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II. Taylor expansions on singular sets

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NILPOTENT GELFAND PAIRS AND SPHERICAL TRANSFORMS OF SCHWARTZ FUNCTIONS II. TAYLOR EXPANSIONS ON SINGULAR SETS

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Abstract. This paper is a continuation of [FRY], in the direction of proving the conjecture that the spherical transform on a nilpotent Gelfand pair (N, K) establishes an isomorphism between the space of K-invariant Schwartz functions on N and the space of Schwartz functions restricted to the Gelfand spectrum $\Sigma_{\mathcal{D}}$, properly embedded in a Euclidean space.

We prove a result, of independent interest for the representation theoretical problems that are involved, which can be viewed as a generalised Hadamard lemma for K-invariant functions on N. The context is that of nilpotent Gelfand pairs satisfying Vinberg's condition. This means that the Lie algebra $\mathfrak{n}$ of N (which is step 2) decomposes as $\mathfrak{v} \oplus [\mathfrak{n}, \mathfrak{n}]$ with $\mathfrak{v}$ irreducible under K.

Contents

1. Outline and formulation of the problem

We follow the notation of [7]. We say that (N, K) is a *nilpotent Gelfand pair* (n.G.p. in short) if N is a connected, simply connected nilpotent Lie group, K is a compact group of automorphisms of N, and the convolution algebra $L^1(N)^K$ of K-invariant integrable functions on N is commutative. This is the same as saying that $(K \ltimes N, K)$ is a Gelfand pair according to the common terminology.

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By $\mathbb{D}(N)^K$ we denote the left-invariant and K-invariant differential operators on N, and by

$$\mathcal{D} = (D_1, \dots, D_d) \ ,$$

a d-tuple of self-adjoint generators of $\mathbb{D}(N)^K$. To each bounded K-spherical function φ on N we associate (injectively) the d-tuple $\xi(\varphi) = \big(\xi_1(\varphi), \dots, \xi_d(\varphi)\big)$ of eigenvalues of φ as an eigenfunction of $D_1, \dots, D_d$ respectively.

The d-tuples $\xi(\varphi)$ form a closed subset $\Sigma_{\mathcal{D}}$ of $\mathbb{R}^d$ which is homeomorphic to the Gelfand spectrum of $L^1(N)^K$ [5]. If φ_ξ is the spherical function corresponding to $\xi \in \Sigma_{\mathcal{D}}$, the spherical transform

$$\mathcal{G}fF(\xi) = \int_N F(x)\varphi_\xi(x^{-1})\, dx \ , \tag{1}$$

can be viewed as a function on $\Sigma_{\mathcal{D}}$.

The following conjecture has been formulated in [6].

Conjecture. *The spherical transform maps the space $\mathcal{S}(N)^K$ of K-invariant Schwartz functions on N isomorphically onto*

$$\mathcal{S}(\Sigma_{\mathcal{D}}) \stackrel{\text{def}}{=} \mathcal{S}(\mathbb{R}^d)/\{f : f_{|\Sigma_{\mathcal{D}}} = 0\} \ .$$

The inclusion $\mathcal{G}\big(\mathcal{S}(N)^K\big) \supseteq \mathcal{S}(\Sigma_{\mathcal{D}})$ is known to hold in general [2, 6], so that the conjecture only concerns the opposite inclusion. Moreover, the validity of the conjecture does not depend on the choice of $\mathcal{D}$ [6].

In a nilpotent Gelfand pair (N, K) the group N is at most step-two. We denote by $\mathfrak{n}$ its Lie algebra and by $\mathfrak{v}$ a K-invariant complement of the derived algebra $[\mathfrak{n}, \mathfrak{n}]$. We consider $\mathfrak{n}$ endowed with a K-invariant scalar product.

The conjecture is known to be true when N is abelian or the Heisenberg group [1, 2], and when the following conditions are satisfied [6, 7]:

(i) the K-orbits in $[\mathfrak{n}, \mathfrak{n}]$ are full spheres,
(ii) K acts irreducibly on $\mathfrak{v}$.

In this paper we remove condition (i), still keeping condition (ii). The pairs for which (ii) holds have been classified by E. Vinberg in [14], and for this reason we call (ii) *Vinberg's condition.* Notice that, under Vinberg's condition, $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$, the centre of $\mathfrak{n}$.

We mention here that the classification of nilpotent Gelfand pairs has been completed in [16, 17], see also [15, Chapters 13, 15].

Assuming Vinberg's condition and disregarding the pairs for which the conjecture has already been proved, the basic list of n.G.p. to look at is that contained in Table 1. The space $\mathfrak{z}_0$ which appears in the last column is the unique irreducible component of $\mathfrak{z}$ on which K acts non-trivially.

All other nilpotent Gelfand pairs satisfying Vinberg's condition are obtained from those in Table 1 by either of the following operations:

(a) normal extensions of K: replace K by a larger group $K^\#$ of automorphisms of N with $K \triangleleft K^\#$;

	K	$\mathfrak{v}$	$\mathfrak{z}$	notes	$\mathfrak{z}_0$ (if $\neq \mathfrak{z}$)
1	SO_n	$\mathbb{R}^n$	$\mathfrak{so}_n$	$n \geq 4$	
2	SU_{2n+1}	$\mathbb{C}^{2n+1}$	$\Lambda^2\mathbb{C}^{2n+1}$	$n \geq 2$	
3	$\mathrm{Sp}_2 \times \mathrm{Sp}_n$	$\mathbb{H}^2 \otimes \mathbb{H}^n$	$\mathfrak{sp}_2$	$n \geq 2$	
4	U_{2n+1}	$\mathbb{C}^{2n+1}$	$\Lambda^2\mathbb{C}^{2n+1} \oplus \mathbb{R}$	$n \geq 1$	$\Lambda^2\mathbb{C}^{2n+1}$
5	SU_{2n}	$\mathbb{C}^{2n}$	$\Lambda^2\mathbb{C}^{2n} \oplus \mathbb{R}$	$n \geq 2$	$\Lambda^2\mathbb{C}^{2n}$
6	U_n	$\mathbb{C}^n$	$\mathfrak{u}_n$	$n \geq 2$	$\mathfrak{su}_n$
7	Sp_n	$\mathbb{H}^n$	$HS^2_0\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H}$	$n \geq 2$	$HS^2_0\mathbb{H}^n$
8	$\mathrm{U}_2 \times \mathrm{SU}_n$	$\mathbb{C}^2 \otimes \mathbb{C}^n$	$\mathfrak{u}_2$	$n \geq 2$	$\mathfrak{su}_2$
9	$\mathrm{U}_2 \times \mathrm{Sp}_n$	$\mathbb{C}^2 \otimes \mathbb{H}^n$	$\mathfrak{u}_2$	$n \geq 2$	$\mathfrak{su}_2$
10	$\mathrm{U}_1 \times \mathrm{Spin}_7$	$\mathbb{C} \otimes \mathbb{O}$	$\mathrm{Im}\,\mathbb{O} \oplus \mathbb{R}$		$\mathrm{Im}\,\mathbb{O}$

TABLE 1

(b) central reductions: if $\mathfrak{z}$ has a nontrivial proper K-invariant subspace $\mathfrak{s}$, replace $\mathfrak{n}$ by $\mathfrak{n}/\mathfrak{s}$.

In [7] we proved that if $N, K, K^\#$ are as in (a), and the conjecture is true for (N, K), then it is also true for $(N, K^\#)$. It will be proved elsewhere that, applying (b) to pairs for which the conjecture is true, the resulting pair also satisfies the conjecture. We will therefore concentrate our attention on the pairs in Table 1.

In this paper we do not give a proof of the conjecture for these pairs (this will be done elsewhere), but we focus on one specific point, the proof of Theorem 1.1 below, which is crucial in the proof of the conjecture, and which is rather involved in itself. It requires a detailed analysis of the action of K on the polynomial algebras over $\mathfrak{v}$ and $\mathfrak{z}$ and on tensor products of their irreducible components. In order to formulate Theorem 1.1, we need to describe some aspects of the structure of $\Sigma_{\mathcal{D}}$.

In $\Sigma_{\mathcal{D}}$ one can distinguish a relatively open and dense "regular set" from a "singular set", and singular points may have different levels of singularity. The highest level of singularity is reached by those bounded spherical functions which factor to the quotient group $N' = N/\exp_N \mathfrak{z}_0$. Call $\Sigma^0_{\mathcal{D}}$ this subset of $\Sigma_{\mathcal{D}}$.

At this point it is convenient to introduce a preferred system $\mathcal{D}$ of generators of $\mathbb{D}(N)^K$, obtained, via symmetrisation, from the bases of fundamental K-invariants on $\mathfrak{n}$ listed, case by case, in Section 7 of [7]. We denote by $\rho = (\rho_1, \dots, \rho_d)$ the d-tuple of these polynomials.

The polynomials ρ_j have the property of being homogeneous in each of the variables $v \in \mathfrak{v}$, $z \in \mathfrak{z}_0$, $t \in \mathfrak{z}'$ where $\mathfrak{z}'$ is the orthogonal complement of $\mathfrak{z}_0$ in $\mathfrak{z}$; notice that K acts trivially on $\mathfrak{z}'$. For each j, we denote by $[j]$ the degree of ρ_j in the $\mathfrak{z}_0$-variables.

Notice that $[j] > 0$ if and only if D_j annihilates all the spherical functions which factor to N'. At the same time, the polynomials ρ_j with $[j] = 0$ provide a system of fundamental K-invariants on the Lie algebra of N', $\mathfrak{n}' \cong \mathfrak{v} \oplus \mathfrak{z}'$, where $[\mathfrak{n}', \mathfrak{n}'] = \mathfrak{z}'$. Symmetrising ρ_j on N' produces an operator $D'_j \in \mathbb{D}(N')^K$, which is the push-forward of D_j via the canonical projection.

Suppose that the $D_j \in \mathcal{D}$ have been ordered so that $D_1, \dots, D_{d_0}$ are the operators with $[j] = 0$. Then $\Sigma^0_{\mathcal{D}}$ can be realised as the intersection of $\Sigma_{\mathcal{D}}$ with the coordinate subspace

$$\Sigma^0_{\mathcal{D}} = \{\xi \in \Sigma_{\mathcal{D}} : \xi_{d_0+1} = \cdots = \xi_d = 0\} \ .$$

What has been said above shows that there is a natural identification of $\Sigma^0_{\mathcal{D}}$ with the Gelfand spectrum $\Sigma_{\mathcal{D}'}$ of the pair (N', K), with $\mathcal{D}' = \{D'_1, \dots, D'_{d_0}\}$.

We will decompose the variables of $\mathbb{R}^d$ as $\xi = (\xi', \xi'')$, with $\xi' = (\xi_1, \dots, \xi_{d_0})$, $\xi'' = (\xi_{d_0+1}, \dots, \xi_d)$. To have a consistent notation, multi-indices α'' will have components indexed from $d_0 + 1$ to d, so that monomials $\xi^{\alpha''}$ only depend on ξ'' and, similarly,

$$D^{\alpha''} = D_{d_0+1}^{\alpha_{d_0+1}} \cdots D_d^{\alpha_d} \ .$$

We set $[\alpha''] = \sum_{j=d_0+1}^d [j]\alpha_j$. Of course, $[\alpha'']$ equals the order of derivation of $D^{\alpha''}$ in the $\mathfrak{z}_0$-variables.

Let us go back to the conjecture. Given a function $F \in \mathcal{S}(N)^K$ we are interested in proving that its spherical transform (1) extends from $\Sigma_{\mathcal{D}}$ to a Schwartz function on $\mathbb{R}^d$. In [7], one of the crucial points in the proof was Proposition 5.1, providing a Taylor development of $\mathcal{G}F$ along the singular set; in that situation, there was just one level of singularity.

Recast in our present situation, that result can be phrased as follows: given $k \in \mathbb{N}$, there exist K-invariant Schwartz functions $\{F_{\alpha''}\}_{[\alpha''] \le k-1}$ on N, with $\mathcal{G}F_{\alpha''}$ only depending on ξ', and such that

$$F = \sum_{[\alpha''] \le k-1} D^{\alpha''} F_{\alpha''} + \sum_{|\beta| = k} \partial_z^\beta R_\beta \ , \tag{2}$$

with $R_\beta \in \mathcal{S}(N)$ for every β.

It is clear, by induction, that it will be sufficient to show that the remainder term

$$\Phi_k(v, z, t) = \sum_{|\beta| = k} \partial_z^\beta R_\beta(v, z, t)$$

can be further expanded as

$$\Phi_k = \sum_{[\alpha''] = k} D^{\alpha''} F_{\alpha''} + \sum_{|\gamma| = k+1} \partial_z^\gamma S_\gamma \tag{3}$$

for some new functions $F_{\alpha''} \in \mathcal{S}(N)^K$, $[\alpha''] = k$, and some new $S_\gamma \in \mathcal{S}(N)$.

We sketch, without proof, the basic ideas that allow to reduce the problem to proving Theorem 1.1 below, skipping many technical details that will be presented elsewhere.

It is convenient to introduce modified versions of the operators D_j, an operation that corresponds to replacing the group N with the direct product $\tilde{N} = N' \times \mathfrak{z}_0$ of N' with the additive group $\mathfrak{z}_0$. We remark that $(\tilde{N}, K)$ is also a Gelfand pair (not satisfying Vinberg's condition), as it can be checked from the classification in [17] or, through a direct argument, from the fact that the Lie algebra $\tilde{\mathfrak{n}}$ is a contraction of $\mathfrak{n}$.

From the same system of invariants ρ_j used to generate the differential operators D_j on N, we produce, by symmetrisation, a system $\tilde{\mathcal{D}} = \{\tilde{D}_1, \dots, \tilde{D}_d\}$ of generators of $\mathbb{D}(\tilde{N})^K$. We also use the same coordinates $(v, z, t) \in \mathfrak{v} \times \mathfrak{z}_0 \times \mathfrak{z}'$ on $\tilde{N}$, via the exponential map $\exp_{\tilde{N}}$.

Taking advantage of this common coordinate system for N and $\tilde{N}$, we can compare D_j and $\tilde{D}_j$ as follows: the left-invariant vector field corresponding to the basis element $e_\nu \in \mathfrak{v}$ is

$$X_\nu = \partial_{v_\nu} + \sum_i b_i(v)\partial_{z_i} + \sum_\ell c_\ell(v)\partial_{t_\ell}$$

on N, and

$$\tilde{X}_\nu = \partial_{v_\nu} + \sum_\ell c_\ell(v)\partial_{t_\ell}$$

on $\tilde{N}$. Therefore,

$$D_j - \tilde{D}_j = \sum_{\alpha,\beta,\gamma\,:\,|\beta|\geq 1} a^j_{\alpha,\beta,\gamma}(v)\partial_v^\alpha\partial_z^\beta\partial_t^\gamma \ ,$$

where each term contains at least one derivative in the z-variables.

This implies that, if $[\alpha''] = k$, then each term in $D^{\alpha''} - \tilde{D}^{\alpha''}$ contains at least $k+1$ derivatives in z. Then it will be sufficient to prove (2) with each $D^{\alpha''}$ replaced by $\tilde{D}^{\alpha''}$, since the difference can be absorbed in the remainder term. Therefore (3) is equivalent to

$$\Phi_k = \sum_{[\alpha'']=k} \tilde{D}^{\alpha''} F_{\alpha''} + \sum_{|\gamma|=k+1} \partial_z^\gamma S_\gamma \ . \tag{4}$$

To both sides of (4) we apply Fourier transform in the z-variables, that we denote by "$\widehat{\ }$", e.g.,

$$\widehat{F_{\alpha''}}(v,\zeta,t) = \int_{\mathfrak{z}_0} F_{\alpha''}(v,z,t)e^{-i\langle z,\zeta\rangle}\,dz \ ,$$

where $\langle\ ,\ \rangle$ is the given K-invariant scalar product on $\mathfrak{z}_0$. We obtain:

$$\widehat{\Phi}_k(v,\zeta,t) = \sum_{[\alpha'']=k} \tilde{D}_\zeta^{\alpha''}\widehat{F_{\alpha''}}(v,\zeta,t) + \sum_{|\gamma|=k+1} (i\zeta)^\gamma \widehat{S_\gamma}(v,\zeta,t) \ , \tag{5}$$

where each $\tilde{D}_{j,\zeta}$ is obtained from $\tilde{D}_j$ by replacing each derivative ∂_{z_ℓ} by $i\zeta_\ell$.

Modulo error terms that involve higher-order powers of ζ, we are left with proving that the k-th order term in the Taylor expansion in ζ of $\widehat{\Phi_k}(v,\zeta,t)$, i.e.,

$$\sum_{|\gamma|=k} \frac{\zeta^\gamma}{\gamma!}\partial_z^\beta \widehat{R_\beta}(v,0,t) \ ,$$

equals the k-th order term in the Taylor expansion in ζ of (5), i.e.,

$$\sum_{[\alpha'']=k} \tilde{D}_\zeta^{\alpha''}\widehat{F_{\alpha''}}(v,0,t) \ .$$

This last point is the subject of our main theorem.

Theorem 1.1. *Let G be a K-invariant function on $\tilde{N}$ of the form*

$$G(v,\zeta,t) = \sum_{|\gamma|=k} \zeta^\gamma G_\gamma(v,t) \ , \tag{6}$$

with $G_\gamma \in \mathcal{S}(N')$. *Then there are* $H_{\alpha''} \in \mathcal{S}(N')^K$, *for* $[\alpha''] = k$, *such that*

$$G = \sum_{[\alpha'']=k} \tilde{D}_\zeta^{\alpha''} H_{\alpha''} \ . \tag{7}$$

More precisely, given a Schwartz norm $\| \ \|_{(p)}$, *the functions* $H_{\alpha''}$ *can be found so that, for some* $q = q(k,p)$, $\|H_{\alpha''}\|_{(p)} \le C_{k,p} \sum_{|\gamma|=k} \|G_\gamma\|_{(q)}$, *for every* α'', $[\alpha''] = k$.

In Section 2, we prove Theorem 1.1 for the pairs in the first block of Table 1. Indeed, in these cases the group N' is reduced to $\mathfrak{v}$ and is abelian.

The rest of the article will be devoted to the proof of Theorem 1.1 for the other pairs, where N' is a Heisenberg group, with the exception of line 7, where it is a "quaternionic Heisenberg group" with Lie algebra $\mathbb{H}^n \oplus \mathrm{Im}\,\mathbb{H}$.

In Section 3, we develop a careful analysis of the structure of the K-invariant polynomials on $\mathfrak{v} \oplus \mathfrak{z}_0$, describing the K-invariant irreducible subspaces of the symmetric algebras over $\mathfrak{v}$ and $\mathfrak{z}$ that are involved.

In Section 4, we reduce the proof of Theorem 1.1 to an equivalent problem of representing vector-valued K-equivariant functions in terms of K-equivariant differential operators applied to K-invariant scalar functions (Proposition 4.3). Then we analyse the images of these differential operators in the Bargmann representations of N', identifying the K-invariant irreducible subspaces of the Fock space on which they vanish. This analysis reveals interesting connections between these operators and the natural action of K itself on the Fock space, once both are realised to be part of the metaplectic representation.

Finally, in Sections 5 and 6, we complete the proof of Theorem 1.1 for the pairs with N' nonabelian.

2. Proof of Theorem 1.1 for N' abelian

In this section, we consider the pairs in the first block of Table 1, where $\mathfrak{z}_0 = \mathfrak{z}$ and, therefore, $N' = \mathfrak{v}$ is abelian. We call $(\mathfrak{v}, K)$ an *abelian pair*. In this case, one can prove that Theorem 1.1 is true with the additional property that the functions H_α can be chosen independently of p and depending linearly on the G_γ.

Via Fourier transform in $\mathfrak{v}$, this statement is equivalent to the Proposition 2.1 below. We first explain the notation. We split the set ρ of fundamental invariants into the two subsets ρ', ρ'', where ρ' contains the polynomials depending only on $v \in \mathfrak{v}$, and ρ'' those which contain $z \in \mathfrak{z}$ at a positive power. This notation matches with the splitting of coordinates (ξ', ξ'') on the Gelfand spectrum introduced in Section 1.

Proposition 2.1. *Let* $G \in C^\infty(N)^K$ *satisfying*

$$G(v,\zeta) = \sum_{|\gamma|=k} \zeta^\gamma G_\gamma(v) \ ,$$

with $G_\gamma \in \mathcal{S}(\mathfrak{v})$. Then there exist $g_{\alpha''} \in \mathcal{S}(\mathbb{R}^{d_0})$, $[\alpha''] = k$, depending linearly and continuously on $\{G_\gamma\}_\gamma$ and such that

$$G(v,\zeta) = \sum_{[\alpha'']=k} \rho(v,\zeta)^{\alpha''} g_{\alpha''} \circ \rho'(v) \ .$$

The proof is quite simple and relies on two adapted versions of Hadamard's Lemma on one side, and of the Schwarz-Mather theorem [12, 13] on the other side. Hadamard's Lemma states that if a function of two variables $f(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ satisfies $f(0,y) = 0$ for every y, then there exist C^∞-functions $g_j(x,y)$, $j = 1,\dots,n$, such that

$$f(x,y) = \sum_{j=1}^n x_j g_j(x,y) \ .$$

Adapting the proof of Hadamard's lemma given in Proposition 5.3 in [7], it is easy to show the following.

Lemma 2.2. *Let $f(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$ and $k \in \mathbb{N}$. Then there exist smooth functions $g_\alpha(y) \in C^\infty(\mathbb{R}^m)$, $|\alpha| \le k$, and $R_\alpha(x,y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, $|\alpha| = k+1$, such that*

$$f(x,y) = \sum_{|\alpha|\le k} x^\alpha g_\alpha(y) + \sum_{|\alpha|=k+1} x^\alpha R_\alpha(x,y) \ .$$

Furthermore if $f(x,y) \in C^\infty(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^m)$ in the sense that, for every L,

$$\sup_{\substack{|\alpha|,|\beta|,|x| \le L \\ y \in \mathbb{R}^n}} (1+|y|)^L |\partial_x^\alpha \partial_y^\beta f(x,y)| < \infty \ ,$$

then the functions $g_\alpha(y)$, $|\alpha| \le k$, and $R_\alpha(x,y)$, $|\alpha| = k+1$, can be chosen in $\mathcal{S}(\mathbb{R}^m)$ and $C^\infty(\mathbb{R}^n)\hat{\otimes}\mathcal{S}(\mathbb{R}^m)$ respectively, and depending linearly and continuously on f.

Proof of Proposition 2.1. All the polynomials ρ_j are homogeneous in v and z and, for $j = 1,\dots,d_0$, they only depend on v. Hence it is easy to adapt the proof of Theorem 6.1 in [2] to show that there exists a continuous linear operator $\tilde{\mathcal{E}} : \big(\mathcal{S}(\mathfrak{v})\hat{\otimes}C^\infty(\mathfrak{z})\big)^K \to \mathcal{S}(\mathbb{R}^{d_0})\hat{\otimes}C^\infty(\mathbb{R}^{d-d_0})$ such that $\tilde{\mathcal{E}}(g) \circ \rho = g$ for every $g \in \big(\mathcal{S}(\mathfrak{v})\hat{\otimes}C^\infty(\mathfrak{z})\big)^K$. So let $h = \tilde{\mathcal{E}}(G)$. Using Lemma 2.2, we obtain that, for any $\xi = (\xi', \xi") \in \mathbb{R}^d$,

$$h(\xi) = \sum_{|\alpha''|\le k} \xi^{\alpha''} g_{\alpha''}(\xi') + \sum_{|\alpha''|=k+1} \xi^{\alpha''} U_{\alpha''}(\xi) \ ,$$

where each $g_{\alpha''}$ depends linearly and continuously on $h \in \mathcal{S}(\mathbb{R}^{d_0})\hat{\otimes}C^\infty(\mathbb{R}^{d-d_0})$, hence on $\{G_\gamma\}_\gamma$. Composing with ρ, we get:

$$G(v,\zeta) = h\circ\rho(v,\zeta) = \sum_{|\alpha''|\le k} \rho(v,\zeta)^{\alpha''} g_{\alpha''}\big(\rho'(v)\big) + \sum_{|\alpha''|=k+1} \rho(v,\zeta)^{\alpha''} U_{\alpha''}\big(\rho(v,\zeta)\big) \ .$$

As G is a polynomial of degree k in ζ, we have:

$$G(v,\zeta) = \sum_{[\alpha'']=k} \rho(v,\zeta)^{\alpha''} g_{\alpha''}\big(\rho'(v)\big) \ . \quad \square$$

3. N' nonabelian: structure of K-invariant polynomials on $\mathfrak{v}\oplus\mathfrak{z}_0$

From Table 1 we isolate the last two blocks, i.e., the cases where N' is not abelian. To each line we add the list of fundamental K-invariants on $\mathfrak{v}\oplus\mathfrak{z}_0$ as it appears in Theorem 7.5 of [7]. We split the set ρ of these invariants into three subsets, $\rho_{\mathfrak{v}}$, $\rho_{\mathfrak{z}_0}$, $\rho_{\mathfrak{v},\mathfrak{z}_0}$, containing the polynomials which depend, respectively, only on $v\in\mathfrak{v}$, only on $z\in\mathfrak{z}_0$, or on both v and z. We call the last ones the "mixed invariants". We convene to use the letters r,q,p to denote, respectively, elements of $\rho_{\mathfrak{v}}$, $\rho_{\mathfrak{z}_0}$, $\rho_{\mathfrak{v},\mathfrak{z}_0}$.

The result is Table 2. Note that expressions like z^k refer to the k-th power of a matrix z. For unexplained notation at lines 9 and 10, we refer to [7].

	K	$\mathfrak{v}$	$\mathfrak{z}_0$	$r_k(v)$	$q_k(z)$	$p_k(v,z)$
4	U_{2n+1}	$\mathbb{C}^{2n+1}$	$\Lambda^2\mathbb{C}^{2n+1}$	$\lvert v\rvert^2$	$\mathrm{tr}\,\big((\bar z z)^k\big)$ $(1\le k\le n)$	$v^*(\bar z z)^k v$ $(1\le k\le n)$
5	SU_{2n}	$\mathbb{C}^{2n}$	$\Lambda^2\mathbb{C}^{2n}$	$\lvert v\rvert^2$	$\mathrm{tr}\,\big((\bar z z)^k\big)$ $(1\le k\le n-1)$ $\mathrm{Pf}(z)\,,\ \overline{\mathrm{Pf}(z)}$	$v^*(\bar z z)^k v$ $(1\le k\le n-1)$
6	U_n	$\mathbb{C}^n$	$\mathfrak{su}_n$	$\lvert v\rvert^2$	$\mathrm{tr}\,\big((iz)^k\big)$ $(2\le k\le n)$	$v^*(iz)^k v$ $(1\le k\le n-1)$
7	Sp_n	$\mathbb{H}^n$	$HS^2_0\mathbb{H}^n$	$\lvert v\rvert^2$	$\mathrm{tr}\,z^k$ $(2\le k\le n)$	$v^*z^k v$ $(1\le k\le n-1)$
8	$\mathrm{U}_2\times\mathrm{SU}_n$	$\mathbb{C}^2\otimes\mathbb{C}^n$	$\mathfrak{su}_2$	$\mathrm{tr}\,\big((vv^*)^k\big)$ $(k=1,2)$	$\lvert z\rvert^2$	$i\mathrm{tr}\,(v^*zv)$
9	$\mathrm{U}_2\times\mathrm{Sp}_n$	$\mathbb{C}^2\otimes\mathbb{C}^{2n}$	$\mathfrak{su}_2$	$\mathrm{tr}\,\big((vv^*)^k\big)$ $(k=1,2)$ $\lvert x\rvert^2\lvert y\rvert^2-({}^txy)^2$	$\lvert z\rvert^2$	$i\mathrm{tr}\,(v^*zv)$
10	$\mathrm{U}_1\times\mathrm{Spin}_7$	$\mathbb{C}\otimes\mathbb{O}$	$\mathrm{Im}\,\mathbb{O}$	$\lvert v\rvert^2$ $\lvert v_1\rvert^2\lvert v_2\rvert^2-\big(\mathrm{Re}\,(v_1\bar v_2)\big)^2$	$\lvert z\rvert^2$	$\mathrm{Re}\,\big(z(v_1\bar v_2)\big)$

Table 2

If X is a real vector space, we call $\mathcal{P}(X)$ the polynomial algebra over X, and $\mathcal{P}^k(X)$ the subspace of homogeneous polynomials of degree k. When X is endowed by a complex structure, we denote by $\mathcal{P}^{k_1,k_2}(X)$ the terms in the splitting of $\mathcal{P}(X)$ according to bi-degrees; for example $\mathcal{P}^{k,0}$ is the space of holomorphic polynomials in $\mathcal{P}^k$.

This applies in particular to $\mathfrak{v}$, which always carries a complex structure, and to $\mathfrak{z}_0$ at lines 4 and 5. At line 7, in fact, $\mathfrak{v}$ admits a different complex structure for every choice of a unit quaternion.

The indexing of the elements $p_k(v,z)$ of $\rho_{\mathfrak{v},\mathfrak{z}_0}$ is assumed to match with the notation of Table 2 when there is more than one element in the family.

Coherently with the notation used in the previous sections, if $p^\alpha(v,z)$ is a monomial in the p_k, we denote by $|\alpha|$ its usual length, and by $[\alpha]$ its degree in z. When $\mathfrak{z}_0$ is a complex space, we denote by $[\![\alpha]\!]$ the bi-degree of p^α in $z, \bar{z}$. The same convention on the use of $[\ ]$ and $[\![\ ]\!]$ applies to monomials in the q_k.

It follows from [7, Corollary 7.6] that the algebra $\mathcal{P}(\mathfrak{v} \oplus \mathfrak{z}_0)^K$ is freely generated by $\rho = \rho_\mathfrak{v} \cup \rho_{\mathfrak{z}_0} \cup \rho_{\mathfrak{v},\mathfrak{z}_0}$. The pairs in Table 2 are distinguished by the property that we can add a subspace $\mathfrak{z}'$, of dimension one or three, to $\mathfrak{z}_0$ keeping (K, N) as a nilpotent Gelfand pair and $\mathfrak{v}\oplus\mathfrak{z}$ is a Heisenberg Lie algebra or a quaternionic Heisenberg Lie algebra. Another observation will be of particular importance in the future.

Remark 1. Fix $\zeta \in \mathfrak{z}_0$ and let K_ζ be its stabiliser in K. Then the pair (N', K_ζ) is also a nilpotent Gelfand pair. The result goes back to Carcano's characterisation of nilpotent Gelfand pairs in terms of multiplicity free actions [4]. An alternative proof can be found, e.g., in [14, Ch.2,§4].

The first dividend we get is the following. Evaluating K-invariants at ς, we get K_ς-invariant polynomials on $\mathfrak{n}'$, more precisely, on $\mathfrak{v}$. These polynomials have the same degree in v and in $\bar{v}$ [8, Section 4]. Hence the expressions of the polynomials $r_k(v)$ and $p_k(v, z)$ must also have the same degree in v and $\bar{v}$ (this can be seen directly from Table 2). Therefore we have the splitting

$$\mathcal{P}(\mathfrak{v} \oplus \mathfrak{z}_0)^K = \sum_{m,k\geq 0} \left(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\right)^K .$$

We want to refine this decomposition, by putting special attention on the mixed invariants. Any mixed invariant $p(v, z)$ in $\left(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\right)^K$ can be expanded as

$$p = \sum_j p_{V_j,W_j} \ , \tag{8}$$

where, for each j, V_j and W_j are K-invariant, irreducible subspaces of $\mathcal{P}^{m,m}(\mathfrak{v})$ and $\mathcal{P}^k(\mathfrak{z}_0)$ respectively, with $V \sim W$ equivalent to W as a K-module, and

$$p_{V_j,W_j}(v, z) = \sum_h a_h(v) b_h(z) \ , \tag{9}$$

with $\{a_h\}$ and $\{b_h\}$ being orthonormal dual bases.

In a rather canonical way, we will now replace the basis of monomials $p^\alpha(v, z)q^\beta(z)r^\gamma(v)$ by a new basis, obtained by replacing each p^α by a new polynomial $\widetilde{p^\alpha}$ which is "irreducible", in the sense that it equals p_{V_α,W_α} for appropriate irreducible V_α, W_α.

Before going into this construction, we remark some useful aspects of the list of pairs and invariants in Table 2.

Remark 2.

(a) The first block of Table 2 contains four infinite families, with both $\dim \mathfrak{v}$ and $\dim \mathfrak{z}_0$ increasing with the parameter n. Each pair admits a single invariant in $\rho_\mathfrak{v}$, and several in $\rho_{\mathfrak{z}_0}$ and $\rho_{\mathfrak{v},\mathfrak{z}_0}$.

(b) Inside the first block, the pairs at lines 4 and 5 have a special feature, in that $\mathfrak{n}_0$ is a complex Lie algebra and $\mathfrak{z}_0$ is a complex space. Each pair in these two lines is "twinned" with a pair in line 6, the one with the same $\mathfrak{v}$. The invariants for a pair in line 4 or 5 coincide with the lower degrees invariants for the twin pair at line 6 evaluated at $(v, -i\bar{z}z)$ instead of (v,z).

(c) Each line in the second block contains either an "exceptional" isolated pair (line 10), or an infinite family (lines 8, 9), but with $\dim \mathfrak{z}_0$ fixed. Each pair admits a single invariant in $\rho_{\mathfrak{z}_0}$ and in $\rho_{\mathfrak{v},\mathfrak{z}_0}$, but several in $\rho_{\mathfrak{v}}$.

(d) For each pair, the k-th mixed polynomial $p_k(v,z)$ is a finite sum

$$p_k(v,z) = \sum_{j=1}^{\nu_1} \ell_j(v) b_{jk}(z) \ , \tag{10}$$

with $\nu_1 = \dim \mathfrak{z}_0$ and the ℓ_j independent of k.

(e) For the pairs at lines 6-10, the polynomials $b_{j1}(z)$ appearing in the expression (10) of p_1 are the coordinate functions on $\mathfrak{z}_0$. The real span of the polynomials $\ell_j(v)$ is a K-invariant subspace of $\mathcal{P}^{1,1}(\mathfrak{v})$ equivalent to $\mathfrak{z}_0$.

(f) At lines 6, 7 and for $k > 1$, $p_k(v,z)$ (resp. $q_k(z)$) equals, up to a power of i, $p_1(v, z^k)$ (resp. $q_1(z^k)$). Here again z^k is the k-th power of a matrix.

At this point we must isolate the two special families of lines 4 and 5, and restrict our attention to the pairs of lines 6-10.

For given m,k, we look at the structure of $\big(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$, the space of K-invariant polynomials on $\mathfrak{v} \oplus \mathfrak{z}_0$ of bi-degree (m,m) in v and degree k in z.

Inside $\mathcal{P}^{m,m}(\mathfrak{v})$ consider the subspace generated by polynomials which are divisible by elements of $\rho_{\mathfrak{v}}$, and let $\mathcal{H}^{m,m}(\mathfrak{v})$ its orthogonal complement. More explicitly, if $r^\gamma(v)$ is a monomial in the r_j of bi-degree $(\delta_\gamma, \delta_\gamma)$, then

$$\mathcal{H}^{m,m}(\mathfrak{v}) = \Big(\sum_{1 \le \delta_\gamma \le m} r^\gamma \mathcal{P}^{m-\delta_\gamma, m-\delta_\gamma}(\mathfrak{v}) \Big)^\perp \ .$$

With an abuse of language, we call $\mathcal{H}^{m,m}(\mathfrak{v})$ the *harmonic subspace* of $\mathcal{P}^{m,m}(\mathfrak{v})$. By the K-invariance of each $\mathcal{H}^{m,m}(\mathfrak{v})$,

$$\big(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K = \sum_{0 \le \delta \le m}^{\oplus} \sum_{\delta_\gamma = \delta}^{\oplus} r^\gamma \big(\mathcal{H}^{m-\delta_\gamma, m-\delta_\gamma}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K \ .$$

Similarly, we set

$$\mathcal{H}^k(\mathfrak{z}_0) = \Big(\sum_{1 \le [\beta] \le k} q^\beta \mathcal{P}^{k-[\beta]}(\mathfrak{z}_0) \Big)^\perp \ .$$

For an element p of $\big(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$ we denote by $\tilde{p}$ its $\mathfrak{v}$-*harmonic component*, i.e., its component in $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$.

Finally, we denote by $\mathcal{P}^m(\ell) \subset \mathcal{P}^{m,m}(\mathfrak{v})$ the space generated by the monomials of degree m in the ℓ_j.

Proposition 3.1. *Let K, $\mathfrak{v}$, $\mathfrak{z}_0$ be as in Table 2, lines 6-10.*

(i) *If $k < m$, $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$ is trivial.*

(ii) *For $k = m$, $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^m(\mathfrak{z}_0)\big)^K$ is one-dimensional, and it is generated by $\widetilde{p_1^m}$.*

(iii) *Let* $V_m = \mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)$*. Then* V_m *is absolutely irreducible, i.e., it stays irreducible as a representation of* $K^{\mathbb{C}}$ *after the complexification* $V_m \otimes_{\mathbb{R}} \mathbb{C}$*. We fix an orthonormal basis* $a_j^{(m)}$*,* $1 \le j \le \nu_m$*, of* V_m*. Then*

$$\widetilde{p_1^m} = \sum_{j=1}^{\nu_m} a_j^{(m)}(v) b_j^{(m)}(z) \ , \tag{11}$$

with the $b_j^{(m)}$ *non-trivial.*

Let W_m *denote the linear span of the* $b_j^{(m)}$*,* $1 \le j \le \nu_m$*. Then* $W_m \sim V_m$ *and*

$$W_m \subset \mathcal{H}^m(\mathfrak{z}_0) \ .$$

(iv) *If* $|\alpha| = m$*, then* $\widetilde{p^\alpha} \neq 0$ *and*

$$\widetilde{p^\alpha} = \sum_{j=1}^{\nu_m} a_j^{(m)}(v) b_j^{(\alpha)}(z) \ .$$

(v) *For every* m *and* k*, the products* $\widetilde{p^\alpha} q^\beta$ *with* $|\alpha| = m$ *and* $[\alpha] + [\beta] = k$ *form a basis of* $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$*. In particular,*

$$\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K = \big(V_m \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K \ .$$

(vi) *The spaces* V_m *are mutually* K*-inequivalent.*

Proof. (i) is a consequence of the structure of the p_j. If $k < m$, a monomial in the p, q, r must necessarily contain some r-factor.

(ii) follows from the fact that p_1^m is the only monomial in $\big(\mathcal{P}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^m(\mathfrak{z}_0)\big)^K$ which does not contain r-factors. If we had $\widetilde{p^\alpha} = 0$, this would establish an algebraic relation among the fundamental invariants, in contrast with [7, Corollary 7.6]. This last remark also proves (v) and the first statement in (iv).

The proof of (iii) requires some discussion of $\mathcal{P}^m(\ell)$. First of all, every element of $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$ necessarily belongs to the smaller space $\Big(\big(\mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)\big) \otimes \mathcal{P}^k(\mathfrak{z}_0)\Big)^K$, by the structure of the invariants.

The second fact is that the equivariant map of Remark 2 (e), from $\mathfrak{z}_0$ to the span of the ℓ_j, induces a surjective equivariant map from $\mathcal{P}^m(\mathfrak{z}_0)$ to $\mathcal{P}^m(\ell)$.

Consider now $\mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)$. For every irreducible K-invariant subspace V of it, there must be an equivalent irreducible subspace W in $\mathcal{P}^m(\mathfrak{z}_0)$. This gives rise to an invariant $p_{V,W}$ of (9), belonging to $(V \otimes W)^K \subset \big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^m(\mathfrak{z}_0)\big)^K$. But by (ii), this space is one-dimensional. Therefore there exists a unique $V \subset \mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)$ and a corresponding unique $W \subset \mathcal{P}^m(\mathfrak{z}_0)$ equivalent to V. This forces V to be all of $\mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)$, and it must coincide with V_m. The equality $\dim (V \otimes V)^K = 1$ implies also that V is absolutely irreducible.

Decompose now $\widetilde{p_1^m}$ as

$$\widetilde{p_1^m} = p^\sharp(v,z) + \sum_{[\beta]>0} q^\beta(z) p^\flat_\beta(v,z) \ ,$$

with $p^\sharp \in V_m \otimes \mathcal{H}^m(\mathfrak{z}_0)$ and $p^\flat_\beta \in V_m \otimes \mathcal{H}^{m-[\beta]}(\mathfrak{z}_0)$. Then $p^\sharp$ and the $p^\flat_\beta$'s are all K-invariant. It follows from (i) that $p^\flat_\beta = 0$ for every β, i.e., $\widetilde{p_1^m} = p^\sharp \in V_m \otimes \mathcal{H}^m(\mathfrak{z}_0)$.

To complete the proof of (iv), take any element p of $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$. By (8),

$$p = \sum_j p_{V_j,W_j} \ ,$$

with the p_{V_j,W_j} as in (9). Repeating the same argument used above, each V_j gives rise to an invariant polynomial in $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^m(\mathfrak{z}_0)\big)^K$. By (iii), $V_j = \mathcal{H}^{m,m}(\mathfrak{v}) \cap \mathcal{P}^m(\ell)$ for every j.

We prove (vi) by contradiction. If we had $V_m \sim V_{m'}$ with $m < m'$, the polynomial $\sum_j a_j^{(m')}(v) b_j^{(m)}(z)$ would be a non-zero element of $\big(\mathcal{H}^{m',m'}(\mathfrak{v}) \otimes \mathcal{P}^m(\mathfrak{z}_0)\big)^K$, contradicting (i). □

Consider now the pairs of lines 4, 5. Introducing bi-degrees for polynomials on $\mathfrak{z}_0$, we obtain the following rather obvious variants, on the basis of Remark 2 (b).

Proposition 3.2. *Let K, $\mathfrak{v}$, $\mathfrak{z}_0$ be as in Table 2, lines 4, 5.*

(i') *If $k < m$, $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K$ is trivial.*

(ii') *The polynomials p^α, $\widetilde{p^\alpha}$ coincide with those of line 6, evaluated at $(v, -i\bar{z}z)$. In particular,* (ii), (iii), (iv), (v), (vi) *of Proposition 3.1 have the same formulation (up to the obvious notational changes), with the same $a_j^{(m)}$ and V_m as for the twin pair of line 6.*

(iii') *For $k \geq m$, $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K = \operatorname{span}\{\widetilde{p^\alpha} q^\beta : |\alpha| = m\,,\ [\![\alpha]\!] + [\![\beta]\!] = (k,k)\}$. In particular,*

$$\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K = \big(V_m \otimes \mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K \ .$$

(iv') *If $k_1 \neq k_2$, $\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k_1,k_2}(\mathfrak{z}_0)\big)^K$ is trivial, except at line 5, for $k_1 - k_2 = jn$, $j \in \mathbb{Z}$. In this case,*

$$\big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k_1,k_2}(\mathfrak{z}_0)\big)^K = \begin{cases} (\operatorname{Pf} z)^j \big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k_2,k_2}(\mathfrak{z}_0)\big)^K & \text{if } j > 0\ , \\ (\overline{\operatorname{Pf} z})^{-j} \big(\mathcal{H}^{m,m}(\mathfrak{v}) \otimes \mathcal{P}^{k_1,k_1}(\mathfrak{z}_0)\big)^K & \text{if } j < 0\ . \end{cases}$$

Notice that Propositions 3.1 and 3.2 show that, for every α,

$$\widetilde{p^\alpha} = p_{V_m,W_\alpha} \ , \tag{12}$$

with $m = |\alpha|$ and $W_\alpha \subset \mathcal{P}^{[\alpha]}(\mathfrak{z}_0)$ (resp. $W_\alpha \subset \mathcal{P}^{[\![\alpha]\!]}(\mathfrak{z}_0)$) equivalent to V_m.

Corollary 3.3. *The polynomials $\widetilde{p^\alpha} q^\beta r^\gamma$ form a basis of $\mathcal{P}(\mathfrak{v} \oplus \mathfrak{z}_0)^K$.*

4. Fourier analysis of K-equivariant functions on N'

We start from a function G as in Theorem 1.1,

$$G(v,\zeta,t)=\sum_{|\gamma|=k}\zeta^\gamma G_\gamma(v,t) \qquad \text{(lines 6-10)}\ ,$$
$$G(v,\zeta,t)=\sum_{|\gamma_1|+|\gamma_2|=k}\zeta^{\gamma_1}\bar\zeta^{\gamma_2}G_\gamma(v,t) \qquad \text{(lines 4, 5)}\ ,$$

which is K-invariant, and with $G_\gamma\in\mathcal{S}(N')$ (we use the variable ζ as a reminder that, in the course of the argument, we have taken a Fourier transform in z).

The following statement follows from Proposition 2.1 and Corollary 3.3.

Lemma 4.1.

(i) *(lines 6-10) A function $G\in\big(\mathcal{S}(N')\otimes\mathcal{P}^k(\mathfrak{z}_0)\big)^K$ can be uniquely decomposed as*

$$G(v,\zeta,t)=\sum_{[\alpha]+[\beta]=k}q^\beta(\zeta)\widetilde{p^\alpha}(v,\zeta)g_{\alpha\beta}(v,t)\ , \tag{13}$$

with $g_{\alpha\beta}\in\mathcal{S}(N')^K$ depending continuously on G.

(ii) *(lines 4, 5) A function $G\in\big(\mathcal{S}(N')\otimes\mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K$ can be uniquely decomposed as*

$$G(v,\zeta,t)=\sum_{[\![\alpha]\!]+[\![\beta]\!]=(k,k)}q^\beta(\zeta)\widetilde{p^\alpha}(v,\zeta)g_{\alpha\beta}(v,t)\ , \tag{14}$$

with $g_{\alpha\beta}\in\mathcal{S}(N')^K$ depending continuously on G.

(iii) *For the pair at line 5, $\big(\mathcal{S}(N')\otimes\mathcal{P}^{k+jn,k}(\mathfrak{z}_0)\big)^K$ equals $(\mathrm{Pf}\,z)^j\big(\mathcal{S}(N')\otimes\mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K$ for $j>0$, $(\overline{\mathrm{Pf}\,z})^{-j}\big(\mathcal{S}(N')\otimes\mathcal{P}^{k,k}(\mathfrak{z}_0)\big)^K$ for $j<0$.*

From the right-hand side of (13), or of (14), we extract the single term

$$\widetilde{p^\alpha}(v,\zeta)\tilde g_{\alpha\beta}(v,t)=\sum_{j=1}^{\nu_m}a_j^{(m)}(v)g_{\alpha\beta}(v,t)b_j^{(\alpha)}(\zeta)\ ,$$

with $m=|\alpha|\geq 1$.

In order to emphasize that the following analysis depends only on m and not on the specific multi-index α, it is convenient to introduce an abstract representation space $\mathcal{V}_m$ of K, equivalent to V_m, and denote by $\{e_j^{(m)}\}_{1\leq j\leq\nu_m}$ an orthonormal basis corresponding to the basis $\{a_j^{(m)}\}$ of V_m via an intertwining operator.

We denote by τ_m the representation of K on $\mathcal{V}_m$.

We replace $\widetilde{p^\alpha}g_{\alpha\beta}$ by the $\mathcal{V}_m$-valued function

$$G_{\alpha\beta}(v,t)=g_{\alpha\beta}(v,t)\sum_{j=1}^{\nu_m}a_j^{(m)}(v)\,e_j^{(m)}\ .$$

Since the $b_j^{(\alpha)}$ form an orthonormal basis of the space W_α in (12) and $W_\alpha \sim V_m \sim \mathcal{V}_m$, it follows that $G_{\alpha\beta}$ is K-equivariant, i.e.,

$$G_{\alpha\beta}(kv,t) = \tau_m(k)G_{\alpha\beta}(v,t) \ , \qquad (k \in K) \ .$$

In fact, we have the following characterisation of K-equivariant $\mathcal{V}_m$-valued smooth functions.

Lemma 4.2. *Let H be a $\mathcal{V}_m$-valued, K-equivariant smooth (resp. Schwartz) function H on N'. Then H can be expressed as*

$$H(v,t) = h(v,t)\sum_{j=1}^{\nu_m} a_j^{(m)}(v)e_j^{(m)} \ ,$$

with h smooth and K-invariant (resp. $h \in \mathcal{S}(N')^K$), depending continuously on H.

Proof. Reversing the argument above, from a K-equivariant function $H(v,t) = \sum_j H_j(v,t)e_j^{(m)}$ we can construct the K-invariant scalar-valued function $\tilde{H}(v,\zeta,t) = \sum_j H_j(v,t)b_j^{(m)}(\zeta)$.

If H is smooth (resp. Schwartz), so is $\tilde{H}$. By Proposition 2.1, $\tilde{H}$ can be expressed as

$$\tilde{H}(v,\zeta,t) = \sum_{m'\leq m}\Big(\sum_{\substack{[\alpha]+[\beta]=m\\ |\alpha|=m'}} q^\beta(\zeta)\widetilde{p^\alpha}(v,\zeta)h_{\alpha\beta}(v,t)\Big) \ ,$$

with each $h_{\alpha\beta}$ K-invariant and smooth (resp. Schwartz). Each term in parenthesis can be turned into a K-equivariant function with values in $\mathcal{V}_{m'}$. Since the $\mathcal{V}_{m'}$ are mutually inequivalent, the only non-zero term is the one with $m' = m$. □

Remark 3. From this point on, we may completely disregard the special cases of lines 4 and 5, because in this abstract setting they are completely absorbed by those of line 6.

We denote by $A_j^{(m)} \in \mathbb{D}(N')$ the differential operators obtained from the polynomials $a_j^{(m)}$ by symmetrisation. Then

$$M_m = \sum_{j=1}^{\nu_m} e_j^{(m)} A_j^{(m)} \tag{15}$$

is a K-equivariant differential operator mapping scalar valued functions on N' to $\mathcal{V}_m$-valued functions.

The following statement is the key step in the proof of Theorem 1.1.

Proposition 4.3. *Let G be a $\mathcal{V}_m$-valued, K-equivariant Schwartz function on N'. Then G can be expressed as*

$$G(v,t) = M_m h(v,t) \ , \tag{16}$$

with $h \in \mathcal{S}(N')^K$.

More precisely, given a Schwartz norm $\| \ \|_{(p)}$, the function h can be found so that, for some $q = q(m,p)$, $\|h\|_{(p)} \leq C_{m,p}\|G\|_{(q)}$.

The proof requires some representation theoretic considerations that will be developed in the next subsections.

4.1. **The Bargmann representations of N'.**

The proof requires Fourier analysis on N'. As we mentioned already, N' is either a Heisenberg group or (line 7) its quaternionic analogue, with a 3-dimensional centre. It will suffice to restrict attention to the infinite-dimensional representations.

When N' is a Heisenberg group, i.e., $\mathfrak{n}' = \mathfrak{v} \oplus \mathbb{R}$, we see from Table 1 that $\mathfrak{v}$ is a complex space (whose dimension we denote by κ), with K acting on it by unitary transformations. We use the Bargmann-Fock model of its representations, that we briefly describe.

If $(v_1, \ldots, v_\kappa)$ are linear complex coordinates on $\mathfrak{v}$, the $2m$ left-invariant vector fields

$$Z_j = \partial_{v_j} - \frac{i}{4}\bar{v}_j \partial_t \ , \qquad \bar{Z}_j = \partial_{\bar{v}_j} + \frac{i}{4} v_j \partial_t \ , \qquad j = 1, \ldots, \kappa, \tag{17}$$

generate ${\mathfrak{n}'}^{\mathbb{C}}$.

For $\lambda > 0$, the Bargmann representation π_λ acts on the Fock space $\mathcal{F}_\lambda(\mathfrak{v})$, defined as the space of holomorphic functions φ on $\mathfrak{v}$ such that

$$\|\varphi\|^2_{\mathcal{F}_\lambda} = (\lambda/2\pi)^\kappa \int_{\mathfrak{v}} |\varphi(v)|^2 e^{-\frac{\lambda}{2}|v|^2}\, dv < \infty \ ,$$

and is such that

$$d\pi_\lambda(Z_j) = \partial_{v_j} \ , \qquad d\pi_\lambda(\bar{Z}_j) = -\frac{\lambda}{2} v_j \ . \tag{18}$$

For $\lambda < 0$, π_λ acts on $\mathcal{F}_{|\lambda|}$ as $\pi_\lambda(v,t) = \pi_{|\lambda|}(\bar{v}, -t)$, so that the rôles of Z_j and $\bar{Z}_j$ are interchanged:

$$d\pi_\lambda(Z_j) = \frac{\lambda}{2} v_j \ , \qquad d\pi_\lambda(\bar{Z}_j) = \partial_{v_j} \ . \tag{19}$$

By the Stone-von Neumann theorem, the Bargmann representations π_λ, $\lambda \neq 0$, cover the whole dual object $\widehat{N'}$ up to a set of Plancherel measure zero.

The case $\mathfrak{n}' = \mathfrak{v} \oplus \mathrm{Im}\,\mathbb{H}$, with $\mathfrak{v} = \mathbb{H}^n$, requires some modifications. For every $\mu \neq 0$ in $\mathrm{Im}\,\mathbb{H}$, with polar decomposition $\mu = \lambda\varsigma$, $\lambda = |\mu| > 0$, there is an analogous representations $\pi_\mu = \pi_{\lambda,\varsigma}$ which factors to the quotient algebra $\mathfrak{n}'_\varsigma = \mathfrak{v}_\varsigma \oplus (\mathrm{Im}\,\mathbb{H}/\varsigma^\perp)$. This is a Heisenberg algebra, with $\mathfrak{v}_\varsigma$ denoting $\mathfrak{v}$ endowed with the complex structure induced by the unit quaternion ς. Then $\pi_{\lambda,\varsigma}$ is the Bargmann representation of index λ of $\mathfrak{n}'_\varsigma$, acting on the Fock space $\mathcal{F}(\mathfrak{v}_\varsigma)$. Again, the π_μ cover $\widehat{N'}$ up to a set of Plancherel measure zero.

For the sake of a unified discussion, we drop the subscripts λ or μ, and simply write π and $\mathcal{F}$. Only when strictly necessary, we will reintroduce a parameter $\lambda > 0$, leaving to the reader the obvious modifications for the other cases.

In all cases, the fact that K acts trivially on $\mathfrak{z}'$ implies that each representation as above is stabilised by K. In fact, if σ denotes the representation of U_κ on functions on $\mathfrak{v}$ given by

$$\big(\sigma(k)\varphi\big)(v) = \varphi(k^{-1}v) \ , \tag{20}$$

one has the identity

$$\pi(kv,t) = \sigma(k)\pi(v,t)\sigma(k^{-1}) \ .$$

The representation π maps functions $H \in \mathcal{S}(N')\otimes\mathcal{V}_m$ into operators $\pi(H) \in \mathcal{L}(\mathcal{F})\otimes\mathcal{V}_m \cong \mathcal{L}(\mathcal{F},\mathcal{F}\otimes\mathcal{V}_m)$, depending linearly on H and such that

$$\pi(h\otimes w) = \pi(h)\otimes w \ , \qquad h \in \mathcal{S}(N') \ .$$

If H is K-equivariant, then

$$\pi(H)\sigma(k) = \big(\sigma\otimes\tau_m\big)(k)\pi(H) \ , \tag{21}$$

for all $k \in K$. Similarly, the equivariance of M_m implies that, for $k \in K$,

$$d\pi(M_m)\sigma(k) = \big(\sigma\otimes\tau_m\big)(k)d\pi(M_m) \ , \tag{22}$$

i.e., $\pi(H)$ and $d\pi(M_m)$ intertwine σ with $\sigma\otimes\tau_m$.

With an abuse of notation, we denote the restriction of σ to K by the same symbol.

Since (N',K) is a n.G.p., the representation σ decomposes into irreducibles without multiplicities. We can write

$$\mathcal{F} = \sum_{\mu\in\mathfrak{X}}^{\oplus} V(\mu) \ , \tag{23}$$

for some set $\mathfrak{X}$ of dominant weights μ of K. For each μ, we denote by $R(\mu)$ the representation of K with highest weight μ. Each $V(\mu)$ is contained in some $\mathcal{P}^{s,0}(\mathfrak{v})$ with $s = s(\mu)$, since these subspaces are obviously invariant under σ.

In particular, $V(\mu)$ consists of C^∞-vectors for π, so that $d\pi(M_m)$ is well defined on $V(\mu)$.

Notice that, for the pairs in the first block of Table 2, each $\mathcal{P}^{s,0}(\mathfrak{v})$ is itself irreducible. Only for the pairs in the second block, different $V(\mu)$'s may be contained in the same $\mathcal{P}^{s,0}(\mathfrak{v})$.

The following lemma in invariant theory will be important in the next proof.

Lemma 4.4. *Let $R(\mu_1), R(\mu_2), R(\mu_3)$ be three irreducible finite dimensional representations of a complex group G on spaces V_1, V_2, V_3 respectively. Denote by $c_{\mu_i}(\mu_j,\mu_k)$ the multiplicity of $R(\mu_i)$ in $R(\mu_j)\otimes R(\mu_k)$. Then*

$$c_{\mu_i}(\mu_j,\mu_k) = \dim\,(V_i'\otimes V_j\otimes V_k)^G = c_{\mu_j'}(\mu_k,\mu_i') \ ,$$

where μ' stands for the highest weight of the dual representation and V' for the dual vector space of V. Over $\mathbb{R}$ the statement modifies as follows:

$$\dim\,(V_i\otimes V_i')^G c_{\mu_i}(\mu_j,\mu_k) = \dim\,(V_i'\otimes V_j\otimes V_k)^G = c_{\mu_j'}(\mu_k,\mu_i')\dim\,(V_j\otimes V_j')^G \ .$$

Proposition 4.5. *Let Φ be a linear operator, defined on the algebraic sum of the $V(\mu)$, $\mu\in\mathfrak{X}$, with values in $\mathcal{F}\otimes\mathcal{V}_m$, and intertwining σ with $\sigma\otimes\tau_m$. Then*

(i) *for every μ,*

$$\Phi : V(\mu) \longrightarrow V(\mu)\otimes\mathcal{V}_m \ ;$$

(ii) $\Phi_{|_{V(\mu)}} = 0$, *unless* $R(\mu) \subset R(\mu) \otimes \tau_m$;
(iii) $\Phi_{|_{V(\mu)}} = 0$ *if* $s(\mu) < m$.

Proof. Let P_μ be the orthogonal projection of $\mathcal{F}$ onto $V(\mu)$. If $\mu_1 \in \mathfrak{X}$, $(P_{\mu_1} \otimes \mathrm{Id})\Phi_{|_{V(\mu)}}$ intertwines $R(\mu)$ with $R(\mu_1) \otimes \tau_m$. Hence $(P_{\mu_1} \otimes \mathrm{Id})\Phi_{|_{V(\mu)}} = 0$ unless $R(\mu) \subset R(\mu_1) \otimes \tau_m$.

Take W_m, the linear span of the polynomials $b_j^{(m)}(z)$ in (11), as a concrete realisation of τ_m. Take also $V(\mu_1)$ as a concrete realisation of $R(\mu_1)$ and $\overline{V(\mu)}$ as concrete realisation of the (complex) contragredient representation $R(\mu)'$ of $R(\mu)$. By Lemma 4.4,

$$R(\mu) \subset R(\mu_1) \otimes \tau_m \iff \left(\overline{V(\mu)} \otimes V(\mu_1) \otimes W_m\right)^K \neq \{0\} \ . \tag{24}$$

By Remark 1, for a nonzero element $\zeta \in \mathfrak{z}_0$, the pair (N', K_ζ) is also a nilpotent Gelfand pair, so that $\mathcal{F}(\mathfrak{v})$ decomposes without multiplicities under the action of K_ζ. Let $p(v,z)$ be a nonzero element of $\left(\overline{V(\mu)} \otimes V(\mu_1) \otimes W_m\right)^K$, and fix $\zeta \in \mathfrak{z}_0$ such that $p_0(v) = p(v,\zeta)$ is not identically zero. Then p_0 is K_ζ-invariant and contained in $\overline{V(\mu)} \otimes V(\mu_1)$. Hence $V(\mu)$ and $V(\mu_1)$ must contain two K_ζ-invariant, irreducible, equivalent subspaces. By multiplicity freeness, this forces that $\mu = \mu_1$ and we obtain (i).

At this point, (ii) is obvious.

To verify (iii), observe that the subspaces V_m are mutually inequivalent by Propositions 3.1(vi), 3.2(ii'). Hence V_m does not appear in $\mathcal{P}^{s,s}(\mathfrak{v})$ for $s < m$. □

4.2. **Multiplicity of $R(\mu)$ in $R(\mu) \otimes \mathcal{V}_m$.**

We need at this point to obtain, for any m,

(a) a precise description of the "m-admissible" weights μ, i.e., such that $R(\mu) \subset R(\mu) \otimes \mathcal{V}_m$;
(b) that, for such a pair, $R(\mu)$ is contained in $R(\mu) \otimes \mathcal{V}_m$ without multiplicities.

Point (a) above forces us to go into a case by case analysis, from which we will obtain sets of parameters for the m-admissible weights. This analysis will also give us a positive answer to point (b).

For a simple complex (or compact) group, we let ϖ_i denote its fundamental dominant weights.

4.2.1. *Pairs in the first block of Table 2.*

In these cases we know that $V(\mu) = \mathcal{P}^{s,0}(\mathfrak{v})$ for some s.

Proposition 4.6. *Let* $\mathfrak{v} = \mathbb{C}^n$, *with* $K = (\mathrm{S})\mathrm{U}_n$, *or* $\mathfrak{v} = \mathbb{C}^{2n}$ *with* $K = \mathrm{Sp}_n$. *Then* $\mathcal{P}^{s,0}(\mathfrak{v})$ *is contained in* $\mathcal{P}^{s,0}(\mathfrak{v}) \otimes \mathcal{V}_m$ *if and only if* $s \geq m$, *and in this case with multiplicity one.*

Proof. We know from Propositions 3.1 (i) and 3.2 (i'), that $\mathcal{P}^{s,0}(\mathfrak{v})$ is not contained in $\mathcal{P}^{s,0}(\mathfrak{v}) \otimes \mathcal{V}_m$ if $s < m$. We suppose now that $s \geq m$ and apply the equivalence (24). Since the only fundamental invariant depending only on v is $|v|^2$, there is exactly one invariant (up to scalars) in $\mathcal{P}^{s,0}(\mathfrak{v}) \otimes \mathcal{P}^{0,s}(\mathfrak{v}) \otimes W_m$, namely $|v|^{2(s-m)}\widetilde{p_1^m}$.

By Lemma 4.4, this gives existence and uniqueness of a subspace of $\mathcal{P}^{s,0}(\mathfrak{v}) \otimes \mathcal{V}_m$ equivalent to $\mathcal{P}^{s,0}(\mathfrak{v})$. □

4.2.2. *Pair of line 8.*

For convenience we assume that the action of SU_n (resp. $\mathrm{SL}_n(\mathbb{C})$) on the $\mathbb{C}^n$-factor in $\mathfrak{v}$ has the highest weight ϖ_{n-1}. Also let S^i denote the representation of SU_2 on $\mathcal{P}^{i,0}(\mathbb{C}^2)$ and by χ^s the s-th power of the identity character on U_1. Then, cf. [9],

$$\sigma_{|_{\mathcal{P}^{s,0}(\mathfrak{v})}} = \sum_{i+2j=s} R(i\varpi_1 + j\varpi_2) \otimes S^i \otimes \chi^s \ .$$

We call $R_{s,i}$ (with $0 \leq i \leq s$, $s - i \in 2\mathbb{N}$) the i-th summand above, and $V_{s,i}$ the corresponding subspace of $\mathcal{P}^{s,0}(\mathfrak{v})$.

Proposition 4.7. *$R_{s,i}$ is contained in $V_{s,i} \otimes \mathcal{V}_m$ if and only if $i \geq m$, and in this case with multiplicity one.*

Proof. Notice that both SU_n and the centre of U_2 act trivially on $\mathfrak{z}_0$ and that the remaining factor SU_2 of K acts on W_m by S^{2m}. Then we want to find when it is true that $R_{s,i} \subset R_{s,i} \otimes S^{2m}$. We have

$$\begin{aligned} R_{s,i} \otimes S^{2m} &= R(i\varpi_1 + j\varpi_2) \otimes (S^i \otimes S^{2m}) \otimes \chi^s \\ &= R(i\varpi_1 + j\varpi_2) \otimes (S^{i+2m} \oplus S^{i+2m-2} \oplus \cdots \oplus S^{|2m-i|}) \otimes \chi^s \ . \end{aligned} \tag{25}$$

It is quite clear that we find the summand S^i in the sum in parentheses if and only if $i \geq |2m - i|$, i.e., $i \geq m$, and in this case it appears once and only once. ☐

4.2.3. *Pair of line 9.*

With the same notation of the previous case, we have, cf. [9],

$$\sigma_{|_{\mathcal{P}^{s,0}(\mathfrak{v})}} = \sum_{\substack{i+2j\leq s \\ s-i\in 2\mathbb{N}}} R(i\varpi_1 + j\varpi_2) \otimes S^i \otimes \chi^s = \sum_{\substack{i+2j\leq s \\ s-i\in 2\mathbb{N}}} R_{s,i,j} \ .$$

Proposition 4.8. *$R_{s,i,j}$ is contained in $V_{s,i,j} \otimes \mathcal{V}_m$ if and only if $i \geq m$, and in this case with multiplicity one.*

Proof. As before, we want to find when it is true that $R_{s,i,j} \subset R_{s,i,j} \otimes S^{2m}$. The same identity (25) as above holds and we obtain the same conclusion. ☐

4.2.4. *Pair of line 10.*

We can identify $\mathfrak{v}$ with $\mathbb{C}^8$, with Spin_7 acting via the spin representation and U_1 by scalar multiplication.

The spin representation defines an embedding of Spin_7 into SO_8. Under the action of $\mathrm{U}_1 \times \mathrm{SO}_8$, $\mathcal{P}^{s,0}(\mathbb{C}^8)$ decomposes into irreducibles as

$$\mathcal{P}^{s,0}(\mathbb{C}^8) = \sum_{\substack{i\geq 0 \\ s-i\in 2\mathbb{N}}} n(v)^{s-i}\mathcal{H}^i = \sum_{\substack{i\geq 0 \\ s-i\in 2\mathbb{N}}} V_{s,i} \ , \qquad n(v)^2 = v_1^2 + \cdots + v_8^2 \ .$$

Since the compact groups Spin_7 and SO_8 have the same invariants on $\mathbb{C}^8$, this decomposition is also irreducible under Spin_7. Therefore

$$\sigma_{|_{\mathcal{P}^{s,0}(\mathfrak{v})}} = \sum_{\substack{i\geq 0\\ s-i\in 2\mathbb{N}}} R\big(i\varpi_3\big)\otimes\chi^s = \sum_{2i\leq s} R_{s,i}\ .$$

Proposition 4.9. *$R_{s,i}$ is contained in $V_{s,i}\otimes\mathcal{V}_m$ if and only if $i\geq m$, and in this case with multiplicity one.*

Proof. The group Spin_7 acts on $\mathfrak{z}_0$ via $R(\varpi_1)$ (and U_1 acts trivially). The orthogonal projection of W_m on the highest component $R(m\varpi_1)$ of $\mathcal{P}^m(\mathfrak{z}_0)$ must be non-zero, otherwise $\mathcal{V}_m\subset\mathcal{P}^{m-2}(\mathfrak{z}_0)$ and we would have an invariant contradicting Proposition 3.1(i). Therefore, Spin_7 acts on $\mathcal{V}_m$ via $R(m\varpi_1)$.

We follow [11, Example 5.2]: setting $k=m-s+i$, $R(m\varpi_1)\otimes R(k\varpi_3)$ decomposes as a direct sum

$$R(m\varpi_1)\otimes R(k\varpi_3) = \sum_{a_1,a_2,a_3,a_4} R\big(a_1\varpi_1+a_2\varpi_2+(a_3+a_4)\varpi_3\big)\ ,$$

extended over the quadruples $(a_j)_{1\leq j\leq 4}$ of nonnegative integers such that

$$a_1(1,0)+a_2(1,2)+a_3(0,1)+a_4(1,1)=(m,k)\ . \tag{26}$$

We are interested in the solutions of (26) which satisfy the requirement $a_1=a_2=0$ and $a_3+a_4=k$. It is clear that there is one (and only one) solution if and only if $m\leq k$, with $a_3=m$, $a_4=k-m$. □

4.3. Nonvanishing of $d\pi(M_m)$ on m-admissible weight spaces.

We have shown that, if μ is m-admissible, there is a unique subspace $X(\mu,m)\subset V(\mu)\otimes\mathcal{V}_m$ equivalent to $V(\mu)$. Therefore, Proposition 4.5 (i) can be made more precise by saying that an operator Φ intertwining σ with $\sigma\otimes\tau_m$ maps $V(\mu)$ into $X(\mu,m)$ for any m-admissible μ. Moreover, $\Phi_{|_{V(\mu)}}$ is uniquely determined up to a scalar factor.

Assume that the identity (16) holds. Applying π to both sides, we obtain

$$\pi(G)=d\pi(M_m)\pi(h)\ .$$

In this identity, $\pi(G)$ and $d\pi(M_m)$ satisfy the assumptions of Proposition 4.5, whereas $\pi(h)$ maps each $V(\mu)$ into itself by scalar multiplication (this is the special case $m=0$ of Proposition 4.5).

The next proposition, whose proof is postponed to the end of this section, provides a necessary condition for being able to solve equation (16) in u.

Proposition 4.10. *For every m-admissible weight μ, $d\pi(M_m)_{|_{V(\mu)}}\neq 0$.*

Let $C=(c_{jk})$ be a $\kappa\times\kappa$ hermitian matrix (with $\kappa=\dim_{\mathbb{C}}\mathfrak{v}$), and

$$\ell_C(v)=\sum_{j,k} c_{jk}v_j\bar{v}_k$$

the associated quadratic form. The symmetrisation process transforms ℓ_C into the operator $L_C \in \mathbb{D}(N')$,

$$L_C = \frac{1}{2} \sum_{i,k} c_{ik}(Z_j \bar{Z}_k + \bar{Z}_k Z_j) \ ,$$

where the $Z_j, \bar{Z}_j$ are the vector fields in (17).

The image of L_C in the Bargmann representations can be described in terms of the representation σ in (20).

Lemma 4.11. *Let $C = (c_{ik})$ be a $\kappa \times \kappa$ hermitian matrix (so that $iC \in \mathfrak{u}_\kappa$), and let*

$$L_C = \frac{1}{2} \sum_{i,k} c_{ik}(Z_j \bar{Z}_k + \bar{Z}_k Z_j) \in \mathbb{D}(N') \ .$$

Then, for $\lambda > 0$,

$$d\pi_\lambda(iL_C) = \frac{\lambda}{2} d\sigma(iC) \ .$$

This identity extends by $\mathbb{C}$-linearity to $C \in \mathfrak{sl}_\kappa$, understanding L_C as $\frac{1}{2}L_{C+C^*} - \frac{i}{2}L_{i(C-C^*)}$.

For the proof, that we skip, it suffices to verify the identity for $C = E_{ik} + E_{ki}$ and $C = iE_{ik} - iE_{ki}$. Notice that σ is the restriction to U_κ of the metaplectic representation.

Denote by L_j the symmetrisation on N' of the polynomials $\ell_j(v)$ appearing in the expression (10) of the mixed invariants p_k. We want to identify how $d\pi\big(\mathrm{span}\,\{L_j\}\big)$ sits inside $d\sigma(\mathfrak{u}_\kappa)$ and understand the action on $V(\mu)$ of the complex algebra generated by the $d\pi(L_j)$. By Proposition 4.12, this is equivalent to identify

$$\mathfrak{c} = \big\{iC : L_C \in \mathrm{span}\,\{L_j\}\big\}$$

inside $\mathfrak{u}_\kappa$ and study the algebra generated by $d\sigma(\mathfrak{c}^{\mathbb{C}})$.

Proposition 4.12. *As a representation space of K, $\mathfrak{c} \sim \mathcal{V}_1 \sim \mathfrak{z}_0$. Moreover,*

(i) *When $\mathfrak{z}_0 = \mathfrak{su}_r$ (line 6 with $r = n$, or lines 8, 9 with $r = 2$), K contains a factor $K_0 \cong \mathrm{SU}_r$ acting nontrivially on $\mathfrak{z}_0$. Then $\mathfrak{c} = \mathfrak{k}_0$.*

(ii) *For line 7, $\mathfrak{c}$ is the Sp_n-invariant complement of $\mathfrak{sp}_{2n}$ in $\mathfrak{su}_{2n}$.*

(iii) *For line 10, let ι be the inclusion of Spin_7 in SO_8 given by the spin representation $R(\varpi_3)$. Then $\mathfrak{c}$ is the 7-dimensional Spin_7-invariant complement of $d\iota(\mathfrak{so}_7)$ in $\mathfrak{so}_8$.*

Proof. The first statement follows from the equivalence $\mathrm{span}\,\{L_j\} \sim \mathrm{span}\,\{\ell_j\} \sim \mathcal{V}_1$.

After Lemma 4.11, (i) is almost tautological: the symmetrisation of $p_1(\cdot, z)$ is $L_{-id\sigma(z)}$. For (ii), it is basically the same argument.

For (iii), we must recall from [7] that the terms v_1, v_2 in the expression of $p_1(v, z) = \mathrm{Re}\,\big(z(v_1\bar{v}_2)\big)$ are octonions representing the two components of $v = 1 \otimes v_1 + i \otimes v_2$ in the decomposition of $\mathbb{C} \otimes \mathbb{O}$ as the direct sum of $\mathbb{R} \otimes \mathbb{O}$ and $(i\mathbb{R}) \otimes \mathbb{O}$.

For fixed z, $p_1(\cdot, z)$ is a quadratic form satisfying $p_1(\bar{v}, z) = -p_1(v, z)$ (here $\bar{v} = 1 \otimes v_1 - i \otimes v_2$). In complex coordinates, it is then expressed by a hermitian matrix C_z with purely imaginary coefficients. It follows that $iC_z \in \mathfrak{so}_8$, and these elements span a Spin_7-invariant 7-dimensional subspace. This is necessarily the complement of $d\iota(\mathfrak{so}_7)$. □

Notice that either $\mathfrak{c} \subset \mathfrak{k}$ is already a Lie algebra, or $\mathfrak{k} \oplus \mathfrak{c} \subset \mathfrak{u}_\kappa$ is itself a Lie algebra. Set $\mathfrak{g} := \mathfrak{k} + \mathfrak{c}$. In two case, lines 7 and 9, $\mathfrak{g} \neq \mathfrak{k}$, when $\mathfrak{g}$ is either $\mathfrak{su}_{2n}$ or $\mathfrak{g} = \mathfrak{so}_8 \oplus \mathbb{R}$, respectively. Let G be the corresponding compact group with $\mathfrak{g} = \operatorname{Lie} G$. Also notice that if $\mathfrak{g} \neq \mathfrak{k}$, then, up to the summand $\mathbb{R}$, $\mathfrak{k} \oplus \mathfrak{c}$ is the Cartan decomposition of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

Lemma 4.13. *The subspaces $V(\mu)$ in* (23) *are also G-invariant.*

Proof. The action of K on $\mathfrak{c}$ is equivalent to the action of K on $\mathfrak{z}_0$. Therefore for each $iC \in \mathfrak{c}$, the action of the stabiliser K_{iC} on $\mathcal{F}$ is multiplicity free and iC preserves each of the irreducible summands. Since $K_{iC} \subset K$, the action of iC also preserves K-invariant irreducible subspaces in $\mathcal{F}$. □

The statement of Lemma 4.13 can also be verified directly using the fact that K and G have the same invariants on $\mathfrak{v}$.

We can now prove Proposition 4.10.

Proof of Proposition 4.10. First of all, recall that we do not treat lines 4 and 5, because they are completely covered by line 6.

Fix a complex basis $\{u_1, \dots u_{\nu_1}\}$ of $\mathfrak{z}_0^{\mathbb{C}}$ with u_1 being a lowest weight vector (of weight, say, $-\alpha$) and let $(z_1, \dots z_{\nu_1})$ denote coordinates in this basis. Then α is also the highest weight of $\mathfrak{c}^{\mathbb{C}}$, z_1^m is a vector of the highest weight, $m\alpha$, in $\mathcal{P}^m(\mathfrak{z}_0)$, and the weights $\pm m\alpha$ do not appear in lower degree polynomials on $\mathfrak{z}_0$. Hence $\pm m\alpha$ are not among the weights of $\mathcal{P}^s(\ell)$ with $s < m$. Decomposing $p_1(v, z)$ with respect to z_j, one gets

$$p_1(v, z) = \sum_{j=1}^{\nu_1} a_j(v) z_j \ ,$$

where a_1^m is a lowest weight vector in $\mathcal{P}^m(\ell)$. We must have $a_1^m \in \mathcal{H}^{m,m}(\mathfrak{v})$, since otherwise, by Corollary 3.3, the weight $-m\alpha$ would also be contained in lower degrees in ℓ. By Proposition 3.1, the K-invariant space generated by a_1^m is V_m. In turn, this implies that z_1^m belongs to the space W_m of Proposition 3.1(iii).

We regard M_1 in (15) as

$$M_1 = \sum_{j=1}^{\nu_1} A_j(v) z_j \ ,$$

identifying $\mathfrak{z}_0$ with $\mathcal{V}_1$. Then

$$M_1^m = \sum_{|\beta|=m} B_\beta z^\beta \ , \tag{27}$$

where each $B_\beta^{(m)}$ is an m-fold composition of the A_j.

Each B_β is the symmetrisation of a polynomial b_β depending on v and $t \in \mathfrak{z}'$. The polynomial

$$P(v, z, t) = \sum_\beta b_\beta(v, t) z^\beta$$

is K-invariant, and its component of highest degree in v is p_1^m. Therefore, $\widetilde{p_1^m}$ is the highest weight term in the decomposition (8) of P.

In particular, M_1^m and M_m have the same highest weight component. Then

$$\langle M_m, z_1^m\rangle = \langle M_1^m, z_1^m\rangle = A_1^m \ .$$

Let X be the lowest weight element in $\mathfrak{c}^{\mathbb{C}}$ such that $A_1 = L_X$. Lemma 4.11 implies that

$$d\pi_\lambda(A_1^m) = (\lambda/2)^m d\sigma(X)^m \ .$$

Therefore it remains to show that under the identification $\mathfrak{z}_0 = \mathfrak{c}$, the element $d\sigma(X)^m$ does not vanish on $V(\mu)$.

As an illustration, consider first the example of line 6. Here $\mathfrak{c} = \mathfrak{k}_0$ and X is a lowest root vector in $\mathfrak{sl}_n$. The complex group $\mathrm{SL}_n(\mathbb{C})$ acts on $V(\mu)$ via $R(s\varpi_1)$ with $s \geq m$. Clearly $d\sigma(X^m)$ is non-zero on the highest weight vector of $V(\mu)$.

In general, we argue in the following way. The action of X on polynomials on $\mathfrak{v}$ is completely determined by the action of X on $\mathfrak{v}$ itself or by the representations of the group G. If $V(\mu)$ is m-admissible and $d\sigma(X^m)$ is zero on $V(\mu)$, then it is also zero on the contragredient space $\overline{V(\mu)}$, and, hence, $d\sigma(X^{2m})$ vanishes on a copy of V_m sitting inside $V(\mu)\otimes\overline{V(\mu)} \subset \mathcal{P}^{s,s}$.

Now V_m has the highest weight $m\alpha$ and X is of weight $-\alpha$. Since X is a weight vector (with a nonzero weight) of a torus in $\mathfrak{g}^{\mathbb{C}}$, it is necessary a nilpotent element. Therefore one can include it into an $\mathfrak{sl}_2$-triple $\{X, H, Y\} \subset \mathfrak{g}^{\mathbb{C}}$, where the semisimple element H is contained in $\mathfrak{k}^{\mathbb{C}}$. (If $\mathfrak{k} = \mathfrak{g}$ this is Jacobson-Morozov theorem, in the two cases with $\mathfrak{g} \neq \mathfrak{k}$ the claim follows from the fact that $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair, see [10, Prop. 4].)

Then H multiplies a highest weight vector $v \in V_m$ by $2m$, therefore v gives rise to at least one irreducible representation of $\{X, H, Y\}$ of dimension at least $(2m+1)$. By the linear algebra considerations, $d\sigma(X^{2m})v \neq 0$. A more careful analysis can show that $d\sigma(Y)v = 0$ and $d\sigma(X^{2m})v$ is a lowest weight vector of V_m. □

5. Proof of Proposition 4.3

First, let us fix some notation. Let $T = \partial_t$ be the central derivative of N' when N' is the Heisenberg group. For the pair at line 7, where N' is the quaternionic Heisenberg group, we take $T_j = \partial_{t_j}$, $j = 1, 2, 3$, the derivatives in three orthogonal coordinates on $\mathfrak{z}'$.

We can assume that $\mathcal{D}' = (D'_1, \ldots, D_{d_0-1}, i^{-1}T)$ and $\mathcal{D}' = (D'_1, \ldots, D'_{d_0-3}, i^{-1}T_1, i^{-1}T_2, i^{-1}T_3)$ respectively. The first $d_0 - 1$ (resp. $d_0 - 3$) operators come from symmetrisation of the polynomials $\rho_j \in \rho_{\mathfrak{v}}$. We convene that D'_1 is the sublaplacian, i.e., the symmetrisation of $|v|^2$. A point of the spectrum $\Sigma_{\mathcal{D}'}$ of (N', K) can then be written as $\xi' = (\tilde{\xi}, \lambda)$ with λ in $\mathbb{R}$ or $\mathbb{R}^3$, depending on the pair considered. The points of the spectrum with $\lambda \neq 0$ form a dense subset of $\Sigma_{\mathcal{D}'}$ and they are parametrised by λ and $\mu \in \mathfrak{X}$ as $\xi'(\lambda, \mu)$, where $\xi'_j(\lambda, \mu)$ is the scalar such that

$$d\pi_\lambda(D'_j)_{|V(\mu)} = \xi_j(\lambda, \mu)\mathrm{Id} \ . \tag{28}$$

Note that $\xi_{d_0}(\lambda, \mu) = \lambda$ and, if $V(\mu) \subset \mathcal{P}^{s,0}(\mathfrak{v})$, then $\xi'_1(\lambda, \mu) = |\lambda|(2s + \kappa)$, cf., e.g., [1].

By δ_j we denote the degree of homogeneity of the polynomial ρ_j (and hence of D'_j) with respect to the automorphic dilations

$$r \cdot (v,t) = (r^{\frac{1}{2}} v, rt) \tag{29}$$

of $\mathfrak{n}'$ (and of N'); i.e., $\delta_j = \frac{1}{2}\deg \rho_j$ for the first $d_0 - 1$ (resp. $d_0 - 3$) operators, and $\delta_j = 1$ for the T's.

If $\varphi(v,t)$ is a spherical function, then $\varphi_r(v,t) = \varphi(r^{\frac{1}{2}} v, rt)$ is also spherical, and $\xi'_j(\varphi_r) = r^{\delta_j}\xi'_j(\varphi)$. Then $\Sigma_{\mathcal{D}'}$ is invariant under the following dilations of $\mathbb{R}^{d_0}$:

$$r \cdot (\xi'_1, \dots, \xi'_{d_0}) = (r^{\delta_1}\xi'_1, \dots, r^{\delta_{d_0}}\xi'_{d_0}) \ . \tag{30}$$

In terms of the parameters (λ, μ), we have

$$r \cdot \xi'(\lambda, \mu) = \xi'(r^2\lambda, \mu) \ .$$

Now we define the following left-invariant, self-adjoint differential operator on N':

$$U_m = M_m^* M_m = \sum_{j=1}^{\nu_m} A_j^{(m)^*} A_j^{(m)} \ .$$

Note that

$$\ker U_m = \bigcap_{j=1}^{\nu_m} \ker A_j^{(m)} \ .$$

As $a_j^{(m)} \in \mathcal{P}^{m,m}(\mathfrak{v})$, the operators $A_j^{(m)}$ and U_m are homogeneous of degree m and $2m$, respectively, w.r. to the dilation (29). Furthermore as M_m is K-invariant, U_m is also K-invariant. Hence it can be written as $U_m = u_m(D')$ where $u_m \in \mathcal{P}(\mathbb{R}^{d_0})$ is homogeneous of degree $2m$ with respect to the dilations (30) of $\mathbb{R}^{d_0}$.

By (28),

$$\pi_\lambda(U_m)_{|V(\mu)} = u_m(\xi'(\lambda,\mu))\mathrm{Id} \ .$$

Let

$$S_m = \{\xi' \in \Sigma_{\mathcal{D}'}, \ u_m(\xi') = 0\} \ .$$

Then

$$\ker U_m \cap \mathcal{S}(N')^K = \{f : \operatorname{supp} \mathcal{G}f \subset S_m\} \ . \tag{31}$$

Moreover, S_m is invariant under the dilations (30).

The next lemma shows that polynomials which vanish on S_m can be divided by u_m.

Lemma 5.1. *Assume that $p \in \mathcal{P}(\mathbb{R}^{d_0})$ vanishes on S_m. Then p is divisible by u_m.*

Proof. We may assume that p is homogeneous with respect to the dilations (30) of $\mathbb{R}^{d_0}$.

Consider first the pairs in the first block of Table 2.

In this case there is only one invariant in $\rho_\mathfrak{v}$, leading to the sublaplacian on N', and then only one coordinate ξ'_1 besides those corresponding to the T's. The space $V(\mu)$ coincides with $\mathcal{P}^{s,0}(\mathfrak{v})$ and by Proposition 4.6 $\mathcal{P}^{s,0}(\mathfrak{v}) \subset \mathcal{P}^{s,0}(\mathfrak{v}) \otimes \mathcal{V}_m$ if and only if $s \geq m$. By Proposition 4.5, $\pi(M_m)$ vanishes on $\mathcal{P}^{s,0}$, if $s < m$. This is also the case of $\pi(U_m) = \pi(M_m)^*\pi(M_m)$. Hence the set S_m contains all the points of the form $(|\lambda|(2s+\kappa), \lambda)$ for any $\lambda \in \mathbb{R}^{\dim \mathfrak{z}'}$ and $s = 0, \dots, m-1$.

We decompose p into its odd and even part w.r. to ξ'_1 as

$$p(\xi'_1,\lambda)=\xi'_1\,p_1({\xi'_1}^2,\lambda)+p_2({\xi'_1}^2,\lambda)\ ,$$

where p_1 and p_2 are two polynomials with suitable homogeneity.

We claim that p_1 and p_2 must both vanish on the set of points $(|\lambda|^2(2s+\kappa)^2,\lambda)$ with $\lambda\in\mathbb{R}^{\dim\mathfrak{z}'}$ and $s=0,\dots,m-1$. If it were not so, we would have the identity

$$|\lambda|(2s+\kappa)=-\frac{p_2\big(|\lambda|^2(2s+\kappa)^2,\lambda\big)}{p_1\big(|\lambda|^2(2s+\kappa)^2,\lambda\big)}\ .$$

This contrasts with the fact that the right-hand side is a rational function in λ, while the left-hand side is not. Then $p_1(\eta,\lambda)$ and $p_2(\eta,\lambda)$ are both divisible by $\Pi_{s=0}^{m-1}(\eta-(2s+\kappa)^2|\lambda|^2)$. Therefore $p(\xi'_1,\lambda)$ is divisible by $\prod_{s=0}^{m-1}({\xi'_1}^2-(2s+\kappa)^2|\lambda|^2)$. This also holds for $p=u_m$. Hence

$$u_m(\xi'_1,\lambda)=c\prod_{s=0}^{m-1}({\xi'_1}^2-(2s+\kappa)^2|\lambda|^2)\ . \tag{32}$$

We consider next the pairs in the second block of Table 2.

There are two invariants in $\rho_{\mathfrak{v}}$ for the pairs at lines 8 and 10 and three for the pair at line 9. In the notation of Subsection 4.2.3, the space $V(\mu)$ coincides with $V_{s,i}$ or $V_{s,i,j}$ respectively, always with i and s of the same parity. We adopt the notation

$$\xi'(\lambda,\mu)=\begin{cases}\xi'(\lambda,s,i) & \text{(lines 8,10)}\ ,\\ \xi'(\lambda,s,i,j) & \text{(line 9)}\ .\end{cases}$$

More precisely, $\xi'_1=|\lambda|(2s+\kappa)$ only depends on λ and s. For the pair at line 9, ξ'_2 only depends on λ,s,i, because it is invariant under the larger group $\mathrm{U}_2\times\mathrm{SU}_{2n}$.

The homogeneity degrees of the elements of $\mathcal{D}'$ w.r. to the dilations (29) are $(1,2,1)$ at lines 8 and 10, and $(1,2,2,1)$ for the pair at line 9. By (30) and the subsequent comments,

$$\xi'_1(\lambda,s)=|\lambda|\xi'_1(1,s)\ ,\quad \xi'_2(\lambda,s,i)=\lambda^2\xi'_2(1,s,i)\ ,\quad \xi'_3(\lambda,s,i,j)=\lambda^2\xi'_3(1,s,i,j)\ . \tag{33}$$

We split $\Sigma_{\mathcal{D}'}$ as the union of $\Sigma^\flat_{\mathcal{D}'}=\{\xi':\xi'_{d_0}=0\}=\rho_{\mathfrak{v}}(\mathfrak{v})\times\{0\}$, cf. [2], and the sets

$$\tilde S_i=\begin{cases}\{\xi'(\lambda,s,i)\ ,\ \lambda\in\mathbb{R}\ ,\ s\in i+2\mathbb{N}\}\ , & \text{(lines 8, 10)}\\ \{\xi'(\lambda,s,i,j)\ ,\ \lambda\in\mathbb{R}\ ,\ s\in i+2\mathbb{N}\ ,\ 0\le j\le (s-i)/2\}\ , & \text{(line 9)}\end{cases}$$

depending on $i\ge 0$.

By Propositions 4.7, 4.8, and 4.9, $R_{s,i}$ (resp. $R_{s,i,j}$) is contained in $V_{s,i}\otimes\mathcal{V}_m$ (resp. $V_{s,i,j}\otimes\mathcal{V}_m$) if and only if $i\ge m$. By Proposition 4.5 and Proposition 4.10 $\pi(M_m)$ vanishes on $V(\mu)$ if and only if $R(\mu)$ is not included in $V(\mu)\otimes\mathcal{V}_m$, which means $i<m$. This is also the case of $\pi(U_m)=\pi(M_m)^*\pi(M_m)$.

Hence S_m contains the union of sets $\tilde S_i$ for $0\le i\le m-1$. Moreover, each polynomial u_m vanishes on $\tilde S_i$, $i<m$, but is never zero on $\tilde S_i$, $i\ge m$, except for the origin.

We prove recursively the existence of polynomials $\tilde u_i\in\mathcal{P}(\mathbb{R}^{d_0})$, $i\ge 0$, such that

(a) $\tilde u_i(\xi'_1,\xi'_2,\lambda)=c_{1,i}{\xi'_1}^2+\xi'_2+d_i\lambda^2$, resp. (for line 9), $\tilde u_i(\xi'_1,\xi'_2,\xi'_3,\lambda)=c_{1,i}{\xi'_1}^2+\xi'_2+c_{3,i}\xi'_3+d_i\lambda^2$;

(b) each $\tilde{u}_i$ vanishes on $\tilde{S}_i$ but does not vanish on any other $\tilde{S}_{i'}$, $i' \neq i$, except for the origin;
(c) u_m is a scalar multiple of $\prod_{i=0}^{m-1} \tilde{u}_i$.

Once this is done, the proof can be concluded as in the previous case.

Consider the polynomial u_1. Being homogeneous of degree 2, it must be of the form

$$u_1(\xi_1', \xi_2', \lambda) = a_1 {\xi_1'}^2 + a_2 \xi_2' + b\lambda^2 + c\xi_1'\lambda \ , \tag{34}$$

resp.

$$u_1(\xi_1', \xi_2', \xi_3', \lambda) = a_1 {\xi_1'}^2 + a_2 \xi_2' + a_3 \xi_3' + b\lambda^2 + c\xi_1'\lambda \ . \tag{35}$$

For $i = 0$, we have

$$a_1 \lambda^2 {\xi_1'}^2(1,s) + a_2 \lambda^2 \xi_2'(1,s,0) \underbrace{+ a_3 \lambda^2 \xi_3'(1,s,0,j)}_{\text{only for line 9}} + b\lambda^2 + c\lambda|\lambda|\xi_1'(1,s) = 0 \ ,$$

for every $\lambda \neq 0$, s even (and $j \leq s/2$). This forces $c = 0$ by parity in λ.

In any case, we must have $a_2 \neq 0$. Suppose in fact that $a_2 = 0$. In the cases of lines 8, 10, the identity above would hold for every i, and u_1 would vanish on every $\tilde{S}_i$. In the case of line 9, u_1 would not depend on ξ_2' and the polynomial $p(\xi_1', \xi_3') = u_1(\xi_1', \xi_3', 1) = a_1 {\xi_1'}^2 + a_3 \xi_3' + b$ would vanish at all points $\big(2s + \kappa, \xi_3'(1,s,0,j)\big)$, for s even and $j \leq s/2$. Notice that, for s and i fixed, the values $\xi_3'(1,s,i,j)$ must all be different, because ξ_3' is the only coordinate on $\Sigma_{\mathcal{D}'}$ depending on j. Then we would have $p = 0$ and, by homogeneity, $u_1 = 0$.

Thus, we have obtained $\tilde{u}_1 = u_1/a_2$ satisfying (a), (b), (c) above.

Assume now that we have constructed $\tilde{u}_i \in \mathcal{P}(\mathbb{R}^{d_0})$, $i = 0, \dots, i_0 - 1$, satisfying (a), (b), (c) above. Consider the polynomial u_{i_0}. It vanishes on $\tilde{S}_i$, $i < i_0$, but does not vanish on $\tilde{S}_i$, $i \geq i_0$. Hence we can factor out $\tilde{u}_i$, $i = 0, \dots, i_0 - 1$, from u_{i_0} and there exists a polynomial q_{i_0} such that $u_{i_0} = q_{i_0} \prod_{i=0}^{i_0-1} \tilde{u}_i$. Necessarily $q_{i,0}$ is homogeneous of degree 2 with respect to (30), and vanishes on $\tilde{S}_{i_0}$ because the polynomials $\tilde{u}_i$, $i < i_0$, do not vanish on it. Hence the quotient q_{i_0} will have the form (34), resp. (35). Arguing as before, it can be shown that $c = 0$ and $a_2 \neq 0$. Then $\tilde{u}_{i_0} = q_{i_0}/a_2$ has the required properties. □

The higher complexity of the second part of the proof given above was due to the presence of more than one polynomial in $\rho_{\mathfrak{v}}$, but also by the fact that we did not use explicit formulas for $\xi_2'(1,s,i)$ and $\xi_3'(1,s,i,j)$. To find such formulas does not seem an easy task anyhow, cf. [3]. On the other hand, the arguments used in the proof emphasize a pattern which is common to all cases at hand.

Note that we have also proved the following identities:

$$S_m = \begin{cases} \bigcup_{s=0}^{m-1} \{(|\lambda|(2s+\kappa), \lambda),\ \lambda \in \mathbb{R}^{\dim \mathfrak{z}'}\} & \text{(lines 8, 10)} \ , \\ \bigcup_{i=0}^{m-1} \tilde{S}_i & \text{(line 9)} \ . \end{cases}$$

Also note that what prevents S_m from being an algebraic set is the dependence on $|\lambda|$ of ξ_1'. It follows from (32) and (33) that the zero set of u_m in $\mathbb{R}^d$ is $S_m \cup S_m^-$, where

$$S_m^- = \big\{(-\xi_1, \xi_2, \xi_3, \lambda) : (\xi_1, \xi_2, \xi_3, \lambda) \in S_m\big\} \ ,$$

(with the ξ_3'-component omitted for the pairs at lines 8, 10 - this *caveat* will not be repeated in the sequel).

Let now G be a $\mathcal{V}_m$-valued, K-equivariant Schwartz function G on N'. Set $f = M_m^* G$. Then $f \in \mathcal{S}(N')^K$ and f belongs to the orthogonal complement of $\bigcap_{j=1}^{\nu_m} \ker A_j^{(m)} = \ker U_m$. Hence the Gelfand transform of f vanishes on S_m. The following lemma justifies that we can choose Schwartz extensions of $\mathcal{G}'f$ which vanish on S_m.

Proposition 5.2. *Let $f \in \mathcal{S}(N')^K$ be such that its spherical transform $\mathcal{G}'f$ vanishes on S_m. For any $p \in \mathbb{N}$, there exists $\psi = \psi^{(p)} \in \mathcal{S}(\mathbb{R}^{d_0})$ such that:*

(i) $(u_m \psi)_{|\Sigma_{\mathcal{D}'}} = \mathcal{G}'f$

(ii) *there exist $C = C_p > 0$ and $q = q(p)$ such that $\|\psi\|_{(p)} \le C\|f\|_{(q)}$.*

We state first a preliminary lemma.

Lemma 5.3. *Let $P(y)$ be a real polynomial in $y \in \mathbb{R}^n$. If $f(x,y) \in \mathcal{S}(\mathbb{R}\times\mathbb{R}^n)$ vanishes on $\{(P(y),y) : y \in \mathbb{R}^n\}$, then there exists $\tilde f \in \mathcal{S}(\mathbb{R}^{d_0})$ satisfying $f(x,y) = (x - P(y))\tilde f(x,y)$. Furthermore $\tilde f$ depends linearly and continuously on f.*

Proof. The conclusion follows easily from Hadamard's lemma (Lemma 2.2), once we know that the change of variables $(x,y) \longmapsto \big(x - P(y), y\big)$ preserves $\mathcal{S}(\mathbb{R}^{n+1})$ with its topology. This is trivial if $\deg P(y) \le 1$. If $\deg P = m > 1$, it follows from the inequality

$$|x - P(y)| + |y| \ge C\big(|x|^{1/m} + |y|\big) \ ,$$

which can be verified distinguishing between the two cases $|x-P(y)| < |y|$ and $|x-P(y)| \ge |y|$. □

Proof of Proposition 5.2.

Let $\varphi \in \mathcal{S}(\mathbb{R}^{d_0})$ be an extension of $\mathcal{G}'f$. Such an extension exists by [2]. Let P_k be the homogeneous component of degree k with respect to (30) in the Taylor expansion of φ around the origin. Since φ vanishes on S_m, which is invariant under these dilations, P_k vanishes on S_m.

By Lemma 5.1, there exists $Q_k \in \mathcal{P}(\mathbb{R}^{d_0})$ homogeneous of degree k with respect to (30) such that $u_m Q_k = P_{k+2m}$.

Applying Whitney's extension theorem, there exists $\psi_1 \in C^\infty(\mathbb{R}^{d_0})$ with compact support around the origin and Taylor expansion $\sum_{k\in\mathbb{N}} Q_k$ at the origin. Then $\varphi - u_m\psi_1$ vanishes of infinite order at the origin.

We take now a function η, homogeneous of degree 0 w.r. to the dilations (30), C^∞ away from the origin, and equal to 1 on a conic neighbourhood of $\Sigma_{\mathcal{D}'}$ and equal to 0 on a conic neighbourhood of S_m^-. Such a function exists because, by the hypoellipticity of the sublaplacian, $\Sigma_{\mathcal{D}'}$ is contained in a conic region around the positive ξ_1'-semiaxis, cf. e.g. (15) in [7]:

$$\Sigma_{\mathcal{D}'} \subset \Big\{(\xi_1',\xi_2',\xi_3',\lambda) : |\xi_2'|^{\frac12} + |\xi_3'|^{\frac12} + |\lambda| \le C\xi_1'\Big\} \ .$$

Then the function $\omega = (\varphi - u_m\psi_1)\eta$ is Schwartz and vanishes on $S_m \cup S_m^-$. By repeated application of Lemma 5.3, $\omega = u_m\psi_2$, with ψ_2 Schwartz. Take $\psi = \psi_1 + \psi_2$. Then $u_m\psi\eta = \varphi\eta$, so that (i) holds.

Consider now the Schwartz norm $\|\psi\|_{(p)} \leq \|\psi_1\|_{(p)} + \|\psi_2\|_{(p)}$.

By Lemma 5.3, there exist an integer $\nu = \nu(p) \geq p$ and a constant A_p such that

$$\|\psi_2\|_{(p)} \leq A_p\|\omega\|_{(\nu)} \leq A'_p\|\varphi - u_m\psi_1\|_{(\nu)} \leq A''_p\big(\|\varphi\|_{(\nu)} + \|\psi_1\|_{(\nu+2m)}\big) \ .$$

In order to estimate $\|\psi_1\|_{(\nu+2m)}$, we use the fact that the Whitney extension of the jet $\{Q_k\}_{k\in\mathbb{N}}$ can be performed so that the resulting function $\psi_1 = \psi_1^{(p)}$ satisfies, for an integer $r = r(p)$ and a constant B_p,

$$\|\psi_1\|_{(\nu+2m)} \leq B_p \sum_{k\leq r} \|Q_k\| \leq B'_p \sum_{k\leq r} \|P_{k+2m}\| \leq B'_p\|\varphi\|_{(r+2m)} \ ,$$

where the norm of a polynomial is meant as the maximum of its coefficients.

Putting all together,

$$\|\psi\|_{(p)} \leq C_p\|\varphi\|_{(\max(r,\nu)+2m)} \ .$$

By [2], there are an integer $q = q(p)$ and a constant C_p such that it is possible to choose $\varphi = \varphi^{(p)}$ above so that

$$\|\varphi\|_{(\max(r,\nu)+2m)} \leq C'_p\|f\|_{(q)} \ ,$$

and this concludes the proof. □

We resume the proof of Proposition 4.3.

Given G, set $f = M_m^* G \in \mathcal{S}(N')^K \in (\ker U_m)^\perp$. By (31), $\mathcal{G}'f$ vanishes on S_m.

Applying Proposition 5.2, we can choose a Schwartz function ψ such that $u_m\psi$ extends $\mathcal{G}'f$. Defining $h = \mathcal{G}'^{-1}(\psi)$, we easily obtain, on $\Sigma_{\mathcal{D}'}$,

$$\mathcal{G}'(U_m h) = u_m\psi = \mathcal{G}'f \ .$$

This implies

$$M_m^* M_m h = U_m h = f = M_m^* G \ .$$

To factor out M_m^*, observe that for any $\lambda \neq 0$,

$$\pi_\lambda(M_m)^*\pi_\lambda\left(M_m h - H\right) = 0 \ .$$

By Proposition 4.5, both sides are 0 when restricted to a subspaces $V(\mu)$ with μ non-m-admissible. If μ is m-admissible, then Proposition 4.10 implies that $\pi_\lambda\left(M_m h - H\right) = 0$ on $V(\mu)$. Then $M_m h = H$.

It remains to prove the estimates on the Schwartz norms. To the norm estimates given by Proposition 5.2 it is sufficient to add that M_m^* and $\mathcal{G}'^{-1}$ are continuous on the appropriate Schwartz spaces. For $\mathcal{G}'^{-1}$ we refer to [2, 6, 7].

6. Conclusion

We complete the proof of Theorem 1.1.

Let $G \in \big(\mathcal{S}(N') \otimes \mathcal{P}^k(\mathfrak{z}_0)\big)^K$ as in (6). We decompose G as in (13). We realise the representation space $\mathcal{V}_m$ as W_α when $|\alpha| = m$. By Lemma 4.3, for each (α,β), $[\alpha]+[\beta]=k$, there exists $h_{\alpha,\beta} \in \mathcal{S}(N')^K$ such that

$$\widetilde{p^\alpha}(v,\zeta)\tilde{g}_{\alpha\beta} = M_{\alpha,\zeta} h_{\alpha,\beta} \ ,$$

where the operator $M_{\alpha,\zeta} = \sum_{j=1}^{\nu_m} A_j^{(m)} b_j^{(\alpha)}(\zeta)$ is the realisation of M_m on W_α.

In the notation of (5), the operators $\tilde{D}_\zeta^{\alpha''}$ form a basis of $\big(\mathbb{D}(N') \otimes \mathcal{P}(\mathfrak{z}_0)\big)^K$. Therefore, each $M_{\alpha,\zeta}$ can be expressed as a linear combination of the $\tilde{D}_\zeta^{\alpha''}$ with $[\alpha''] = k$, and one can write G as

$$G = \sum_{[\alpha'']=k} D_\zeta^{\alpha''} H_{\alpha''} \ ,$$

where the functions $H_{\alpha''}$ are finite linear combinations of $h_{\alpha,\beta}$.

The norm estimates are obvious after Proposition 4.3.

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