p-adic aspects of Jacobi forms

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1 Introduction

We are interested in understanding and describing the p-adic properties of Jacobi forms. As opposed to the case of modular forms, not much work has been done in this area. The literature includes [?, ?, ?].

In the first section, we follow Serre's ideas from his theory of p-adic modular forms. We study Jacobi forms whose Fourier expansions have integral coefficients and look at congruences between them. Non-trivial examples are given by Jacobi-Eisenstein series. It turns out that two Jacobi forms need to have the same index and satisfy a condition on the weights in order to be congruent.

If we define p-adic Jacobi forms in the natural way in this context, and restrict ourselves to the case of $SL_2(Z)$, we obtain a structure theorem for the space of p-adic Jacobi forms for $SL_2(Z)$ of a given weight $\chi \in Z_p'$ and index $m \in Z$.

Another feature is that p-adic Jacobi forms for $\Gamma_0(p)$ are also forms for $SL_2(Z)$. This parallels the similar result for modular forms, and it will most probably play an important role in defining some p-adic operators that do not arise directly from complex operators.

In the second section, we associate to every Jacobi form with integral coefficients a measure on Z_p with values in the p-adic ring of Katz's generalized modular forms. This is an injection that allows us to interpret Jacobi forms with p-adic coefficients as truly p-adic objects, and this suggests where to look for the adequate "test objects" for a modular p-adic theory. It also provides examples of p-adic analytic families of modular forms.

Finally, we point out that a lot of work remains to be done, starting by finding a modular definition of p-adic Jacobi forms and studying Hecke and other operators. We hope to eventually obtain some results on p-adic properties of 1/2-integral weight modular forms, since Jacobi forms are closely related to them.

Let us define precisely what we refer to as a Jacobi form (usually called a "weak" Jacobi form). The standard reference for Jacobi forms is [?]. For a more recent overview of the topic, see [?]. Let $\mathcal{H}$ be the complex upper-half plane. Let $e(z)$ denote the exponential $e^{2\pi iz}$ for $z \in \mathbb{C}$.

Definition 1.1 *A Jacobi form of weight $k \in \mathbb{N}$ and index $m \in \mathbb{Z}$ on (Γ, L) where $\Gamma \subset SL_2(\mathbb{Z})$ is a congruence subgroup and $L \subset \mathbb{Z}^2$ is a Γ-invariant by right multiplication rank-2 lattice, is a holomorphic function $\Phi : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ satisfying the following:*

1. *First Transformation Law:*

$$\Phi|_{k,m}\gamma = (c\tau + d)^{-k} e(-\frac{mcz^2}{c\tau + d})\Phi(\gamma.\tau, \frac{z}{c\tau + d}) = \Phi$$

   for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$

2. *Second Transformation Law:*

$$\Phi|_m X = e(m(\lambda^2\tau + 2\lambda z))\Phi(\tau, z + \lambda\tau + \mu) = \Phi$$

   for every $X = [\lambda, \mu] \in L,$

3. *Holomorphicity at the cusps: for each $\gamma \in SL_2(\mathbb{Z})$, $\Phi|_{k,m}\gamma$ has a Fourier expansion of the form*

$$\sum_{n \geq 0, r} c(n, r) q^n \zeta^r$$

   where $q = e(\tau)$, $\zeta = e(z)$, and $c(n, r) \in \mathbb{C}$.

The space of all such forms is denoted $J_{k,m}(\Gamma, L)$. If $A \subset \mathbb{C}$ is any subring, we denote by $J_{k,m}(\Gamma, L, A)$ the subspace consisting of those forms $\Phi = \sum c(n, r) q^n \zeta^r$ with $c(n, r) \in A$ for all n and r.

We write $\mathbf{M}_k^{merom}(\Gamma, A)$ for the space of meromorphic elliptic modular forms of weight k for Γ whose Fourier coefficients at ∞ belong to A and $\mathbf{M}_k(\Gamma, A)$ for the subspace of holomorphic forms.

We often omit to mention the ring of coefficients $A = \mathbb{C}$.

Remarks:

- The indexes n and r appearing in the Fourier expansions of Jacobi forms are rational numbers with bounded denominators, the bound depending on Γ and L.
- It follows from the second transformation law of Jacobi forms that $c(n,r) = 0$ if $r^2 > 4nm + m^2$. Therefore, for a fixed n, there are finitely many non-zero $c(n,r)$.
- The standard definition of a Jacobi form includes the only further requirement that the coefficients $c(n,r)$ vanish whenever $r^2 > 4nm$.
- The space $J_{k,m}(\Gamma, L)$ has finite dimension over $\mathbb{C}$.
- The group $\Gamma \triangleright L$, where $(\gamma, X)(\gamma', X') = (\gamma\gamma', X\gamma' + X')$, acts on $J_{k,m}(\Gamma, L, \mathbb{C})$ via (1) and (2) of the previous definition:

$$\Phi|(\gamma, X) = (\Phi|_{k,m}\gamma)\,|_m X.$$

- If $\Phi \in J_{k,m}(\Gamma, L, A)$ and $X = [\lambda, \mu] \in \mathbb{Q}^2$ with $MX \in \mathbf{Z}^2$ for some $M \in \mathbf{Z}$, then $(\Phi|_m X)\,|_{z=0} \in \mathbf{M}_k^{merom}(\Gamma \cap \Gamma(\frac{M^2}{(M,m)}), A)$. Moreover, if $X = [0, \mu]$, then $(\Phi|_m X)\,|_{z=0} \in \mathbf{M}_k(\Gamma \cap \Gamma(\frac{M^2}{(M,m)}), A)$.
- If Φ is a nonzero Jacobi form of index m, and we fix $\tau \in \mathcal{H}$, then $\Phi(\tau, z)$ has exactly $2m$ zeros as a function of the variable z in a fundamental domain for the action of $\tau\mathbf{Z} + \mathbf{Z}$.

2 Congruences and p-adic limits

We first follow Serre's approach to p-adic modular forms, and consider congruences of Fourier coefficients of Jacobi forms.

Let K be a number field, $\mathcal{O}$ its ring of integers. Let $p \geq 5$ be a rational prime and $\wp|p$ a prime ideal of $\mathcal{O}$. Let $K_\wp$ be the completion of K at $\wp$, $\mathcal{O}_\wp$ its ring of integers, $\pi \in \wp$ an uniformizing parameter. We also let $\mathcal{O}^\wp = K \cap \mathcal{O}_\wp$ and $\mathbf{F} = \mathcal{O}_\wp/(\pi)$.

We say that $\Phi \in J_{k,m}(\Gamma, L, \mathcal{O}^{\mathfrak{p}})$ and $\Psi \in J_{k,m}(\Gamma', L', \mathcal{O}^{\mathfrak{p}})$ are congruent modulo π^s, and denote it by

$$\Phi \equiv \Psi \bmod \pi^s$$

when, if $\Phi = \sum c(n,r)q^n\zeta^r$ and $\Psi = \sum c'(n,r)q^n\zeta^r$, then $c(n,r) \equiv c'(n,r) \bmod \pi^s$ for all n and r.[1]

Example: The first non-trivial examples of congruences are given by the Jacobi-Eisenstein series $E_{k,m} \in J_{k,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathbf{Z})$ defined in [?, I.2]. By looking at the explicit coefficients for $E_{k,1}$, we get

$$E_{k,1} \equiv E_{k',1} \bmod p^{s+1}$$

if $k \equiv k' \bmod (p-1)p^s$. (The same congruence also holds for any given index m; this follows easily from the fact that $E_{k,1}|V_m = E_{k,m}$, where V_m is the operator studied in [?, I.4].)

This shows that, not surprisingly, one can have congruences among Jacobi forms of different weights. What about different indexes?

Lemma 2.1 *Let* $\Phi \in J_{k,m}(\Gamma_0(N), L, \mathcal{O}^{\mathfrak{p}})$, $\Psi \in J_{k',m'}(\Gamma_0(N), L, \mathcal{O}^{\mathfrak{p}})$ *and assume that*

$$0 \not\equiv \Phi \equiv \Psi \bmod \pi^s$$

for some $s \geq 1$. *Then* $m = m'$ *and* $k \equiv k' \bmod (p-1)p^{g(s)}$, *for some* $g(s) \to \infty$ *when* $s \to \infty$.

Proof: Let us work in $B = \mathbb{F}[\zeta, \zeta^{-1}]((q))$. The congruence implies

$$\Phi - \Psi = 0 \tag{1}$$

Let $X = [\lambda, \mu] \in L$, $\lambda \neq 0$. We replace z by $z + \lambda\tau + \mu$ in (??) and we use the second transformation law for Jacobi forms. We get

$$\left(q^{\lambda^2}\zeta^{2\lambda}\right)^{-m}\Phi - \left(q^{\lambda^2}\zeta^{2\lambda}\right)^{-m'}\Psi = 0 \tag{2}$$

Equations (??) and (??) form a linear system in B that can only be solved non trivially if $m = m'$.

[1]There is a q-expansion principle for Jacobi forms, but its proof requires some features of Jacobi forms not visited here, and will appear elsewhere.

Now consider $\Phi|_{z=0} \in \mathbf{M}_k(\Gamma_0(N), \mathcal{O}^{\mathfrak{p}})$ and $\Psi|_{z=0} \in \mathbf{M}_{k'}(\Gamma_0(N), \mathcal{O}^{\mathfrak{p}})$. The hypothesis $\Phi \equiv \Psi \bmod \pi^s$ implies that the q-expansions of $\Phi|_{z=0}$ and $\Psi|_{z=0}$ as modular forms are congruent modulo π^s -just replace 1 for ζ in the original congruence. If $\Phi|_{z=0} \equiv \Psi|_{z=0} \equiv 0 \bmod (\pi^s)$, evaluate instead at some other $\mu \in \mathbb{Q}$, $M\mu \in \mathbb{Z}$, $(p, M) = 1$: the forms $\Phi|_{z=\mu} = \Phi|_m[0, \mu]$ and $\Psi|_{z=\mu} = \Psi|_m[0, \mu]$ are forms of weights k and k' for $\Gamma_0(N) \cap \Gamma(M')$, $(p, M') = 1$. Since $2m+1$ well-chosen such evaluations characterize a Jacobi form of index m, they cannot all be congruent to 0 modulo a power of a prime above π without being the original form itself congruent to 0 mod (π^s) too. In any case, we deduce that $k \equiv k' \bmod (p-1)p^{g(s)}$ from a well-known result by Serre and Katz.

Let us now concentrate on the case $\Gamma = SL_2(\mathbb{Z})$, $L = \mathbb{Z}^2$, where we have the following structure theorem for Jacobi forms of even weight. Consider the graded ring $J_{2*,*} = J_{2*,*}(SL_2(\mathbb{Z}), \mathbb{Z}^2)$. Then

$$J_{2*,*} = \mathbf{M}_*[A, B],$$

the polynomial ring in two variables, where $\mathbf{M}_*$ denotes the graded ring of holomorphic modular forms for $SL_2(\mathbb{Z})$ over $\mathbb{C}$, and $A \in J_{-2,1}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathbb{Z})$, $B \in J_{0,1}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathbb{Z})$ are two specific Jacobi forms -for an explicit description of A and B, see [?, III.9, I.3]. The coefficients of A and B are coprime, and

$$A = z^2 + O(z^4) \quad , \quad B = 12 + O(z^2).$$

(There is a similar result for $J_{2*+1,*}$; but let us stick to even weights.)

The forms $\Phi \in J_{k,m}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathcal{O}^{\mathfrak{p}})$, $\Psi \in J_{k',m}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathcal{O}^{\mathfrak{p}})$ can be expressed via the structure theorem as

$$\Phi = \sum_{j=0}^{m} g_j(\tau) A^j B^{m-j} \quad , \quad \Psi = \sum_{j=0}^{m} h_j(\tau) A^j B^{m-j}$$

for unique modular forms $g_j \in \mathbf{M}_{k+2j}(SL_2(\mathbb{Z}), \mathcal{O}^{\mathfrak{p}})$ and $h_j \in \mathbf{M}_{k'+2j}(SL_2(\mathbb{Z}), \mathcal{O}^{\mathfrak{p}})$.

Lemma 2.2 *If $\Phi \equiv \Psi \bmod \pi^s$ for some $s \geq 0$ then $g_j \equiv h_j \bmod \pi^s$ for $j = 0, \ldots, m$.*

Proof: If there is a j with $g_j \not\equiv h_j \bmod \pi^s$, take j_0 to be the first such index. By the properties of A and B, we have

$$\begin{aligned}\Phi &= \sum_{j=0}^{m} g_j \left(12^{m-j} z^{2j} + O(z^{2j+2})\right) \\ \Psi &= \sum_{j=0}^{m} h_j \left(12^{m-j} z^{2j} + O(z^{2j+2})\right)\end{aligned}$$

Since $\Phi \equiv \Psi \bmod \pi^s$, then also

$$\left(\zeta \frac{d}{d\zeta}\right)^{2j_0} |_{\zeta=1} \Phi \equiv \left(\zeta \frac{d}{d\zeta}\right)^{2j_0} |_{\zeta=1} \Psi \bmod \pi^s$$

In terms of the complex variable z:

$$\left(\frac{1}{2\pi i} \frac{d}{dz}\right)^{2j_0} |_{z=0} \Phi \equiv \left(\frac{1}{2\pi i} \frac{d}{dz}\right)^{2j_0} |_{z=0} \Psi \bmod \pi^s$$

More precisely,

$$12^{m-2j_0} g_{j_0} \equiv 12^{m-2j_0} h_{j_0} \bmod \pi^s$$

which contradicts the property of j_0 if $p \geq 5$.

We denote by $\mathbf{M}_k(SL_2(\mathbf{Z}), \mathcal{O}^{\mathfrak{p}}) \bmod \pi$ and $J_{k,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}^{\mathfrak{p}}) \bmod \pi$ the spaces of power series obtained by reducing modπ the Fourier coefficients at ∞ of forms in $\mathbf{M}_k(SL_2(\mathbf{Z}), \mathcal{O}^{\mathfrak{p}})$ and $J_{k,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}^{\mathfrak{p}})$, respectively.

The following follows from Lemma ??.

Corollary:

$$J_{k,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}^{\mathfrak{p}}) \bmod \pi \simeq \bigoplus_{j=0}^{m} [\mathbf{M}_{k+2j}(SL_2(\mathbf{Z}), \mathcal{O}^{\mathfrak{p}}) \bmod \pi] A^j B^{m-j}.$$

The structure of $\mathbf{M}_l(SL_2(\mathbf{Z}), \mathcal{O}^{\mathfrak{p}}) \bmod \pi$ is well known (see [?]).

As a consequence of Lemma ??, we can attach a weight to the limit of a sequence of Jacobi forms. That is, if $\Phi_j \in J_{k_j,m}(\Gamma_0(N), L, \mathcal{O}^{\mathfrak{p}})$ and $\{\Phi_j\}$ converges, then $k_j \to \chi \in (\mathbf{Z}_p^*)' \simeq \mathbf{Z}/(p-1) \times \mathbf{Z}_p$. Here a weight $k \in \mathbf{Z}$ is interpreted as an element of $(\mathbf{Z}_p^*)'$ via $(k \bmod (p-1), k)$.

We next give a definition of p-adic Jacobi forms of a given weight as limits of complex Jacobi forms.

Definition 2.3 *A p-adic Jacobi form of weight $\chi \in (\mathbf{Z}_p^*)'$ and index $m \in \mathbf{Z}$ on $(\Gamma_0(N), L)$ with coefficients in $\mathcal{O}_\mathfrak{p}$ is an element of*

$$J^p_{\chi,m}(\Gamma_0(N), L, \mathcal{O}_\mathfrak{p})$$

$$= \left\{ \begin{array}{l} \Phi \in \mathcal{O}_\mathfrak{p}((\zeta))[[q]], \Phi = \lim_j \Phi_j, \Phi_j \in J_{k_j,m}(\Gamma_0(N), L, \mathcal{O}^\mathfrak{p}), \\ k_j \to \chi \end{array} \right\}.$$

Denote by $\mathbf{M}^p_\xi(SL_2(\mathbf{Z}), \mathcal{O}_\mathfrak{p})$ the space of p-adic modular forms of weight $\xi \in (\mathbf{Z}_p^*)'$ on $SL_2(\mathbf{Z})$ with coefficients in $\mathcal{O}_\mathfrak{p}$. The next fact also follows from Lemma ??.

Corollary: $J^p_{\chi,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}_\mathfrak{p}) = \bigoplus_{j=0}^m \mathbf{M}^p_{\chi+2j}(SL_2(\mathbf{Z}), \mathcal{O}_\mathfrak{p}) A^j B^{m-j}$.

Proof: If $\Phi \in J^p_{\chi,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}_\mathfrak{p})$, then $\Phi = \lim_n \Phi_n$ for some $\Phi_n \in J_{k_n,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}^\mathfrak{p})$. Lemma ?? for the family Φ_n clearly implies that $\Phi \in \bigoplus_{j=0}^m \mathbf{M}^p_{\chi+2j}(SL_2(\mathbf{Z}), \mathcal{O}_\mathfrak{p}) A^j B^{m-j}$.
Choose now forms $f_{\chi+2j} \in \mathbf{M}^p_{\chi+2j}(SL_2(\mathbf{Z}), \mathcal{O}_\mathfrak{p})$. Let $\Phi = \sum_{j=0}^m f_{\chi+2j} A^j B^{m-j}$. We need to prove that $\Phi \in J^p_{\chi,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}_\mathfrak{p})$. By definition, for each j

$$f_{\chi+2j} = \lim_n f^{(n)}_{k_{n_j}+2j}$$

with $f^{(n)}_{k_{n_j}+2j} \in \mathbf{M}_{k_{n_j}+2j}(SL_2(\mathbf{Z}), \mathcal{O}^\mathfrak{p})$ and $k_{n_j} \equiv \chi \bmod (p-1)p^n$. Assume for the time being that all k_{n_j} coincide for $j = 0, \ldots, m$ and relabel them k_n. Then $\Phi_n = \sum_{j=0}^m f^{(n)}_{k_n+2j} A^j B^{m-j} \in J_{k_n,m}(SL_2(\mathbf{Z}), \mathbf{Z}^2, \mathcal{O}^\mathfrak{p})$ and $\Phi = \lim_n \Phi_n$ is a p-adic Jacobi form.

It remains to show that we can assume, without loss of generality, that the k_{n_j} coincide. Since $k_{n_0} \equiv k_{n_1} \equiv \ldots \equiv k_{n_m} \bmod (p-1)p^n$, define k_n to be the largest of these integers, and write $k_n = k_{n_j} + a_j(p-1)p^n$ with $a_j \in \mathbf{Z}$; replace now $f^{(n)}_{k_{n_j}+2j}$ by $f^{(n)}_{k_n+2j} = f^{(n)}_{k_{n_j}+2j} E_{p-1}^{a_j p^n}$. We still have $f_{\chi+2j} = \lim_n f^{(n)}_{k_n+2j}$ because $E_{p-1} \equiv 1 \bmod p$.

This ends the proof.

This already gives a pretty good idea of what a p-adic Jacobi form on $(SL_2(\mathbf{Z}), \mathbf{Z}^2)$ -as defined in ??- looks like. The next example -communicated to us by Rodriguez-Villegas- and theorem show the first step of an expected property of p-adic Jacobi forms, namely: that forms of a certain level Np^r are also forms of level N.

Example : Let $p \equiv 1 \bmod 4$, and let $k \in \mathbb{N}$, $k \equiv 1 + \frac{p-1}{2} \bmod (p-1)$. Then

$$p(k-1)E_{k,1} \equiv \sum_{r,s,r\equiv s \bmod 2} q^{\frac{r^2+ps^2}{4}} \zeta^r \bmod p.$$

The left-hand side form belongs to $J_{k,1}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathbb{Z})$, and the right-hand side form belongs to $J_{1,1}(\Gamma_0(p), \mathbb{Z}^2, \mathbb{Z}, \left(\frac{\cdot}{p}\right))$ –where the symbol $\left(\frac{\cdot}{p}\right)$ affects the First Transformation Law in the expected manner. In accordance to the spirit of the theory of p-adic modular forms, we expect the latter form to have weight $1 + \left(\frac{\cdot}{p}\right)$ on $SL_2(\mathbb{Z})$. The congruence requirement for the weight k now becomes more clear.

This congruence follows from a study of the coefficients of $E_{k,1}$, the Cohen numbers, done in [?].

Theorem 2.4

$$J^p_{\chi,m}(\Gamma_0(p), \mathbb{Z}^2, \mathcal{O}_\mathfrak{p}) \simeq J^p_{\chi,m}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathcal{O}_\mathfrak{p})$$

Proof: Let $\Phi \in J^p_{\chi,m}(\Gamma_0(p), \mathbb{Z}^2, \mathcal{O}_\mathfrak{p})$. We will show that Φ belongs to the closure of $J^p_{\chi,m}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathcal{O}_\mathfrak{p})$. That will imply the theorem.

Let

$$\Phi_j = tr(\Phi g^{p^j})$$

where $g = E_a - p^a E_a(q^p)$ (here E_a is the standard Eisenstein series and $(p-1)|a$) and $tr\Psi \in J_{k,m}(SL_2(\mathbb{Z}), \mathbb{Z}^2, \mathcal{O}^\mathfrak{p})$ if $\Psi \in J_{k,m}(\Gamma_0(p), \mathbb{Z}^2, \mathcal{O}^\mathfrak{p})$ is given by the formula $tr\Psi = \sum_{\gamma\in\Gamma_0(p)\backslash SL_2(\mathbb{Z})} \Psi|\gamma$. Then $\lim_j \Phi_j = \Phi$. For proving this, let us find a more explicit trace formula. If

$$\begin{aligned} \gamma_l &= \begin{pmatrix} 0 & -1 \\ 1 & l \end{pmatrix} = S \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \quad ,1 \le l \le p \\ \gamma_{p+1} &= I \end{aligned}$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$tr\Phi = \Phi + \sum_{l=1}^{p} \Phi|\gamma_l = \Phi + \sum_{l=1}^{p} (\Phi|S)(\tau + l, z)$$

Using the Fourier expansion

$$\Phi|S = \sum_{n\geq 0,r} b(n,r) q^{n/p} \zeta^r,$$

we have

$$\begin{aligned} tr\Phi &= \Phi + \sum_{n\geq 0,r} \sum_{l} b(n,r)q^{n/p}\zeta^r e(nl/p) \\ &= \Phi + p \sum_{n\equiv 0 mod p,r} b(n,r)q^{n/p}\zeta^r \\ &= \Phi + \sum_{n\geq 0,r} b(np,r)q^n\zeta^r \\ &= \Phi + p\Phi|S|U_\tau \end{aligned} \tag{3}$$

where $U_\tau \sum_{n,r} a(n,r)q^n\zeta^r = \sum_{n,r} a(np,r)q^n\zeta^r$.
Now let us prove that $\lim_j \Phi_j = \Phi$.

$$\Phi_j - \Phi = tr(\Phi g^{p^j}) - \Phi g^{p^j} + \Phi(g^{p^j} - 1) \tag{4}$$

Recalling the definition of g, we see that

$$g \equiv 1 \pmod p$$

Therefore, the second term in (??) tends to 0. It is easy to see that $g|S = E_a - E_a(\tau/p)$, so we also have

$$g|S \equiv 0 \pmod p$$

We still need to establish that the first term in (??) tends to 0. If v_p is a p-adic valuation in $\mathcal{O}_p$ normalized in order to satisfy $v_p(p) = 1$,

$$\begin{aligned} v_p\left(tr(\Phi g^{p^j}) - \Phi g^{p^j}\right) &= v_p\left(p(\Phi g^{p^j})|S|U_\tau\right) \qquad \text{, by (??)} \\ &\geq v_p\left(p(\Phi g^{p^j})|S\right) \\ &= 1 + v_p(\Phi|S) + p^j v_p(g|S) \end{aligned}$$

Since $v_p(g|S) > 0$, this valuation approaches ∞ and the second term in (??) tends to 0. This ends the proof.

3 The p-adic measure associated to a Jacobi form

We keep the same notation as before.

In this section, we are going to associate to every $\Phi \in J_{k,m}(\Gamma, L, \mathcal{O}^p)$ a p-adic measure μ_Φ on $\mathbf{Z}_p$ with values in $\mathbf{M}^p(\Gamma, \mathcal{O}_p)$, the p-adic ring of

Katz's p-adic modular forms. The idea behind the definition is as follows. If $\Phi = \sum_{n,r} c(n,r)q^n\zeta^r$, and we evaluate Φ at any root of unity $\zeta \in \mathbb{C}$, we obtain a modular form (in principle of an increased level; see [?, Theorem 1.3]). Moreover, the collection of $2m+1$ evaluations of Φ at different roots of unity characterize Φ. Therefore, taking ζ to be an indeterminate in μ_{p^∞}, the group of roots of unity of order a power of p, still preserves all the information about Φ. One way to formalize this is to interpret Φ as the measure μ_Φ on $\mathbf{Z}_p$ whose Fourier transform is the power series in X:

$$\begin{aligned} \mu_\Phi &= \sum_{l\geq 0}\left(\sum_{n,r}\binom{r}{l} c(n,r)q^n\right)X^l \qquad (5) \\ &= \sum_{n,r} c(n,r)q^n T^r \end{aligned}$$

where $T = X+1$. (Recall that for given n and l, $\sum_{n,r}\binom{r}{l}c(n,r)$ is a finite sum.)

The next theorem states the precise result.

Theorem 3.1 *Let $\Phi = \sum_{n,r} c(n,r)q^n\zeta^r \in J_{k,m}(\Gamma, L, \mathcal{O}^p)$. Then the power series*

$$\sum_{l\geq 0}\left(\sum_{n,r}\binom{r}{l} c(n,r)q^n\right)X^l \qquad (6)$$

where $\binom{r}{l} = (-1)^l\binom{l-r-1}{-r-1}$ if $r<0$, is the Fourier transform of the measure on $\mathbf{Z}_p$ with values in $\mathbf{M}^p(\Gamma, \mathcal{O}_p)$ whose j-moment is

$$m_j = \left(\zeta\frac{d}{d\zeta}\right)^j \Phi|_{\zeta=1} = \left(\frac{d}{2\pi i dz}\right)^j \Phi|_{z=0} \qquad (7)$$

Moreover, the association $\Phi \to \mu_\Phi$ is one to one.

Proof: Let us show that the m_j's defined in (??) are the moments of a measure. Notice that the Fourier expansion of m_j is

$$m_j = \sum_{n,r} r^j c(n,r)q^n \qquad (8)$$

and that

$$\Phi = \sum_{j\geq 0} m_j \frac{(2\pi i z)^j}{j!}.$$

We first prove that $m_j \in \mathbf{M}^p_{k+j}(\Gamma, \mathcal{O}^{\mathfrak{p}})$. One nice way to see this, while at the same time introducing a useful technique, is to show that, if $\tau = x + iy$,

$$e^{\frac{\pi m z^2}{y}} \Phi|_k \gamma = e^{\frac{\pi m z^2}{y}} \Phi \quad , \quad \gamma \in \Gamma$$

where $f(\tau, z)|_k \gamma = (c\tau + d)^{-k} f(\gamma\tau, \frac{z}{c\tau+d})$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This simple computation is left to the reader. This says that, if we write

$$e^{\frac{\pi m z^2}{y}} \Phi = \sum_{j \geq 0} f_j(\tau) \frac{(2\pi i z)^j}{j!}$$

then f_j is a nearly holomorphic -in the sense of [?]- modular form for Γ of weight $k + j$, with coefficients in $\mathcal{O}_{\mathfrak{p}}$. The powers of $\frac{1}{y}$ in each f_j are bounded. Also,

$$f_j = m_j(q) + \frac{1}{y} * .$$

It is a general fact that in such situation, m_j is a p-adic modular form. (Write the Maass-Weil operator $W = q\frac{d}{dq} - \frac{weight}{4\pi y}$. If we replace in f_j the action of W by the action of $q\frac{d}{dq}$, we are left with m_j. On the other hand, being nearly holomorphic forms the closure of modular forms acted on by W, we obtain a form belonging to the closure of modular forms acted on by $q\frac{d}{dq}$, which is known to be a p-adic operator. For a more rigorous exposition, see [?, ?].)

So $m_j \in \mathbf{M}^p_{k+j}(\Gamma, \mathcal{O}_{\mathfrak{p}})$. What follows is a sketch of a standard argument that can be seen in [?, ?, ?]. If we write $\begin{pmatrix} x \\ l \end{pmatrix} = \sum_{j=0}^{l} a_{j,l} x^j$, $a_{j,l} \in \mathbb{Q}$, then the l-coefficient in (??) satisfies

$$\sum_{n,r} \begin{pmatrix} r \\ l \end{pmatrix} c(n,r) q^n = \sum_{j=0}^{l} a_{j,l} m_j$$

and hence belongs to $\mathbf{M}^p(\Gamma, \mathcal{O}_{\mathfrak{p}}) \otimes \mathbb{Q}$; but its q-expansion at ∞ has integral coefficients. We deduce that the l-coefficient of (??) belongs to $\mathbf{M}^p(\Gamma, \mathcal{O}_{\mathfrak{p}})$. Therefore (??) is the Fourier transform of a measure μ_Φ on $\mathbb{Z}_p$ with values in $\mathbf{M}^p(\Gamma, \mathcal{O}_{\mathfrak{p}})$. Its l-moment can be computed by using $T = X - 1$:

$$m_l = \left((X+1)\frac{d}{dX}\right)^l \mu_\Phi|_{X=0} = \left(T\frac{d}{dT}\right)^l \mu_\Phi|_{T=1}.$$

Look at the Fourier expansion you obtain for the moments of μ_Φ by performing this operation to (??); it coincides with the Fourier expansion of m_l in (??).

Finally, we can read off the injectivity of $\Phi \mapsto \mu_\Phi$ from the explicit Fourier expansions for Φ and μ_Φ in (??).

This concludes the proof.

The measures obtained from Jacobi forms via the theorem satisfy the following properties.

- If $\Phi \in J_{k,m}(\Gamma, L, \mathcal{O}^p)$ then m_0, the 0-moment of μ_Φ, belongs to $\mathbf{M}_k^p(\Gamma, \mathcal{O}_\wp)$. Also, m_l, the l-moment of μ_Φ, belongs to $\mathbf{M}_{k+l}^p(\Gamma, \mathcal{O}_\wp)$ for every $l \geq 0$. Hence, we can learn the weight of the original Jacobi form from any of its nonzero moments m_l.

- $\int_{\mathbf{Z}_p} \zeta^x d\mu_\Phi(x) = \Phi(q, \zeta)$ for every $\zeta \in \mu_{p^\infty}$. In fact, $2m+1$ of these values characterize $\Phi \in J_{k,m}(\Gamma, L, \mathcal{O}^p)$. Equivalently, $2m+1$ moments of μ_Φ characterize Φ. This property can probably be restated in a more suitable way for learning what the index m of the original Jacobi form is from its associated measure.

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