Hölder gradient estimates for fully nonlinear elliptic equations.

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1. Introduction.

Local Hölder gradient estimates for classical solutions of fully nonlinear, second order elliptic equations of the form,

$$F[u] = F(x,u,Du,D^2u) = 0 , \tag{1.1}$$

were established by the author [12] as an extension of the corresponding estimates of Ladyzhenskaya and Ural'tseva for the quasilinear case, (see [7], Chapter 8, [1] Chapter 13). In this paper we derive such estimates for weak solutions in the "viscosity" sense of P-L Lions [10], also under reduced smoothness hypotheses of the function F with respect to the independent variables. In a sequel we permit a more general dependence on the gradient variables, enabling us to resolve an outstanding problem concerning obstacles for fully nonlinear equations under natural structure conditions.

To get some grip on the viscosity solutions, we employ the fundamental approximations of Jensen [2]. Our subsequent analysis involves subtle techniques with difference quotients, which had previously been developed by the author in other contexts ([9], [13]). We also establish local boundary estimates for viscosity solutions of the Dirichlet problem (Lemma 3.1, Theorem 3.2) although for these, the method originating in Krylov [4], goes through with only minor modification.

The function F in (1.1) is assumed defined on the set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$, where Ω denotes a domain in Euclidean n-space, $\mathbb{R}^n$, and $\mathcal{S}^n$ denotes the linear space of real, $n \times n$, symmetric matrices. Introducing, for any positive constants K_0, K_1, the subsets,

$$\Gamma_{0,1} = \{(x,z,p,r) \in \Gamma \mid |z| \le K_0, \ |p| \le K_1\} ,$$

we subject the function F to the following structural conditions:

F1. (Uniform ellipticity).

$$\lambda \text{ trace } \eta \le F(x,z,p,r+\eta) - F(x,z,p,r) \le \Lambda \text{ trace } \eta ;$$

F2. $$|F(x,z,p,0)| \le \mu_0 ;$$

F3. $$|F(x,z,p,r) - F(y,t,q,r)| \le \mu_1(|x-y|^\gamma + |z-t|^\gamma)|r| + \mu_2 ,$$

for all (x,z,p,r), $(y,t,q,r) \in \Gamma_{0,1}$, $\eta \ge 0, \in \mathcal{S}^n$, for any $K_0, K_1 > 0$, where $\lambda, \Lambda, \mu_0, \mu_1, \mu_2, \gamma$ are positive constants, depending possibly on K_0, K_1.

We observe that F1 implies the Lipschitz continuity of F with respect to r and hence may be equivalently expressed as

$$\lambda I \leq F_r \leq \Lambda I . \tag{1.2}$$

We also point out that for classical solutions, (or at least solutions in the Sobolev space $W^{2,n}(\Omega)$) , we shall be able to permit μ_0 and μ_2 to lie in certain L^p spaces, (Remark 3 (ii)). The structure conditions, F1, F2, F3, may be usefully viewed in conjunction with the following example, namely the Isaac's equation from stochastic game theory. Let $\{L_{\alpha\beta}\}$ be a family of linear operators, indexed by two parameters $\alpha \in A$, $\beta \in B$ and given by

$$L_{\alpha\beta}u = a^{ij}_{\alpha\beta}D_{ij}u + b^i_{\alpha\beta}D_iu + c_{\alpha\beta}u \ , \tag{1.3}$$

where $a^{ij}_{\alpha\beta}$, $b^i_{\alpha\beta}$, $c_{\alpha\beta}$, $f_{\alpha\beta}$, $i,j = 1, .. n$, $\alpha \in A$, $\beta \in B$ are real functions on Ω . The corresponding Isaac's equation,

$$F[u] = \inf_{\alpha \in A} \sup_{\beta \in B} (L_{\alpha\beta}u - f_{\alpha\beta}) = 0 \ , \tag{1.4}$$

will satisfy F1 to F3 if the operators $L_{\alpha\beta}$ are <u>uniformly elliptic</u> with respect to α and β , that is,

$$\lambda I \leq [a^{ij}_{\alpha\beta}] \leq \Lambda I \tag{1.5}$$

for all $\alpha \in A$, $\beta \in B$ for fixed positive constants λ , Λ , and the coefficients $a^{ij}_{\alpha\beta} \in C^{\gamma}(\bar{\Omega})$, $b^{i}_{\alpha\beta}$, $c_{\alpha\beta}$, $f_{\alpha\beta} \in L^{\infty}(\Omega)$ with norms bounded independently of α and β . When either of the sets A or B are singletons, we obtain the Bellman equations of stochastic control theory. The regularity results of this paper can be used to establish the existence of continuously differentiable solutions of equation (1.4).

Unless otherwise indicated, all notation in this paper follows the book [1].

2. Interior estimates

In order to formulate our estimates, we first define the notion of viscosity solution for equation (1.1). Let u be a continuous function in Ω . The second superdifferential of u at a point x in Ω is defined by

$$(2.1)\quad D_+^{1,2}u(x) = \{(p,r) \in \mathbb{R}^n \times \mathcal{S}^n \,|\, u(x+y) \le u(x)+p\cdot y+ry\cdot y+o(|y|^2)\}$$

while the corresponding subdifferential is defined by

$$(2.2)\quad D_-^{1,2}u(x) = \{(p,r) \in \mathbb{R}^n \times \mathcal{S}^n \,|\, u(x+y) \ge u(x)+p\cdot y+ry\cdot y+o(|y|^2)\} .$$

We then call u a viscosity subsolution of (1.1), (or say that u satisfies $F[u] \ge 0$ in the viscosity sense), if

$$(2.3)\quad F(x,u(x),p,r) \ge 0 \quad \text{for all} \quad (p,r) \in D_+^{1,2}u(x),\ x \in \Omega ,$$

and a viscosity supersolution of (1.1), (or u satisfies $F[u] \le 0$ in the viscosity sense), if

$$(2.4)\quad F(x,u(x),p,r) \le 0 \quad \text{for all} \quad (p,r) \in D_-^{1,2}u(x),\ x \in \Omega .$$

The function u is a viscosity solution of (1.1) if it is both a viscosity subsolution and supersolution. Some basic properties of viscosity solutions are treated in the papers [2], [10] and [15].

We can now state the following interior estimate and regularity assertion.

Theorem 2.1. Let u be a uniformly Lipschitz continuous viscosity solution of equation (1.1) in the domain Ω, where F satisfies the structure conditions, F1, F2 and F3. Then u possesses Hölder continuous first derivatives in Ω and for any subdomain $\Omega' \subset\subset \Omega$, we have the estimate,

$$[Du]_{\alpha;\Omega'} \leq C(1+\delta^{-\alpha}|Du|_0) , \tag{2.5}$$

where α is a positive constant depending only on $n, \Lambda/\lambda$ and γ while C depends also on μ_0/λ, μ_1/λ, μ_2/λ, diam Ω and $|u|_{1;\Omega}$, and $\delta = \text{dist}\,(\Omega',\partial\Omega)$.

Proof of Theorem 2.1.

We first observe that we can, without loss of generality, restrict attention to functions F that are independent of z. Let us recall some basic properties of the Jensen approximations ([2], [15]). Setting, for $\varepsilon > 0$,

$$\Omega_\varepsilon = \{x \in \Omega \mid \text{dist}\,(x,\partial\Omega) > \varepsilon\} ,$$

we define two functions $u_\varepsilon^{\pm} \in C^{0,1}(\Omega_\varepsilon)$ whose graphs have fixed distance ε from the graph of u and which lie respectively above and below the graph of u.

It follows that

$$(2.6) \qquad |Du_\varepsilon^\pm| \le |Du|_0 \quad \text{and}$$

$$\pm D^2 u_\varepsilon^\pm \ge - \frac{(1+|Du|_0^2)^{3/2}}{\varepsilon}$$

in the sense of distributions. Accordingly the functions $u_\varepsilon^\pm$ possess second differentials almost everywhere in Ω_ε and moreover at any point x of twice differentiability,

$$(2.7) \qquad (Du_\varepsilon^\pm,\ D^2u_\varepsilon^\pm) \in D_\pm^{2,1}\ u(x\pm\varepsilon\nu^\pm) \ ,$$

where

$$\nu^\pm = \frac{Du_\varepsilon^\pm}{\sqrt{1+|Du_\varepsilon^\pm|^2}}$$

Consequently if $u \in C^{0,1}(\bar\Omega)$ is a viscosity subsolution (or supersolution) of (1.1) we have the differential inequalities,

$$(2.8) \qquad \pm F(x\pm\varepsilon\nu^\pm,\ Du_\varepsilon^\pm(x),\ D^2u_\varepsilon^\pm(x)) \ge 0$$

almost everywhere in Ω .

We shall approach the estimation of first derivatives, through the approximating difference quotients, (2.9) $$v_\varepsilon(x,\xi) = \frac{1}{h}\{u_\varepsilon^+(x+h\xi) - u_\varepsilon^-(x)\} \quad , \quad h > 0 \ ,$$

which we shall regard as functions of 2n variables on the domain $\Omega' \times \mathbb{R}^n$ where $\Omega' = \Omega_{\varepsilon+h}$. In particular we observe that

(2.10) $$D_{\xi_i} v_\varepsilon(x,\xi) = D_i u_\varepsilon^+(x+h\xi) \ ,$$

$$D_{\xi_i\xi_j} v_\varepsilon(x,\varphi) = hD_{ij} u_\varepsilon^+(x+h\xi) \ .$$

Using the inequalities (2.8) we now obtain

(2.11) $$F(x-\varepsilon\nu^-, Du_\varepsilon^-(x), D^2u_\varepsilon^+(x+h\xi))$$

$$- F(x-\varepsilon\nu^-, Du_\varepsilon^-(x), D^2u_\varepsilon^-(x))$$

$$\geq F(x-\varepsilon\nu^-, Du_\varepsilon^-(x), D^2u_\varepsilon^+(x+h\xi))$$

$$- F(x+h\xi+\varepsilon\nu^+, Du_\varepsilon^+(x+h\xi), D^2u_\varepsilon^+(x+h\xi)) \ ,$$

so that, writing

$$a^{ij}(x) = \int_0^1 F_{r_{ij}}(x-\varepsilon\nu^-, Du_\varepsilon^-(x), D^2W_t(x))\, dt$$

where

$$W_t = tu_\varepsilon^+(x+h\xi) + (1-t)\ u_\varepsilon^-(x) \ ,$$

and using the structural condition F3 , we have

$$(2.12)\quad Lv_\varepsilon \equiv a^{ij}D_{ij}v_\varepsilon$$

$$\geq -h^{-1}\{\mu_1(h+2\varepsilon)^\gamma|D^2u_\varepsilon^+(x+h\xi)| + \mu_2\}$$

$$\geq -2^\gamma h^{\gamma-1}\mu_1|D^2u_\varepsilon^+(x+h\xi)| - h^{-1}\mu_2 \ , \ (\text{if} \quad 2\varepsilon \leq h) \ ,$$

$$\geq -2h^{\gamma-1}\mu_1\sigma^{ij}D_{ij}u_\varepsilon^+(x+h\xi) - h^{-1}\mu_2$$

$$= -2h^{\gamma-1}\mu_1\sigma^{ij}D_{i\xi_j}v_\varepsilon - h^{-1}\mu_2$$

by (2.10), where

$$\sigma^{ij} = \begin{cases} \dfrac{D_{ij}u_\varepsilon^+}{|D^2u_\varepsilon^+|} & \text{if} \quad D^2u_\varepsilon^+ \neq 0 \ , \\ 0 & \text{if} \quad D^2u_\varepsilon^+ = 0 \ . \end{cases}$$

Consequently we obtain a differential inequality in both x and ξ , namely

$$(2.13)\quad a^{ij}D_{ij}v_\varepsilon - 2h^{\gamma-1}\mu_1\sigma^{ij}D_{i\xi_j}v_\varepsilon \geq -h^{-1}\mu_2 \ .$$

The inequality (2.13) can be made uniformly elliptic by addition of a suitable elliptic inequality in the ξ variables. But first, to simplify matters, let us rescale ξ through a transformation

(2.14) $\xi \longmapsto 2h^{\gamma-1}\mu_1\lambda^{-1}\xi$

to give in place of (2.13),

(2.15) $a^{ij}D_{ij}v_\varepsilon - \lambda\sigma^{ij}D_{i\xi_j}v_\varepsilon \geq -h^{-1}\mu_2$.

Next we write the differential inequality (2.8) for u_ε^+ in the form,

$$F(x+\varepsilon\nu^+,\ Du_\varepsilon^+,\ D^2u_\varepsilon^+) - F(x+\varepsilon\nu^+,\ Du_\varepsilon^+,\ 0)$$

$$\geq - F(x+\varepsilon\nu^+,\ Du_\varepsilon^+,\ 0)\ ,$$

so that by (2.10), (2.14) and F2 ,

(2.16) $\alpha^{ij}D_{\xi_i\xi_j}v_\varepsilon = 4h^{2\gamma-1}(\mu_1/\lambda)^2\alpha^{ij}D_{ij}u_\varepsilon^+(x+h\xi)$

$$\geq -4h^{2\gamma-1}\mu_0(\mu_1/\lambda)^2\ ,$$

where the coefficients α^{ij} are given by

$$\alpha^{ij}(x) = \int_0^1 F_{r_{ij}}(x+\varepsilon\nu^+,\ Du_\varepsilon^+,\ tD^2u_\varepsilon^+)\,dt\ .$$

Therefore, we now have

$$(2.17)\quad \tilde{L}v_\varepsilon \equiv a^{ij}D_{ij}v_\varepsilon + \lambda\sigma^{ij}D_{i\xi_j}v_\varepsilon + \alpha^{ij}D_{\xi_i\xi_j}v_\varepsilon$$

$$\geq -4h^{2\gamma-1}\mu_0(\mu_1/\lambda)^2 - h^{-1}\mu_2$$

$$\equiv \mu_3 ,$$

with $\tilde{L}$ uniformly elliptic in $\Omega' \times \mathbb{R}^n$.

We are now essentially ready to invoke the <u>weak Harnack inequality</u> for non-negative supersolutions ([11], Theorem 2, [1] Theorem 9.22). To do this, we first fix points $x_0 \in \Omega$, $\xi_0 \in \mathbb{R}^n$ with $|\xi_0| = 1$ and let $R < 1/3\ \mathrm{dist}(x_0,\partial\Omega)$, $h < R/2$. Denoting

$$\tilde{B}_R = B_R(x_0) \times B_R(\xi_0) ,$$

we then set

$$(2.18)\quad \tilde{M}_2 = \sup_{\tilde{B}_{2R}} v_\varepsilon , \quad \tilde{M}_1 = \sup_{\tilde{B}_R} v_\varepsilon$$

and apply the weak Harnack inequality to the function $w = \tilde{M}_2 - v_\varepsilon$ in the set $\tilde{B}_{2R}$. We mention here that although w does not necessarily lie in the Sobolev space $W^{2,2n}(\tilde{B}_{2R})$, the weak Harnack inequality is still applicable by virtue of the upper bound on D^2w resulting from (2.6), [15]. Consequently

$$(2.19)\quad \Big(R^{-2n}\int_{\tilde{B}_R}(\tilde{M}_2-v_\varepsilon)^p\Big)^{1/p} \leq C(\tilde{M}_2 - \tilde{M}_1 + \frac{1}{\lambda}\mu_3R^2)$$

where p and C are positive constants depending only on n, Λ/λ . At this stage we can let $\varepsilon \to 0$, thereby obtaining (2.19) for the difference quotient,

$$v(x,\xi) = v_0(x,\xi) = \frac{1}{h}\{u(x+h\xi) - u(x)\} .$$

By replacing u by $-u$, we see that (2.19) also holds for the function $-v$ and addition of the two inequalities yields the oscillation estimate,

$$\text{(2.20)} \quad \operatorname*{osc}_{\tilde{B}_R} v \le \chi \operatorname*{osc}_{\tilde{B}_{2R}} v + \frac{1}{\lambda}\mu_3 R^2$$

where $\chi = 1-C^{-1}$. Next using (2.6) and (2.14), we can reduce (2.20) to an estimate in the x variables only, namely

$$\text{(2.21)} \quad \operatorname*{osc}_{B_R} v(x,\xi_0) \le \chi \operatorname*{osc}_{B_{2R}} v(x,\xi_0) + \frac{1}{\lambda}(12h^{\gamma-1}\mu_1 R|Du|_0 + \mu_3 R^2) ,$$

and hence we obtain, for any $R < R_0$, $\mathrm{dist}(x_0,\partial\Omega) < R_0/2$, $h \le R$, the Hölder estimate

$$\text{(2.22)} \quad \operatorname*{osc}_{B_R} v \le C\left(\frac{R}{R_0}\right)^{\alpha}\{\operatorname*{osc}_{B_{R_0}} v + \frac{1}{\lambda}[\mu_0(\mu_1/\lambda)^2 h^{2\gamma-1}R_0^2 + \mu_1 h^{\gamma-1}R_0|Du|_0 + \mu_2 h^{-1}R_0^2]\}$$

where C and α depend on n and Λ/λ . Of course we cannot send h to zero in (2.22) but we can proceed with the aid of the following trick. Fix a subdomain $\Omega' \subset\subset \Omega$ and scale x

so that $\delta = \text{dist}(\Omega', \partial\Omega) = 2$. Choosing in (2.22), $R = h < 1$ $R_0 = h^\beta$, for some $\beta < 1$, we obtain

$$\frac{1}{h}\{u(x_0 + h\xi_0) + u(x_0 - h\xi_0) - 2u(x_0)\}$$

$$\leq C\, h^{\alpha(1-\beta)}\{|Du|_0 + \frac{1}{\lambda}[\mu_0(\mu_1/\lambda)^2 h^{2(\gamma+\beta)-1}$$

$$+ \mu_1 h^{\gamma+\beta-1}|Du|_0 + \mu_2 h^{2\beta-1}]\}$$

so that, with $\beta + \gamma = 1$, $\gamma \leq 1/2$ we have

$$\text{(2.23)} \quad \frac{1}{h^{1+\alpha\gamma}}\{u(x_0+h\xi_0) + u(x_0-h\xi_0) - 2u(x_0)\}$$

$$\leq C\{|Du|_0 + \frac{1}{\lambda}[\mu_0(\mu_1/\lambda)^2 + \mu_1|Du|_0 + \mu_2]\}$$

for all $h \leq 1$, $x_0 \in \Omega'$, $|\xi_0| = 1$. Consequently $u \in C^{1,\alpha\gamma}(\Omega)$ and eliminating the normalization $\delta = 1$, we obtain the estimate

$$\text{(2.24)} \quad [Du]_{\alpha\gamma;\Omega'} \leq C\,\delta^{-\alpha\gamma}\{(1+\mu_1\delta^\gamma/\lambda)|Du|_0$$

$$+ \frac{1}{\lambda}(\mu_0\mu_1^2\delta^{1+2\gamma} + \mu_2\delta)\}$$

from which (2.5) follows. ||

<u>Remarks</u>. (i) If condition F3 is strenghtened to the Lipschitz condition,

$$(2.25)\quad |F(x,z,p,r)-F(y,t,q,r)| \le \mu_1(|x-y|+|z-t|)|r| + \mu_2(|x-y|+|z-t|+|p-q|) ,$$

then we can send h to zero in the proof of Theorem 2.1, with the effect that the constant C in the estimate (2.5) will be independent of μ_0.

(ii) If the solution u lies in the Sobolev space $W^{2,n}(\Omega)$, there is no need for the approximations $u_\varepsilon^\pm$, whence the functions μ_0 and μ_2 can be permitted to lie in certain L^p spaces. In particular we can allow $\mu_0 \in L^n(\Omega)$, while setting $h = x-y$, $\mu_2 = \mu_2(x,h)$ we can assume $\mu_2 \in L^\infty\{L^q(\Omega);\ |h| \le h_0\}$ for some $q > n$ and $h_0 > 0$. The estimate (2.5) then depends on $\|\mu_0\|_n$, $\|\mu_2\|_{q,\infty}$, h_0 instead of μ_0 and μ_2.

(iii) If in conditions F1, F2, F3, the quantities λ,Λ,γ are independent of K_1, while

$$(2.26)\quad \mu_0 \le \mu_0'(1+K_1^2),\ \mu_1 \le \mu_1'(1+K_1^\gamma),\ \mu_2 \le \mu_2'(1+K_1^2)$$

for constants μ_0', μ_1', μ_2' depending only on K_0, then by interpolation ([14], Lemma 1) we can conclude an interior bound for the gradient of u, namely

$$(2.27)\quad |Du|_{0;\Omega'} \le C(1+\delta^{-1}|u|_0)$$

where C depends on n, Λ/λ, γ, μ_0'/λ, μ_1'/λ, μ_2'/λ, diam Ω and $|u|_{0;\Omega}$. For this we need also the local Hölder estimate [15] but for the Isaac's equation (1.4) where we have a linear structure,

(2.28) $\mu_0 \leq \mu_0'(1+K_1)$, $\mu_1 \leq \mu_1'$, $\mu_2 \leq \mu_2'(1+K_1)$,

the estimate (2.27) follows by the standard Hölder interpolation, ([1], Lemma 6.32).

3. Boundary estimates

As indicated in [15], certain pointwise estimates for classical solutions of (1.1), such as the Harnack and Hölder estimate of Krylov and Safonov [6], (see [1], Chapter 9), carry over to viscosity solutions. This can be demonstrated by writing the inequalities (2.8) to the form,

$$(3.1)\qquad \pm\, \alpha_{\pm}^{ij} D_{ij} u_{\varepsilon}^{\pm} \geq -F(x \pm \varepsilon\nu^{\pm},\ Du_{\varepsilon}^{\pm}(x),\ 0)$$

$$\geq -\mu_0$$

by F2 , where the coefficients α^{ij} , given by

$$\alpha_{\pm}^{ij}(x) = \int_0^1 F_{r_{ij}}(x \pm \varepsilon\nu^{\pm},\ Du_{\varepsilon}^{\pm}(x),\ t\, D^2 u_{\varepsilon}^{\pm}(v))\,dt \ ,$$

satisfy the uniform elliptic condition,

$$(3.2)\qquad \lambda I \leq [\alpha_{\pm}^{ij}] \leq \Lambda I \ ,$$

by F1 . In particular we infer the following local boundary estimate, from the proof of the corresponding result for classical solutions, due to Krylov [4], [5] (see also [8], [13]).

Lemma 3.1. Let $u \in C^{0,1}(\bar{\Omega})$ be a viscosity solution of equation (1.1) where F satisfies the structure conditions

F1 and F2 . Let T be an open $C^{1,\gamma}$ boundary portion of $\partial\Omega$ with $u = \varphi$ on T for some $\varphi \in C^{1,\gamma}(\Omega \cup T)$, $0 < \gamma \leq 1$. Then for any $x_0 \in T$, $R \leq R_0 < \mathrm{dist}(x_0, \partial\Omega - T)$, the function v given by

$$(3.3) \qquad v(x) = \frac{u(x)-\varphi(x)}{\mathrm{dist}(x,T)}$$

satisfies the estimate

$$(3.4) \qquad \underset{\Omega\cap B_R}{\mathrm{osc}}\, v \leq C \left(\frac{R}{R_0}\right)^{\alpha} \left\{ \underset{\Omega\cap B_{R_0}}{\mathrm{osc}}\, v + R_0^{\gamma}[Dg]_{\gamma} + R_0\mu_0 \right\}$$

where $\alpha > 0$ depends on $n, \Lambda/\lambda$ and γ and C depends also on T .

Lemma 3.1 provides a Hölder estimate for the normal derivative of v restricted to T . By combination of Theorem 2.1 and Lemma 3.1, we arrive at the following global estimate.

Theorem 3.2. Let u be a $C^{0,1}(\bar{\Omega})$ viscosity solution of the Dirichlet problem.

$$(3.5) \qquad F[u] = 0 \text{ in } \Omega \text{ , } u = \varphi \text{ on } \partial\Omega \text{ ,}$$

where F satisfies the structural conditions F1, F2, F3 and $\partial\Omega \in C^{1,\gamma}(\bar{\Omega})$. Then $u \in C^{1,\alpha}(\bar{\Omega})$ for some positive α depending only on n, Λ/λ and γ and we have the estimate

$$(3.6) \qquad [Du]_{\alpha;\Omega} \leq C \text{ ,}$$

where C depends on n, Λ/λ, γ, μ_0/λ, μ_1/λ, μ_2/λ, $|u|_{1;\Omega}$ and Ω .

Remarks. (i) An examination of the form of the estimate (2.24) and (3.1) shows that we can replace μ_0, μ_1, μ_2 in conditions F2, F3 by $\mu_0 d^{\gamma-1}$, μ_1 by $\mu_1 d^{-\gamma}$ and μ_2 by $\mu_2 d^{-1}$ where $d = \mathrm{dist}(x, \partial\Omega)$.

(ii) From Remark (iii) of the proceeding section, if the constants μ_0, μ_1, μ_2 satisfy (2.26), we deduce a global gradient bound, so that

$$(3.7) \qquad |u|_{1,\gamma;\Omega} \leq C$$

where C depends on n, Λ/λ, γ, μ_0'/λ, μ_1'/λ, μ_2'/λ, $|u|_{0;\Omega}$ and Ω . Again for the case of the Isaac's equation, this can be deduced directly from (3.6) by the standard Hölder interpolation, ([1], Lemma 6.35).

(iii) It follows from [15], that if F1 holds, F is non-decreasing in z , and F3 is strenghtened so that

$$(3.8) \qquad |F(x,z,p,r) - F(y,t,q,r)| \leq \mu_1\{(|x-y|+|z-t|)|r|+|p-q|\} + \nu(|x-y|+|z-t|)$$

where $\nu(a) \to 0$ as $a \to 0$, then viscosity solutions of the Dirichlet problem (3.5) are unique.

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