# ON GALOIS COHOMOLOGY OF SOME $P$-ADIC REPRESENTATIONS AND ÉTALE $K$-THEORY OF CURVES

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Abstract. In the first part of this paper we investigate reduction of nontorsion elements in the étale $K$-theory of a curve $X$. We show that the reduction map on the even étale $K$-groups has finite kernel. Assuming a special case of the weak Leopoldt conjecture [PR1] we prove that the cokernel of the bundary map in the localization sequence in étale $K$-theory of the curve consists of infinitely divisible elements in the Galois cohomology group $H^2(G_F;\, T_p(J)(n+1))$. Here, $T_p(J)(n+1)$ is the twisted Tate module of the Jacobian variety of $X$. In the last section of the paper we investigate the Galois cohomology of some $p$-adic representations $T$, main examples of which come from Tate modules of CM elliptic curves and Jacobians of some Fermat curves. We show that $p$-torsion part of the group $H^2(G_S;\, T(n+1))$, for these representations is generated by special elements introduced by Deligne and Soulé.

## 1. Introduction.

Let $F$ be a number field, $\mathcal{O}_F$ its ring of integers and let $S$ be a finite set of primes containing a fixed odd prime $p$. Let $G_S$ be the Galois group of the maximal extension of $F$ which is unramified outside $S$. The weak Leopoldt conjecture, (cf. [PR1, p. 63] and also [J1, p. 317]) says that the group $H^2(G_S;\, T(n+1))$ is finite for some $G_S$-modules $T$ which are free and finitely generated over $\mathbb{Z}_p$. This conjecture holds true for trivial $G_S$ module $T = \mathbb{Z}_p$ and $n > 0$, due to well-known results of Borel, Quillen and Soulé. For $F = \mathbb{Q}$ we can prove this fact in another way using special elements in $K$-theory and the Stickelberger ideal. The method was developed in [BG1]; the key point is to compute the image of the special elements via the boundary map in the localization sequence:

$$\partial_m :\, H^1(\mathbb{Q};\, \mathbb{Z}/p^m(n+1)) \to \bigoplus_{v \notin S} H^0(k_v;\, \mathbb{Z}/p^m(n)).$$

For other cases where the weak Leopoldt conjecture holds true see Remark 4.1 below. The Galois cohomology groups $H^*(G_S;\, T(n+1))$, for $T = T_p(J)$ the Tate module of the Jacobian variety of $X$, are crucial for investigation of étale $K$-theory of curves. The proofs of the following two theorems provide examples of this situation.

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**Theorem A.** *[Th. 3.1]*
*Assume that $X$ is a smooth, proper and geometrically irreducible curve defined over a global field $F$. Let $p$ be an odd rational prime, $p \neq char(F)$. Let $n > 0$ and let $S_0$ be the set of primes of $F$ which consists of primes of bad reduction for $X$ and primes lying over $p$. Let $S \supseteq S_0$ be a finite set of primes. Assume thet $\mathcal{X}$ is the smooth and proper model of $X$ over the ring of $S$-integers of $F$. Assume that $x \in K_{2n}^{et}(\mathcal{X})$ is a nontorsion element. Given $M_1 = p^{m_1}$ a fixed power of $p$, there exist infinitely many primes $v \notin S$ such that $x$ reduces in $K_{2n}^{et}(\mathcal{X}_v)$ to an element of order at least $M_1$.*

As a consequence we obtain:

**Theorem B.** *[Th. 3.4]*
*For $X$, $p$ and $n$ as in Theorem A the kernel of the reduction map:*

$$K_{2n}^{et}(\mathcal{X}) \to \prod_{v \notin S} K_{2n}^{et}(\mathcal{X}_v)$$

*is a finite group.*

Theorem A and Theorem B rely on a result of Jannsen [J1] which concerns the reduction map in Galois cohomology of $T_p(J)(n+1)$. As for odd $K$-groups, we prove an analogue of Theorem B assuming in addition the weak Leopoldt conjecture, namely that $H^2(G_{S_0}; T_p(J)(n+1))$ is a finite group. The same assumption is needed in the proof of Theorem C which we formulate below. Namely, let $C_m$ denote the cokernel of the boundary map:

$$\partial_m : K_{2n+2}^{et}(X; \mathbb{Z}/p^m) \to \bigoplus_{v \notin S} K_{2n+1}^{et}(\mathcal{X}_v; \mathbb{Z}/p^m).$$

In [B] the first author and M. Kolster considered the boundary map in the localization sequence in $K$-theory of a number field. In the proof of [B, Theorem 3] it was shown that the cokernel of the boundary map is isomorphic to the subgroup of infinitely divisible elements in $H^2(G_F; \mathbb{Z}_p(n+1))$ cf. [B, Diagram 2.4, p. 291]. In section 4 of the paper we compare the subgroup of infinitely divisible elements in $H^2(G_F; T_p(J)(n+1))_p$ with $C_m$.

**Theorem C.** *[Th. 4.3]*
*Let $p$ be a prime such that $p \nmid w_n(F)w_{n+1}(F)v_{n+1}(F, J)$. Assume that the group $H^2(G_{S_0}; T_p(J)(n+1))$ is finite. Then for $m$ big enough, we have a natural isomorphism:*

$$C_m \cong div\, H^2(G_F; T_p(J)(n+1))_p.$$

*In particular, the group $C_m$ embeds into the group $H^2(G_F; T_p(J)(n+1))$.*

In the last part of the paper we describe the $p$-torsion part of $H^2(G_S; T(n+1))$ for some $p$-adic representations $T$. Main result of this section generalizes a theorem of two of us [BG2, Theorem 2.4] about generation of $H^2(G_S; \mathbb{Z}_p(n+1))_p$ by cyclotomic elements introduced by Deligne and Soulé cf. [So3] to a class of higher dimensional modules $T$. We find conditions under which the $p$-torsion part of the cohomology group with $\mathbb{Z}_p$ replaced by $T$ is generated by special elements which are modeled after those introduced by Soulé in [So2]. At the end of section 5 we give examples of representations for which our method applies. The examples are provided by Tate modules of CM elliptic curves and Jacobians of some Fermat curves. The following theorem states our result.

**Theorem D.** *[Th. 5.5]*
*Let $T$ satisfy conditions (1)-(4) of the beginning of section 5 and, in addition let $T$ satisfy assumptions of lemma 5.3. There exists a finite set of primes $S \supseteq S_0$ such that every element of the group $H^2(G_{S_0}; T(n+1))_p$ can be written in the form:*

$$\mathfrak{b}[Tr(\alpha'_m(u \otimes \beta^n \otimes v))],$$

*for an $S$-unit $u \in \mathcal{O}^\times_{E',S}$ and some $v \in T/p^m$. The maps $\mathfrak{b}$ and $Tr$ are the Bockstein homomorphism:*

$$\mathfrak{b}:\, H^1(G_S; T/p^m(n+1)) \to H^2(G_S; T(n+1))[p^m]$$

*and the transfer map:*

$$Tr:\, H^1(G'; T/p^m(n+1)) \to H^1(G_S; T/p^m(n+1)),$$

*respectively.*

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**Notation.**

(1) $p$ is an odd prime number.
(2) $A_p$ denotes $p$-torsion part of an abelian group $A$ and $A[p^m]$ its subgroup of elements of order dividing $p^m$. By $Div\, A$ we denote the maximal divisible subgroup of $A$ and $div\, A = \bigcap_{r \geq 1} rA$ the subgroup of infinitely divisible elements in $A$.
(3) $F$ is a global field, $\mathcal{O}_F$ its ring of integers and $G_F = G(\bar{F}/F)$ the absolute Galois group.
(4) $S$ denotes a finite set of prime ideals in $\mathcal{O}_F$ containing primes above $p$ and $\mathcal{O}_{F,S}$ is the ring of $S$-integers in $\mathcal{O}_F$.
(5) $F_S$ denotes the maximal extension of $F$ unramified outside primes over $S$ and $G_{F,S} = G_S$ is the Galois group $G(F_S/F)$.

(6) $F_v$ is the completion of $F$ at a finite prime $v$, $\mathcal{O}_v$ is the ring of integers in $F_v$, $\kappa_v = \mathcal{O}_F/v$ is the residue field of $v$.
(7) $X$ is a smooth, proper and geometrically irreducible curve defined over $F$. We assume that $p$ is different from the characteristic of $F$.
(8) $S_0$ denotes the set which consists of primes of $F$ over $p$ and primes of bad reduction for $X$. For a finite set of primes $S$ containing $S_0$, we choose a smooth, proper model $\mathcal{X}$ of the curve $X$ over $Spec\,\mathcal{O}_{F,S}$. We denote by $J$ the Jacobian variety of $X$.
(9) $T$ will denote a free $\mathbb{Z}_p$-module of finite rank with a continuous, $\mathbb{Z}_p$-linear action of $G_F$ which is given by a representation $\rho : G_F \to GL(T)$. We denote by $\bar{\rho}_m$ the representation $\rho \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^m : G_F \to GL(T/p^m)$, where $m \geq 1$. For $n \geq 0$, $T(n) = T \otimes \mathbb{Z}_p(n)$ is the $n$-th Tate twist of $T$.
(10) $K_*(-)$ $(K_*(-;\mathbb{Z}/M),\ )$ denotes the Quillen $K$-theory (the $K$-theory with finite coefficients defined by Browder and Karoubi).
(11) $K_*^{et}(-)$ denotes étale $K$-theory and $K_*^{et}(-;\mathbb{Z}/M)$ denotes the étale $K$-theory with finite coefficients defined in [DF].

## 2. Preliminary Computations.

In this section for the convenience of the reader we collect some basic facts about the continuous étale cohomology groups of $\mathcal{X}$ and $p$-adic étale $K$-groups, which will be often used in this paper. Our main references are: [J1], [J2], [M1] and [M2]. Let $(\mathcal{F}_m)$ be a projective system of $\mathbb{Z}/p^m$-sheaves on $\mathcal{X}$. The functor which sends $(\mathcal{F}_m)$ to $\varprojlim H^0(\mathcal{X};(\mathcal{F}_m))$ is left exact and one defines $H^i_{cts}(\mathcal{X};(\mathcal{F}_m))$ to be its $i$th right derived functor cf. [J2, p.216]. When the projective system is $\mathbb{Z}_p(k)$ we denote these groups by $H^i_{cts}(\mathcal{X};\mathbb{Z}_p(k))$. We will use the following properties of continous cohomology.

(a) There is an exact sequence [J2, p.216, (3.1)]:

$$(2.1) \qquad 0 \to \varprojlim{}^1 H^{i-1}(\mathcal{X};\mathcal{F}_m) \to H^i_{cts}(\mathcal{X};(\mathcal{F}_m)) \to \varprojlim H^i(\mathcal{X};\mathcal{F}_m) \to 0.$$

In particular, if $H^i(\mathcal{X};\mathcal{F}_m)$ are finite for any $i$ and $m$, then by the Mittag-Leffler condition $H^i_{cts}(\mathcal{X};(\mathcal{F}_m)) \cong \varprojlim H^i(\mathcal{X},\mathcal{F}_m)$.

(b) Let $f : \mathcal{X} \to Spec\,\mathcal{O}_{F,S}$ be the natural map. Then there is a spectral sequence [J2, p. 219, (3.10)]:

$$(2.2) \qquad E_2^{r,s} = H^r_{cts}(\mathcal{O}_{F,S}, R^s f_*(\mathcal{F}_m)) \Rightarrow H^{r+s}_{cts}(\mathcal{X},(\mathcal{F}_m))$$

Note that the projective system $R^j f_*(\mathcal{F}_m)$ above is isomorphic to $(R^j f_*\mathcal{F}_m)$, [J2, Prop. 1.2, p. 210]. ¿From now on $\mathcal{F}_m$ will be one of the sheaves: $\mathbb{Z}/p^m(k)$ or $R^j f_*\mathbb{Z}/p^m(k)$.

(c) There is the Dwyer-Friedlander spectral sequence, which calculates étale $K$-theory in terms of continuous cohomology, [DF, Prop. 5.1, p.260]:

$$(2.3) \qquad E_2^{r,-s} = H^r_{cts}(\mathcal{X},\mathbb{Z}_p(s/2)) \Rightarrow K^{et}_{s-r}(\mathcal{X}),$$

where $\mathbb{Z}_p(s/2)$ is defined as 0 if $s$ is odd.

**Lemma 2.1.**

(a) *If $A$ is a locally constant constructible $p$-torsion étale sheaf on $Spec\,\mathcal{O}_{F,S}$, then $H^i(\mathcal{O}_{F,S}; A)$ is finite for $i = 0, 1, 2$ and vanish for $i > 2$.*

(b) *If $A$ is a locally constant constructible $\mathbb{Z}_p$-sheaf on $Spec\,\mathcal{O}_{F,S}$, then the groups $H^i_{cts}(\mathcal{O}_{F,S}; A)$ are finitely generated $\mathbb{Z}_p$-modules for $i = 0, 1, 2$ and vanish for $i > 2$.*

*Proof.* cf. [J2, p.219] □

**Lemma 2.2.** *Let $\bar{\mathcal{X}}_v = \mathcal{X} \times_{\mathcal{O}_{F,S}} \bar{\kappa}_v$ and $\bar{X} = X \times_{\mathcal{O}_{F,S}} \bar{F}$. Let $x$ be a point of $Spec\,\mathcal{O}_{F,S}$ and $\bar{x}$ be the geometric point over $x$. Then the following isomorphisms hold.*

$$R^0 f_* \mathbb{Z}/p^m(k) \cong \mathbb{Z}/p^m(k)$$

$$R^1 f_* \mathbb{Z}/p^m(k)_{\bar{x}} \cong \begin{cases} J[p^m] \otimes \mathbb{Z}/p^m(k-1), & for\ x = Spec\,F \\ J_v[p^m] \otimes \mathbb{Z}/p^m(k-1), & for\ x = Spec\,\kappa_v \end{cases}$$

$$R^2 f_* \mathbb{Z}/p^m(k) \cong \mathbb{Z}/p^m(k-1)$$

$$R^j f_* \mathbb{Z}/p^m(k) = 0,\ for\ j > 2.$$

*Here, $J_v$ is the Jacobian variety of $\bar{\mathcal{X}}_v$. In particular, the sheaves $R^j f_* \mathbb{Z}/p^m(k)$ are locally constant and constructible.*

*Proof.* The first isomorphism is obvious. For the second and fourth use [M1, VI, Cor. 2.5]. The third isomorphism is the trace map isomorphism of sheaves [SGA 4, Exposé XTIII, Section 1.1]. □

As a corollary we obtain

**Lemma 2.3.** *We have the following isomorphisms*

$$R^0 f_* \mathbb{Z}_p(k) \cong \mathbb{Z}_p(k)$$

$$R^1 f_* \mathbb{Z}_p(k)_{\bar{x}} \cong \begin{cases} T_p(J) \otimes \mathbb{Z}_p(k-1), & for\ x = Spec\,F \\ T_p(J_v) \otimes \mathbb{Z}_p(k-1), & for\ x = Spec\,\kappa_v \end{cases}$$

$$R^2 f_* \mathbb{Z}_p(k) \cong \mathbb{Z}_p(k-1)$$

$$R^j f_* \mathbb{Z}_p(k) = 0,\ for\ j > 2.$$

*In particular, the $R^j f_* \mathbb{Z}_p(k)$ are locally constant and constructible $\mathbb{Z}_p$-sheaves.*

The Leray spectral sequence of the map $f : \mathcal{X} \to Spec\,\mathcal{O}_{F,S}$ gives the following useful description of the groups $H^i_{cts}(\mathcal{X}; \mathbb{Z}_p(k))$ in terms of Galois cohomology.

**Proposition 2.4.**

(a)

$$H^0_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \cong H^0_{cts}(\mathcal{O}_{F,S};\mathbb{Z}_p(k)) = 0$$

$$H^1_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \cong H^1_{cts}(\mathcal{O}_{F,S};\mathbb{Z}_p(k)) \cong H^1(G_S;\mathbb{Z}_p(k))$$

(b) *There are exact sequences:*

$$0 \to H^2(G_S;\mathbb{Z}_p(k)) \to H^2_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \to H^1(G_S;T_p(J)(k-1)) \to 0$$

$$0 \to H^2(G_S;T_p(J)(k-1)) \to H^3_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \to H^1(G_S;\mathbb{Z}_p(k-1)) \to 0.$$

(c)

$$H^4_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \cong H^2(G_S;\mathbb{Z}_p(k-1))$$

$$H^i_{cts}(\mathcal{X};\mathbb{Z}_p(k)) = 0\, for\, i > 4.$$

*Proof.* We give a detailed proof for the convenience of the reader and for further reference.

(a) For the group $H^0_{cts}$ the Leray spectral sequence gives immediately:

$$H^0_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \cong H^0(\mathcal{O}_{F,S};R^0f_*\mathbb{Z}_p(k)) \cong H^0(\mathcal{O}_{F,S};\mathbb{Z}_p(k)) = \mathbb{Z}_p(k)^{G_S} = 0.$$

As for $H^1_{cts}$, we have the low terms exact sequence:

$$(2.7)\quad 0 \to H^1_{cts}(\mathcal{O}_{F,S};\mathbb{Z}_p(k)) \to H^1_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \to H^0_{cts}(\mathcal{O}_{F,S};R^1f_*\mathbb{Z}_p(k)) \to \dots$$

On the other hand, for any $i$:

$$(2.8)\quad H^i_{cts}(\mathcal{O}_{F,S};R^jf_*\mathbb{Z}_p(k)) \cong H^i(G_S;R^jf_*\mathbb{Z}_p(k)_{\bar{x}}) \cong H^i(G_S;H^j(\bar{X};\mathbb{Z}_p(k))).$$

To prove (2.8) one first checks the same isomorphisms for the étale cohomology with coefficients in $R^jf_*\mathbb{Z}/p^m(k)$ which hold true by [M2, Prop. 2.9, p. 209] and [M1, Prop 1.13, p. 88], respectively. The isomorphisms (2.8) follow then by Lemma 2.3 and the Mittag-Leffler condition applied to the sequence (2.1) (where we put $\mathcal{F}_m = R^jf_*\mathbb{Z}/p^m(k)$). In particular, (2.8) implies: $H^0_{cts}(\mathcal{O}_{F,S};R^1f_*\mathbb{Z}_p(k)) \cong H^0(G_S;H^1(\bar{X};\mathbb{Z}_p(k)) \cong H^0(G_S;T_p(J)(k-1))$ which vanishes by a result of Suslin [Su, Prop 2.4, p. 11]. This proves the second isomorphism in (a).

(b) We have: $H^0_{cts}(\mathcal{O}_{F,S};R^2f_*\mathbb{Z}_p(k)) \cong H^0(\mathcal{O}_{F,S};\mathbb{Z}_p(k-1)) = 0$ by Lemma 2.3 and (a) above. The Leray spectral sequence and (2.8) imply that there is an exact sequence:

$$0 \to H^2_{cts}(\mathcal{O}_{F,S};R^0f_*\mathbb{Z}_p(k)) \to H^2_{cts}(\mathcal{X};\mathbb{Z}_p(k)) \to H^1(G_S;T_p(J)(k-1)) \to 0.$$

It also follows by (2.8) that the kernel in the last sequence is: $H^2(G_S; R^0 f_* \mathbb{Z}_p(k)) \cong H^2(G_S; \mathbb{Z}_p(k))$. This gives the first exact sequence in (b). To prove that the second sequence is exact it is enough to check that in the Leray spectral sequence (2.2) (with $\mathcal{F}_m = R^s f_* \mathbb{Z}/p^m(k)$) the groups $E_2^{3,0}$ and $E_2^{0,3}$ vanish. For $E_2^{3,0}$ the exact sequence (2.1) gives :

$$0 \to \varprojlim{}^1 H^2(\mathcal{O}_{F,S}; \mathbb{Z}/p^m(k)) \to H^3_{cts}(\mathcal{O}_{F,S}; \mathbb{Z}_p(k)) \to \varprojlim H^3(\mathcal{O}_{F,S}; \mathbb{Z}/p^m(k)) \to 0.$$

The $lim^1$-term vanishes by the Mittag-Leffler condition and Lemma 2.1 (a). Because $cd_p(\mathcal{O}_{F,S}) = 2$ we have: $E_2^{3,0} = 0$. $E_2^{0,3} = H^0_{cts}(\mathcal{O}_{F,S}; R^3 f_* \mathbb{Z}_p(k)) = 0$ follows by part (a) and Lemma 2.3.

(c) The same argument as in the proof of part (b) above shows that all groups $E_2^{r,s} = H^r_{cts}(\mathcal{O}_{F,S}; R^s f_* \mathbb{Z}_p(k))$ with $r+s = 4$ vanish except $E_2^{2,2}$. By (2.8) and Lemma 2.3 we have $E_2^{2,2} \cong H^2(G_S; \mathbb{Z}_p(k-1))$. This proves the first isomorphism in part (c). The second isomorphism in (c) follows in the same way by Lemma 2.3. $\square$

The Leray spectral sequence of the map $f_v : \mathcal{X}_v \to Spec\, \kappa_v$ gives the following Galois-theoretic description of the groups $H^i_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k))$.

**Proposition 2.5.**

(a)

$$H^0_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) = 0$$

$$H^1_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) \cong H^1(\kappa_v; \mathbb{Z}_p(k))$$

(b)

$$H^2_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) \cong H^1(\kappa_v; T_p(J_v)(k-1))$$

(c)

$$H^3_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) \cong H^1(\kappa_v; \mathbb{Z}_p(k-1))$$

$$H^i_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) = 0 \, for\, i > 3.$$

*Proof.* One can use the same reasoning as in the proof of Proposition 2.4. In particular, the isomorphism:

$$H^1_{cts}(\mathcal{X}_v; \mathbb{Z}_p(k)) \cong H^1(\kappa_v; \mathbb{Z}_p(k))$$

follows from the low terms exact sequence of the Leray spectral sequence, since by the result of Suslin [Su, Prop. 2.4] the group $H^0(\kappa_v; R^1 f_{v*} \mathbb{Z}_p(k))$ vanishes. We leave details to the reader. $\square$

**Proposition 2.6.**

(a) *There are exact sequences:*

$$0 \to H^4_{cts}(\mathcal{X};\mathbb{Z}_p(n+2)) \to K^{et}_{2n}(\mathcal{X}) \to H^2_{cts}(\mathcal{X};\mathbb{Z}_p(n+1)) \to 0$$

$$0 \to H^3_{cts}(\mathcal{X};\mathbb{Z}_p(n+2)) \to K^{et}_{2n+1}(\mathcal{X}) \to H^1_{cts}(\mathcal{X};\mathbb{Z}_p(n+1)) \to 0.$$

(b) *There is an isomorphism and an exact sequence:*

$$K^{et}_{2n}(\mathcal{X}_v) \cong H^2_{cts}(\mathcal{X}_v;\mathbb{Z}_p(n+1))$$

$$0 \to H^3_{cts}(\mathcal{X}_v;\mathbb{Z}_p(n+2)) \to K^{et}_{2n+1}(\mathcal{X}_v) \to H^1_{cts}(\mathcal{X}_v;\mathbb{Z}_p(n+1)) \to 0.$$

*Proof.* (a) By the Dwyer-Friedlander spectral sequence (2.3) and Proposition 2.4. we have an exact sequence:

$$\begin{array}{l} 0 \longrightarrow H^3_{cts}(\mathcal{X};\mathbb{Z}_p(n+2)) \longrightarrow K^{et}_{2n+1}(\mathcal{X}) \longrightarrow H^1_{cts}(\mathcal{X};\mathbb{Z}_p(n+1)) \xrightarrow{d} \\ \xrightarrow{d} H^4_{cts}(\mathcal{X};\mathbb{Z}_p(n+2)) \longrightarrow K^{et}_{2n}(\mathcal{X}) \longrightarrow H^2_{cts}(\mathcal{X};\mathbb{Z}_p(n+1)) \longrightarrow 0. \end{array}$$

Hence, it suffices to show that the differential $d = 0$. We have a commutative diagram:

$$\begin{array}{ccc} H^1_{cts}(\mathcal{X};\mathbb{Z}_p(n+1)) & \xrightarrow{d} & H^4_{cts}(\mathcal{X};\mathbb{Z}_p(n+2)) \\ \cong\uparrow & & \uparrow \\ H^1(\mathcal{O}_{F,S};\mathbb{Z}_p(n+1)) & \longrightarrow & H^4(\mathcal{O}_{F,S};\mathbb{Z}_p(n+2)) = 0 \end{array} \tag{2.9}$$

where the left vertical map is an isomorphism by Proposition 2.4 (a).
(b) It follows in the same way as (a) since $cd_p(\mathcal{X}_v) = 3$, compare [DF, p. 273]. □

## 3. Reduction of nontorsion elements in étale $K$-theory of a curve.

The algebraic $K$-groups of arithmetic schemes are expected to carry deep arithmetic and geometric information, according to Beilinson's striking conjectures which relate them to special values of $L$-functions. Yet still little is known on their structure. In the case of the ring of integers $\mathcal{O}_F$ of a number field $F$, Quillen proved that the groups $K_i(\mathcal{O}_F)$ are finitely generated and Borel computed their ranks. In [AB] D.Arlettaz and the first author investigated the kernel of the natural map

$$K_i(\mathcal{O}_F) \to \prod_{\mathfrak{p}\subseteq\mathcal{O}_F} K_i(\kappa_{\mathfrak{p}})$$

where $\mathfrak{p}$ runs through the maximal ideals of $\mathcal{O}_F$, and proved that this kernel is finite. This is trivial for $i$ even, since then $K_i(\mathcal{O}_F)$ itself is finite; the difficulty is for $i$ odd. We note that the same proof yields the corresponding result for rings of integers of global fields of positive characteristic. In this section, we consider the case of next dimension, that is of a curve $X$ over a global field $F$.

**Theorem 3.1.** *Assume that $X$ is a smooth, proper and geometrically irreducible curve defined over a global field $F$. Let $p$ be an odd rational prime, $p \neq char(F)$. Let $n > 0$ and let $S$ be a finite set of primes of $F$ which contains primes of bad reduction for $X$ and primes lying over $p$. Let $\mathcal{X}$ be the smooth and proper model of $X$ over the ring of $S$-integers of $F$. Assume that $x \in K_{2n}^{et}(\mathcal{X})$ is a nontorsion element. Given $M_1 = p^{m_1}$ a fixed power of $p$, there exist infinitely many primes $v \notin S$ such that $x$ reduces in $K_{2n}^{et}(\mathcal{X}_v)$ to an element of order at least $M_1$.*

*Proof.* By Proposition 2.6, Proposition 2.4 (c) we have the following commutative diagram.

$$
\begin{tikzcd}
0 \arrow[d] & \\
H^2(G_S;\mathbb{Z}_p(n{+}1)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
K_{2n}^{et}(\mathcal{X}) \arrow[r] \arrow[d] & K_{2n}^{et}(\mathcal{X}_v) \arrow[d, "\cong"'] \\
H^2_{cts}(\mathcal{X};\mathbb{Z}_p(n{+}1)) \arrow[r] \arrow[d] & H^2_{cts}(\mathcal{X}_v;\mathbb{Z}_p(n{+}1)) \\
0 &
\end{tikzcd}
\tag{3.1}
$$

Moreover, Proposition 2.4 (b), Proposition 2.5 (b) and the isomorphisms (2.8) give the following commutative diagram.

$$
\begin{tikzcd}
0 \arrow[d] & \\
H^2(G_S;\mathbb{Z}_p(n{+}1)) \arrow[r] \arrow[d] & 0 \arrow[d] \\
H^2_{cts}(\mathcal{X};\mathbb{Z}_p(n{+}1)) \arrow[r] \arrow[d] & H^2_{cts}(\mathcal{X}_v;\mathbb{Z}_p(n{+}1)) \arrow[d, "\cong"'] \\
H^1(G_S;T_p(J)(n)) \arrow[r] \arrow[d] & H^1(\kappa_v;T_p(J_v)(n)) \\
0 &
\end{tikzcd}
\tag{3.2}
$$

Here $J$ and $J_v$ denote the Jacobian varieties of $\bar{X}$ and $\bar{X}_v$, respectively. Since the group $H^2(G_S;\mathbb{Z}_p(n{+}1))$ is finite [So1, Theorem 4 and Remark], the nontorsion element $x \in K_{2n}^{et}(\mathcal{X})$, by the composition of the lower vertical arrows in the left columns of the diagrams (3.1) and (3.2), is sent to a nontorsion element of the group $H^1_{cts}(\mathcal{O}_{F,S};R^1 f_* \mathbb{Z}_p(n{+}1)) \cong H^1(G_S\ T_p(J)(n))$. The inflation map injects the last group into $H^1(G_F; T_p(J)(n))$. By the commutativity of the inflation and restriction maps:

$$\begin{array}{ccc} H^1(G_S; T_p(J)(n)) & \longrightarrow & H^1(\kappa_v; T_p(J_v)(n)) \\ \downarrow & & \downarrow \\ H^1(G_F; T_p(J)(n)) & \longrightarrow & H^1(G_v; T_p(J_v)(n)) \end{array} \tag{3.3}$$

the proof of Theorem 3.1 reduces to the following result which we apply for $A = J$.

**Lemma 3.3.** *Assume that $A$ is an abelian variety defined over a global field $F$. Let $p$ be an odd rational prime, $p \neq char(F)$ and $k > 0$. Let $S$ be a finite set of primes of $F$ which contains primes of bad reduction for $A$ and primes over $p$. Let $G_F = G(\bar{F}/F)$. Fix a choice of the decomposition group $G_v$ for every place of $F$. Let further $x \in H^1(G_F; T_p(A)(k))$ be a nontorsion element. Given $M_1 = p^{m_1}$ a fixed power of $p$, there exist infinitely many primes $v \notin S$ such that $x$ restricts in $H^1(G_v; T_p(A_v)(k))$ to an element of order at least $M_1$.*

*Proof.* By [J1, Th. 3(a), p.337] the kernel of the reduction map:

$$H^1(G_F; T_p(A)(k)) \to \prod_{v \notin S} H^1(G_v; T_p(A_v)(k)) \tag{3.4}$$

is a finite group. Consider the exact sequence:

$$0 \to H^1(\kappa_v;\, T_p(A)(k)) \to H^1(G_v;\, T_p(A)(k)) \to H^0(\kappa_v;\, T_p(A)(k-1)) \to 0, \tag{3.5}$$

which by the Weil conjectures implies that the group $H^1(G_v;\, T_p(A)(k))$ is finite for $v \notin S$ and positive $k$. Suppose that there are only finitely many $v \notin S$ such that the reduction $red_v(x) \in H^1(G_v;\, T_p(A_v)(k))$ has order bigger or equal to $M_1$. Then by (3.5) there exists an $n$ such that in the group $H^1(G_v;\, T_p(A_v)(k))$, $red_v(p^n x) = 0$ for all $v \notin S$. Hence, $p^n x$ belongs to the kernel of the reduction map (3.4). So it is a torsion element, a contradiction. $\square$

**Theorem 3.4.** *For $X$, $p$ and $n$ as in Theorem 3.1 the kernel of the reduction map:*

$$K_{2n}^{et}(\mathcal{X}) \to \prod_{v \notin S} K_{2n}^{et}(\mathcal{X}_v)$$

*is a finite group.*

*Proof.* It follows from Proposition 2.6, Proposition 2.4 (b) and Lemma 2.1 (b) that $K_{2n}^{et}(\mathcal{X})$ is a finitely generated $\mathbb{Z}_p$-module. Now, Theorem 3.4 is an immediate corollary of Theorem 3.1. $\square$

*Remark 3.5.* Let us consider the reduction map for the odd dimensional étale $K$-theory.

$$K_{2n+1}^{et}(\mathcal{X}) \to \prod_{v \notin S} K_{2n+1}^{et}(\mathcal{X}_v) \tag{3.6}$$

By Proposition 2.6 (a) we have the exact sequence:

$$0 \to H^3_{cts}(\mathcal{X};\, \mathbb{Z}_p(n{+}2)) \to K^{et}_{2n+1}(\mathcal{X}) \to H^1_{cts}(\mathcal{X};\, \mathbb{Z}_p(n{+}1)) \to 0. \tag{3.7}$$

By Proposition 2.4 (b) there is another exact sequence:

$$0 \to H^2(G_S; T_p(J)(n{+}1)) \to H^3_{cts}(\mathcal{X}; \mathbb{Z}_p(n{+}2)) \to H^1(G_S; \mathbb{Z}_p(n{+}1)) \to 0. \tag{3.8}$$

The group $H^2(G_S; T_p(J)(n{+}1))$ is predicted to be finite for any curve $X$ and $n > 0$. This follows from the weak Leopoldt conjecture cf. Remark 4.1 below. If the conjecture holds true, then one can prove an analog of Theorem 3.1 for the odd dimensional étale $K$-theory using the argument of this paper. This is implied by the following result.

**Proposition 3.6.** *For any curve $X$, the kernel of the reduction map:*

$$H^1_{cts}(\mathcal{X};\, \mathbb{Z}_p(n{+}1)) \to \prod_{v \notin S} H^1(\mathcal{X}_v;\, \mathbb{Z}_p(n{+}1))$$

*is a finite group.*

*Proof.* It is an immediate corollary of Proposition 2.4 (a), Proposition 2.5 (a) and [AB, Prop. 1]. □

## 4. On the group $\mathbf{H}^2(G_{S_0};\, T_p(J)(n{+}1))$ and divisible elements.

In the remainder of the paper $F$ will be a number field. In papers [BG1] and [BG2] two of us investigated arithmetic of the group of divisible elements $D_{n+1}(F) = div\,[K_{2n}(F)_p]$, which serves as an analogue for higher $K$-theory of the ideal class group of classical number theory. For $m$ big enough the group $D_{n+1}(F)$ can be expressed as the cokernel of the boundary map:

$$\partial_m :\, K_{2n+1}(F;\, \mathbb{Z}/p^m) \longrightarrow \bigoplus\nolimits_{v \nmid p} K_{2n}(\kappa_v;\, \mathbb{Z}/p^m).$$

In this section we consider the boundary map in the localization sequence in étale $K$-theory of a curve $X$:

$$\partial_m :\, K^{ét}_{2n+2}(X;\, \mathbb{Z}/p^m) \longrightarrow \bigoplus\nolimits_{v \notin S_0} K^{ét}_{2n+1}(\mathcal{X}_v;\, \mathbb{Z}/p^m) \tag{4.1}$$

The curve $X$ is defined over the number field $F$. Let $\mathcal{X}$ denote its model over $Spec\, \mathcal{O}_{F,S_0}$.

**Definition.** $C_m := Coker\, \partial_m$

The following condition is our working assumption in this section.

**Assumption.** *The group $H^2(G_{S_0};\, T_p(J)(n{+}1))$ is finite for $n > 0$.*

*Remark 4.1.* This is a straigthforward consequence of the weak Leopoldt conjecture for the $p$-adic representation $V = T_p(J) \otimes \mathbb{Q}_p$ formulated by Schneider and Perrin-Riou [PR1, p.63], see also [J1, p.317]. It follows by [PR2] and [K1, (9.6.7), p.57] that if there exists a $p$-adic Euler system for the representation $V = T_p(J) \otimes \mathbb{Q}_p$, then the Assumption holds true for infinitely many $n$. In the case when $X = E$ is a modular elliptic curve defined over the rationals, Kato has constructed in [K2] such an Euler system using Beilinson symbols of modular units. See [Ru1] for a survey of this work. If $E$ is an elliptic curve with complex multiplication by an imaginary quadratic field, then the Euler system of elliptic units from [Ru2] can be used to check the assumption for infinitely many $n$ cf. [PR2, 1.5.4, p.15]. Moreover, for a CM elliptic curve a result of Wingberg [Wi, Corollary 2, p.147] implies that the group $H^2(G_{S_0};\, T_p(E)(n{+}1))$ is finite for all positive $n$ and primes $p$ which are regular for $E$ cf. [So2, p.257].

We introduce the following notation.

**Definition.** *Let:*

$$w_j(F) = |H^0(G_F;\, \mathbb{Q}_p/\mathbb{Z}_p(j))|$$
$$v_j(F,\, J) = |H^0(G_F;\, J_\infty(j))|$$

*for $j > 1$, where $J_\infty = \bigcup_{k\geq 0} J[p^m]$. The numbers $v_j(F,\, J)$ are finite because of the Weil conjectures.*

**Theorem 4.2.** *Let $p$ be a prime such that $p \nmid w_n(F)w_{n+1}(F)v_{n+1}(F,\, J)$. For $m$ big enough we have:*

$$C_m \cong div\, H^2(G_F;\, T_p(J)(n{+}1))_p.$$

*In particular, $C_m$ embeds into the group $H^2(G_F;\, T_p(J)(n{+}1))$.*

We split the proof of the theorem into two lemmas.

**Lemma 4.3.** *If $p$ does not divide the number $w_n(F)w_{n+1}(F)v_{n+1}(F,\, J)$, then*

$$C_m = Ker[H^2(G_{S_0};\, T_p(J)(n{+}1))/p^m \to H^2(G_F;\, T_p(J)(n{+}1))/p^m].$$

*Proof.* Consider the following commutative diagram with exact rows:

(4.2)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^3_{cts}(\mathcal{X};\, \mathbb{Z}/p^m(n{+}2)) & \longrightarrow & K^{ét}_{2n+1}(\mathcal{X};\, \mathbb{Z}/p^m) & \longrightarrow & H^1_{cts}(\mathcal{X};\, \mathbb{Z}/p^m(n{+}1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^3_{cts}(X;\, \mathbb{Z}/p^m(n{+}2)) & \longrightarrow & K^{ét}_{2n+1}(X;\, \mathbb{Z}/p^m) & \longrightarrow & H^1_{cts}(X;\, \mathbb{Z}/p^m(n{+}1)) & \longrightarrow & 0
\end{array}$$

which is obtained by the naturality of the Dwyer-Friedlander spectral sequence for étale $K$-theory with finite coefficients:

$$E_2^{r,-s} = H^r_{cts}(\mathcal{X}, \mathbb{Z}/p^m(s/2)) \Rightarrow K^{et}_{s-r}(\mathcal{X};\, \mathbb{Z}/p^m)$$

[DF, Prop. 5.2, p.260]. Exactness of the rows in (4.2) follow in a similar way as for $\mathbb{Z}_p$-coefficients (cf. proof of Proposition 2.6 (a)) except for the vanishing of the differential $d$ in the diagram (2.9), which for finite coefficients is guarranteed by the assumption $p \nmid w_n(F)v_{n+1}(F,\, J)$. The right vertical map in (4.2) is an injection, which follows by the Leray spectral sequence (2.2) of the map $X \to \mathcal{X}$ for $\mathbb{Z}/p^m(n{+}1)$ and the assumption $p \nmid w_{n+1}(F)$. We leave details of the computation to the reader. The diagram (4.2) gives:

$$C_m = Ker[H^3_{cts}(\mathcal{X};\, \mathbb{Z}/p^m(n{+}2)) \to H^3_{cts}(X;\, \mathbb{Z}/p^m(n{+}2))].$$

Using the naturality of the Leray spectral sequence for $\mathbb{Z}/p^m(n{+}2)$ and the assumption $p \nmid w_{n+1}(F)$ we easily obtain the following commutative diagram with exact rows (cf. proof of Proposition 2.4 (b)).

(4.3)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(G_{S_0};\, J[p^m](n{+}1)) & \longrightarrow & H^3_{cts}(\mathcal{X};\, \mathbb{Z}/p^m(n{+}2)) & \longrightarrow & H^1(G_{S_0};\, \mathbb{Z}/p^m(n{+}1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(G_F;\, J[p^m](n{+}1)) & \longrightarrow & H^3_{cts}(X;\, \mathbb{Z}/p^m(n{+}2)) & \longrightarrow & H^1(G_F;\, \mathbb{Z}/p^m(n{+}1)) & \longrightarrow & 0
\end{array}$$

The right vertical arrow in (4.3) is an inflation map which is an injection by the Hochschild-Serre spectral sequence. This shows:

$$C_m = Ker[H^2(G_{S_0};\, J[p^m](n{+}1)) \to H^2(G_F;\, J[p^m](n{+}1))].$$

The claim follows now, because:

$$H^2(G;\, J[p^m](n{+}1)) = H^2(G;\, T_p(J)/p^m(n{+}1)) = H^2(G;\, T_p(J)(n{+}1))/p^m,$$

for $G = G_{S_0}$ ($G = G_F$, respectively) by properties of the Tate module and since we have $cd_p(G) = 2$. $\square$

**Lemma 4.4.**

(a) *If $H^2(G_{S_0};\, T_p(J)(n{+}1))$ is finite, then the group $H^2(G_S;\, T_p(J)(n{+}1))$ is finite for any finite $S \supseteq S_0$.*

(b) *For any finite $S \supseteq S_0$ the inflation map:*

$$H^2(G_S;\, T_p(J)(n{+}1)) \to H^2(G_F;\, T_p(J)(n{+}1))$$

*is injective.*

(c)
$$\varinjlim_{S} H^2(G_S;\, T_p(J)(n{+}1)) = H^2(G_F;\, T_p(J)(n{+}1))_p$$

*where the direct limit is over the inflation maps for finite sets $S \supseteq S_0$.*

*Proof.* (a) To simplify notation let us put $T = T_p(J)$. By finite generation of the cohomology groups of $G_S$ it is enough to show that:

$$H^2(G_S;\, T/p^m(n{+}1)) \cong H^2(G_S;\, T(n{+}1))/p^m$$

has order bounded independently of $m$. By Assumption this holds true for the set $S_0$. Consider the exact sequence:

(4.4)
$$\to H^2(G_{S_0};\, T/p^m(n{+}1)) \to H^2(G_S;\, T/p^m(n{+}1)) \to \bigoplus_{v\in S\setminus S_0} H^1(G_v;\, T/p^m(n)) \to 0$$

cf. [So3, exact sequence, p.117]. The sequence implies that it suffices to check that for $v \in S \setminus S_0$ the group $H^1(G_v;\, T/p^m(n))$ has order bounded independently of $m$. Consider the obvious exact sequence:

$$(4.5)\quad 0 \to H^1(G_v;\, T(n))/p^m \to H^1(G_v;\, T/p^m(n)) \to H^2(G_v;\, T(n))[p^m] \to 0.$$

Let $I_v$ denote the inertia group of $v$ and $\kappa_v$ the residue field. By the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(\kappa_v;\, H^q(I_v;\, T(n))) \Rightarrow H^{p+q}(G_v;\, T(n))$$

and the result of Soulé, [So1, III.1.3., p.168] we obtain that the groups $H^i(G_v;\, T(n))$, for $i = 1, 2$ are finite. Hence, the conclusion follows by the exact sequence (4.5).

(b) Consider the exact sequence of Galois modules:

$$0 \to T(n{+}1) \to V(n{+}1) \to V/T(n{+}1) \to 0$$

where for simplicity we have put $T = T_p(J)$ and $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Observe that in this notation $V/T = J_\infty = \bigcup J[p^m]$ with the usual Galois action. The above exact sequence induces the following commutative diagram.

$$(4.6)\quad \begin{array}{ccc} H^2(G_S;\, T(n{+}1))_p & \xrightarrow{inf} & H^2(G_F;\, T(n{+}1))_p \\ \simeq\uparrow & & \simeq\uparrow \\ H^1(G_S;\, V/T(n{+}1))/Div & \longrightarrow & H^1(G_F;\, V/T(n{+}1))/Div \end{array}$$

The vertical maps in the diagram (4.6) are isomorphisms by [T, Proposition 2.3, p.261]. Consider the exact sequence:

$$(4.7)\quad 0 \to H^1(G_S;\, V/T(n{+}1)) \xrightarrow{inf} H^1(G_F;\, V/T(n{+}1)) \to \bigoplus_{v\notin S} H^0(G_v;\, V/T(n))$$

cf. [So3, p.117]. Since by the Weil conjectures the last group on the right in (4.7) is a direct sum of finite groups, we conclude that the groups on the left have the same maximal divisible subgroup. This shows that the lower horizontal map in the diagram (4.6) is injective.

In order to prove part (b), we pass with $S$ to the direct limit in the sequence (4.7). This gives:

$$\varinjlim_{S} H^1(G_S;\, V/T(n{+}1))/Div = H^1(G_F;\, V/T(n{+}1))/Div.$$

Now, taking the direct limit over $S$ in (4.6) completes the proof. $\square$

*Proof of Theorem 4.2.* By the diagram (4.6) and the exact sequence (4.7) applied to $S = S_0$ we get:

$$div\, H^2(G_F;\, T(n{+}1))_p \subseteq H^2(G_{S_0};\, T(n{+}1))_p. \tag{4.8}$$

Let us choose $m$ so big that $p^m$ is divisible by the exponent of $H^2(G_{S_0};\, T_p(J)(n{+}1))$. By Lemma 4.4 (b) the kernel of the natural map:

$$H^2(G_{S_0};\, T(n{+}1)) \to H^2(G_F;\, T(n{+}1))_p/p^m$$

equals:

$$H^2(G_{S_0};\, T(n{+}1)) \cap [p^m H^2(G_F;\, T(n{+}1))_p].$$

By the finiteness of the group $H^2(G_{S_0};\, T(n{+}1))$, if $p^m$ is big enough, then the last intersection equals $div\, H^2(G_F;\, T(n{+}1))_p$. The claim of the theorem follows now by Lemma 4.3. $\square$

## 5. Soulé elements and Galois representations.

Let $\mathfrak{b} :\, H^1(G; T/p^m(n{+}1)) \to H^2(G; T(n{+}1))[p^m]$ be the Bockstein map which is induced in cohomology by the exact sequence of Galois modules ($G = G_F,\, G_S$):

$$0 \longrightarrow T(n{+}1) \xrightarrow{\times p^m} T(n{+}1) \longrightarrow T/p^m(n{+}1) \longrightarrow 0.$$

In this section we use the map $\mathfrak{b}$ to construct elements in $H^2(G; T(n{+}1))[p^m]$. From now on we fix a power $p^m$ which is divisible by exponent of the $p$-torsion part of the group $H^2(G_{S_0}; T(n{+}1))$. This is possible, since the group is a finitely generated $\mathbb{Z}_p$-module. We consider $p$-adic representations

$$\rho : G_F \to GL(T)$$

with the following properties.

(1) The representation $\rho : G_F \to GL(T)$ is unramified except for a finite set of primes $S_0$ which contains primes above $p$.
(2) For the mod $p$ representation $\bar{\rho}_1 : G_F \to GL(T/p)$, let $E = (\bar{F})^{ker\, \bar{\rho}_1}$ be the field fixed by the kernel of $\bar{\rho}_1$. We assume that $\mu_p \subset E$ and that the degree $[E : F]$ is relatively prime to $p$.
(3) For the mod $p^m$ representation $\bar{\rho}_m : G_F \to GL(T/p^m)$, let $E' = (\bar{F})^{ker\, \bar{\rho}_m}$ be the field fixed by the kernel of $\bar{\rho}_m$. We assume that: $\mu_{p^m} \subset E'$.
(4) We assume that the mod $p^m$ representation $\bar{\rho}_m : G_F \to GL(T/p^m)$ has solvable image.

For a representation $\rho$ with properties (1)-(4), we describe the $p$-torsion part of the group $H^2(G_{S_0}; T(n{+}1))$ in terms of special elements in Galois cohomology which come from products, units and corestriction maps. The elements were introduced by Deligne and independently by Soulé in a different context, [So3]. Our method is based on the construction from [BG2], where the special case of $T(n{+}1) = \mathbb{Z}_p(n{+}1)$ was considered. The Galois action in this case is given by the cyclotomic character $\rho : G_{\mathbb{Q}} \to \mathbb{Z}_p^{\times}$ for which the assumptions (1)-(4) are clearly satisfied. We leave the proof of the following elementary lemma for the reader.

**Lemma 5.1.** *Let $S$ be a finite of primes of $F$ which contains $S_0$.*

(1) *Representations $\rho$ and $\bar{\rho}_m$ factor through the group $G_S = G(F_S/F)$. We denote by $\rho_S : G_S \rightarrow GL(T)$ and $\bar{\rho}_{S,m} : G_S \rightarrow GL(T/p^m)$, the representations of $G_S$ which are obtained in this way. We continue to denote the fixed fields $(F_S)^{ker \bar{\rho}_{S,1}}$ and $(F_S)^{ker \bar{\rho}_{S,m}}$ by $E$ and $E'$, respectively.*
(2) *The conditions (1)-(4) above are satisfied for $\rho_S$, if they hold true for the representation $\rho$.*

The proof of Theorem 5.5 below, which is the main result of this section relies on the following two lemmas.

**Lemma 5.2.** *Let $L/K$ be a finite Galois extension of number fields. Let $K_1$, $K_2$, $\dots$, $K_r$ be Galois extensions of $K$ such that for every $i$:*

$$L \quad \supseteq \quad K_i \quad \supseteq \quad K.$$

*Assume that the Galois groups $G(L/K_i)$ are generated by inertia subgroups of finite primes of $\mathcal{O}_L$. Then there is a finite set of primes $S$ of $\mathcal{O}_K$ such that $Pic\, \mathcal{O}_{K_i,S} = 0$ for every $1 \leq i \leq r$ and every prime from $S$ splits completely in $L$.*

*Proof.* We consider one subextension $K_i$ at the time, e.g.:

$$L \quad \supseteq \quad K_1 \quad \supseteq \quad K.$$

Let $H_{K_1}$ denote the Hilbert class field of $K_1$. The extension $H_{K_1}/K$ is Galois because $H_{K_1}$ is the maximal unramified extension of $K_1$ and $K_1/K$ is Galois. We use the Artin isomorphism:

$$Pic\, \mathcal{O}_{K_1} \xrightarrow{\sim} G(H_{K_1}/K_1)$$

which sends class $[\mathfrak{p}']$ of ideal $\mathfrak{p}'$ to its Frobenius automorphism $\widetilde{Fr_{\mathfrak{p}'}}$. By the assumptions we have:

$$L \cap H_{K_1} = K_1.$$

Hence we can find an element $\gamma \in G(H_{K_1}L/K_1)$ such that:

$$\gamma|_L = id \quad \text{and} \quad \gamma|_{H_{K_1}} = \widetilde{Fr_{\mathfrak{p}'}}.$$

Since $G(H_{K_1}L/K_1) \subseteq G(H_{K_1}L/K)$ we think of $\gamma$ as an element $G(H_{K_1}L/K)$. By the Chebotarev density theorem [La] there is an ideal $\mathfrak{P}$ in the field $H_{K_1}L$ over some ideal $v$ (unramified in $H_{K_1}L$) such that in the extension $H_{K_1}L/K$ we have: $Fr_{\mathfrak{P}} = \gamma$. Let us put:

$$\mathfrak{h} = \mathfrak{P} \cap \mathcal{O}_{H_{K_1}}, \quad \mathfrak{p} = \mathfrak{P} \cap \mathcal{O}_{K_1}, \quad v = \mathfrak{P} \cap \mathcal{O}_K \quad \text{and} \quad \mathfrak{q} = \mathfrak{P} \cap \mathcal{O}_L$$

the corresponding ideals lying below $\mathfrak{P}$.

$$\begin{tikzcd}
& H_{K_1}L \arrow[dl, dash] \arrow[dr, dash] & \\
L \arrow[dr, dash] & & H_{K_1} \arrow[dl, dash] \\
& K_1 \arrow[d, dash] & \\
& K &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathfrak{P} \arrow[dl, dash] \arrow[dr, dash] & & \\
\mathfrak{q} \arrow[dr, dash] & & \mathfrak{h} \arrow[dl, dash] & \mathfrak{h}' \arrow[dl, dash] \\
& \mathfrak{p} \arrow[d, dash] & \mathfrak{p}' \arrow[d, dash] & \\
& v & v' &
\end{tikzcd}$$

Clearly, we have:

$$Fr_{\mathfrak{h}} = Fr_{\mathfrak{P}}|_{H_{K_1}} = \gamma|_{H_{K_1}} = \widetilde{Fr_{\mathfrak{p}'}}$$

and

$$Fr_{\mathfrak{p}} = Fr_{\mathfrak{h}}|_{K_1} = Fr_{\mathfrak{P}}|_{K_1} = \gamma|_{K_1} = id.$$

We get:

$$\widetilde{Fr_{\mathfrak{p}'}} = Fr_{\mathfrak{h}} = \widetilde{Fr_{\mathfrak{p}}}.$$

So we get:

$$[\mathfrak{p}] = [\mathfrak{p}'].$$

We also have:

$$Fr_{\mathfrak{q}} = Fr_{\mathfrak{P}}|_L = \gamma|_L = id.$$

It follows that the prime $v$ splits completely in $L$. Let us find such primes $v$ for all ideal classes $[\mathfrak{p}'] \in Pic\,\mathcal{O}_{K_1}$. Let $S_1$ denote the set of the all primes $v$. We obtain $Pic\,\mathcal{O}_{K_1,S_1} = 0$. To finish the proof we choose the set of primes $S_i$ as above, for every field $K_i$ and we take $S = S_0 \cup S_1 \cup S_2 \cup \cdots \cup S_r$. $\square$

**Lemma 5.3.** *Let us assume that for every $p^m$ the corresponding extension $E'/E$ has the following property: there exists a solvable tower*

$$E' \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_{r-1} \supseteq F_r \supseteq E$$

*such that for each $i$ the degree $[F_{i-1}; F_i]$ is a prime number and that in $F_{i-1}/F_i$ only one prime over primes in $S_0$ has nontrivial decomposition group. Then there exists a finite set of primes $S$ of $\mathcal{O}_E$ such that $S_0 \subseteq S$ and the norm map:*

$$N : \mathcal{O}_{E',S}^{\times} \to \mathcal{O}_{E,S}^{\times}$$

*is surjective.*

*Proof.* It follows by Lemma 5.2 that there exists a finite set of primes $S$ of $\mathcal{O}_E$ such that $Pic\,\mathcal{O}_{F_i,S} = 0$ for any $i$, and all primes from $S$ split completely in $E'$. The claim follows now by induction over $i$, where the inductive step is a repetition of the Euler characteristic calculation from the proof of Lemma 2.1 in [BG2]. □

*Remark 5.4.* In particular, the last lemma holds true for an extension $E'/E$ where $S_0$ contains only one prime which in addition is totally ramified in $E'$

¿From now on we work with the set $S$ from Lemma 5.3. To simplify notation let us put $G' = G(F_S/E')$ and $G = G(F_S/E)$. Observe that for any $G'$-module $N$ and for any trivial, discrete $G'$-module $W$, which is a free $\mathbb{Z}_p$-module of finite rank (is a free $\mathbb{Z}/p^m$-module of finite rank, respectively), the cup product in group cohomology induces an isomorphism:

$$H^*(G'; N) \otimes H^0(G'; W) = H^*(G'; N \otimes W).$$

It follows easily by assumptions (1)-(4) from the beginning of the section and Lemma 5.1 that $G'$ acts trivially on modules $\mathbb{Z}/p^k(n)$ and $T/p^k$ for $1 \leq k \leq m$. Thus, we get that for any $1 \leq k \leq m$ the following map induced by the cup product is an isomorphism.

(5.1)
$$H^1(G'; \mathbb{Z}/p^k(1)) \otimes H^0(G'; \mathbb{Z}/p^k(n)) \otimes H^0(G'; T/p^k) \longrightarrow H^1(G'; T/p^k(n+1))$$

For the same reason we have:

$$H^0(G'; \mathbb{Z}/p^k(n)) \otimes H^0(G'; T/p^k) \cong T/p^k(n). \tag{5.2}$$

Let

$$\alpha'_k : \mathcal{O}^\times_{E',S} \otimes T/p^k(n) \to H^1(G'; T/p^k(n+1)) \tag{5.3}$$

be the map obtained by precomposing the map (5.1) with the map which is induced by the Kummer map: $\mathcal{O}^\times_{E',S} \to H^1(G'; \mathbb{Z}/p^k(1))$ and the isomorphism (5.2). To formulate the next theorem we need additional notation. Let $\beta = \xi_{p^m} \in H^0(G', \mathbb{Z}/p^m(1))$ be a fixed generator. Consider the inflation map:

$$H^2(G_{S_0}; T(n+1)) \longrightarrow H^2(G_S; T(n+1))$$

which is defined by the extension:

$$1 \to G(F_S/F_{S_0}) \to G_S \to G_{S_0} \to 1.$$

It follows by diagram (4.6) that the inflation map is injective on $p$-torsion, cf. [T, Proposition 2.3, p. 261]. We identify the group $H^2(G_{S_0}; T(n+1))_p$ with its image by the inflation in $H^2(G_S; T(n+1))_p$.

**Theorem 5.5.** *Every element of the group $H^2(G_{S_0};T(n+1))_p$ can be written in the form:*

$$\mathfrak{b}[Tr(\alpha'_m(u\otimes\beta^n\otimes v))]$$

*for some $S$-unit $u\in\mathcal{O}^\times_{E',S}$ and some $v\in T/p^m$. The maps $\mathfrak{b}$ and $Tr$ are the Bockstein homomorphism:*

$$\mathfrak{b}:\, H^1(G_S;T/p^m(n+1))\to H^2(G_S;T(n+1))[p^m]$$

*and the transfer map:*

$$Tr:\, H^1(G';T/p^m(n+1))\to H^1(G_S;T/p^m(n+1))$$

*respectively.*

*Proof.* We adapt the argument from [BG2, proof of Theorem 2.4] to the current situation. Consider the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{O}^\times_{E',S}\otimes T/p(n) & \xrightarrow{N\otimes id} & \mathcal{O}^\times_{E,S}\otimes T/p(n)\\
{\scriptstyle\alpha'_1}\big\downarrow & & {\scriptstyle\alpha_1}\big\downarrow{\scriptstyle\cong}\\
H^1(G';T/p(n+1)) & \xrightarrow{Tr} & H^1(G;T/p(n+1))\\
{\scriptstyle\mathfrak{b}}\big\downarrow & & {\scriptstyle\mathfrak{b}}\big\downarrow\\
H^2(G';T(n+1))[p] & \xrightarrow{Tr} & H^2(G;T(n+1))[p]\\
{\scriptstyle Tr}\big\downarrow & & {\scriptstyle Tr}\big\downarrow\\
H^2(G_S;T(n+1))[p] & \xrightarrow{=} & H^2(G_S;T(n+1))[p]
\end{array}\tag{5.4}$$

The map $\alpha_1$ is defined in the same way as $\alpha'_1$, since by assumptions (1)-(4) from the beginning of the section $G$-modules $\mathbb{Z}/p(n)$ and $T/p$ are trivial. It follows by the Kummer exact sequence that the map $\alpha_1$ is an isomorphism because by Lemma 5.3 the group $Pic\,\mathcal{O}_{E,S}$ vanishes. Commutativity of the diagram follows easily by the projection formula and because transfer commutes with Bockstein maps. By Lemma 5.2 the upper horizontal map in (5.4) is surjective. By assumption (2) the degree $[E:F]$ is relatively prime with $p$, hence the transfer map:

$$Tr:\, H^2(G;T(n+1))[p]\to H^2(G_S;T(n+1))[p]$$

is surjective. This implies that the composition of upper horizontal nad right vertical maps in the diagram (5.4) is a surjective map. Using commutativity of $\mathfrak{b}$ and $Tr$ we conclude that the composition:

$$\mathcal{O}^\times_{E',S}\otimes T/p(n)\xrightarrow{\alpha'_1} H^1(G';T/p(n+1))\xrightarrow{Tr}$$
$$\xrightarrow{Tr} H^1(G_S;T/p(n+1))\xrightarrow{\mathfrak{b}} H^2(G_S;T(n+1))[p]$$

is a surjection. By induction over $k$ (with $1 \leq k \leq m$) which goes the same way as in the proof of Lemma 2.3 in [BG2], we obtain that the composition:

$$\begin{aligned} \mathcal{O}_{E',S}^{\times} \otimes T/p^m(n) &\xrightarrow{\alpha'_m} H^1(G'; T/p^m(n{+}1)) \xrightarrow{Tr} \\ &\xrightarrow{Tr} H^1(G_S; T/p^m(n{+}1)) \xrightarrow{\mathfrak{b}} H^2(G_S; T(n{+}1))[p^m] \end{aligned}$$

is also a surjective map. $\square$

*Remark 5.6.* It is easy to check that for any finite abelian extension $L$ of the number field $F$ the following coditions are equivalent.

(a) $G(L/F)$ is generated by its inertia subgroups.
(b) Every subextension $F \subseteq F' \subseteq L$ is ramified over $F$.
(c) $H_F \cap L = F$, where $H_F$ denotes the Hilbert class field of $F$.

*Example 5.7.* Let $A$ be a selfdual abelian variety which has complex multiplication by a CM field $K$ and is defined over $K$. Let us consider representations:

$$\rho:\, G_K \to GL(T_p(A))$$

$$\rho':\, G_K \to GL(H^2(\bar{A}; \mathbb{Z}_p(2)))$$

where we have put $\bar{A} = A \times_K \bar{K}$. Recall that we have:

$$H^j(\bar{A};\, \mathbb{Z}_p(i)) = \bigwedge^j H^1(\bar{A};\, \mathbb{Z}_p(1))(i{-}j) = \bigwedge^j T_p(A)(i{-}j).$$

by the identification of the $H^1$ above with the Tate module of the dual variety. By the main theorem of complex multiplication [La2, Theorem 6.1, p.78] $\rho$ and $\rho'$ factor through abelian quotients. Observe that:

$$\bar{K}^{ker\,\rho'} \subset \bar{K}^{ker\,\rho}. \tag{5.9}$$

In addition, we make the following assumptions.

(i) There is only one prime in $K$ over $p$.
(ii) $A$ has good reduction outside the prime above $p$. Hence, in this case $S_0$ contains only one element.
(iii) $\bar{K}^{ker \rho} \cap H_K = K$

By (5.9) we have $\bar{K}^{ker \rho'} \cap H_K = K$. Assumptions (i), (ii), (iii) and Remark 5.6 imply that conditions of Remark 5.4 are fulfilled. The assumptions (1)-(4) of the beginning of this section are met for such representations $\rho$ and $\rho'$. Indeed (1) holds by (i) and by the theorem of Serre and Tate [La2, Theorem 3.4, p. 46]. Observe that $G(K(A[p])/K)$ imbeds into $(\mathcal{O}_K/p)^{\times}$ by the main theorem of complex multiplication [La2, p. 84-89]. If the number of elements of $(\mathcal{O}_K/p)^{\times}$ is prime to $p$, then the assumption (2) follows. If it is not, then we can replace the field of definition $K$ by an extention $F$ contained in $K(A[p])$ so (2) is met. Note that condition (3) holds by the Weil pairing and (4) is obvious. Now we apply Lemma 5.3 to choose $S$. By Theorem 5.5 we conclude that the groups:

$$H^2(G_{F,S};\, T_p(A)(n{+}1))_p \quad \text{and} \quad H^2(G_{F,S};\, H^2(\bar{A};\, \mathbb{Z}_p(n{+}1)))_p$$

are generated by the elements of the form:

$$\mathfrak{b}[Tr(\alpha'_m(u \otimes \beta^n \otimes v))] \quad \text{and} \quad \mathfrak{b}[Tr(\alpha'_m(u' \otimes \beta^{n-2} \otimes v'))],$$

respectively.

The following two examples are special cases of Example 5.7.

*Example 5.8.* Let $X$ be the Fermat curve given in $\mathbb{P}^2_{\mathbb{Q}}$ by the equation:

$$x^k{+}y^k = z^k.$$

Clearly $X$ is defined over $\mathbb{Q}$ and it has bad reduction only at primes dividing $k$. The Jacobian variety $J$ of $X$ is defined over $\mathbb{Q}$ and has complex multiplication by the cyclotomic field $K = \mathbb{Q}(\mu_k)$. We consider $J$ over the field $K$. Take $k = p^r$ and put $F = K(J[p])$, hence $K \subset F$. Setting $A = J$ in Example 5.7 we only need to check for which prime powers $k$ the condition:

$$\bar{K}^{ker\,\rho} \cap H_K = K$$

holds, since for such $k$'s the only prime in $F$ over $p$ is totally ramified in $\bar{K}^{ker\,\rho}$ and in $\bar{K}^{ker\,\rho'}$. By tables in [Wa, p. 352-360] it holds true, for example for $k = 3$, 5, 7, 9, 11, 13, 17, 19, 25, 27 and 81. Observe that for all these values of $k$, except for $k = 81$, we have $H_K = K$. We could not replace $F$ by $K$ in this case, since the assumption (2) may not hold. It is easy to see that the number of elements of $(\mathcal{O}_K/p)^\times$ is not prime to $p$.

*Example 5.9.* Let us consider an elliptic curve $E$ defined over $\mathbb{Q}$ with CM [Si, p.483] by a quadratic field $K$. We consider curves from second table on this page with discriminat a prime power and corresponding to quadratic fields: $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-11})$, $\mathbb{Q}(\sqrt{-19})$, $\mathbb{Q}(\sqrt{-43})$, $\mathbb{Q}(\sqrt{-67})$, $\mathbb{Q}(\sqrt{-163})$. All these elliptic curves are over $\mathbb{Q}$, hence the Hilber class field for every one of these curves is just $K$. We can now put $A = E$ in Example 5.7.

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