

ELEMENTARY COMBINATORICS OF THE HOMFLYPT POLYNOMIAL

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ABSTRACT. We explore a state model for the HOMFLYPT polynomial. Reformulated in the language of Gauss diagrams it admits an elementary proof of existence of two HOMFLYPT polynomials for virtual string links which coincide in the case of classical links. In particular, for classical knots it gives a new simple combinatorial proof of existence of the HOMFLYPT polynomial. In the second part of the paper we obtain Gauss diagram formulas for Vassiliev invariants coming from the HOMFLYPT polynomial. These formulas are new already for invariants of degree 3.

INTRODUCTION

The *HOMFLYPT polynomial* $P(L)$ is an invariant of oriented link L . It is defined as the Laurent polynomial in two variables a and z with integer coefficients satisfying the following skein relation and the initial condition:

$$(1) \quad aP(\text{crossing}) - a^{-1}P(\text{crossing}) = zP(\text{two crossings}); \quad P(\text{circle}) = 1.$$

If L is an unlink with m components then $P(L) = \left(\frac{a-a^{-1}}{z}\right)^{m-1}$. The existence of such an invariant is a long and cumbersome theorem. It was established simultaneously and independently by five groups of authors [HOM, PT].

The proofs of [H, PT, LM] are based on the following scheme: for a fixed link diagram, an ordering of the components and a choice of base points is made. The polynomial $P(L)$ is then defined by a recursive use of the skein relation to transform the diagram to a descending form. The hardest part of the proof is then to show the independence of the choice of base points and the order. This is then used to prove the invariance of $P(L)$ under the Reidemeister moves.

An extension of this scheme to virtual links breaks down at the independence of the base points, since even for the simplest virtual knot diagram with two (classical) crossings the polynomial defined in this way depends on the base point. However, it turns out that (both in the classical and virtual case) it is possible to prove the invariance under the Reidemeister moves directly, with no induction and without any assumption of its independence on the ordering and the choice of basepoints.

In this paper we give a short and elementary proof of existence of two HOMFLYPT polynomials for ordered virtual links (which coincide in the classical case). As a byproduct, it implies the existence of the HOMFLYPT polynomial for long

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(and hence also for closed) classical knots. It enables us to avoid the difficulties with the independence of orderings in the approaches of [H, LM, PT].

The second part of the paper is devoted to Gauss diagram formulas for Vassiliev invariants coming from the HOMFLYPT polynomial. It is known [GPV] that any Vassiliev knot invariant may be presented by a Gauss diagram formula. This type of formulae is the simplest for computation purposes; however, the algorithm for producing them is complicated and until recently only few lower degree cases were described explicitly. The first description of such formulas for an infinite family of Vassiliev invariants was given in [CKR], where the coefficients of the Conway polynomial were considered. This paper generalizes the result of [CKR] to the HOMFLYPT polynomial.

We use a non-standard change of variables (used formerly in [G2]), leaving z alone and plugging in $a = e^h$ to obtain a power series $\sum_{k,l} p_{k,l} h^k z^l$. The coefficients $p_{k,l}$ are Vassiliev invariants of degree $\leq k + l$, see [G2]. We give the Gauss diagram formulas for $p_{k,l}$ for arbitrary k, l . These formulas are new already for invariants of degree 3.

The paper is organized in the following way. In Section 1 we start from the scheme of [H, LM, PT], extracting from it an explicit state model for the HOMFLYPT following [Ja] in Section 2. We then briefly review the notions of Gauss diagrams and virtual links in Section 3 and reformulate the state model in these terms in Section 4. Section 5 is dedicated to the formulation and proof of the invariance of $P(L)$ under the Reidemeister moves. The expansion of $P(L)$ into power series in h and z is considered in Section 6. In the same section we remind the definition of the Gauss diagram formulas for Vassiliev invariants. Finally, we describe the Gauss diagram formulas for $p_{k,l}$ in Section 7. In the last Section 8 we analyse low degree cases in details.

Note that using instead of (1) the skein relation for the two-variable Kauffman polynomial, one gets a similar state model. We plan to consider the resulting Gauss diagram formulas in a forthcoming paper.

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1. HOMFLYPT AND DESCENDING DIAGRAMS

The skein relation (1) allows one to calculate the HOMFLYPT polynomial of a link. Following [H, LM, PT], this can be done by ordering a link diagram and then transforming it into a descending diagram. We call a diagram D *ordered*, if its components D_1, D_2, \dots, D_m are ordered and on every component a (generic) base point is chosen. An ordered diagram is *descending*, if D_i is above D_j for all $i < j$ and if for every i as we go along D_i starting from its base point along the orientation we pass each self-crossing first on the overpass and then on the underpass.

An elementary step of the algorithm computing $P(L)$ consists of the following procedure. Suppose that D is an ordered diagram and that the subdiagram D_1, \dots, D_{i-1} is already descending. We go along D_i (starting from the base point) looking for the first crossing which fails to be descending. At such a crossing x we change it using the skein relation. Namely, depending on the sign ε (the local

writhe) of the crossing, we express $P(D)$ as

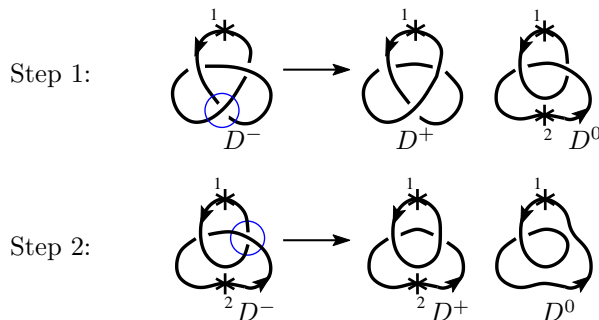
$$(2) \quad \begin{aligned} P(\text{crossing}) &= a^{-2}P(\text{crossing}) + a^{-1}zP(\text{two crossings}) \\ P(\text{crossing}) &= a^2P(\text{crossing}) - azP(\text{two crossings}) \end{aligned}$$

Denote the corresponding diagrams $D^\varepsilon, D^{-\varepsilon}, D^0$.

The ordering of $D = D^\varepsilon$ induces the ordering of $D^{-\varepsilon}$ (in an obvious way); the ordering of D^0 requires some explanation. If x was a crossing of D_i with $D_j, j > i$, then these two components merge into a single component D_i^0 of D^0 , with a base point being the base point of D_i . If x was a self-crossing of D_i , then D_i splits into two components: D_i^0 , which contains the base point of D_i , and D_{i+1}^0 , where we choose the base point in a neighborhood of x . In both cases the order of remaining components shifts accordingly.

The diagrams $D^{-\varepsilon}, D^0$ are “more” descending than D^ε . At the next step we apply the same procedure to each of them.

Example 1a. For the trefoil 3_1 the algorithm consists of two steps, illustrated in the figure below. The diagram D^+ appearing in the first step is already descending; the diagram D^0 is not, so the second step is needed to transform it.



$$\text{Hence } P(3_1) = a^2 \cdot 1 - az \left(a^2 \cdot \frac{a-a^{-1}}{z} - az \cdot 1 \right) = (2a^2 - a^4) + a^2z^2.$$







2. STATE MODEL REFORMULATION

The state model of [Ja] for the HOMFLYPT polynomial is a convenient reformulation of the algorithm of Section 1.

A *state* S on a link diagram D is a subset of its crossings. The HOMFLYPT polynomial is going to be a sum over the states. Let $D(S)$ be the link diagram obtained by smoothing every crossing in S according to orientation and $c(S)$ be the number of its components. We will not use the topology of $D(S)$, however its combinatorics will determine the contribution of the state S to the state sum. The contribution will be a product of a global weight of the state as a whole, $\left(\frac{a-a^{-1}}{z}\right)^{c(S)-1}$ and local weights of crossings of the diagram.

The ordering of D induces an ordering of $D(S)$ (in the way explained in Section 1 above) and thus determines a tracing of the link $D(S)$. The local weight $\langle x|D|S \rangle$ of a crossing x of D depends on the first passage of a neighborhood of x in the tracing and on the sign ε of x . Namely, if x is in S and we approach x first time on an overpass of D then $\langle x|D|S \rangle = 0$ (since such a situation does not occur in the

above algorithm). If we approach x on an underpass of D then $\langle x|D|S \rangle = \varepsilon a^{-\varepsilon} z$ (i.e., the coefficient of D^0 in (2)). In the case if x does not belong to S and we approach x first time on an overpass then $\langle x|D|S \rangle = 1$ (since in the above algorithm we do not apply the skein relation to x). If we approach x on an underpass then $\langle x|D|S \rangle = a^{-2\varepsilon}$ (i.e., the coefficient of $D^{-\varepsilon}$ in (2)). These assignments can be summarized in the following figure.

First passage:				
	0	$a^{-1}z$	1	a^{-2}
	$-az$	0	a^2	1

Denote by $\langle D|S \rangle := \prod_x \langle x|D|S \rangle$ the product of local weights of all crossings. For a link L with a diagram D we have [Ja, Proposition 2]:

$$P(L) = \sum_S \langle D|S \rangle \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}$$

Example 1b. Consider a based trefoil diagram D and a state S consisting of one crossing $\{x_1\}$.



The tracing of $D(S)$ first approaches the crossing x_1 on the strand which was an underpass in D . So its weight will be $\langle x_1|D|S \rangle = -az$. Similarly the weights of the other two crossings are $\langle x_2|D|S \rangle = a^2$ and $\langle x_3|D|S \rangle = 1$. So the total contribution from this state will be equal to $-a^3 z \left(\frac{a - a^{-1}}{z} \right) = a^2 - a^4$. The next table shows the contributions from all eight states. Non-zero weights come from states corresponding to descending diagrams appearing in the end of the algorithm, see Example 1a.

\emptyset	$\{x_1\}$	$\{x_2\}$	$\{x_3\}$	$\{x_1, x_2\}$	$\{x_1, x_3\}$	$\{x_2, x_3\}$	$\{x_1, x_2, x_3\}$
a^2	$a^2 - a^4$	0	0	$a^2 z^2$	0	0	0

So we recover the result of Example 1a: $P(3_1) = (2a^2 - a^4) + z^2 a^2$.

Remarks. 1. Smoothing a crossing from a state S changes the number of components by one. Hence the cardinality $|S|$ and the difference $m - c(S)$ (where k is the number of components of D) are congruent modulo 2. Therefore the HOMFLYPT polynomial $P(L)$ is even in each of the variables a and z if m is odd, and it is an odd polynomial if m is even.

2. The negative powers of z come from the factors $\left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}$. A smoothing of a crossing $x \in S$ may increase $c(S)$ by one, however this increment might be compensated by a local weight $\langle x|D|S \rangle$. As a consequence we have that the lowest power of z in the HOMFLYPT polynomial of L is equal to z^{-m+1} with some coefficient (depending on a). In particular the HOMFLYPT polynomial $P(K)$ of a knot K is a genuine polynomial in z , i.e., does not contain terms with negative powers of z .

3. GAUSS DIAGRAMS

Definition. Gauss diagrams provide an alternative and more combinatorial way to present links. For a link diagram D consider a collection of (counterclockwise) oriented circles parameterizing it. Two preimages of a crossing of D we unite in a pair and connect them by an arrow pointing from the overpassing preimage to the underpassing one. To each arrow we assign a sign ± 1 of the corresponding crossing. The result is called the *Gauss diagram* G_D of the link diagram D . A link can be uniquely reconstructed from the corresponding Gauss diagram [GPV].

For example, a Gauss diagram of the trefoil looks as follows.

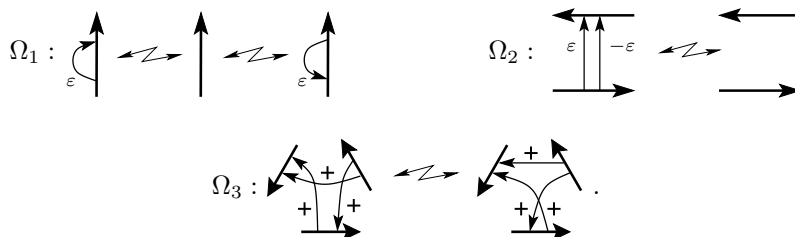


A knot and its Gauss diagram

Not every diagram with arrows is realizable as a Gauss diagram of a classical link. For example, is not realizable regardless of signs of its arrows. An *abstract Gauss diagram*, or an *arrow diagram* is a generalization of a notion of Gauss diagram, in which we forget about realizability. In other words, an arrow diagram consists of a number of oriented circles with several arrows connecting pairs of distinct points on them. The arrows are equipped with signs ± 1 . We consider these diagrams up to orientation preserving diffeomorphisms of the circles.

We are going work with *ordered Gauss diagrams*, i.e. Gauss diagrams with ordered circles and a base point $\star_1, \star_2, \dots, \star_m$ on each circle corresponding to an ordering of D . Similarly, an *ordered arrow diagram* is an arrow diagram equipped with an ordering of the circles and a base point (different from the end points of the arrows) on each of them.

Two Gauss diagrams represent isotopic links if and only if they are related by a finite number of Reidemeister moves (see, for example, [GPV, Öll, CDBooK]).



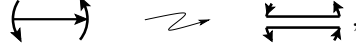
Note that the segments involved in Ω_2 or Ω_3 may lie on the different components of the link. So the order in which they are traced along the link may be arbitrary.

A *virtual link* [Ka2, GPV] is an abstract (not necessarily realizable) Gauss diagram considered modulo Reidemeister moves. Similarly, we define a *ordered virtual link* as an equivalence class of ordered abstract Gauss diagrams modulo Reidemeister moves which do not involve base points. In particular, virtual string links are ordered. For one-component links this notion is equivalent to the notion of long (virtual) knots. It is well known that for classical knots the theories of long and closed knots coincide, which however is not true for virtual knots.

4. STATE MODELS ON GAUSS DIAGRAMS

All notions and constructions of Section 2 have a straightforward translation to the language of Gauss diagrams.

A state S on an abstract Gauss diagram G is a subset of its arrows. Let $G(S)$ be the abstract Gauss diagram obtained by doubling every arrow in S as in the figure



and let $c(S)$ be the number of its circles. The ordering of G induces an ordering of $G(S)$. The local weight $\langle \alpha | G | S \rangle$ of an arrow α of G in general depends on whether α belongs to S , on the first passage in a neighborhood of α in the tracing of $G(S)$, and on the sign ε of α . Given a table of such local weights, we denote by $\langle G | S \rangle := \prod_{\alpha} \langle \alpha | G | S \rangle$ the product of local weights of all arrows and define a polynomial $P(G)$ by

$$(3) \quad P(G) := \sum_S \langle G | S \rangle \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}$$

The table of local weights for the HOMFLYPT state model (readily taken from Section 2) is shown below.

First passage:				
(4)	$a^{-1}z$	0	a^{-2}	1
	$-az$	0	a^2	1

Example 1c. For the Gauss diagram of the trefoil the states with non-zero weights are the following.

States of		:					
Weights :	$1 \cdot a^2 \cdot 1$:	$1 \cdot (-az) \cdot a^2 \cdot \left(\frac{a - a^{-1}}{z} \right)$		$1 \cdot (-az) \cdot (-az)$		$1 \cdot (-az) \cdot (-az)$

Hence, $P(G) = (2a^2 - a^4) + z^2a^2$.

5. HOMFLYPT FOR VIRTUAL LINKS

It turns out that directly - without any assumption about the existence of the HOMFLYPT polynomial - one may prove the following

Theorem 1. *The expression (3) is invariant under Reidemeister moves of ordered Gauss diagrams and thus defines an invariant of ordered virtual links.*

Remark. In particular, this theorem gives a direct elementary proof of an existence of the HOMFLYPT polynomial of (long, and thus also closed) classical knots. Here we diverge from the scheme of proof of [H, LM, PT]. Indeed, their proofs of invariance under Reidemeister moves are based on the independence of

the ordering. The later is the hardest part of the proof (and fails for virtual links). Thus we avoid the difficulties arising in the case of [H, LM, PT].

Corollary 1. *HOMFLYPT extends to an invariant of ordered virtual links. In particular, HOMFLYPT polynomial for long virtual knots is well-defined. Since one may use ascending diagrams instead of descending, there are two (descending and ascending) HOMFLYPT polynomials of ordered virtual links, which coincide on classical links.*

Proof. Let Gauss diagrams G and G' be related by an Ω_1 move, so that G' contains a new isolated arrow α . Every state S of G corresponds to two states of G' : S and $S \cup \alpha$. Depending on the orientation of α , their weights will be either $\langle G|S \rangle$ and 0, or $\varepsilon a^{-\varepsilon} z \frac{a-a^{-1}}{z} \langle G|S \rangle$ and $a^{-2\varepsilon} \langle G|S \rangle$. In both cases they sum up to $\langle G|S \rangle$ (since $\varepsilon a^{-\varepsilon} (a - a^{-1}) + a^{-2\varepsilon} = 1$).

If Gauss diagrams G and G' are related by an Ω_2 move, G' contains a pair of extra arrows α_1 and α_2 with signs ε and $-\varepsilon$. Every state S of G corresponds to four states of G' : S , $S \cup \alpha_1$, $S \cup \alpha_2$, and $S \cup \alpha_1 \cup \alpha_2$. There are three cases to consider: (1) the first passage of this fragment in S is on the bottom segment (so we encounter both arrow tails first); (2a) the first passage is on the top segment and both segments belong to the same circle of $G(S)$; (2b) the first passage is on the top segment and the segments belong to two different circles of $G(S)$. The contribution of these arrows in each case are shown in the table.

States of G' :				
(1)	1	0	0	0
(2a)	1	$\varepsilon a^\varepsilon z$	$-\varepsilon a^{-\varepsilon} z$	$(a - a^{-1})z$
(2b)	1	$\varepsilon a^\varepsilon z$	$-\varepsilon a^\varepsilon z$	0

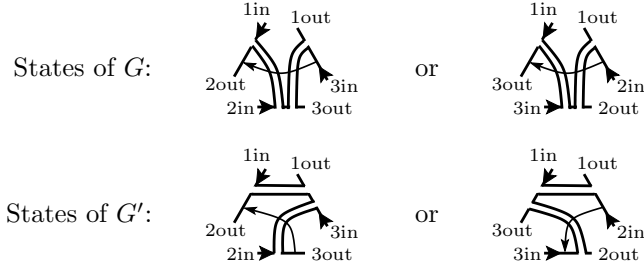
In all cases $\langle G'|S \rangle = \langle G|S \rangle$, while the contributions of the last three states of G' cancel out, i.e., $\langle G'|S \cup \alpha_1 \rangle + \langle G'|S \cup \alpha_2 \rangle + \langle G'|S \cup \alpha_1 \cup \alpha_2 \rangle = 0$.

Finally, if Gauss diagrams G and G' are related by an Ω_3 move, there is a bijective correspondence between states of G and G' , depending on the order in which three segments are passed in the tracing. This correspondence preserves the weights and the combinatorics of the order in which the tracing enters and leaves the neighborhood of these arrows. The table below summarises the correspondence of states with non-zero weights.

$a^{-3}z$	$a^{-3}z$	$a^{-3}z$	$a^{-5}z$	$a^{-3}z$	$a^{-1}z$	$a^{-1}z$

$a^{-3}z$	$a^{-5}z$	$a^{-4}z^2$	$a^{-2}z^2$	$a^{-4}z^2$	$a^{-2}z^2$	$a^{-3}z^3$

For a better understanding of this table, let us explain one of the cases in details. Denote the top, left, and right arrows in the fragment by α_t , α_l , and α_r respectively. Consider a state $S \cup \alpha_l \cup \alpha_r$ which contains two arrows of the fragment. The order of tracing the fragment depends on S . Only two orders of tracing may result in a non-zero weight:



Here the three consecutive entries and exits from the fragment are indicated by 1in, 1out, 2in, 2out, 3in, 3out. In the first case, the local weight of this fragment is $a^{-1}z \cdot a^{-1}z \cdot a^{-2} = a^{-4}z^2$. The corresponding state of G' is $S \cup \alpha_t \cup \alpha_r$. Note that the pattern of entries and exits from the fragment is indeed the same. Its local weight is also the same. In the second case, the local weight of this fragment is $a^{-1}z \cdot a^{-1}z \cdot 1 = a^{-2}z^2$. Now the corresponding state of G' is $S \cup \alpha_t \cup \alpha_l$. Both the pattern of entries and exits and its local weight are again the same as in G . \square

6. VASSILIEV INVARIANTS COMING FROM THE HOMFLYPT POLYNOMIAL

6.1. HOMFLYPT power series. A standard way [BN, BL] to relate Vassiliev invariants to the HOMFLYPT polynomial is to make a substitution $a = e^{Nh}$, $z = e^h - e^{-h}$ and then take the Taylor expansion of $P(L)$ in the variable h . The coefficient at h^n turns out to be a Vassiliev invariant of order $\leq n$ which depends on a parameter N .

In this paper we are working in a different way, following [G2]. Namely, we substitute $a = e^h$ and take the Taylor expansion in h . The result will be a Laurent polynomial in z and a power series in h . Let $p_{k,l}(L)$ be its coefficient at $h^k z^l$. It is not difficult to see that for any link L the total degree $k + l$ is not negative. (It also follows from the Jaeger model in section 2.)

Lemma ([G2]). $p_{k,l}(L)$ is a Vassiliev invariant of order $\leq k + l$.

Indeed, plugging $a = e^h$ into the skein relation we get

$$P\left(\begin{array}{c} \curvearrowright \\ \diagdown \\ \diagup \\ \curvearrowleft \end{array}\right) - P\left(\begin{array}{c} \curvearrowleft \\ \diagdown \\ \diagup \\ \curvearrowright \end{array}\right) = zP\left(\begin{array}{c} \curvearrowright \\ \curvearrowright \\ \curvearrowright \\ \curvearrowright \end{array}\right) + h(\text{some terms}).$$

Since all terms of the HOMFLYPT polynomial have non-negative total degree in z and h , the terms of the right hand side has degree at least 1. Therefore, if we change $n + 1$ crossings in different places then the alternating sum of the 2^n polynomials will have the degree of its monomials $\geq n + 1$. Hence the coefficient at any degree n term will be zero. \square

Remark. After substitution $a = e^h$ and the Taylor expansion in h the factor $\frac{a-a^{-1}}{z}$ becomes $\frac{2h+\dots}{z}$. In other words its total degree in h and z is not negative. Therefore, the total degree $k + l$ of the monomial $h^k z^l$ of $P(L)$ is not negative, however the exponent l of z may be as negative as $-k + 1$.

Our next goal is to describe the Gauss diagram formulas for $p_{k,l}(L)$. Note that the case $k = 0$ corresponds to the substitution $a = 1$ into the HOMFLYPT polynomial, i.e. to the Conway polynomial. Thus $p_{0,l}(L)$ are coefficients of the Conway polynomial for which the Gauss diagram formulas were found in [CKR]. This part of our work may be considered as a generalization of [CKR].

6.2. Gauss diagram formulas for Vassiliev invariants. Let \mathcal{A} be a free \mathbb{Z} -module generated by ordered arrow diagrams with m circles. Define a map $I : \mathcal{A} \rightarrow \mathcal{A}$ by $I(G) := \sum_{A \subseteq G} A$ for any (abstract, ordered) Gauss diagram G , and extend it to \mathcal{A} by linearity. Here $A \subseteq G$ means the arrow subdiagram A containing the same circles as the whole diagram G but only a subset of arrows of G with their signs. A natural scalar product on \mathcal{A} is given by $(A, B) := 0$ if A is not equal to B , and $(A, B) := 1$ if $A = B$ for a pair of arrow diagrams A and B . Let us define a pairing $\langle A, G \rangle := (A, I(G))$.

Definition. Let A be a fixed element of \mathcal{A} . By a *Gauss diagram formula* we mean a function \mathcal{I}_A on abstract Gauss diagrams defined by $\mathcal{I}_A : G \mapsto \langle A, G \rangle$.

If A is chosen at random then $\mathcal{I}_A(G)$ usually changes under Reidemeister moves and thus does not define any link invariant. However, for some special choice of A it might be a link invariant. Due to certain special features of Gauss diagrams for classical links, it may even happen that a formula gives an invariant of classical links, but is not invariant for virtual links. According to [GPV] any Vassiliev invariant of long knots can be expressed by a Gauss diagram formula. In the following sections we describe an algorithm for finding such formulas for invariants $p_{k,l}(L)$ coming from the HOMFLYPT polynomial.

For shortness of notation, further we will use *unsigned arrow diagrams*, understanding by that a linear combination of arrow diagrams with all possible choices of signs and appearing with a coefficients ± 1 depending on whether even or odd number of negative signs were chosen.

Examples. If $m = 2$ and

$$A = \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} := \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array} - \begin{array}{c} \text{---} \circ \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \circ \text{---} \end{array},$$

then $\mathcal{I}_A(G)$ is equal to the linking number of components.

If $m = 1$ and

$$A = \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} := \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} - \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} - \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} + \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array},$$

then $\mathcal{I}_A(G)$ is equal to the second coefficient of the Conway polynomial, $p_{0,2}(G)$ (see [PV]).

7. GAUSS DIAGRAM FORMULAS FOR HOMFLYPT COEFFICIENTS

Our aim is to figure out contributions of various arrow subdiagrams to $p_{k,l}$, using the state model from Section 4.

Consider a state model on an arrow diagram A with the following table of local weights $\langle \alpha | A | S \rangle$:

	First passage:				
(5)		$e^{-h}z$	0	$e^{-2h} - 1$	0
		$-e^h z$	0	$e^{2h} - 1$	0

Let $\langle A | S \rangle = \prod_{\alpha \in A} \langle \alpha | A | S \rangle$ and define a power series in h and z by

$$(6) \quad W(A) = \sum_S \langle A | S \rangle \left(\frac{e^h - e^{-h}}{z} \right)^{c(S)-1}$$

Denote by $w_{k,l}$ the coefficient of $h^k z^l$ in $W(A)$, so that $W(A) = \sum_{k,l} w_{k,l}(A) h^k z^l$.

Definition. Now the linear combination $A_{k,l} \in \mathcal{A}$ can be defined as follows.

$$A_{k,l} := \sum w_{k,l}(A) \cdot A$$

Theorem 2. Let G be a Gauss diagram of an ordered (possibly virtual) link L . Then

$$p_{k,l}(L) = \mathcal{I}_{A_{k,l}}(G) = \langle A_{k,l}, G \rangle.$$

Proof. According to 4, the HOMFLYPT is equal to

$$P(G) = \sum_{S \subset G} \langle G | S \rangle \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}.$$

We have

$$\begin{aligned} \langle G | S \rangle &= \prod_{\alpha \in G} \langle \alpha | G | S \rangle = \prod_{\alpha \in S} \langle \alpha | G | S \rangle \prod_{\alpha \in G \setminus S} \langle \alpha | G | S \rangle = \\ &= \prod_{\alpha \in S} \langle \alpha | G | S \rangle \sum_{A \supset S} \left(\prod_{\alpha \in A \setminus S} (\langle \alpha | G | S \rangle - 1) \prod_{\alpha \in G \setminus A} 1 \right) = \\ &= \sum_{A \supset S} \left(\prod_{\alpha \in S} \langle \alpha | G | S \rangle \prod_{\alpha \in A \setminus S} (\langle \alpha | G | S \rangle - 1) \right). \end{aligned}$$

Therefore

$$\begin{aligned} P(G) &= \sum_{S \subset G} \sum_{A \supset S} \left(\prod_{\alpha \in S} \langle \alpha | G | S \rangle \prod_{\alpha \in A \setminus S} (\langle \alpha | G | S \rangle - 1) \right) \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1} = \\ &= \sum_{A \subset G} \sum_{S \subset A} \left(\prod_{\alpha \in S} \langle \alpha | G | S \rangle \prod_{\alpha \in A \setminus S} (\langle \alpha | G | S \rangle - 1) \right) \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1} \end{aligned}$$

Comparing tables (4) and (5) of local weights, we get

$$\prod_{\alpha \in S} \langle \alpha | G | S \rangle \prod_{\alpha \in A \setminus S} (\langle \alpha | G | S \rangle - 1) = \prod_{\alpha \in A} \langle \alpha | A | S \rangle = \langle A | S \rangle$$

Thus

$$P(G) = \sum_{ACG} \sum_{S \subset A} \langle A | S \rangle \cdot \left(\frac{a - a^{-1}}{z} \right)^{c(S)-1}$$

And the theorem follows. \square

7.1. Contributions of various diagrams to $A_{k,l}$. A state S of an arrow diagram A is called *ascending*, if in the tracing of $A(S)$ we approach a neighborhood of every arrow (not only the ones in S) first at the arrow head. As easy to see from the weight table, only ascending states contribute to $W(A)$. In particular, the first end point of an arrow in A (as we move from the base point along the orientation) must be an arrow head.

Note that since $e^{\pm 2h} - 1 = \pm 2h +$ (higher degree terms) and $\pm e^{\mp h} z = \pm z +$ (higher degree terms), the power series $W(A)$ starts with terms of degree at least $|A|$, the number of arrows of A . Moreover, the z -power of $\langle A | S \rangle \left(\frac{e^h - e^{-h}}{z} \right)^{c(S)-1}$ is equal to $|S| - c(S) + 1$. Therefore, for fixed k and l , the weight $w_{k,l}(A)$ of an arrow diagram may be non-zero only if A satisfies the following conditions:

- (i) $|A|$ is at most $k + l$;
- (ii) there is an ascending state S such that $c(S) = |S| + 1 - l$.

For diagrams of the highest degree $|A| = k + l$, the contribution of an ascending state S to $w_{k,l}(A)$ is equal to $(-1)^{|A|-|S|} 2^k \varepsilon(A)$, where $\varepsilon(A)$ is the product of signs of all arrows in A . If two such arrow diagrams A and A' with $|A| = k + l$ differ only by signs of arrows, their contributions to $A_{k,l}$ differ by the sign $\varepsilon(A)\varepsilon(A')$. Thus all such diagrams may be combined to the unsigned diagram A , appearing in $A_{k,l}$ with the coefficient $\sum_S (-1)^{|A|-|S|} 2^k$ (where the summation is over all ascending states of A with $c(S) = |S| + 1 - l$).

Arrow diagrams with isolated arrows do not contribute to $A_{k,l}$. Indeed, all ascending states cancel out in pairs similarly to the proof of invariance under Ω_1 in Theorem 1.

7.2. Coefficients of the Conway polynomial. The Conway polynomial is obtained from the HOMFLYPT polynomial by setting $h = 0$. So our formulas for $A_{0,l}$ are the Gauss diagram formulas for coefficients of the Conway polynomial, discovered earlier by Michael Khoury and Alfred Rossi [CKR]. Indeed, only states with $|S| = |A|$ and $c(S) = 1$ contribute to $w_{0,l}(A)$. Since these are diagrams of the highest degree, according to 7.1 they may be combined into unsigned ascending diagrams which appear with coefficients 1.

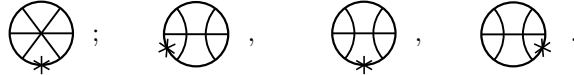
For example, in the case $m = 1$ of long knots, states with $c(S) = 1$ exist only for even number l of arrows. For $l = 2$ and $l = 4$ the resulting linear combinations $A_{0,l}$ are

$$\begin{aligned}
 A_{0,2} &= \text{Diagram 1} ; & A_{0,3} &= 0 ; \\
 A_{0,4} &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} + \\
 &+ \text{Diagram 6} + \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} + \text{Diagram 11} + \text{Diagram 12} + \\
 &+ \text{Diagram 13} + \text{Diagram 14} + \text{Diagram 15} + \text{Diagram 16} + \text{Diagram 17} + \text{Diagram 18} + \text{Diagram 19} .
 \end{aligned}$$

8. LOW DEGREE EXAMPLES

Let us describe the corresponding formulas for degree 2 and 3 invariants of knots, i.e. $k + l = 2, 3, m = 1$. The case $A_{0,2}$ was described above. A direct check shows that $A_{2,0} = 0$. Let us explicitly find the formula for $A_{1,2}$. The maximal number of arrows is equal to 3. To get Z^2 in $W(A)$ we need ascending states with either $|S| = 2$ and $c(S) = 1$, or $|S| = 3$ and $c(S) = 2$. In the first case the equation $c(S) = 1$ means that the two arrows of S must intersect. In the second case the equation $c(S) = 2$ does not add any restrictions on the relative position of arrows. In cases $|S| = |A| = 2$ or $|S| = |A| = 3$, since S is ascending, A itself must be ascending as well.

For diagrams of the highest degree $|A| = 1 + 2 = 3$, we should count ascending states of unsigned arrow diagrams with the coefficient $(-1)^{3-|S|}2$, i.e. -2 for $|S| = 2$ and $+2$ for $|S| = 3$. There are only four types of (unsigned) 3-arrow diagrams with no isolated arrows:



Diagrams of the same type differ by directions of arrows.

For the first type, recall that the first arrow should be oriented towards the base point; this leaves 4 possibilities for directions of the remaining two arrows.

One of them, namely does not have ascending states with $|S| = 2, 3$. The remaining possibilities, together with their ascending states, are shown in the table:

The final contribution of this type of 3-arrow diagrams to $A_{1,2}$ is equal to

$$-2 \text{Diagram 1} - 2 \text{Diagram 2} .$$


The remaining three types of 3-arrow diagrams differ by the location of the base point. A similar consideration shows that 5 out of the total of 12 arrow diagrams of these types, namely



do not have ascending states with $|S| = 2, 3$. The remaining possibilities, together with their ascending states, are shown in the table:


The final contribution of this type of 3-arrow diagrams to $A_{1,2}$ is equal to

$$-2 \left(\text{Diagram 1} \right) - 2 \left(\text{Diagram 2} \right) - 2 \left(\text{Diagram 3} \right) + 2 \left(\text{Diagram 4} \right) - 2 \left(\text{Diagram 5} \right).$$

Besides 3-arrow diagrams, some 2-arrow diagrams contribute to $A_{1,2}$ as well. Since $|A| = 2 < k + l = 3$, contributions of 2-arrow diagrams depend also on their signs. Such diagrams must be ascending (since $|S| = |A| = 2$) and should not have isolated arrows. There are four such diagrams, looking like , but with different signs $\varepsilon_1, \varepsilon_2$ of arrows. For each of them $\langle A|S \rangle = \varepsilon_1 \varepsilon_2 e^{-(\varepsilon_1 + \varepsilon_2) h z^2}$. If $\varepsilon_1 = -\varepsilon_2$, then $\langle A|S \rangle = -z^2$, so the coefficient of $h z^2$ vanishes and such diagrams do not occur in $A_{1,2}$. For two remaining diagrams with $\varepsilon_1 = \varepsilon_2 = \pm$, coefficients of $h z^2$ in $\langle A|S \rangle$ are equal to ∓ 2 respectively.

Combing all the above contributions, we finally get

$$A_{1,2} = -2 \left(\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5} - \text{Diagram 6} - \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} - \text{Diagram 10} \right).$$

At this point we can see the difference between virtual and classical long knots. For classical knots $\mathcal{I}_{A_{1,2}} = \langle A_{1,2}, \cdot \rangle$ can be simplified further. Note that for any classical Gauss diagram G , $\langle \text{Diagram 1}, G \rangle = \langle \text{Diagram 2}, G \rangle$. This follows from the symmetry of the linking number. Indeed, supposed we have matched two vertical arrows (which are the same in both diagrams) with two arrows of G . Let us consider the orientation preserving smoothings of the corresponding two crossings of the link diagram D associated with G . The smoothed diagram \tilde{D} will have three components. Matchings of the horizontal arrow of our arrow diagrams with an arrow of G both measure the linking number between the first and the third components of \tilde{D} , using crossings when the first component overpasses (underpasses, respectively) the third one. Thus, as functions on classical Gauss diagrams,  is equal to



and we have

$$p_{1,2}(G) = -2 \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} , G \right) .$$

For virtual Gauss diagrams this is no longer true.

In a similar way one may check that $A_{3,0} = -4A_{1,2}$.

Example 1d. Let us compute the coefficients of hz^2 and h^3 of the HOMFLYPT polynomial on the trefoil from page 5.

$$\langle A_{1,2}, G \rangle = 2 \langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} , G \rangle = 2 \quad \text{and} \quad \langle A_{3,0}, G \rangle = -8 ,$$

It is easy to verify these coefficients in the Taylor expansion of $P(3_1) = (2e^{2h} - e^{4h}) + e^{2h}z^2$.

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