4a(at^2+bt+c)}{4\sqrt{(at^2+bt+c)^3}} =$$

$$\frac{4ac-b^2}{4\sqrt{(at^2+bt+c)^3}}$$

and

$$4ac-b^2 = 4r^4(\delta_0)^2+$$

$$4r^2\left((\delta_0)^2\left(r^2-\left(\frac{\delta_0}{2}\right)^2\right)^2-\left(r^2+\left(\frac{\delta_0}{2}\right)^2\right)^2\right)=$$

$$-4r^2(1-(\delta_0)^2)\left(r^2-\left(\frac{\delta_0}{2}\right)^2\right)^2\leq 0,$$

the lemma is proved.

Let $\varepsilon^3=\varepsilon$; according to (3.1.24),

$$\eta_k^{\wedge}(r,\psi,\delta_0)+\varepsilon=-\frac{2\exp(i\psi)}{1+\delta_0}\times \tag{3.1.30}$$

$$\left(r+\frac{\exp(-i\psi)}{2}-\varepsilon\frac{1+\delta_0}{2}\exp(-i\psi)+(-1)^kR_0(r,\psi,\delta_0)\right),$$

where $\varepsilon^3=\varepsilon, k=0,1$. Let

$$R_\varepsilon^*(r,\psi,\delta_0)=r+\frac{\exp(-i\psi)}{2}-\varepsilon\frac{1+\delta_0}{2}\exp(-i\psi), \tag{3.1.31}$$

where $\varepsilon^3=\varepsilon$.

Lemma 3.1.2. *The values*

$$|\eta_k^{\wedge}(r,\psi,\delta_0)+\varepsilon|^2,$$

where $k=0,1,r\geq 1,\psi\in[0,\pi],r-1+\pi-\psi>0,\varepsilon^3=\varepsilon,$ *are nonnegative zeros of the polynomial*

$$x^2-xp_{1+\varepsilon}(r,\psi,\delta_0)+q_{1+\varepsilon}(r,\psi,\delta_0), \tag{3.1.32}$$

where $\varepsilon^3=\varepsilon,$ *and*

$$p_{1+\varepsilon}(r,\psi,\delta_0)=8(|R_\varepsilon^*(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2, \tag{3.1.33}$$

$$q_{1+\varepsilon}(r,\psi,\delta_0)=|D^{\vee}(z,-\varepsilon)|^2. \tag{3.1.34}$$

Proof. According to theorem about the sum of the squares of the diagonales of the parallelogramm, the following equality holds:

$$\sum_{k=0}^{1}|\eta_k^{\wedge}(r,\psi,\delta_0)+\varepsilon|^2= \tag{3.1.35}$$

$$8(|R^*_\varepsilon(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2.$$

According to (3.1.1),

$$\prod_{k=0}^{1}|\eta^\wedge_k(r,\psi,\delta_0)+\varepsilon|^2=|D^\vee(z,-\varepsilon)|^2. \tag{3.1.36}$$

The lemma is proved.

Lemma 3.1.3. *Let* $\varepsilon^2=\varepsilon$. *With increasing* $\psi\in(0,\pi)$, *the value* $|\eta^\wedge_0(r,\psi,\delta_0)+\varepsilon|$ *monotonically decreases, but the value*

$$|\eta^\wedge_1(r,\psi,\delta_0)+\varepsilon|$$

monotonically increases and

$$|\eta^\wedge_1(r,\psi,\delta_0)+\varepsilon|<|\eta^\wedge_0(r,\psi,\delta_0)+\varepsilon|, \tag{3.1.37}$$

where $\psi\in[0,\pi)$;

$$|\eta^\wedge_1(r,\pi,\delta_0)+\varepsilon|\le\frac{1-\delta_0}{1+|\delta_0|}+\varepsilon\le \tag{3.1.38}$$

$$\frac{1+|\delta_0|}{1+\delta_0}+\varepsilon\le|\eta^\wedge_0(r,\pi,\delta_0)+\varepsilon|;$$

moreover, the first and the last inequalities (3.1.38) become equalities only if $r=1$, *and the middle inequality (3.1.38) becomes equality only if* $\delta_0=0$; *consequently,*

$$|\eta^\wedge_1(r,\pi,\delta_0)+\varepsilon|=|\eta^\wedge_0(r,\pi,\delta_0)+\varepsilon|=1+\varepsilon \tag{3.1.39}$$

only if $r=1$ *and* $\delta_0=0$.

Proof. According to Lemma 3.1.2, if $\varepsilon^3=\varepsilon$, then

$$|\eta^\wedge_k(r,\psi,\delta_0)+\varepsilon|^2=\frac{p_{1+\varepsilon}}{2}\pm\sqrt{\frac{p^2_{1+\varepsilon}}{4}-q_{1+\varepsilon}}, \tag{3.1.40}$$

where $k=0,1,\psi\in[0,\pi]$. According to (3.1.31), (3.1.33) and Lemma 3.1.2 ,

$$p_1=8(|R^*_0(r,\psi,\delta_0)|^2+|R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2= \tag{3.1.41}$$

$$8\left(r^2+rt+\frac{1}{4}+|D_0(r,\psi,\delta_0)|)/(1+\delta_0\right)^2,$$

$$p_2 = 8(|R_1^*(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2 = \tag{3.1.42}$$

$$8\left(r^2 - r\delta_0 t + \frac{(\delta_0)^2}{4} + |D_0(r,\psi,\delta_0)|\right)/(1+\delta_0)^2.$$

In view of (3.1.34) and (3.1.3),

$$q_1(r,\psi,\delta_0) = \left(\frac{1-\delta_0}{1+\delta_0}\right)^2, q_2(r,\psi,\delta_0) = \left(\frac{4r}{1+\delta_0}\right)^2. \tag{3.1.43}$$

In view of (3.1.41) and Lemma 3.1.1, the value p_1 increases with increasing $t = \cos(\psi) \in (-1,1)$. According to the assertion of the Lemma 3.1.1 and the proof of this Lemma,

$$\frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|\Big|_{t=\cos(\psi)} > \frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|\Big|_{t=1} =$$

$$\frac{2a+b}{2|D_0(r,0,\delta_0)|},$$

where

$$a = r^2(\delta_0)^2, b = 2r(r^2 + (\delta_0/2)^2), 0 < \psi \le \pi.$$

Therefore, according to (3.1.42),

$$\frac{\partial}{\partial t}p_2 = \frac{8(-r\delta_0 + \frac{\partial}{\partial t}|D_0(r,\psi,\delta_0)|)}{(1+\delta_0)^2}$$

and

$$-r\delta_0 + \frac{\partial}{\partial t}|D_0(r,\psi,\delta_0 >$$

$$-r\delta_0 + \frac{r^2(\delta_0)^2 + r(r^2 + (\delta_0/2)^2)}{r^2 + r + (\delta_0/2)^2} = r\times$$

$$\frac{-\delta_0(r^2 + r + (\delta_0/2)^2) + r(\delta_0)^2 + r\delta_0 + (r-\delta_0/2)^2}{r^2 + r + (\delta_0/2)^2} =$$

$$\frac{r(1-\delta_0)(r-\delta_0/2)^2}{r^2 + r + (\delta_0/2)^2} > 0.$$

Consequently, the value of p_2 increases with increasing $t = \cos(\psi) \in (-1,1)$. The value of

$$\Delta^*_{1+\varepsilon}(r,\psi,\delta_0) = (p_{1+\varepsilon}/2)^2 - q_{1+\varepsilon}$$

for some $\varepsilon \in \{0, 1\}$ is equal to 0 if and only if $p_{1+\varepsilon}/2 = (q_{1+\varepsilon})^{1/2}$, i.e. if

$$r^2 + rt + \frac{1}{4} + |D_0(r, \psi, \delta_0)| = (1 - \delta_0^2)/4, \tag{3.1.44}$$

when $\varepsilon = 0$, and

$$r^2 - \delta_0 rt + \left(\frac{\delta_0}{2}\right)^2 + |D_0(r, \psi, \delta_0)| = r(1 + \delta_0), \tag{3.1.45}$$

when $\varepsilon = 1$. The equation (3.1.44) is equivalent to the system consisting of the inequality

$$r^2 + rt + \delta_0^2/4 \leq 0 \tag{3.1.46}$$

and the equation

$$(r^2 + \delta_0^2/4)^2 + 2r(r^2 + \delta_0^2/4)t + r^2t^2 = \tag{3.1.47}$$

$$= r^2 + (r^2 - \delta_0^2/4)^2 \ + \ 2r(r^2 + \delta_0^2/4)t + r^2\delta_0^2t^2.$$

The solutions of the equation (3.1.47) are the roots of the polynomial

$$r^2(1 - \delta_0^2)(1 - t^2). \tag{3.1.48}$$

The equation (3.1.45) is equivalent to the system consisting of the inequality

$$\left(r - \frac{\delta_0}{2}\right)^2 - r - \delta_0 rt \leq 0, \tag{3.1.49}$$

and the equation

$$\left(r - \frac{\delta_0}{2}\right)^2 \left(\left(r - \frac{\delta_0}{2}\right)^2 - 2r\right) + r^2 - \tag{3.1.50}$$

$$2\left(\left(r - \frac{\delta_0}{2}\right)^2 - r\right)\delta_0 rt + (\delta_0 rt)^2 =$$

$$= r^2 + (r^2 - \delta_0^2/4)^2 \ + \ 2r(r^2 + \delta_0^2/4)t + r^2\delta_0^2t^2.$$

The solutions of the equation (3.1.50) are the roots of the polynomial

$$2r(1 + \delta_0)(r - \delta/2)^2(1 + t). \tag{3.1.51}$$

Since, for $t = 1$, the inequality (3.1.46) is not fulfilled, it follows, according to (3.1.48) and (3.1.51), that the value

$$\Delta^*_{1+\varepsilon}(r, \psi, \delta_0)$$

is not equal to 0, when $\varepsilon^2 = \varepsilon$ and $t \in (-1, 1]$; according to (3.1.30) and (3.1.17), if $\varepsilon^3 = \varepsilon$ and $k^2 = k$, then

$$\eta_k^\wedge(r, 0, \delta_0) + \varepsilon = -\frac{2\left(r + \frac{1}{2} - \varepsilon\frac{1+\delta_0}{2} + (-1)^k\sqrt{r^2 + r + (\frac{\delta_0}{2})^2}\right)}{1 + \delta_0}, \tag{3.1.52}$$

$$|\eta_0^\wedge(r, 0, \delta_0) + \varepsilon| = \frac{2\left(r + \frac{1}{2} - \varepsilon\frac{1+\delta_0}{2} + \sqrt{r^2 + r + (\frac{\delta_0}{2})^2}\right)}{1 + \delta_0} < \tag{3.1.53}$$

$$|\eta_1^\wedge(r, 0, \delta_0) + \varepsilon| = \frac{2\left|r + \frac{1}{2} - \varepsilon\frac{1+\delta_0}{2} - \sqrt{r^2 + r + (\frac{\delta_0}{2})^2}\right|}{1 + \delta_0}.$$

Therefore, (3.1.37) is fulfilled and, if

$$\varepsilon^2 = \varepsilon,$$

then instead (3.1.40), one may write a more exact equality

$$|\eta_k^\wedge(r, \psi, \delta_0) + \varepsilon|^2 = \frac{p_{1+\varepsilon}}{2} + (-1)^k\sqrt{\left(\frac{p_{1+\varepsilon}}{2}\right)^2 - q_{1+\varepsilon}}; \tag{3.1.54}$$

consequently, for $\varepsilon \in \{0, 1\}$ with increasing $t = cos(\psi) \in (-1, 1)$, the value

$$|\eta_0^\wedge(r, \psi, \delta_0) + \varepsilon|$$

monotonically increases and, in view of (3.1.43), the value

$$|\eta_1^\wedge(r, \psi, \delta_0) + \varepsilon|$$

monotonically decreases. According to (3.1.30), if

$$\varepsilon^3 = \varepsilon$$

and $k^2 = k$, then

$$\eta_k^\wedge(r,\pi,\delta_0)+\varepsilon = \frac{2\left(r-\frac{1}{2}+\varepsilon\frac{1+\delta_0}{2}+(-1)^k\sqrt{r^2-r+\left(\frac{\delta_0}{2}\right)^2}\right)}{1+\delta_0}. \tag{3.1.55}$$

If $\varepsilon^2=\varepsilon$, then the following from (3.1.55) and (3.1.54) relations

$$|\eta_0^\wedge(r,\pi,\delta_0)+\varepsilon| = \eta_0^\wedge(r,\pi,\delta_0)+\varepsilon =$$

$$\frac{2\left(r-\frac{1}{2}+\sqrt{r^2-r+(\frac{\delta_0}{2})^2}\right)}{1+\delta_0}+\varepsilon \geq \frac{1+|\delta_0|}{1+\delta_0}+\varepsilon,$$

together with (3.1.43), entail the inequality

$$|\eta_1^\wedge(r,\pi,\delta_0)|+\varepsilon \leq \frac{1-\delta_0}{1+|\delta_0|}+\varepsilon =$$

$$\frac{1-|\delta_0|}{1+\delta_0}+\varepsilon$$

and therefore also entail (3.1.38) and (3.1.39). The lemma is proved.

Corollary 1. *If* $-1<\delta_0<1, r\geq 1, \psi\in(0,\pi)$, *then*

$$|\eta_0^\wedge(r,\psi,\delta_0)| > |\eta_0^\wedge(r,\pi,\delta_0)| = \eta_0^\wedge(r,\pi,\delta_0) = \tag{3.1.56}$$

$$\frac{2}{1+\delta_0}(r-1/2+(r^2-r+\delta_0^2/4)^{1/2}) \geq$$

$$\frac{2}{1+\delta_0}(r-1/2-(r^2-r+\delta_0^2/4)^{1/2}) =$$

$$\eta_1^\wedge(r,\pi,\delta_0) = |\eta_1^\wedge(r,\pi,\delta_0)| > |\eta_1^\wedge(r,\psi,\delta_0)|,$$

$$|\eta_1^\wedge(r,\psi,\delta_1)+1| \leq |\eta_1^\wedge(r,\psi,\delta_1)|+1 < |\eta_1^\wedge(r,\pi,\delta_0)|+1 = \eta_1^\wedge(r,\pi,\delta_0)+1 =$$

$$\frac{2}{1+\delta_0}(r-1/2-(r^2-r+\delta_0^2/4)^{1/2})+1 =$$

$$\frac{2}{1+\delta_0}(r+\delta_0/2-(r^2-r+\delta_0^2/4)^{1/2}) \leq$$

$$\frac{2}{1+\delta_0}(r+\delta_0/2+(r^2-r+\delta_0^2/4)^{1/2}) =$$

$$\eta_0^\wedge(r,\pi,\delta_0)+1 = |\eta_0^\wedge(r,\pi,\delta_0)+1| =$$

$$q_2(r,\pi,\delta_0)/|\eta_1^{\wedge}(r,\pi,\delta_0)+1| = \frac{4r}{(1+\delta_0)|\eta_1^{\wedge}(r,\pi,\delta_0)+1|} <$$

$$\frac{4r}{(1+\delta_0)|\eta_1^{\wedge}(r,\psi,\delta_0)+1|} = q_2(r,\psi,\delta_0)/|\eta_1^{\wedge}(r,\psi,\delta_0)+1| =$$

$$|\eta_0^{\wedge}(r,\psi,\delta_0)+1|.$$

Proof. Since, according to the Lemma, for $\varepsilon \in \{0,1\}$ with increasing $\psi \in (0,\pi)$, the value $|\eta_0^{\wedge}(r,\psi,\delta_0)+\varepsilon|$ monotonically decreases, but the value

$$|\eta_1^{\wedge}(r,\psi,\delta_0)+\varepsilon|$$

monotonically increases, it follows that the first, the seventh, the ninth and the eighteenth relations of the assertions of the corollary 1 hold; the second, the third, the fifth, the sixth, the tenth, the twelfth the fourteenth and the fifteenth relations of the assertions of the corollary 1 follow directly from (3.1.55); the fourth, the eighth, the thirteenth, the sixteenth and twentieth relations of the assertions of the corollary 1 are obvious; the seventeenth and the nineteenth relations of the assertions of the corollary 1 follow from (3.1.43). The corollary is proved.

Corollary 2. *If* $r \geq 1, -1 < \delta_0 < 1, \psi \in [0,\pi]$, *then*

$$|\eta_0^{\wedge}(r,\psi,\delta_0)| \geq |\eta_0^{\wedge}(r,\pi,\delta_0)| = \frac{2}{1+\delta_0}(r-1/2+(r^2-r+\delta_0^2/4)^{1/2}) \geq \tag{3.1.57}$$

$$\frac{1+|\delta_0|}{1+\delta_0} \geq \frac{1-|\delta_0|}{1+\delta_0} \geq$$

$$|\eta_1^{\wedge}(r,\pi,\delta_0)| \geq |\eta_1^{\wedge}(r,\psi,\delta_0)|;$$

$$|\eta_1^{\wedge}(r,\psi,\delta_0)+1| \leq |\eta_1^{\wedge}(r,\psi,\delta_0)|+1 \leq$$

$$\frac{2-|\delta_0|+\delta_0}{1+\delta_0};$$

$$|\eta_0^{\wedge}(r,\psi,\delta_0)+1| = \frac{4r}{((1+\delta_0)|\eta_1^{\wedge}(r,\psi,\delta_0)+1|)} \geq \frac{4r}{2-|\delta_0|+\delta_0}.$$

Proof. The first relation (3.1.57) follows from the first relation (3.1.56); the second relation (3.1.57) coincides with the first relation (3.1.56); since $r \geq 1$, the third relation (3.1.57) is obvious; the fourth and seventh relations (3.1.57) are also obvious; since

$$(1+|\delta_0|)(1-|\delta_0|) = (1-(\delta_0)^2),$$

the fifth relation (3.1.57) follows from the second relation (3.1.57), from the third relation (3.1.57) and from (3.1.43); the sixth relation (3.1.57) follows from the seventh relation (3.1.56); the eighth relation (3.1.57) follows from the fifth relation (3.1.57); the ninth relation (3.1.57) follows from (3.1.43); the tenth relation (3.1.57) follows from the eighth relation (3.1.57). The corollary is proved.

Since

$$|\eta_k^\wedge(r,\psi,\delta_0)-1|^2+|\eta_k^\wedge(r,\psi,\delta_0)+1|^2=2|\eta_k^\wedge(r,\psi,\delta_0)|^2+2, \tag{3.1.58}$$

in view of (3.1.53) and (3.1.43), it follows that

$$\begin{gathered}|\eta_k^\wedge(r,\psi,\delta_0)-1|^2=2+2|\eta_k^\wedge(r,\psi,\delta_0)|^2-|\eta_k^\wedge(r,\psi,\delta_0)+1|^2= \\ =\frac{2p_1-p_2}{2}+ \\ (-1)^k2\sqrt{\left(\frac{p_1}{2}\right)^2-q_1}-(-1)^k\sqrt{\left(\frac{p_2}{2}\right)^2-q_2}.\end{gathered} \tag{3.1.59}$$

Since, according to (3.1.48) and (3.1.51), the value

$$\Delta^*_{1+\varepsilon}(r,\psi,\delta_0)$$

is not equal to 0, when $\varepsilon^2=\varepsilon$ and $t\in(-1,1]$, it follows that

$$\begin{gathered}\frac{\partial}{\partial t}(|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2)= \\ (-1)^k\left(\frac{\partial}{\partial t}\frac{p_{1+\varepsilon}}{2}\right)|\eta_k^\wedge(r,\psi,\delta_0)+\varepsilon|^2\Big/\sqrt{\left(\frac{p_{1+\varepsilon}}{2}\right)^2-q_{1+\varepsilon}},\end{gathered} \tag{3.1.60}$$

where $\varepsilon^2=\varepsilon, k^2=k,\ |\delta_0|<1,\ \psi\in[0,\pi)$. Let

$$s=\sqrt{\frac{(r-1)^2}{4}+r\cos^2\left(\frac{\psi}{2}\right)}, \tag{3.1.61}$$

where $\psi\in[0,\pi]$. Then

$$\frac{r-1}{2}\le s\le\frac{r+1}{2}, \tag{3.1.62}$$

$$t=\cos(\psi)=\frac{4s^2-r^2-1}{2r} \tag{3.1.63}$$

Lemma 3.1.4. *The following inequalities hold:*

$$|\eta_0^\wedge(r,\psi,\delta_0)-1|>|\eta_1^\wedge(r,\psi,\delta_0)-1|, \tag{3.1.64}$$

where $r > 1 + \frac{\delta_0}{2}$, $\frac{r-1}{2} \le s \le \frac{r+1}{2}$;

$$|\eta_0^\wedge(r,\psi,\delta_0) - 1| > |\eta_1^\wedge(r,\psi,\delta_0) - 1|, \tag{3.1.65}$$

where $1 \le r \le 1 + \delta_0/2$, $\delta_0/4 < s \le (r+1)/2$;

$$|\eta_0^\wedge(r,\psi,\delta_0) - 1| = |\eta_1^\wedge(r,\psi,\delta_0) - 1|, \tag{3.1.66}$$

where $1 \le r \le 1 + \delta_0/2$, $s = \delta_0/4$;

$$|\eta_0^\wedge(r,\psi,\delta_0) - 1| < |\eta_1^\wedge(r,\psi,\delta_0) - 1|, \tag{3.1.67}$$

where $1 \le r \le 1 + \delta_0/2$, $(r-1)/2 \le s < \delta_0/4$;

Proof. According to (3.1.31),

$$R^*_{-1}(r,\psi,\delta_0) = r + e^{-i\psi}(2+\delta_0)/2.$$

Therefore,

$$|R^*_{-1}(r,\psi,\delta_0)|^2 = r^2 + (2+\delta_0)^2/4 + r(2+\delta_0)\cos(\psi) = \tag{3.1.68}$$

$$r^2 + (2+\delta_0)^2/4 + r(2+\delta_0)(4s^2 - r^2 - 1)/2r =$$

$$2(2+\delta_0)s^2 - \delta_0\frac{r^2-1}{2} + \left(\frac{\delta_0}{2}\right)^2$$

decreases with increasing $\psi \in [0,\pi]$. According to (3.1.10) and (3.1.61),

$$|R_0(r,\psi,\delta_0)|^4 = |D_0(r,\psi,\delta_0)|^2 = \tag{3.1.69}$$

$$r^2 + \left(r^2 - \left(\frac{\delta_0}{2}\right)^2\right)^2 + 2r\left(r^2 + \left(\frac{\delta_0}{2}\right)^2\right)t + r^2(\delta_0)^2t^2 =$$

$$r^2(\delta_0)^2\left(\frac{4s^2 - r^2 - 1}{2r}\right)^2 +$$

$$2r\left(r^2 + \left(\frac{\delta_0}{2}\right)^2\right)\frac{4s^2 - r^2 - 1}{2r} + r^2 + \left(r^2 - \left(\frac{\delta_0}{2}\right)^2\right)^2 =$$

$$4(\delta_0)^2s^4 + (4r^2 - (\delta_0)^2(2r^2+1))s^2 + r^2(r^2-1)\left(\frac{\delta_0}{2}\right)^2 + \left(\frac{\delta_0}{2}\right)^4.$$

According to (3.1.33), (3.1.68) and (3.1.69),

$$p_0 = 8\frac{|R^*_{-1}(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2}{(1+\delta_0)^2} = \tag{3.1.70}$$

$$\frac{8}{(1+\delta_0)^2}\left(2(2+\delta_0)s^2 - \delta_0\frac{r^2-1}{2} + \left(\frac{\delta_0}{2}\right)^2\right) + \frac{8}{(1+\delta_0)^2}\times$$

$$\sqrt{4(\delta_0)^2s^4 + (4r^2-(\delta_0)^2(2r^2+1))s^2 + r^2(r^2-1)\left(\frac{\delta_0}{2}\right)^2 + \left(\frac{\delta_0}{2}\right)^4}.$$

According to (3.1.34), (3.1.1) and (3.1.63),

$$q_0(r,\psi,\delta_0) = |D^{\vee}(z,1)|^2 = \tag{3.1.71}$$

$$16\frac{r^2+1+2r cos(\psi)}{(1+\delta_0)^2} =$$

$$\frac{64s^2}{(1+\delta_0)^2}.$$

The value

$$\Delta^*_0(r,\psi,\delta_0) = (p_0/2)^2 - q_0$$

is equal to 0 if and only if $p_0/2 = (q_0)^{1/2}$, i.e. if

$$2s(1+\delta_0) = 2(2+\delta_0)s^2 - \delta_0\frac{r^2-1}{2} + \left(\frac{\delta_0}{2}\right)^2 + \tag{3.1.72}$$

$$\sqrt{4(\delta_0)^2s^4 + (4r^2-(\delta_0)^2(2r^2+1))s^2 + r^2(r^2-1)\left(\frac{\delta_0}{2}\right)^2 + \left(\frac{\delta_0}{2}\right)^4}.$$

The equation (3.1.72) is equivalent to the system consisting of the inequality

$$2(2+\delta_0)s^2 - \delta_0\frac{r^2-1}{2} + \left(\frac{\delta_0}{2}\right)^2 - \tag{3.1.73}$$

$$2s(1+\delta_0) \le 0$$

and the equation

$$P_0(s) = 0, \tag{3.1.74}$$

where

$$P_0(s) = a_0 s^4 + b_0 s^3 + c_0 s^2 + d_0 s + r_0,$$

$$a_0 = 4(2+\delta_0)^2 - 4(\delta_0)^2 = 16(1+\delta_0),$$

$$b_0 = (1+\delta_0)(-16-8\delta_0),$$

$$c_0 = 4(2+\delta_0)(-\delta_0(r^2-1)/2 + (\delta_0/2)^2) +$$

$$4(1+\delta_0)^2 - (4r^2 - (\delta_0)^2(2r^2+1)) =$$

$$-4r^2(\delta_0+1) + 4(1+\delta_0)^2 + 4\delta_0 + 5\delta_0^2 + \delta_0^3 =$$

$$(\delta_0+1)\times$$

$$(-4(r^2-1) + \delta_0^2 + 8\delta_0),$$

$$d_0 = -4(1+\delta_0)(-\delta_0(r^2-1)/2 + (\delta_0/2)^2),$$

$$r_0 = (\delta_0/2)^2(r^2-1-\delta_0/2)^2 -$$

$$(\delta_0/2)^2 r^2(r^2-1) - (\delta_0/2)^4 =$$

$$-(\delta_0/2)^2(1+\delta_0)(r^2-1).$$

Therefore,

$$P(s) = (16s^2 - 8s\delta_0 + \delta_0^2)\times$$

$$(s^2 - s - (r^2-1)/4)(1+\delta_0) =$$

$$(1+\delta_0)(4s-\delta_0)^2(s-(r+1)/2)(s+(r-1)/2).$$

The left part of (3.1.73) for $s = (r+1)/2 \geq 1$ is equal to the positive number

$$2(2+\delta_0)s^2 - \delta_0(r^2-1)/2 + (\delta_0/2)^2 -$$

$$2s(1+\delta_0) = 2(s^2-s)(1+\delta_0) +$$

$$(r+1)^2/2 - \delta_0(r^2-1)/2 + (\delta_0/2)^2;$$

therefore, $s = (r+1)/2$ is not the solution of the system consisting of the inequality (3.1.73) and the equation (3.1.74). According to (3.1.62) and (3.1.63), the equality $s = -(r-1)/2$ is only possible when when the equalities $r = 1$ and $\psi = \pi$ are fulfilled at the same time. According to (3.1.30) and (3.1.6),

$$\eta_k^{\wedge}(r, \pi, \delta_0) - 1 = (2/(1+\delta_0))\times$$

$$(r - 1 - \delta_0/2 + (-1)^k\sqrt{r^2 - r + (\delta_0/2)^2}),$$

where $k = 0, 1$. Therefore,

$$|\eta_0^\wedge(r, \pi, \delta_0) - 1| > |\eta_1^\wedge(r, \pi, \delta_0) - 1|, \tag{3.1.75}$$

if $r > 1 + \delta_0/2$;

$$|\eta_0^\wedge(r, \pi, \delta_0) - 1| = |\eta_1^\wedge(r, \pi, \delta_0) - 1|, \tag{3.1.76}$$

if $r = 1 + \delta_0/2$;

$$|\eta_0^\wedge(r, \pi, \delta_0) - 1| < |\eta_1^\wedge(r, \pi, \delta_0) - 1|, \tag{3.1.77}$$

if $1 \leq r < 1 + \delta_0/2$. In view of (3.1.77), if $1 \leq r < 1 + \delta_0/2$, then the number $s = -(r-1)/2$ is not the solution of the system consisting of the inequalities (3.1.62) and (3.1.73) and the equation (3.1.74). For $s = (r-1)/2$, the inequality (3.1.64) follows from (3.1.75). For $s = (r-1)/2$, i.e. for $r = 1 + \delta_0/2$, the inequality (3.1.66) follows from (3.1.76). For $s = (r-1)/2$, the inequality (3.1.67) follows from (3.1.77). Now we consider the number

$$s = \delta_0/4 \tag{3.1.78}$$

as a possible solution of the system consisting of the inequalities (3.1.62) and (3.1.73) and the equation (3.1.74). As before, we consider only $r \geq 1$. According to (3.1.62), the equality (3.1.78) is only possible if

$$r \leq 1 + \delta_0/2. \tag{3.1.79}$$

Therefore, according to (3.1.75),the inequality (3.1.64) is fulfilled.

The case when equality is achieved in (3.1.79) and $s = \delta_0/4$, was considered above. It remains to consider the case when

$$r - 1 < \delta_0/2 = 2s. \tag{3.1.80}$$

But (3.1.73) for $s = \delta_0/4$ turns into inequality

$$\delta_0((\delta_0)^2/4 - r^2) < 0.$$

So, if (3.1.79) is fulfilled, then $s = \delta_0/4$ is the solution of the system consisting of inequalities (3.1.62) and (3.1.73) and equation (3.1.74); consequently, (3.1.66) holds. From the equalities

$$\eta_k^\wedge(r, 0, \delta_0) - 1 = (2/(1 + \delta_0)) \times \tag{3.1.81}$$

$$(r+1+\delta_0/2+(-1)^k\sqrt{r^2+r+(\delta_0/2)^2}),$$

where $k=0,1$; from the continuity of the sum

$$\sum_{k=0}^{1} |\eta_k^{\wedge}(r,0,\delta_0)-1|$$

as function of s; from (3.1.77); and from the necessity of (3.1.79) for $s=\delta_0/4$ to satisfy the system consisting of inequalities (3.1.62) and (3.1.73) and equation (3.1.74), follow (3.1.65) and (3.1.67). The lemma is proved.

Proof of Theorem 3.1.2. Since now $r=1$, it follows, according to (3.1.61), that

$$s=cos^2\left(\frac{\psi}{2}\right),$$

where $\psi=(2l-1)\pi/3$ and $l=0,1,2$. If $l=0,1$, then $s=1/\sqrt{3}>1/52=\delta_0/4$. From Lemma 3.1.4 it follows that $|\eta^*_{0,l}-1|>|\eta^*_{1,l}-1|$. This inequalty and Lemma 3.1.3 have as consequence the first assertion of Theorem 3.1.2. If $l=2$, then $s=0<\delta_0/4=1/52$; the second assertion of Theorem 3.1.2 follows from Lemma 3.1.4, or directly from (3.1.55) with $\varepsilon=-1$. Theorem is proved.

For $l=0,1,2$ we denote by $F^*_l(\eta)$ the polynomial (3.1.1) as polynomial of η with $r=1, \psi=(2l-1)\pi/3$ and $\Delta=13$. It is clear that

$$\prod_{l=0}^{2} F^*_l(\eta)=D^{\wedge}(1,\eta)=(\eta+1)^3(\eta+\gamma_1)^3-2^3(1+\gamma_1)^3\eta^3 \tag{3.1.82}$$

with $\gamma=6/7$.

Lemma 3.1.5. *For $l=0,1$, the polynomial $F^*_l(\eta)$ is irreducible over the field $K_0=\mathbb{Q}[\exp(2\pi i/3)]$.*

Proof. Clearly, it is sufficient to consider the case $l=1$. Since the degree of the polynomial $F^*_1(\eta)$ is equal to 2, the contrary means that it has a zero in the field K_0; then there is $a\in K_0$ such that for $D_0(1,\pi/3,\delta_0)$ from (3.1.6) the following equality holds:

$$D_0(1,\pi/3,\delta_0)=1-\exp(2i\pi/3)-(\delta_0/2)^2(1+\exp(2i\pi/3))=$$

$$1-(\delta_0/2)^2-\exp(2i\pi/3)(1+(\delta_0/2)^2)=a^2.$$

Then

$$\mathrm{Nm}_{K_0/\mathbb{Q}}(D_0(1,\pi/3,\delta_0)=3+(\delta_0/2)^4)=\frac{48\Delta^4+1}{16\Delta^4} \tag{3.1.83}$$

$$(\mathrm{Nm}_{K_0/\mathbb{Q}}(a))^2 = 3 + (\delta_0/2)^4.$$

Therefore,

$$16\Delta^4(\mathrm{Nm}_{K_0/\mathbb{Q}}(a))^2 = 48\Delta^4 + 1 \equiv -1 \mod 11,$$

but -1 is a non-quadratic residue $\mod 11$. This contradiction proves the lemma. The lemma is proved.

Let $F_l^{\sim}(\eta) = \prod_{k=0}^{1}(\eta - h^{\sim}(\eta_{k,l}^*))$.

Lemma 3.1.6. *Let $\Delta = 13$. Then, for $l = 0, 1$, the polynomial $F_l^{\sim}(\eta)$ is irreducible over the field $K_0 = \mathbb{Q}[\exp(2\pi i/3)]$, and the polynomial*

$$F_0(\eta) = \prod_{l=0}^{1}\prod_{k=0}^{1}(\eta - h^{\sim}(\eta_{k,l}^*))$$

is irreducible over the field $\mathbb{Q}$.

Proof. According to Lemma 3.1.5, for $k = 0, 1, l = 0, 1$, the field $K_{l+1} = K_0[\eta_{k,l}^*]$ is the quadratic extension over the field K_0. In view of (3.1.1),

$$\exp(-i(2l-1)2\pi/3) = \frac{(\eta_{k,l}^* + 1)(\eta_{k,l}^* + \gamma_1)}{2(1+\gamma_1)\eta_{k,l}^*}, \tag{3.1.84}$$

where $k = 0, 1,\ l = 0, 1$. Therefore,

$$K_{l+1} = K_0[\eta_{k,l}^*] = \mathbb{Q}[\eta_{k,l}^*] \tag{3.1.85}$$

According to (3.1.82), if $\Delta = 13$, then the numbers $\eta_{k,l}^*$ with $k = 0, 1,\ l = 0, 1$ are $5\mathbb{Z}_{K_{l+1}}$ - integer numbers of the field K_{l+1} (the ideal $5\mathbb{Z}_{K_{l+1}}$, as it will be shown below, is not prime). For $l = -1, 0, 1$, let $\mathfrak{O}_{l+1}$ be the ring of all $5\mathbb{Z}_{K_{l+1}}$ - integer numbers of the field K_{l+1}; let further $K_3 = \mathbb{Q}$ and $\mathfrak{O}_3$, the ring of all $5\mathbb{Z}_{K_3}$ - integer numbers of the field K_3.

Since the polynomial $\eta^2 + \eta + 1$ is irreducible over the field $\mathbb{Z}/5\mathbb{Z}$, it follows that the ideal $5\mathbb{Z}_{K_0}$ is the prime ideal of the second degree.

Below, for the prime ideal $\mathfrak{p}$ of the number field K, we denote by $v_{\mathfrak{p}}$ the additive $\mathfrak{p}-$adic valuation of the field K. According to (3.1.83), if $\Delta = 13$, then

$$v_{5\mathbb{Z}_{K_0}}(\mathrm{Nm}_{K_0/\mathbb{Q}}(D_0(1, \pi/3, \delta_0)) = 0.$$

Therefore, for any $k = 0, 1, l = 0, 1$, the numbers $1, \eta_{k,l}^*$ form the $5\mathfrak{O}_0-$integer basis of the extension K_{l+1}/K_0 and the four numbers

$$(\eta_{k,l}^*)^{\alpha} \exp\left(2\pi\beta\frac{2l-1}{3}i\right), \tag{3.1.86}$$

where $\alpha = 0, 1, \beta = 0, 1$, form $5\mathfrak{O}_3-$integer basis of the extension K_{l+1}/K_3. Further we have

$$F_l^*(\eta) = \left(\eta + \frac{6}{7}\right)(\eta+1) + \frac{26}{7}\eta \exp\left(\pi\frac{2l-1}{3}i\right) \equiv \tag{3.1.87}$$

$$(\eta+3)(\eta+1) + 3\eta\exp\left(\pi\frac{2l-1}{3}i\right) \equiv$$

$$\eta^2 + \left(3\exp\left(\pi\frac{2l-1}{3}i\right) - 1\right)\eta + 3 \equiv$$

$$\eta^2 + \left(2\exp\left(4\pi\frac{2l-1}{3}i\right) - 1\right)\eta + 3 \equiv$$

$$\eta^2 + \left(-2\exp\left(2\pi\frac{2l-1}{3}i\right) + 2\right)\eta - 2 \equiv$$

$$\left(\eta + \exp\left(2\pi\frac{2l-1}{3}i\right)\right)\left(\eta + 2\exp\left(2\pi\frac{2l-1}{3}i\right) + 2\right) \equiv$$

$$\left(\eta + \exp\left(2\pi\frac{2l-1}{3}i\right)\right)\left(\eta - 2\exp\left(4\pi\frac{2l-1}{3}i\right)\right) \mod 5\mathfrak{O}_0[\eta],$$

where $l = 0, 1$; therefore,

$$\mathrm{Nm}_{K_{l+1}/K_0}(\eta_{k,l}^* + \exp{(2l-1)2i\pi/3}) =$$

$$F_l^*(-\exp((2l-1)2i\pi/3)) \in 5\mathfrak{O}_0,$$

$$\mathrm{Nm}_{K_{l+1}/K_0}(\eta_{k,l}^* - \exp((2l-1)4i\pi/3)) =$$

$$F_l^*(\exp{(2l-1)4i\pi/3}) \in 5\mathfrak{O}_0,$$

and for each l=0,1, by transition from the field K_0 to the field K_{l+1}, the prime ideal $5\mathfrak{O}_0$ turns into the product

$$5\mathfrak{O}_{l+1} = \mathfrak{p}_{1,l+1}\mathfrak{p}_{2,l+1} \tag{3.1.88}$$

of two prime ideals

$$\mathfrak{p}_{1,l} = \left(\eta_{0,l}^* + \exp\left(2\pi\frac{2l-1}{3}i\right)\right)\mathfrak{O}_{l+1} + 5\mathfrak{O}_{l+1} \tag{3.1.89}$$

and

$$\mathfrak{p}_{2,l} = \left(\eta_{0,l}^* - 2\exp\left(4\pi\frac{2l-1}{3}i\right)\right)\mathfrak{O}_{l+1} + 5\mathfrak{O}_{l+1}. \tag{3.1.90}$$

According to (3.1.87),

$$\eta^2 \equiv (2\exp((2l-1)2i\pi/3)-2)\eta+2 \mod (5\mathfrak{O}_0[\eta]+F_l^*(\eta)\mathfrak{O}_0[\eta]), \tag{3.1.91}$$

where $l=0,1$,

$$\eta^2\exp\left(2\pi\frac{2l-1}{3}i\right)\equiv \tag{3.1.92}$$

$$(-2+\exp\left(2\pi\frac{2l-1}{3}i\right))\eta+2\exp\left(2\pi\frac{2l-1}{3}i\right) \mod 5\mathfrak{O}_0[\eta]+F_l^*(\eta)\mathfrak{O}_0[\eta],$$

where $l=0,1$.

Let $\mathfrak{M}$ denote the $\mathfrak{O}_0[\eta]-$module consisting of all columns with four elements contained in the ring $\mathfrak{O}_0[\eta]$. For $l=0,1$, according to (3.1.87), (3.1.91) and (3.1.92),

$$\eta\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \exp((2l-1)2i\pi/3))\eta\end{pmatrix}\equiv$$

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\2&0&3&2\\0&2&3&1\end{pmatrix}\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \eta\exp((2l-1)2i\pi/3))\end{pmatrix} \mod (5\mathfrak{M}+F_l^*(\eta)\mathfrak{M}),$$

$$\eta^2\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \eta\exp((2l-1)2i\pi/3))\end{pmatrix}\equiv \tag{3.1.93}$$

$$\begin{pmatrix}0&0&1&0\\0&0&0&1\\2&0&3&2\\0&2&3&1\end{pmatrix}^2\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \eta\exp((2l-1)2i\pi/3))\end{pmatrix}\equiv$$

$$\begin{pmatrix}2&0&3&2\\0&2&3&1\\1&4&2&3\\1&2&2&4\end{pmatrix}\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \eta\exp((2l-1)2i\pi/3))\end{pmatrix} \mod (5\mathfrak{M}+F_l^*(\eta)\mathfrak{M}),$$

$$\eta^4\begin{pmatrix}1\\ \exp((2l-1)2i\pi/3))\\ \eta\\ \eta\exp((2l-1)2i\pi/3))\end{pmatrix}\equiv$$

$$\begin{pmatrix} 2 & 0 & 3 & 2 \\ 0 & 2 & 3 & 1 \\ 1 & 4 & 2 & 3 \\ 1 & 2 & 2 & 4 \end{pmatrix}^2 \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \equiv$$

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 4 & 3 & 4 & 0 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \mod (5\mathfrak{M} + F_l^*(\eta)\mathfrak{M}),$$

$$\eta^8 \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \equiv$$

$$\begin{pmatrix} 4 & 1 & 1 & 1 \\ 4 & 3 & 4 & 0 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & 1 & 1 \end{pmatrix}^2 \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \equiv$$

$$\begin{pmatrix} 0 & 4 & 4 & 4 \\ 1 & 1 & 1 & 0 \\ 3 & 3 & 4 & 1 \\ 2 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \mod (5\mathfrak{M} + F_l^*(\eta)\mathfrak{M}),$$

$$\eta^{12} \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \equiv \begin{pmatrix} 4 & 1 & 1 & 1 \\ 4 & 3 & 4 & 0 \\ 2 & 2 & 0 & 4 \\ 3 & 0 & 1 & 1 \end{pmatrix} \times \tag{3.1.94}$$

$$\begin{pmatrix} 0 & 4 & 4 & 4 \\ 1 & 1 & 1 & 0 \\ 3 & 3 & 4 & 1 \\ 2 & 0 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \equiv$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \eta\exp((2l-1)2i\pi/3)) \end{pmatrix} \mod (5\mathfrak{M} + F_l^*(\eta)\mathfrak{M}).$$

Let me explain the relation (3.1.94). According to (3.1.88),

$$\mathfrak{O}_{l+1}/5\mathfrak{O}_{l+1} = \mathfrak{O}_{l+1}/\mathfrak{p}_{1,l+1} \oplus \mathfrak{O}_{l+1}/\mathfrak{p}_{2,l+1}. \tag{3.1.95}$$

In view of (3.1.89),

$$\eta^*_{0,l} \equiv -\exp\left(2\pi\frac{2l-1}{3}i\right) \mod \mathfrak{p}_{1,l+1}$$

and, therefore, the order of $\eta^*_{0,l} + \mathfrak{p}_{1,l+1}$ in the multiplicative group of the field $\mathfrak{O}_{l+1}/\mathfrak{p}_{1,l+1}$ is equal to 6; in view of (3.1.90),

$$\eta^*_{0,l} \equiv 2\exp\left(4\pi\frac{2l-1}{3}i\right) \mod \mathfrak{p}_{2,l+1}$$

and, therefore, the order of $\eta^*_{0,l} + \mathfrak{p}_{2,l+1}$ in the multiplicative group of the field $\mathfrak{O}_{l+1}/\mathfrak{p}_{2,l+1}$ is equal to 12.

For $l = 0, 1$, according to (3.1.93) and (3.1.94),

$$(\eta^2-1)\eta^{12}\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \equiv \tag{3.1.96}$$

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \mod (5\mathfrak{M} + F^*_l(\eta)\mathfrak{M}),$$

$$(\eta^2-1)^2\eta^{24}\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \equiv$$

$$\begin{pmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 4 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{pmatrix}^2\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \equiv$$

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 4 & 0 & 3 & 3 \\ 0 & 4 & 2 & 3 \\ 1 & 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \mod (5\mathfrak{M} + F^*_l(\eta)\mathfrak{M}),$$

$$(\eta^2-1)^3\eta^{36}\begin{pmatrix} 1 \\ \exp((2l-1)2i\pi/3)) \\ \eta \\ \exp((2l-1)2i\pi/3))\eta \end{pmatrix} \equiv \tag{3.1.97}$$

$$\begin{pmatrix}1&1&0&2\\4&0&3&3\\0&4&2&3\\1&1&2&4\end{pmatrix}\times$$

$$\begin{pmatrix}1&0&3&2\\0&1&3&1\\1&4&1&3\\1&2&2&3\end{pmatrix}\begin{pmatrix}1\\\exp((2l-1)2i\pi/3))\\\eta\\\exp((2l-1)2i\pi/3))\eta\end{pmatrix}\equiv$$

$$\begin{pmatrix}3&0&0&4\\0&3&1&1\\0&3&0&4\\2&2&1&1\end{pmatrix}\begin{pmatrix}1\\\exp((2l-1)2i\pi/3))\\\eta\\\exp((2l-1)2i\pi/3))\eta\end{pmatrix}\quad\bmod (5\mathfrak{M}+F_l^*(\eta)\mathfrak{M}),$$

(3.1.98)
$$(\eta^2-1)^6\eta^{72}\begin{pmatrix}1\\\exp((2l-1)2i\pi/3))\\\eta\\\exp((2l-1)2i\pi/3))\eta\end{pmatrix}\equiv$$

$$\begin{pmatrix}3&0&0&4\\0&3&1&1\\0&3&0&4\\2&2&1&1\end{pmatrix}^2\begin{pmatrix}1\\\exp((2l-1)2i\pi/3))\\\eta\\\eta\exp((2l-1)2i\pi/3))\end{pmatrix}\equiv$$

$$\begin{pmatrix}2&3&4&1\\2&4&4&3\\3&2&2&2\\3&1&3&0\end{pmatrix}^2\begin{pmatrix}1\\\exp((2l-1)2i\pi/3))\\\eta\\\eta\exp((2l-1)2i\pi/3))\end{pmatrix}\quad\bmod (5\mathfrak{M}+F_l^*(\eta)\mathfrak{M}).$$

In correspondence to (3.1.28), let

(3.1.99)
$$h_1^{\sim}(\eta)=(\eta^2-1)^3\eta^36=h^{\sim}(\eta)(12/13)^{36}2^6.$$

From (3.1.97) and (3.1.98), it follows that

(3.1.100)
$$h_1^{\sim}(\eta_{k,l}^*)\equiv 3+4\exp\left(2\pi\frac{2l-1}{3}i\right)\eta_{k,l}^*\quad\bmod 5\mathfrak{O}_{l+1}.$$

(3.1.101)
$$(h_1^{\sim}(\eta_{k,l}^*))^2\equiv 2+3\exp\left(2\pi\frac{2l-1}{3}i\right)+$$
$$\left(4+\exp\left(2\pi\frac{2l-1}{3}i\right)\right)\eta_{k,l}^*\quad\bmod 5\mathfrak{O}_{l+1},$$

where $k = 0, 1, l = 0, 1$; therefore, $h_1^{\sim}(\eta^*_{k,l}) \notin K_0$ and, since $[K_{l+1} : K_0] = 2$ for any $l = 0, 1$, it follows, according to (3.1.99), that

$$K_0[h^{\sim}(\eta^*_{k,l})] = K_{l+1}, \tag{3.1.102}$$

where $l = 0, 1$. Consequently, the polynomial

$$F_l(\eta)^{\sim} = \prod_{k=0}^{1}(\eta - h^{\sim}(\eta^*(k, l))) \tag{3.1.103}$$

is irreducible over the field K_0. Moreover, according to (3.1.100) and (3.1.101),

$$h_1^{\sim}(\eta^*(k, l)) + 2 = a_1 + b_1 \exp\left(2\pi\frac{2l-1}{3}i\right) +$$

$$c_1\eta^*(k, l) + d_1 \exp\left(2\pi\frac{2l-1}{3}i\right)\eta^*(k, l),$$

$$h_1^{\sim}(\eta^*(k, l)) + (h_1^{\sim}(\eta^*(k, l)))^2 = a_2 +$$

$$b_2 \exp\left(2\pi\frac{2l-1}{3}i\right) + c_2\eta^*(k, l) + d_2 \exp\left(2\pi\frac{2l-1}{3}i\right)\eta^*(k, l),$$

where $k = 0, 1,\ l = 0, 1$,

$$a_1 \in \mathfrak{O}_3, b_1 \in \mathfrak{O}_3, c_1 \in \mathfrak{O}_3, d_1 \in \mathfrak{O}_3,$$

$$a_2 \in \mathfrak{O}_3, b_2 \in \mathfrak{O}_3, c_2 \in \mathfrak{O}_3, d_2 \in \mathfrak{O}_3,$$

$$a_1 \equiv b_1 \equiv c_1 \equiv 0 \mod 5\mathfrak{O}_3, d_1 \equiv 4 \mod 5\mathfrak{O}_3$$

$$a_2 \equiv d_2 \equiv 0 \mod 5\mathfrak{O}_3,$$

$$b_2 \equiv 3 \mod 5\mathfrak{O}_3, c_2 \equiv 4 \mod 5\mathfrak{O}_3.$$

Therefore, the system of numbers

$$1, h_1^{\sim}(\eta^*(k, l)) + 2, h_1^{\sim}(\eta^*(k, l)) + h_1^{\sim}(\eta^*(k, l))^2$$

is linearly independent over $\mathbb{Q}$. In fact, the contrary means that there exist α, β, γ in $\mathbb{Q}$, not all of which are equal to zero, such that

$$\alpha + \beta a_1 + \gamma a_2 = 0,$$

$$\beta b_1 + \gamma b_2 = 0,$$

$$\beta c_1 + \gamma c_2 = 0,$$

$$\beta d_1 + \gamma d_2 = 0;$$

but this is impossible, because

$$\det\left(\begin{pmatrix} 1 & 0 & 0 \\ a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \end{pmatrix}\right) \equiv 3 \mod 5\mathfrak{O}_3.$$

Therefore,

$$[\mathbb{Q}(h_1^{\sim}(\eta^*(k,l))) : \mathbb{Q}] \geq 3$$

and, since $4/[\mathbb{Q}(h_1^{\sim}(\eta^*(k,l))) : \mathbb{Q}] \in \mathbb{N}$, it follows that

$$[\mathbb{Q}(h_1^{\sim}(\eta^*(k,l))) : \mathbb{Q}] = 4.$$

Consequently, according to (3.1.99),

$$[\mathbb{Q}(h^{\sim}(\eta^*(k,l))) : \mathbb{Q}] = 4,$$

where $k = 0,1, l = 0,1$, and the polynomial

$$F_0(\eta) = \prod_{l=0}^{1}\prod_{k=0}^{1}(\eta - h^{\sim}(\eta^*_{k,l}))$$

is irreducible over the field $\mathbb{Q}$. The lemma is proved.

Corollary. *The assertion of the Theorem 3.1.3 holds.*

Proof. Since

$$\mathbb{Q}(h^{\sim}(\eta^*(k,l))) = K_{l+1},$$

where $k = 0,1, `l = 0,1$, it follows that, for $k = 0,1,\ l = 0,1,\ h^{\sim}(\eta^*(k,l) \in \mathbb{R}$ is impossible. The corollary is proved.

So, Theorem 3.1.1 is also proved.

As in the first part of this paper, we shall work with the set $\Omega^* = \{z \in \mathbb{C}\colon |z| \geq 1, -\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}\}$. For any $\varepsilon > 0$, let $D_\varepsilon^{\sim}$ denote the domain $\{z \in \mathbb{C}\colon |z| < 1 + \varepsilon, \mathrm{Re}\,(z) > 1\}$. Clearly, $D_\varepsilon^{\sim}$ lies in the circle $|z-1| < \sqrt{\varepsilon}$, and $|\arg(z)| < \arctan(\sqrt{\varepsilon})$ for any $z \in D_\varepsilon$. Let further $D_\varepsilon^{\wedge}$ be the closure of $D_\varepsilon^{\sim}$. For any $k = 0,1,\ l = 0,1,2$, and $z \in D_\varepsilon^{\wedge}$, let

$$\eta^{**}_{k,l}(z) = \eta^{\wedge}_k(|z|, arg z + (2l-1)\pi/3, \delta_0),$$

where, as above, $\delta_0 = 1/13$. Let further

$$F_0^{\wedge}(\eta; z) = \prod_{l=0}^{2}\prod_{k=0}^{1}(\eta - h^{\sim}(\eta^{**}_{k,l}(z)))$$

and $\mathfrak{D}_0^{\wedge}(z)$ be the discriminant relatively to η of the polynomial $F_0^{\wedge}(\eta; z)$. Then $\mathfrak{D}_0^{\wedge}(z)$ and

$$\eta_{k,l}^{**}(z)$$

with any $k = 0, 1, l = 0, 1, 2$ are continuos functions in $D_\varepsilon^{\wedge}$; moreover,

$$\eta_{k,l}^{**}(1) = \eta_{k,l}^{*},$$

where $k = 0, 1,\ l = 0, 1, 2,$ and, according to Theorem 3.1.1, $\mathfrak{D}_0^{\wedge}(1) = \mathfrak{D}_0 \neq 0$. Therefore, there exist $\varepsilon > 0$ such that $\mathfrak{D}_0^{\wedge}(z) \neq 0$ for any $z \in D_\varepsilon^{\wedge}$; we fix this ε for further considerations.

According to (3.1.43) and (3.1.71),

$$h^{\sim}(\eta_{k,l}^{**}(z)) \neq 0$$

for any $z \in D_\varepsilon$ and any $k = 0, 1,\ l = 0, 1$. Let $\varepsilon_1 \in (0, \varepsilon), \varepsilon_3 \in (\varepsilon_1, \varepsilon)$. Let further

$$D_{\varepsilon_1,\varepsilon_3}^{\sim,\sim} = \{z \in \mathbb{C}\colon |z| < 1 + \varepsilon_3, \mathrm{Re}\,(z) > 1 + \varepsilon_1\},$$

$$D_{\varepsilon_1,\varepsilon_3}^{\wedge,\wedge} = \{z \in \mathbb{C}\colon |z| \leq 1 + \varepsilon_3, \mathrm{Re}\,(z) \geq 1 + \varepsilon_1\}.$$

I wish to apply Lemma 2.4.5 of Part 2 of this work.

For the role of the compact set F of this lemma, I take the set $D_{\varepsilon_1,\varepsilon_3}^{\wedge\wedge}$, which is a compact subset of the domain $D_\varepsilon^{\sim} \subset \mathbb{C}\backslash\{1\}$. The polynomial $D^{\sim}(z, w)$ in (2.4.3) is connected to $D^{\wedge}(z, \eta)$ from (3.1.0) by means of the equality (2.4.52) and therefore, according to the substitution (2.4.51), Lemma 3.1.3 and Lemma 3.1.5, has only simple roots. It will play the role of the polynomial $D^{\sim}(z, \lambda)$ in (2.4.45) of part 2. The independent of z polynomials (2.4.4) and (2.4.5) will play the role of the polynomials $P^{\sim}(z, \lambda)$ and $Q^{\sim}(z, \lambda)$ of Lemma 2.4.5. According to (2.4.55) and Theorem 3.1.1, the map (2.4.46) is injective on the set of all the roots of the polynomial

$$D^{\sim}(z, \lambda)$$

(i.e. the polynomial (2.4.3)). The matrix $B^{\sim}(z)$ of Lemma 2.4.5 will now play the role of the matrix $B^{\sim}(z)$ from (2.4.6). The matrix (2.4.16) will play the role of the matrix $A(z; \nu^{-1})$ from (2.4.44); according to (2.4.10) and (2.4.11), the following equality holds: $A(z, 0) = A^{\sim}(z)$, with $A^{\sim}(z)$ from (2.4.15); therefore, the condition (2.4.47) is also fulfilled. So, all the conditions of Lemma 2.4.5 are fulfilled. Consequently, according to the assertion of Lemma 2.4.5, for every $z \in F$, all the roots of the polynomial (2.4.20) are simple and there exists a matrix $C = C(z) \in Mat_6(\mathbb{C}^F)$ satisfying to the following conditions:

1) the relations (2.4.21) are fulfilled;
2) for every $z \in F$, the matrix $(C(z))^{-1}A(z, 0)C(z)$ is diagonal;

3) for every $z \in F$, no element in the first row of the matrix $C(z)$ is equal to sero.

Therefore, we can now apply Lemma 2.4.4 of the Part 2 of this work. As above, the set $D^{\wedge\wedge}_{\varepsilon_1,\varepsilon_3}$ plays the role of the compact F, the matrix (2.4.16) plays the role of the matrix $A(z;\nu^{-1})$ from (2.4.44) and (2.4.18), the column $X_k(z;\nu)$ from (2.4.17) with $k = 1,2,3,5$ and $\nu \in \mathbb{N} + [1/\varepsilon_0(F)]$ plays the role of the column $X_\nu(z)$ from (2.4.19). Consequently, according to the assertion of Lemma 2.4.4, there is $\varepsilon_2 \in (0,\varepsilon_0)$ such that, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and $k = 1,2,3,5$, the first component $f^*_k(z;\nu)$ of the column $X_k(z;\nu)$ satisfies the equation

$$f^*_k(z;\nu+6) + \sum_{j=0}^{5} q^*_j(z;\nu^{-1}) f^*_k(z;\nu+j) = 0,$$

where $q^*_j(z;\nu^{-1}) \in \mathfrak{H}(F,\varepsilon_2)$ for $j = 0,\ldots,5$; moreover the polynomial

$$\lambda^6 + \sum_{j=0}^{5} q^*_j(z;0)\lambda^j,$$

coincides with polynomial (2.4.20), the roots of which, in view of (2.4.55), coincide with the roots of the polynomial

$$F^\vee_0(\eta) = \prod_{l=0}^{2}\prod_{k=0}^{1}(\eta - h^\sim(\eta^*_{k,l})),$$

having, according to Theorem 3.1.1, only simple roots.

So, the functions $f^*_k(z;\nu)$, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ and $k = 1,2,3,5$, satisfy the equation

$$x_{\nu+6} + \sum_{j=0}^{5} q^*_j(z;\nu^{-1}) x_{\nu+j} = 0. \tag{3.1.104}$$

In view of (2.11) and (2.15) of the Part 1 of this work, $x_\nu = f^*_4(z;\nu)$ is the solution of the equation (3.1.104). In view of (2.13) and (2.15) of the Part 1 of this work, $x_\nu = f^*_6(z;\nu)$, is also the solution of the equation (3.1.104).

Clearly, the rational functions $q^*_j(z;\nu^{-1})$ from (3.1.104) with $j = 0,\ldots,5$ admit the representation in the form

$$q^*_j(z;\nu^{-1}) = q_j(z;\nu^{-1})/q_6(z;\nu^{-1}), \tag{3.1.105}$$

where $q_j(z;\nu^{-1}) \in \mathbb{Q}[z;\nu^{-1}]$ for $j = 0,\ldots,6$ have no common divisors with the exception of different from zero constants, and $q_6(z;\nu^{-1})$ is different from zero for any $z \in F$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$. Consequently, the equation (3.1.104) is equivalent to the equation

$$q_6(z;\nu^{-1})x_{\nu+6} + \sum_{j=0}^{5} q_j(z;\nu^{-1})x_{\nu+j} = 0, \tag{3.1.106}$$

where $z \in F, \nu \in \mathbb{N} + [1/\varepsilon_2(F)]$; and $x_\nu = f_k^*(z;\nu)$, for $k = 1, 2, 4, 6$, is the solution of this equation. In view of (2.19) - (2.28) of the first part of this work, the functions $f_k^*(z;\nu)$, for $k = 1, 2, 4, 6$ and $\nu \in \mathbb{N}$, are regular in the domain $|z| > 1$ and, since the equality (3.1.106) for $x_\nu = f_k^*(z;\nu)$ with $k = 1, 2, 4, 6$, $\nu \in \mathbb{N} + [1/\varepsilon_2(F)]$ is fulfiiled for all z in the domain

$$D^{\sim;\sim}_{\varepsilon_1,\varepsilon_3} = \{z \in \mathbb{C}\colon\ |z| < 1 + \varepsilon_3, \mathrm{Re}\,(z) > 1 + \varepsilon_1\},$$

it follows that, according to the uniqueness theorem, it is fulfilled for all z with $|z| > 1$.

§2. Sextette comes to the foreground.

Lemma 3.2.1. *Let $r \in \mathbb{N} + 1$,*

$$L_k(z) = \sum_{y=1}^{\infty} z^y/y^k, \tag{3.2.1}$$

where $k \in \mathbb{N}$.

Then the functions

$$1, L_1(z)\,, \ldots\,, L_r(z) \tag{3.2.2}$$

form a linear idependent system over $\mathbb{C}(z)$.

Proof. Let

$$P_0(z) + \sum_{k=1}^{r} P_k(z)L_k(z) = 0, \tag{3.2.3}$$

where $P_k(z) \in \mathbb{C}[z]$ with $k = 0\,, \ldots\,, r$. Let further $P_\kappa(z) = 0$ for $\kappa \in \mathbb{N} + r$. If we direct $z \in (0, 1)$ to 1, then we see that

$$P_1(1) = 0. \tag{3.2.4}$$

Therefore, $P_1(z) = (1-z)Q_1(z)$ with $Q_1(z) \in \mathbb{C}[z]$. Applying the operator $\delta = z\frac{\partial}{\partial z}$ to (3.2.3), we obtain the equality

$$(\delta P_0)(z) + zQ_1(z) + \sum_{k=1}^{r}((\delta P_k)(z) + P_{k+1}(z))L_k(z) = 0.$$

Repeating previous arguments m times, we obtain, instead of (3.2.4), the equality

$$\sum_{\kappa=0}^{\min(m,r-1)} \binom{m}{\kappa}((\delta)^{m-\kappa}P_{1+\kappa})(1) = 0, \tag{3.2.5}$$

where $m \in \mathbb{N}-1$. Let $k = \kappa+1$, where $\kappa = 0, \ldots, \min(m, r-1)$. Then

$$\sum_{k=1}^{\min(m+1,r)} \binom{m}{k-1}((\delta)^{m-k+1}P_k)(1) = 0, \tag{3.2.6}$$

where $m \in \mathbb{N}-1$. We fix $n \in \mathbb{N}$, such that, for any $k \in \mathbb{N} \cap [1, r]$, if $P_k(z)$ is not a zero polynomial, then $\deg(P_k(z)) \le n$; let

$$P_k(z) = \sum_{l=0}^{n} a_{k,l}z^l,$$

where $a_{k,l} \in \mathbb{C}$ for $k = 1, \ldots, r$ and $1 = 1, \ldots, n$. Then the equality (3.2.6) for $m \ge r$ may be written in the form

$$\sum_{k=1}^{r}\binom{m}{k-1}\sum_{l=0}^{n} l^{m-k+1}a_{k,l} = \tag{3.2.7}$$

$$\sum_{l=1}^{n}\sum_{k=1}^{r}\binom{m}{k-1}l^{m-k+1}a_{k,l} = 0.$$

Since $y_{\kappa,l}(m) = \binom{m}{k-1}l^{m-k+1}$, where $k = 1, \ldots, r$ and $l = 1, \ldots, n$, form a basis for the space of the solutions of the difference equation

$$(\prod_{l=1}^{n}(\bigtriangledown - l)^r y)(m),$$

where, as usually, the operator $\bigtriangledown$ turns $y(m)$ into $y(m+1)$, it follows from (3.2.7) that $a_{k,l}(m) = 0$, where $k = 1, \ldots, r$ and $l = 1, \ldots, n$. Therefore, $P_k(z) = a_{k,0}$, where $k = 1, \dots, r$. In (3.2.6), taking successively $m = 0, \dots, r-1$, we convince that $a_{k,0} = 0$, where $k = 1, \dots, r$. The lemma is proved.

Corollary. *Let $r \in \mathbb{N} + 1$. Then the functions*

$$1, L_1(1/z), \dots, L_r(1/z) \tag{3.2.8}$$

form a linear idependent system over $\mathbb{C}(z)$.

Proof. Let

$$T_0(z) + \sum_{k=1}^{r} T_k(z) L_k(1/z) = 0, \tag{3.2.9}$$

where $T_k(z) \in \mathbb{C}[z]$ with $k = 0, \dots, r$. We fix $n \in \mathbb{N}$, such that, for any $k \in \mathbb{N} \cap [1, r]$, if $T_k(z)$ is not a zero polynomial, then $\deg(T_k(z)) \leq n$. From (3.2.9), it follows that

$$P_0(\zeta) + \sum_{k=1}^{r} P_k(\zeta) L_k(\zeta) = 0,$$

where $\zeta = 1/z$ and $P_k(\zeta) = \zeta^n T_k(1/\zeta)$ with $k = 1, \dots, r$. According to Lemma 3.2.1, $P_k(\zeta) = 0$ for $k = 1, \dots, r$ and, therefore, $T_k(z) = 0$ for $k = 1, \dots, r$. The Corollary is proved.

Since $x_\nu = f_k^*(z; \nu)$ with $k = 1, 2, 4, 6, \nu \in \mathbb{N} + [1/\varepsilon_2(F)], |z| > 1$ is the solution of the equality (3.1.106), it follows that, in view of (2.19) - (2.28) of the first part of this work and Corollary of Lemma 3.2.1, the six sequences of the polynomials

$$x_\nu = \alpha^*(z; \nu), x_\nu = \beta^*(z; \nu), x_\nu = \gamma^*(z; \nu), \tag{3.2.10}$$

$$x_\nu = \phi^*(z; \nu), x_\nu = \psi^*(z; \nu), x_\nu = \xi^*(z; \nu)$$

with $\nu \in \mathbb{N} + [1/\varepsilon_2(F)],\ |z| > 1$ are also the solutions of the equation (3.1.106).

I will start the next part by studying the sequences of the polynomials (3.2.10).

References

[1] Hata M., *Legendre type polynomials and irrationality measures*, J. Reine Angew. Math. **407** (1990), 99 – 105.

[2] Gutnik L., *On a measure of the irrationality of dilogarithms at rational points,* **4345-84** (1984), VINITI, 1 – 74.

[3] ______, *On the rank over $\mathbb{Q}$ of some real matrices,* **5736-84** (1984), VINITI, 1 – 29.

[4] ______, *To the question of the smallness of some linear forms,* **2413-B93** (1993), VINITI, 1 – 94.

[5] ______, *About systems of vectors, whose components are linear combinations of logarithms of algebraic numbers with algebraic coefficients.* **3122-B94** (1994), VINITI, 1 – 158.

[6] ———, *About linear forms, whose coefficients are* $\mathbb{Q}$*-proportional to the number* $\log 2$, *and the values of* $\zeta(s)$ *for integer* s, **3258-B96** (1996), VINITI, 1 – 70.
[7] ———, *The lower estimate for some linear forms with coefficients* $\mathbb{N}\zeta(1+\mathbb{N})$, **3072-B97** (1997), VINITI, 1 – 77.
[8] ———, *About the behaviour of solutions of difference equations of Poincaré type,* **3384-B97** (1997), VINITI, 1 – 41.
[9] ———, *On some linear forms with coefficients in* $\mathbb{N}\zeta(1+\mathbb{N})$, **3** (2000), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 13.
[10] ———, *On some linear forms with coefficients in* $\mathbb{N}\zeta(1+\mathbb{N})$, *(the detailed version,part 1)* **15** (2001), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 20.
[11] ———, *On some linear forms with coefficients in* $\mathbb{N}\zeta(1+\mathbb{N})$ *(the detailed version,part 2)* **104** (2001), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 36.
[12] ———, *On the rank over* $\mathbb{Q}$ *of some real matrices,* **27** (2002), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 1 – 32.
[13] Harry Bateman H. and Arthur Erdélyi ., *Higher transcendental functions*, New-York – Toronto – London, Mc. Grow-Hill Book Company, Inc., 1953.
[14] Titchmarsh E. C., *The Theory of Functions*, Oxford University Press, 1939.
[15] E.T. Whittaker and G.N.Watson, *A course of modern analysis*, Cambridge University Press, 1927.

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