Moduli of Sheaves on Surfaces and Action of the Oscillator Algebra

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Introduction

Let S be a smooth complex projective surface and $Hilb^n(S)$ the Hilbert scheme of all length n zero-dimensional subschemes of S. It is known (cf. [Fo]) that $Hilb^n(S)$ is a smooth projective variety of dimension $2n$. The structure of the cohomology ring of $Hilb^n(S)$ for a fixed n is rather difficult to understand. However, when we consider the direct sum $\bigoplus_{n\geq 0} H^*(Hilb^n(S))$ (all cohomology in this paper will be with complex coefficients) the picture becomes more comprehensible.

Firstly, for any complex smooth algebraic variety X of dimension d, let $P_t(X)$ be the *shifted* Poincaré polynomial $\sum_{i=0}^{d} t^{i-d}\cdot \dim_{\mathbb{C}} H^i(X)$. It was shown by Göttsche [Gö1] that, for any smooth quasi-projective surface S

$$\sum_{n\geq 0} q^n P_t(Hilb^n(S)) = \prod_{l=1}^{\infty} \frac{(1+t^{-1}q^l)^{b_1(S)}(1+tq^l)^{b_3(S)}}{(1-t^{-2}q^l)^{b_0(S)}(1-q^l)^{b_2(S)}(1-t^2q^l)^{b_4(S)}},$$

where $b_i(S)$ are the Betti numbers of S.

Vafa and Witten [VW] have noticed that the right hand side of the formula above is an irreducible character of the oscillator (or Heisenberg/Clifford) algebra $\mathcal{H}$ defined for a smooth projective surface S as follows:

(a) $\mathcal{H}$ is generated by elements p_i^α, $\alpha \in H^*(S)$, $i \in \mathbb{Z}\setminus 0$;

(b) $[p_i^\alpha, p_j^\beta] = \langle \alpha, \beta\rangle i\cdot \delta_{i+j,0}$;

where the commutator above is to be understood in the graded sense and $\langle \cdot,\cdot\rangle$ is the intersection form on S. The action of the oscillator algebra can be interpreted as follows: there is a standard realization of Fock representation in the space of all symmetric polynomials in infinitely many variables. This realization appears from the cohomology of symmetric powers $Sym^n(S)$ which are related to Hilbert schemes of points via the Hilbert-Chow morphism $Hilb^n(S) \to Sym^n(S)$.

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And indeed, Nakajima and Grojnowski (cf. [Na], [Gr]) have constructed the expected action of $\mathcal{H}$ on $\bigoplus_{n\geq 0} H^*(Hilb^n(S), \mathbb{C})$ using some explicit cycles in the products of Hilbert schemes.

The Hilbert scheme can be viewed as a moduli space of rank one sheaves on S with trivial determinant and $c_2 = n$. It was conjectured by Vafa and Witten in [VW] (see also [Na], [Gr]) that there should be a higher rank extension of the results above. This paper provides such a generalization (cf. Theorem 4.1). The key obsevation which stands behind our arguments is that in certain cases a non-compact moduli space $\mathcal{M}$ may admit two different natural compactifications $\mathcal{M}_1$, $\mathcal{M}_2$ and a map $\mathcal{M}_1 \to \mathcal{M}_2$ which (partially) resolves the singularities of $\mathcal{M}_2$ and restricts to identity on the copies of $\mathcal{M}$ in $\mathcal{M}_1$ and $\mathcal{M}_2$.

The list of examples includes moduli of maps of curves to flag varieties ($\mathcal{M}_1$ and $\mathcal{M}_2$ are Laumon and Drinfeld compactifications, respectively, cf [Ku]), instantons on ALE spaces (quiver variety and the Uhlenbeck-type compactification cf. [N]), moduli of abelian varieties (Voronoi and Satake compactifications) and, finally, moduli of stable bundles on surfaces (Gieseker and Uhlenbeck compactifications). Thus, the Hilbert scheme is replaced by the *Gieseker moduli space* $M^G(r,n)$ of stable torsion-free sheaves and the role of the symmetric power is played by the Uhlenbeck compactification $M^U(r,n)$.

The first key result of this paper is that the fibers of the natural map $M^G(r,n) \to M^U(r,n)$ are irreducible of expected dimension (this generalization of the results by Briançon and Iarrobino (cf. [Br], [Ia]) was also proved independently by Ellingsrud and Lehn, cf. [EL]). This leads to the second important theorem saying that, in the coprime case, the natural morphism $M^G(r,n) \to M^U(r,n)$ is strictly semismall in the sense of Goresky-MacPherson (if a certain technical condition is satisfied: the surface S is allowed, for instance, to be rational, birationally ruled, K3 or abelian). This result was originally conjectured by V. Ginzburg and the corresponding statement is true for some of the moduli spaces mentioned above. We proceed further to give a natural generalization of the rank one correspondences yielding an action of the oscillator algebra on $\bigoplus_{n\geq 0} H^*(M^G(r,n))$. We show how to extend Nakajima's proof of commutation relations to the higher ranks case. Some parts of the proof require a more detailed analysis of the geometry of moduli spaces, than in the case of Hilbert schemes (see Sections 5 and 6).

It was communicated to us by G. Moore that L^2-cohomology of Uhlenbeck compactifications have recently appeared in string theory. Since L^2-cohomology coincides in several interesting cases with intersection homology, this provides another motivation for studying these homology groups. This paper can be regarded as a first step in application of intersection homology to moduli spaces of sheaves on surfaces.

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Contents

1 Moduli spaces.

Let S be a smooth complex projective surface. We denote by $Sym^n S$ the n-th symmetric power of S. Choose and fix an ample line bundle H, an arbitrary line bundle L and a positive integer r.

Consider the moduli space $N(r,n)$ of Gieseker H-stable vector bundles E of rank r with fixed determinant L and $c_2(E) = n$ (cf. [Ma]). We will only consider the case when $gcd(r, c_1(L) \cdot c_1(H)) = 1$. Since the bundles L and H will be fixed we drop them from notation.

The moduli space $N(r,n)$ is non-compact and it can be compactified in two different ways. The first compactification is the *Gieseker moduli space* $M^G(r,n)$ of all H-stable torsion-free sheaves E of rank r with $det(E) = L$ and $c_2(E) = n$ (cf. [Ma]). The second compactification is the *Uhlenbeck moduli space* $M^U(r,n)$ which can be described as follows. Take the disjoint union $\coprod_{s=0}^{\infty} N(r, n-s) \times Sym^s S$. Then Uhlenbeck's theorem [Uh] says that any sequence of points in one piece of this disjoint union has a subsequence that converges (in some sense) to a point in another piece. This endows $\bigcup_{s=0}^{\infty} N(r, n-s) \times Sym^s S$ with a topology of a compact space. Following [DK] one can show that this topology is even metrizable. This topological space is called the Uhlenbeck compactification $M^U(r,n)$ of $N(r,n)$.

Note that the union above is in fact finite since a necessary condition for non-emptiness of $N(r,k)$ is the Bogomolov inequality $2rk - (r-1)(c_1(L))^2 > 0$. We can always tensor all our bundles and sheaves with a high power of H and achieve $c_1^2(L) > 0$. Then Bogomolov inequality implies that $k = n - s$ should be at least positive.

Hence without loss of generality we can assume that $0 \leq s < n$.

In general the two compactifications above may be quite difficult to investigate: $N(r,n)$ may not be dense in $M^G(r,n)$ or $M^U(r,n)$, some components may be of dimension higher than expected, etc. For this reason we introduce a

Technical Condition.

(a) The integers r and $d := c_1(L) \cdot c_1(H)$ are coprime.
(b) Either the canonical bundle K_S is trivial or $c_1(K_S) \cdot c_1(H) < 0$.

The main reason for imposing (a) and (b) is that they ensure (cf. [HL]) that for any $n \geq 0$ the moduli space $M^G(r,n)$ is either empty or smooth of expected dimension

$$\dim M^G(r,n) = 2rn - (r-1)(c_1(L))^2 - (r^2-1)\chi(\mathcal{O}_S) + h^1(\mathcal{O}_S). \qquad (1)$$

This condition (b) is automatic when $(-K)$ is represented by an effective curve. Note that since $N(r,n)$ is an open subset of $M^G(r,n)$ (*loc. cit.*), it also has to be smooth.

Since r and d are assumed to be coprime, the two possible notions of stability (Mumford-Takemoto and Gieseker) coincide.

Examples. If we require that K_S is trivial or effective then S can be a Del Pezzo surface, K3 or abelian surface. If we make a special choice of H as above the list of examples will extend to all rational surfaces and birationally ruled surfaces.

It is known (e.g. [DK 10.3.4]) that for some elliptic surfaces or surfaces of general type the condition (1) may fail.

2 Gieseker to Uhlenbeck.

The Gieseker and Uhlenbeck compacifications are higher rank analogues of the Hilbert scheme of all length n subschemes $Hilb^n(S)$ and the symmetric power $Sym^n(S)$, respectively. Here we prefer to think of $Hilb^n(S)$ as parametrizing rather the ideal sheaves, i.e. torsion-free sheaves of rank one with $c_2 = n$ and trivial determinant. One has a natural map $a_1 : Hilb^n(S) \to Sym^n(S)$:

$$a_1 : J_\xi \mapsto \sum_{x_i \in Supp\ \xi} mult_{x_i}(\xi) \cdot x_i$$

where J_ξ is an ideal sheaf of a subscheme ξ and the formal sum above is viewed as an element of $Sym^n(S)$.

There exists a natural map $a_r : M^G(r,n) \to M^U(r,n)$ generalizing a_1. To define a_r first note that, for any torsion-free sheaf $\mathcal{F}$, the double-dual $\mathcal{F}^{**}$ is reflexive. It is known (cf. [OSS]) that any reflexive sheaf on a smooth variety is locally free on the complement of a closed subset of codimension ≥ 3. Hence in our case ($\dim S = 2$) the double-dual is necessarily locally free.

Consider the short exact sequence:

$$0 \to \mathcal{F} \to \mathcal{F}^{**} \to A_{\mathcal{F}} \to 0.$$

The quotient sheaf $A_{\mathcal{F}} = \mathcal{F}^{**}/\mathcal{F}$ is supported on finitely many points (i.e. $A_{\mathcal{F}}$ is an Artin sheaf). Denote by $l_{\mathcal{F}}$ the length of $A_{\mathcal{F}}$. We define $a_r(\mathcal{F})$ by

$$a_r : \mathcal{F} \mapsto (\mathcal{F}^{**}, \sum_{x_i \in Supp\ A_{\mathcal{F}}} mult_{x_i}(A_F) \cdot x_i)$$

where the image is a point in $N(r, n - l_{\mathcal{F}}) \times Sym^{l_{\mathcal{F}}}(S) \subset M^U(r,n)$.

Remark. Another consequence of r and $d = c_1(H) \cdot c_1(L)$ being *coprime* is that any semistable sheaf is necessarily stable. This ensures that $\mathcal{F}^{**}$ is stable if $\mathcal{F}$ is, hence the definition above makes sense. If r and d are not coprime, the moduli space $M^G(r,n)$ parametrizes only S-equivalence classes (cf. [Ma]) of semistable sheaves and one has to be more careful to make sure that a_r is well-defined on such classes (cf. [Li]).

It takes quite a lot of technical work (see [Li], [Mo]) to show that the map a_r is continuous (this is because one has to use gauge-theoretic Uhlenbeck Compactness Theorem and consider $M^U(r,n)$ as a topological space). We will not attempt to repeat the argument here referring the interested reader to the references cited. Moreover, from now on we *assume that $M^U(r,n)$ has a structure of a projective algebraic variety and a_r is algebraic.* In rank 2 case this was proved in [Li] and we intend to give an alternative construction for the general case in a forthcoming paper.

We need a description of the fiber of a_r over an arbitrary point $p = (E, \sum_i m_i x_i)$ in $M^U(r,n)$. Here m_i are positive integers and the points x_i are pairwise distinct. The fiber $a_r^{-1}(p)$ parametrizes all the quotients $E \to A \to 0$ where the sheaf A is supported at the points x_i with prescribed multiplicities m_i. Since the question is local, the fiber does not depend on E and points x_i but only on r and m_i. To be more precise, let $Quot(r,n)$ denote the *punctual* Quot scheme of all quotients $\mathcal{O}^{\oplus r} \to A$ of fixed length n which are supported at a fixed point x. This scheme depends only on completion $\widehat{\mathcal{O}}_x$ of the local ring $\mathcal{O}_x$ of x. Hence different points x lead to isomorphic Quot schemes and we drop x from notation. The following statement is immediate

Proposition 2.1 *The fiber $a_r^{-1}(p)$ over the point $p = (E, \sum_i m_i x_i)$ is isomorphic to the product of punctual Quot schemes $\prod_i Quot(r, m_i)$.* □

The proposition above motivates the following theorem (cf. [Ba] and [EL]):

Theorem 2.2 *The punctual Quot scheme $Quot(r,n)$ is irreducible of dimension $rn - 1$.*

Proof. See Appendix. □

The first application of Theorem 2.2 is to show that, under the technical assumption above, the moduli spaces $M^G(r,n)$ and $M^U(r,n)$ are well-behaved.

Theorem 2.3 *Assume that the technical condition of Section 1 is satisfied. Then the following statements hold* *(a) $N(r,n)$ is dense in $M^G(r,n)$;*

(b) $N(r,n)$ is dense in $M^U(r,n)$;

(c) Any irreducible component of $N(r,n-s)\times Sym^s(S)$ intersects the closure of a unique component of $N(r,n)$ (and hence by (b) is contained in it).

Proof. Note that (a) implies (b) since a_r is surjective and one-to-one on the copies of $N(r,n)$ in $M^G(r,n)$ and $M^U(r,n)$, respectively.

To prove (a) suppose that $N(r,n)$ is not dense in $M^G(r,n)$. Then there exists a component of $M^G(r,n)$ such that generic point of it corresponds to a non-locally free sheaf. This means that the image of this component in $M^U(r,n)$ is a subset of $\bigcup_{i\geq 1} N(r,n-s)\times Sym^s(S)$. Since all the components of $N(r,n)$ have expected dimension (and not more than that), Theorem 2.2 above implies that the dimension of $a_r^{-1}(\bigcup_{i\geq 1} N(r,n-s)\times Sym^s(S))$ is strictly less than the expected dimension of $M^G(r,n)$ which is impossible.

Finally, if (c) were false, some irreducible component X of $N(r,n-s)\times Sym^s(S)$ would intersect the closures of at least two different components of $N(r,n)$. Since $M^G(r,n)$ is smooth, these would mean that X would belong to the image of two different connected components of $M^G(r,n)$. But this is impossible since all fibers of a_r are irreducible. $\square$

Corollary 2.4 *$M^U(r,n)$ is a disjoint union of the closures of irreducible components of $N(r,n)$.* $\square$

3 Stratifications and semi-small maps.

From now on we will assume that the technical condition on the pair (S,H) is satisfied.

Recall briefly the results on symmetric products and Hilbert schemes.

Let μ be a partition of n. Any such μ can be represented either by a non-increasing sequence $\mu=(\mu_1\geq\mu_2\geq\ldots\geq\mu_m>0)$ with $\sum\mu_i=n$ or in the form $1^{m_1}2^{m_2}\ldots n^{m_n}$, where m_i is the number of parts μ which are equal to i (hence $m_i\geq 0$). Note that $m_1+m_2+\ldots+m_n$ is equal to the number m of non-zero parts of μ.

The symmetric product $Sym^n(S)$ has a natural stratification by locally closed strata labeled by partitions of n.

The stratum $Sym^n_\mu(S)$ is the set of formal sums of the type $\sum\mu_i x_i$ with x_i pairwise distinct (veiwed as elements of $Sym^n S$). Note that $Sym^n_{(1,\ldots,1)}$ is a dense open subset of $Sym^n(S)$ and in general $Sym^n_\mu(S)$ is isomorphic to a dense open subset of $Sym^{m_1}(S) \times \ldots \times Sym^{m_n}(S)$. A generic element $(y_1, \ldots, y_n)$ in the latter product corresponds to the point $y_1+2y_2+\ldots+ny_n$ in $Sym^n_\mu(S)$.

Let $\pi : Z \to Y$ be a proper projective morphism of algebraic varieties. Suppose that Y decomposes into a finite number of locally closed strata: $Y = \bigcup_\mu Y_\mu$ and choose an arbitrary point $y_\mu \in Y_\mu$. Assume that the restriction $\pi : \pi^{-1}(Y_\mu) \to Y_\mu$ is a topological fiber bundle with fiber $\pi^{-1}(y_\mu)$.

Definition. (cf. [BM] or [CG, Chapter 8]) The map π is called strictly semi-small if it satisfies

$$2\ dim\ \pi^{-1}(y_\mu) = codim\ Y_\mu$$

for any stratum Y_μ.

The following proposition is an immediate consequence of results of Briançon and Iarrobino (cf. [Br], [Ia]).

Proposition 3.1 *(cf. [GS])*

The morphism $a_1 : Hilb^n(S) \to Sym^n(S)$ is strictly semi-small, with respect to the stratification given by $Sym^n_\mu(S)$.

We will give a generalization of this proposition to the case of arbitrary rank. Exactly as in [GS] this will lead to some formulas for Poincaré polynomials. These formulas will be later interpreted in representation-theoretic terms.

The first step is to stratify $M^U(r,n)$ appropriately since the natural strata coming from the definition of $M^U(r,n)$ are too big. The new finer strata will be labeled by pairs (s,μ) where $s < n$ is a non-negative integer such that $N(r, n-s)$ is non-empty, and $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m \geq 0)$ is a partition of s. For each pair (s,μ) we consider $M^U_{s,\mu}(r,n) = N(r,n-s) \times Sym^n_\mu(S)$ which is naturally a locally closed subset of $M^U(r,n)$. Of course, this is nothing but a "common refinement" of the natural partition of $M^U(r,n)$ and the above partition above of the symmetric product.

Part (b) of the next theorem was originally conjectured by V. Ginzburg. It provides a starting point for our generalization of Nakajima's construction.

Proposition 3.2 *Let s and $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m > 0)$ be as above and let $M^U_{s,\mu}(r,n)$ be the stratum associated to the pair (s,μ). Then*

(a) For any point $x_{s,\mu} \in M^U_{s,\mu}(r,n)$ the dimension of the fiber $a_r^{-1}(x_{s,\mu})$ is equal to $(rs - m)$ where m is the number of non-zero parts of μ.

(b) The morphism $M^G(r,n) \to M^U(r,n)$ is strictly semi-small with respect ot the stratification given by $M^U_{s,\mu}(r,n)$.

Proof. To prove (a) note that by Proposition 1 the fiber is isomorphic to $\prod_{i=1}^m Quot(r,m_i)$ and by Theorem 2 its dimension is equal to $\sum_{i=1}^m (r\mu_i - 1) = r(\sum_{i=1}^m \mu_i) - m = rs - m$.

To prove (b) we have to show that $codim\ M^U_{s,\mu} = 2 dim\ a_r^{-1}(x_{s,\mu}) = 2(rs-m)$. In fact, $codim\ M^U_{s,\mu}(r,n) = dim\ N(r,n) - dim\ N(r,n-s) - dim\ Sym^n_\mu(S) = 2rs - 2m = 2(rs-m)$. $\square$

The semi-smallness result will allow us to relate homological invariants of $M^G(r,n)$ and $M^U(r,n)$. Recall (cf. [BBD]) that for any algebraic variety X, there exists a remarkable complex of sheaves IC_X (*intersection cohomology* complex) such that its cohomology groups $IH^*(X) = H^*(X, IC_X)$ (*intersection* or *Goresky-MacPherson* homology) satisfy Poincaré duality. If X is a smooth algebraic variety (such as $M^G(r,n)$) or a quotient of a smooth variety by a finite group action (such as $Sym^n(S)$) then up to shift IC_X is just the constant sheaf $\mathbb{C}$ (cf. [BBD] of [GS]).

Following [Sa, 1.13] one can define pure Hodge structure on the intersection homology of any complex algebraic variety X.

We will need a simplified version of Borho-MacPherson formula for the direct image of the intersection homology complex under a projective semismall morphism. This formula is a direct application of the Decomposition Theorem due to Beilinson-Berstein-Deligne-Gabber (cf. [BBD]).

Proposition 3.3 *Suppose that a projective morphism $Z \to Y$ of algebraic varieties is strictly semi-small with respect to some stratification $Y = \bigcup Y_\mu$. Suppose further that for any point $y \in Y$ the fiber $\pi^{-1}(y)$ is irreducible. Then*

$$\pi_*(IC_Z) = \bigoplus_\mu IC_{\overline{Y}_\mu}. \qquad \square$$

Corollary 3.4 *One has the following direct sum decomposition in the derived category of complexes of sheaves:*

$$(a_r)_* IC_{M^G(r,n)} = \bigoplus_{s \in \{0,\dots n\},\, \mu \in P(s)} IC_{\overline{M}^U_{s,\mu}(r,n)}$$

where $P(s)$ is the set of all partitions of s and $IC(\overline{M}^U_{s,\mu})$ denotes the IC-complex on the closure of the stratum $M^U_{s,\mu} \subset M^U(r,n)$. $\square$

We intend to use the formula above by taking the cohomology of both sides. The IC-complexes supported on the closures of the smaller strata can be understood with the help of the following

Proposition 3.5 *Let (s, μ) and m_i be as above and denote by $Sym^{\mu}(S)$ the direct product $Sym^{m_1}(S) \times \ldots \times Sym^{m_n}(S)$. Then there exists a finite birational morphism respecting the induced stratifications:*

$$\pi_{n,s,\mu} : M^U(r, n-s) \times Sym^{\mu}(S) \to \overline{M}^U_{s,\mu}(r, n)$$

Proof. Left as an exercise to the reader. □

Now [GS, Lemma 1] implies that in this situation one can deduce a

Corollary 3.6 *In the notation of the previous proposition, one has the following equality in the derived category of sheaves:*

$$(\pi_{n,s,\mu})_*(IC_{M^U(r,n-s)\times Sym^{\mu}(S)}) = IC_{\overline{M}^U_{s,\mu}(r,n)}. \qquad \square$$

Recall that intersection homology complex of a space X is defined in such a way that it has non-trivial (global) hypercohomology in the range between $(-k)$ and k where $2k$ is the real dimension of X.

Definition. The shifted intersection homology Poincaré polynomial $P_t(X)$ is defined by the formula

$$P_t(X) := \sum_{-\dim_{\mathbb{C}} X}^{\dim_{\mathbb{C}} X} (\dim_{\mathbb{C}} H^i(X, IC_X)) \cdot t^i$$

Note that for a smooth X this is just the usual Poincaré polynomial multiplied by $t^{-\dim_{\mathbb{C}} X}$.

Similarly, one defines a shifted Hodge polynomial $P_{x,y}(X)$ using the pure Hodge structure on $IH^*(X)$. When X is smooth this coincides with the usual Hodge polynomial multiplied by $(xy)^{-\dim_{\mathbb{C}} X/2}$. However, in general it will *not* be a shift of the virtual Hodge polynomial of X.

With this preparation, we can deduce the first main result of the paper.

Theorem 3.7 *One has the following identity between Poincaré polynomials of M^G and M^U:*

$$\frac{\sum_{n=0}^{\infty} q^n P_t(M^G(r,n))}{\sum_{n=0}^{\infty} q^n P_t(M^U(r,n))} = \prod_{l=1}^{\infty} \frac{(1+t^{-1}q^l)^{b_1(S)}(1+t^1q^l)^{b_3(S)}}{(1-t^{-2}q^l)^{b_0(S)}(1-q^l)^{b_2(S)}(1-t^2q^l)^{b_4(S)}}$$

In short, the series for M^G is obtained from the series for M^U when multiplied by the Göttsche's formula.

Proof. Following Nakajima's notations, denote by $a_m(t)$ the (shifted) Poincaré polynomial $P_t(Sym^m(S))$. Then a formula due to MacDonald [Mc1] says that

$$\sum_{m=0}^{\infty} q^m a_m(t) = \frac{(1+t^{-1}q)^{b_1(S)}(1+tq)^{b_3(S)}}{(1-t^{-2}q)^{b_0(X)}(1-q)^{b_2(S)}(1-t^2q)^{b_4(S)}}.$$

Taking the cohomology of both sides in the formula of Corollary 3.4 and using Corollary 3.6 one obtains:

$$\sum_{k=0}^{\infty} q^k P_t(M^G(r,k)) = \sum_{k=0}^{\infty}\sum_{s=0}^{k}\sum_{\mu\in P(s)} q^k P_t(M^U(r,k-s)) \cdot P_t(Sym^{\mu}S) =$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{0\le s\le k\\ \mu\in P(s)}} q^{k-s} P_t(M^U(r,k-s)) \cdot q^s P_t(Sym^{\mu}S) =$$

$$= \Big(\sum_{n=0}^{\infty} q^n P_t(M^U(r,n))\Big) \cdot \Big(\sum_{\substack{s\ge 0\\ \mu\in P(s)}} q^s P_t(Sym^{\mu}S)\Big).$$

Represent all partitions μ in the formula above as $(1^{m_1}, 2^{m_2}, \dots, s^{m_s})$. Then

$$\sum_{\substack{s\ge 0\\ \mu\in P(s)}} q^s P_t(Sym^{\mu}S) = \sum_{\substack{s\ge 0\\ \mu\in P(s)}} a_{m_1}(t)(q)^{m_1} a_{m_2}(t)(q^2)^{m_2} \dots a_{m_s}(t)(q^s)^{m_s} =$$

$$= \prod_{l=1}^{\infty}\Big(\sum_{m=0}^{\infty} a_m(t)(q^l)^m\Big) = \prod_{l=1}^{\infty} \frac{(1+t^{-1}q^l)^{b_1(S)}(1+t^1q^l)^{b_3(S)}}{(1-t^{-2}q^l)^{b_0(S)}(1-q^l)^{b_2(S)}(1-t^2q^l)^{b_4(S)}}. \square$$

Remarks.

(1) A very similar formula was discovered by Göttsche in [Gö22]. He computes the ratio of generating functions for $M^G(r,n)$ and $N(r,n)$ and obtains a similar product (involving more factors).

(2) In [LQ] Li and Qin derive a formula relating cohomology of Gieseker spaces for S and its blowup $\widetilde{S}$. Applying Theorem 3.7 we get a relation between the intersection homology of Uhlenbeck compactifications for S and $\widetilde{S}$. This formula is different form the relation between usual homology of $M^U(r,n)$ for S, $\widetilde{S}$ (also found by Li and Qin), since Uhlenbeck compactifications are quite singular hence their intersection homology is different from the ordinary (co)homology.

Example. When $S=\mathbb{P}^2$ and $r=2$ the generating function for $M^G(r,n)$ was computed by Yoshioka in [Y]. In this case

$$\sum_{i=0}^{\infty} q^n P_t(M^G(r,n)) =$$

$$= \frac{\sum_{b\in\mathbb{Z}} \frac{t^{-2b}q^{b^2}}{1-t^4q^{2b-1}}}{\sum_{n\in\mathbb{Z}} t^{-2n}q^{n^2}} \cdot \frac{1}{t^4(t^2-1)\prod_{l=1}^{\infty}(1-t^{-2}q^l)^2(1-q^l)^2(1-t^2q^l)^2}$$

Let $q=e^{2\pi i\tau}$ and $t=e^{2\pi i z}$. Recall the classical theta functions:

$$\theta_{\mu,\nu}(\tau,z) := \sum_{n\in\mathbb{Z}} (-1)^{n\nu} q^{(n+\mu/2)^2/2} t^{n+\mu/2} \quad (\mu,\nu\in\{0,1\}).$$

Then by the product formula

$$\theta_{1,1}(\tau,z) = q^{1/8}(t^{1/2}-t^{-1/2})\prod_{l>0}(1-t^{-1}q^l)(1-q^l)(1-tq^l)$$

for theta functions one can write

$$\sum_{i=0}^{\infty} q^n P_t(M^G(r,n)) = \sum_{b\in\mathbb{Z}} \frac{t^{-2b}q^{b^2}}{1-t^4q^{2b-1}} \cdot \frac{q^{1/4}(t-t^{-1})}{t^5\,\theta_{0,0}(2\tau,2z)\theta_{1,1}(\tau,2z)^2}$$

By the Theorem above we immediately deduce that

$$\sum_{i=0}^{\infty} q^n P_t(M^U(r,n)) = \sum_{b\in\mathbb{Z}} \frac{t^{-2b}q^{b^2}}{1-t^4q^{2b-1}} \cdot \frac{q^{1/8}}{t^5\,\theta_{0,0}(2\tau,2z)\theta_{1,1}(\tau,2z)}$$

Remark. Conjecturally, the theta functions in the generating series above appear from extra symmetries of cohomology of the moduli spaces.

Using Saito's theory of mixed Hodge modules (cf. [Sa]) and the formula for Hodge polynomials of symmetric powers (cf. [Bu]) one can apply the arguments above to prove the following

Proposition 3.8 *(a) There exists an isomorpshism of pure Hodge structures:*

$$IH^*(M^G(r,n)) \simeq \bigoplus_{\substack{s\geq 0 \\ \mu\in P(s)}} IH^*(M^U(r,n-s)) \otimes IH^*(Sym^{\mu}(S));$$

(b) One has the following identity between the (shifted) intersection homology Hodge polynomials of M^G and M^U:

$$\frac{\sum_{n\geq 0} P_{x,y}(M^G(r,n))q^n}{\sum_{n\geq 0} P_{x,y}(M^U(r,n))q^n} = \prod_{l\geq 0}\prod_{i,j=-1}^{1} (1+(-1)^{i+j+1}x^i y^j q^l)^{(-1)^{i+j+1}h^{i+1,j+1}(S)};$$

where $h^{r,s}(S)$ are the (usual) Hodge numbers of S. $\square$

Remark. Since S is compact, one of course has $b_0 = b_4 = 1$ and $b_1 = b_3$. However, Göttsche-Soergel's approach is valid for Hilbert schemes of any quasi-projective surface. It is interesting to know if the computaion above can be extended to the quasi-projective case.

4 Correspondences.

In this section we give a very natural generalization of correspondences used by Nakajima and Grojnowski and prove that their action on the cohomology satisfies the correct commutation relations. Of course, as one might guess from the formula of Theorem 3.7, the corresponding module over the oscillator algebra is not irreducible any more. In fact, the space of vacuum vector can be naturally identified with intersection homology of Uhlenbeck compactifications.

Recall [CG] that, for any two smooth projective varieties M_1, M_2, a closed subvariety $Z \subset M_1 \times M_2$ (more generally, any cohomology class in

$H^*(M_1 \times M_2))$, defines a map $[Z] : H^*(M_1) \to H^*(M_2)$ given by $[Z](\alpha) = (p_2)_*([Z] \cap (p_1)^*)$, where p_1 and p_2 are the projections onto M_1 and M_2, respectively. If M_3 is a third smooth projective variety and $Z' \subset M_2 \times M_3$ then the composition $[Z'] \circ [Z] : H^*(M_1) \to H^*(M_3)$ can be described as follows. Let $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$ be the projection to the product of the i-th and j-th factors. Then $[Z'] \circ [Z]$ is given by the cohomology class $(p_{13})_*(p_{12}^*[Z] \cup p_{23}^*[Z']) \in H^*(M_1 \times M_3)$.

We want to apply this construction by introducing certain correspondences in the products of moduli spaces.

Definition. For $i > 0$, define a subvariety $P_{-i} \subset \coprod_{n=0}^{\infty} M^G(r,n) \times M(n+i)$ as a set of all pairs $(\mathcal{F}_1, \mathcal{F}_2)$ such that

1) $\mathcal{F}_1^{**} \simeq \mathcal{F}_2^{**}$,
2) $\mathcal{F}_2 \subset \mathcal{F}_1$ as subsheaves of the common double dual,
3) $Supp(\mathcal{F}_1/\mathcal{F}_2) = \{x\}$ for some $x \in X$.

Note that the second condition makes sense since the double dual is stable and therefore simple.

Similarly, we define P_i for $i > 0$, exchanging $\mathcal{F}_1$ and $\mathcal{F}_2$. Let us define a morphism $\Pi : P_i \to S$ by

$$\Pi(\mathcal{F}_1, \mathcal{F}_2) = x \quad \text{for } (\mathcal{F}_1, \mathcal{F}_2) \in P_i,$$

where x is the unique element of $Supp(\mathcal{F}_1/\mathcal{F}_2)$. One can give a rigorous definition of Π using universal sheaves on moduli spaces and [HL, Example 4.3.6] but we will not do it here.

Denote the dimension of $M^G(r,n)$ by $2rn + a$ where a is a number depending on S, r, L and H but not on n, cf. (1.1). Then the dimension of $P_{-i} \cap M^G(n-i) \times M^G(n)$ is equal to $2rn + a - ri + 1$ (this follows from Nakajima's argument [Na 8.3] and Theorem 2.2).

Let $P_{\pm i}^{\alpha} \in \prod_n H^*(M^G(r, n \mp i)) \otimes H^*(M^G(r,n))$, $\alpha \in H^*(S)$, be the images of the cohomology classes $\Pi^*\alpha$ on $P_{\pm i}$. The cycles $P_{\pm i}^{\alpha}$ induce the maps $\bigoplus_n H^*(M^G(r,n)) \to \bigoplus_n H^*(M^G(r \mp i))$ via the convolution construction given above. Computing dimensions we can see that P_{-i}^{α} raises cohomological degree by $2(ri-1) + deg\ \alpha$.

The next theorem asserts that $\bigoplus_{n=0}^{\infty} H^*(M^G(r,n))$ becomes a representation space of the Heisenberg/Clifford algebra generated by P_i^{α} and $P_{-i}^{\beta}/(-1)^{ri-1}r$. It is a direct generalization of [Na, Theorem 8.13].

Theorem 4.1 *The following relations hold,*

$$[P_i^\alpha, P_j^\beta] = (-1)^{ri-1} ri\delta_{i+j,0}\langle\alpha,\beta\rangle\ Id \qquad if(-1)^{deg\alpha\cdot\deg\beta} = 1,$$

$$\{P_i^\alpha, P_j^\beta\} = (-1)^{ri-1} ri\delta_{i+j,0}\langle\alpha,\beta\rangle\ Id \quad otherwise,$$

where $\langle\alpha,\beta\rangle$ is the intersection form on S.

Exactly as in [Na Corollary 8.16] we deduce a

Corollary 4.2 *The cohomology groups $\oplus_{n=0}^{\infty} H^*(M^G(r,n))$ form a tensor product of an irreducible highest weight representation of the oscillator algebra, with a trivial representation on the intersection homology groups of Uhlenbeck compactifications.*

Example. Let $r = 2$, $i = 1$, $S = \mathbb{P}^2$ and $L \simeq H \simeq \mathcal{O}(1)$. Choose α to be the fundamental class and β the class of a point $x \in \mathbb{P}^2$. Then the formula above predicts that $[P_1^\alpha, P_{-1}^\beta] = (-2)Id$. This can be seen as follows.

It is known that $M^G(2,1)$ is a single point corresponding to the twist $T_{\mathbb{P}^2}(-1)$ of the tangent bundle (cf. [OSS, 3.2]). The Uhlenbeck compactification $M^U(2,2)$ can be described as follows. Let V be the 6-dimensional space of symmetric 3×3 matrices. Then $M^U(2,2)$ can be identified with a hypersurface H in $P(V)$ correspoding to all rank ≤ 2 symmetric matrices (cf. [OSS, 4.3]). Its singular locus $P \simeq \mathbb{P}^2$ corresponds to symmetric matrices of rank one. It can be deduced from (cf. [OSS, 4.3]) that $M^G(2,2)$ coincides with the blowup of P.

Let $D \subset M^G(2,2)$ be the exceptional divisor. Each one-dimensional fiber of $D \to P$ corresponds to the set of sheaves $\mathcal{F}$ which fit a short exact sequence

$$0 \to \mathcal{F} \to T_{\mathbb{P}^2}(-1) \to \mathbb{C}_x \to 0,$$

where $x \in \mathbb{P}^2 \simeq P$. Let l be such a fiber and $[pt]$ be the generator of $H^0(M^G(2,1)) = H^0(pt)$. Since $P_1^\alpha([pt]) = 0$ and $P_{-1}^\beta([pt]) = [l]$ we expect that $P_1^\alpha([l]) = -2[pt]$. It follows from definitions that this is equivalent to $[D]\cdot[l] = -2$. The last equality is true since H has an A_2-singularity at any point of P.

Beginning of the proof of 4.1: Most of Nakajima's proof can be repeated word-by-word with almost no changes. In particular, we have the following result:

Theorem 4.3 *For any $v \in H^*(M^G(r,n))$, the following relations hold:*

$$[P_i^\alpha, P_j^\beta](v) = c_{i,n}\delta_{i+j,0}\langle\alpha,\beta\rangle\, v \qquad if (-1)^{\deg\alpha\cdot\deg\beta} = 1,$$

$$\{P_i^\alpha, P_j^\beta\}(v) = c_{i,n}\delta_{i+j,0}\langle\alpha,\beta\rangle\, v \quad otherwise,$$

where $\langle\alpha,\beta\rangle$ is the intersection form on $H^*(S)$ and $c_{i,n}$ is some constant depending apriori *on i and n but not on the classes α and β.* $\square$

The proof is a mere repetition of the argument presented in [Na 8.4] (where all the dimension statements are proved using Theorem 2.2).

Lemma 4.4 *For each i, the constants $c_{i,n}$ are independent of n.*

Proof. Let $k \neq \pm i$. By Theorem 4.3 we have $[P^\gamma_{-k}, [P_i^\alpha, P_{-i}^\beta]] = 0$. Take $v \in H^*(M^G(r,n))$. Applying the double commutator to v, one gets

$$\langle\alpha,\beta\rangle(c_{i,n} - c_{i,n+k})P^\gamma_{-k}v = 0.$$

One can choose v, α, β, γ in such a way that $\langle\alpha,\beta\rangle \neq 0$ and $P_\gamma[-k](v) \neq 0$. Therefore, one has $c_{i,n} = c_{i,n+k}$ if $k \neq \pm i$. If $i \neq \pm 1$ we can take $k = 1$ and obtain $c_{i,n} = c_{i,n+1}$. If $i = \pm 1$ we take $k = 2, 3$ and get $c_{1,n} = c_{1,n+2} = c_{1,n+3}$ which also implies that $c_{1,n}$ is independent of n. $\square$

In view of this lemma we will be writing c_i instead of $c_{i,n}$.

Recall the general setup of Nakajima's Chapter 9 (or rather its adaptation to our case). We choose and fix two smooth transversal curves C and C' in one very ample linear system on S. Take such an s that $M^G(r,s)$ is non-empty and fix a locally free sheaf $\mathcal{E}$ correspoding to a point in $M^G(r,s)$.

We will compute numbers c_i using the formalizm of generating functions and a connection with symmetric functions discovered by Nakajima. The actual idea of using embedded curves is originally due to Grojnowski.

To that end, consider the subvariety $L^{k,s} \subset M^G(r,n)$ formed by all sheaves $\mathcal{F}$ such that

(a) $\mathcal{F}^{**} \simeq \mathcal{E}$;

(b) The quotient $A_\mathcal{F} := \mathcal{E}/\mathcal{F}$ is an Artin sheaf of length $k = n - s$ supported at finitely many points of C;

Definition. For each partition $\mu = (\mu_1 \geq \ldots \geq \mu_l > 0)$ of k let $(L^{\mu,s})^\circ$ be the set of all sheaves $\mathcal{F}$ such that

(a) $\mathcal{F}^{**} \simeq \mathcal{E}$;

(b) the quotient $A_{\mathcal{F}} = \mathcal{E}/\mathcal{F}$ is supported at pairwise distinct points $x_j \in C$, $1 \le j \le l$ with multiplicities μ_j.

Denote also by $L^{\mu,s}$ the closure of $(L^{\mu,s})^\circ$.

Definition. Similarly, we define $\widehat{L}^{\mu,s}$ to be the closure of the subset of all sheaves $\mathcal{F}$ satisfying:

(a) $\mathcal{F} \subset \mathcal{F}'$ for some $\mathcal{F}' \in M^G(r,s)$;

(b) the quotient $\mathcal{F}'/\mathcal{F}$ is supported at pairwise distinct points $x_j \in C$, $1 \le j \le l$ with multiplicities μ_j.

Proposition 4.5 *The closed subvarieties $L^{\mu,s}$ are the irreducible components of $L^{k,s}$. Each component $L^{\mu,s}$ is of dimension rk. Similarly, $\widehat{L}^{\mu,s}$ is the set of all irreducible components of $\widehat{L}^{k,s}$ and all $\widehat{L}^{\mu,s}$ are of the same dimension.*

Proof. If we fix all the points x_j from the definition of $L^{\mu,s}$ then by Theorem 2.2 the variety of all sheaves with the required condition has dimension $\sum_{i=1}^{l}(r\mu_i - 1) = rk - l$. Since the points x_j, $j = 1, \dots, l$ are allowed to move in an l-dimensional family, we obtain $\dim L^{\mu,s} = rk$. By the irreducibility part of Theorem 2.2 we conclude that each $L^{\mu,s}$ is irreducible.

It follows from the definition of $L^{k,s}$ that it is a union of the closed subsets $L^{\mu,s}$. Hence, $L^{\mu,s}$ are the irreducible components of $L^{k,s}$. □

Note that P^C_{-i} maps classes of (cohomological) degree b on $M^G(r,n)$ to classes of degree $ri + b$ on $M^G(r, n+i)$. In particular, the subspace $\bigoplus H^{rn}(M^G(r,n))$ is preserved by P^C_{-i}. Moreover, the next theorem shows that in fact even the subspace generated by the classes of $L^{\mu,s}$, is invariant under P^C_{-i} $(i > 0)$.

Theorem 4.6 *(cf. [Na, Theorem 9.14]) For $i > 0$, we have $P^C_{-i}[L^{\mu,s}] = \sum_\lambda a_{\lambda\mu}[L^{\lambda,s}]$, where the summation is over partitions λ of $|\mu| + i$ which are obtained as follows:*

(a) add i to a term in μ, say μ_j (possibly 0), and then

(b) arrange it in descending order.

The coefficient $a_{\lambda\mu}$ is $\#\{l \mid \lambda_l = \mu_j + i\}$.

A similar statement is true for the classes of $\widehat{L}^{\mu,s}$.

This theorem will be proved in Section 5.

Let $[vac]$ be the cohomology class of the point $\mathcal{E} \in M^G(r,s)$ and $[Vac]$ be the fundamental class of $M^G(r,s)$. Theorem 4.6 allows us to establish a connection with the theory of symmetric functions as follows. Let L (resp. $\widehat{L}$) be the subspace in $\bigoplus_n H^*(M^G(r,n))$ generated by the classes of $L^{\mu,s}$ and $[vac]$ (resp. $\widehat{L}^{\mu,s}$ and $[Vac]$). We define a $\mathbb{C}$-linear isomorphism from L (resp. $\widehat{L}$) onto the space Λ of symmetric functions in infinitely many variables (cf. [Na, 9.1], [Mc2]). This isomorphism sends $[vac]$ (resp. $[Vac]$) to $1 \in \Lambda$ and $[L^{\mu,s}]$ (resp. $[\widehat{L}^{\mu,s}]$) to the orbit sum function m_λ (cf. *loc. cit.*).

Theorem 4.6 means that the operator P^C_{-i} corresponds under the isomorphism above to multiplication by the i-th power sum (or Newton function) $p_i \in \Lambda$ (*loc. cit.*). This provides us with several identities between cohomology classes coming from classical identities between symmetric functions.

Note that for $\mu = (1^k)$ the corresponding variety $L^{\mu,s}$ is isomorphic to the global Quot scheme $Quot^k_C(\mathcal{E})$ on the curve C. This scheme parametrizes quotient sheaves $\mathcal{E}|_C \to A$ on C, where A is of length k. Every such quotient on C defines a sheaf on S, namely the kernel of the composition $\mathcal{E} \to \mathcal{E}|_C \to A$. The class of $Quot^k_C(\mathcal{E})$ corresponds to the k-th elementary symmetric function $e_k = m_{(1^k)}$.

Similarly, we will slightly abuse notation by writing $Quot^k_C(s)$ instead of $\widehat{L}^{(1^k),s}$. This variety is a birational image of the family of schemes $Quot^k_C(\mathcal{F}')$ parametrized by points $\mathcal{F}'$ of $M^G(r,s)$.

Repeating the arguments of Nakajima, (see [Na, formula 9.16]) we obtain the formulas

$$\sum_{n=0}^{\infty} z^n [Quot^n_C(\mathcal{E})] = exp\Big(\sum_{i=1}^{\infty} \frac{z^i P^C_i}{(-1)^{i-1} i} \Big) \cdot [vac]. \tag{2}$$

$$\sum_{n=0}^{\infty} z^n [Quot^n_C(s)] = exp\Big(\sum_{i=1}^{\infty} \frac{z^i P^C_i}{(-1)^{i-1} i} \Big) \cdot [Vac].$$

These formulas arise from the classical identity between Newton symmetric functions and elementary symmetric functions (cf. [Mc2]). They are the first ingredient in our computation of c_i. The second ingredient is provided by

Theorem 4.7 *(cf. [Na, Exercise 9.23]) The following relation holds:*

$$\sum_{n=0}^{\infty} z^{2n} \langle Quot^n_C(\mathcal{E}), Quot^n_{C'}(s)\rangle = (1-(-1)^r z^2)^{r\langle C,C'\rangle}.$$

The proof of this theorem will be given in Section 6 for the case when the intersection $C \bigcap C'$ is transversal (which is sufficient for our purposes).

End of Proof of Theorem 4.1: Following Nakajima, we introduce the following notations:

$$C_-(z) = \sum_{i=1}^{\infty} \frac{P^C_{-i} z^i}{(-1)^{i-1} i}; \qquad C_+(z) = \sum_{i=1}^{\infty} \frac{P^C_i z^i}{(-1)^{i-1} i}.$$

Note that $C_-(z)$ is adjoint to $C_+(z)$ with respect to the intersection form on $\bigoplus_n H^*(M^G(r,n))$ (the cohomology groups for different n are orthogonal to each other). We extend this form to power series in z by z-linearity. By Theorem 4.7 and (2) one has:

$$(1-(-1)^r z^2)^{r\langle C,C'\rangle} = \sum_{n=0}^{\infty} z^{2n} \langle Quot^n_C(\mathcal{E}), Quot^n_{C'}(s)\rangle =$$

$$= \langle \sum_{n=0}^{\infty} z^n Quot^n_C(\mathcal{E}),\ \sum_{n=0}^{\infty} z^n Quot^n_{C'}(s)\rangle =$$
$$= \langle exp(C_-(z)) \cdot [vac],\ exp(C'_-(z)) \cdot [Vac]\rangle =$$
$$= \langle exp(C'_+(z))\ exp(C_-(z)) \cdot [vac], [Vac]\rangle$$

where the last equation follows from adjointness.

For any pair of operators A and B, one has an identity

$$exp(-A)\ B\ exp(A) = exp(-ad\ A)(B) = 1 - (ad\ A)(B) + \frac{(ad\ A)^2}{2!}(B) - \ldots$$

Therefore

$$\langle exp(C_+(z))\ exp(C_-(z)) \cdot [vac],\ [Vac]\rangle =$$

$$= \langle exp(C_-(z)) \Big[exp(-ad\ C_-(z)) \Big(exp(C'_+(z)) \Big) \Big] \cdot [vac], [Vac]\rangle$$

An explicit computation shows that

$$[C_-(z),\ exp(C_+(z))] = -\Big(\sum_{n=1}^{\infty} \frac{c_n}{n^2}\langle C, C'\rangle z^{2n}\Big)\ exp(C_+(z))$$

Denote the expression $\sum\limits_{n=1}^{\infty} \frac{c_n}{n^2}\langle C, C'\rangle z^{2n}$ by $\Phi(z)$. Then by the previous formula:

$$exp(-ad\ C_-(z))\Big(exp(C_+(z))\Big) = exp(\Phi(z))exp(C_+(z)).$$

Collecting the results of computations, we obtain:

$$\begin{gathered}(1-(-1)^r z^2)^{r\langle C,C'\rangle} = \\ = \langle exp(C_-(z))\Big[exp(-ad\ C_-(z))(exp(C_+(z)))\Big]\cdot[vac],[Vac]\rangle = \\ = exp(\Phi(z))\langle exp(C_-(z))\ exp(C_+(z))\cdot[vac],[Vac]\rangle = \\ = exp(\Phi(z))\langle exp(C_-(z))\cdot[vac],[Vac]\rangle = exp(\Phi(z))\end{gathered}$$

The last equality holds since all P_{-i}^C involved in the definition of $C_+(z)$, map $[vac]$ to the orthogonal complement of $[Vac]$. By definition of $\Phi(z)$ we have

$$\sum_{n=1}^{\infty} \frac{c_n}{n^2}\langle C, C'\rangle z^{2n} = r\langle C, C'\rangle log(1-(-1)^r z^2).$$

Hence $c_n = (-1)^{rn-1}rn$ which completes the proof of Theorem 4.1 $\square$

5 A transversality result.

In $r = 1$ case Theorem 4.6 has a simple proof (cf. [Na]) using local coordinates on S. While this result is still true in higher ranks the proof of it requires a more detailed analysis of the geometry of the moduli space to be provided below.

First consider the situation at the set-theoretic level. It follows for the definitions that if $(\mathcal{F}_1, \mathcal{F}_2) \in P_{-i}^C$ then $\mathcal{F}_1^{**} \simeq \mathcal{F}_2^{**}$ and $Supp(\mathcal{F}_1/\mathcal{F}_2) = \{x\} \in C$. In particular, if $\mathcal{F}_1 \in L^{\mu,s}$ then $\mathcal{F}_2$ necessarily belongs to one of the $L^{\lambda,s}$ where the partition λ is as described in the theorem. Hence $P_{-i}^C[L^{\mu,s}]$ is a linear combination of $[L^{\lambda,s}]$ with integral coefficients.

Our next step is to show that for generic $\mathcal{F}_2 \in L^{\lambda,s}$ the intersection

$$p_1^{-1}(L^{\mu,s}) \bigcap P^C_{-i} \bigcap p_2^{-1}(\mathcal{F}_2) \subset M^G(r, s+|\mu|) \times M^G(r, s+|\lambda|)$$

is finite and consists of exactly $a_{\lambda\mu}$ points.

Consider the quotient $A_{\mathcal{F}_2} = \mathcal{E}/\mathcal{F}_2$. If $\lambda = (\lambda_1 \geq \ldots \lambda_p > 0)$ then $A_{\mathcal{F}_2}$ is supported at some points $x_j \in C$, $1 \leq j \leq p$ with multiplicities λ_j. Therefore one can write $A_{\mathcal{F}_2}$ as a direct sum $\bigoplus (A_{\mathcal{F}_2})_{x_j}$.

We claim that for generic $\mathcal{F}_2$ there exist local coordinates (ζ_j, ξ_j) at x_j such that $(A_{\mathcal{F}_2})_{x_j} \simeq \mathbb{C}[\zeta_j]/(\zeta_j)^{\lambda_j}$.

In fact, choose a trivialization $\mathcal{E} \simeq \mathcal{O}^{\oplus r}$ in the neighbourhood of x_j and fix some system of coordinates (ζ'_j, ξ'_j) centered at x_j. Consider such quotients $\mathcal{O}^{\oplus r} \to A$ of length λ_j at x_j that the first component $\mathcal{O} \to A$ is surjective and its kernel is generated by $m_{x_j}^{\lambda_j}$ and some element $\xi_j = \xi'_j + \sum_{k=1}^{\lambda_j - 1} a_k (\zeta'_j)^k$, where $a_1, \ldots, a_{\lambda_j - 1}$ are arbitrary complex numbers. The other components $\mathcal{O}^{\oplus(r-1)} \to A$ can be chosen arbitrarily. If we take $\zeta_j = \zeta'_j$ then A has the required form in the local coordinates (ζ_j, ξ_j). Thus we obtain a $(r\lambda_j - 1)$-dimensional family of non-isomorphic quotients. Here $\lambda_j - 1$ parameters come from the choice of ξ_j and the other $(r-1)\lambda_j$ parameters come from the choice of the map $\mathcal{O}^{\oplus(r-1)} \to A$. By Theorem 2.2 this family forms a dense subset in $Quot(r, \lambda_j)$ and this implies our assertion.

Hence we can assume that $\mathcal{E}/\mathcal{F}_2$ satisfies the conditions described above. Then a choice of $\mathcal{F}_1$ such that $(\mathcal{F}_1, \mathcal{F}_2) \in P^C_{-i}$ amounts to choosing a $(\lambda_j - i)$-dimensional quotient $(A_{\mathcal{F}_1})_{x_j}$ of $(A_{\mathcal{F}_2})_{x_j} \simeq \mathbb{C}[\zeta_j]/(\zeta_j)^{\lambda_j}$. If x_j is fixed, this can be done in a unique way. Therefore the number of points in $p_1^{-1}(L^{\mu,s}) \bigcap P^C_{-i}$ over a generic $\mathcal{F}_2$ is equal to the number of ways to subtract i from one of the parts of partition λ, and obtain partition μ. This number is exactly $a_{\lambda\mu}$.

To finish the proof we need to show that $p^{-1}(L^{\mu,s})$ intersects P^C_{-i} transversly. To that end, we prove the following lemma

Lemma 5.1 *Let $(\mathcal{F}_1, \mathcal{F}_2) \in P^C_{-i} \bigcap M^G(s+|\mu|) \times M^G(s+|\lambda|)$ be as smooth point of P^C_{-i} and assume that $\mathcal{F}_1 \in L^{\mu,s}$ (resp. $\mathcal{F}_2 \in L^{\lambda,s}$) is generic in the sense described above. Then the intersection*

$$W = T_{(\mathcal{F}_1,\mathcal{F}_2)}\, p_1^{-1}(L^{\mu,s}) \bigcap T_{(\mathcal{F}_1,\mathcal{F}_2)}\, P^C_{-i}$$

of tangent spaces to $p_1^{-1}(L^{\mu,s})$ and P^C_{-i} projects isomorphically under p_2 onto the tangent space $T_{\mathcal{F}_2}(L^{\lambda,s})$.

Proof. It suffices to prove that if $(v, w) \in W$ then $w \in T_{\mathcal{F}_2}(L^{\lambda,s})$ and v is uniquely defined by w. Then the dimension count shows that in fact the map $dp_2 : W \to T_{\mathcal{F}_2}(L^{\lambda,s})$ is an isomorphism. Our proof consists of several steps.

Step 1. First we compute the tangent spaces $T_{\mathcal{F}_1}(L^{\mu,s})$ and $T_{\mathcal{F}_2}(L^{\lambda,s})$.

To that end, recall (cf. [Ar]) that the tangent space $T_{\mathcal{F}_1}(M^G(r, s+|\mu|))$ is isomorphic to the kernel of the natural trace map $tr^1 : Ext^1_S(\mathcal{F}_1, \mathcal{F}_1) \to H^1(S, \mathcal{O})$. Infinitesimal deformations of $\mathcal{F}_1$ with fixed $\mathcal{E} = \mathcal{F}_1^{**}$ correspond to the subspace $Hom_S(\mathcal{F}_1, \mathcal{E}/\mathcal{F}_1) \subset Ext^1_S(\mathcal{F}_1, \mathcal{F}_1)$. The embedding of this subspace is just the boundary map which comes from applying $Hom(\mathcal{F}_1, \cdot\,)$ to a short exact sequence $0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{E}/\mathcal{F}_1 \to 0$.

Since $\mathcal{E}/\mathcal{F}_1 = A_{\mathcal{F}_1} = \bigoplus_{x_j \in Supp(A_{\mathcal{F}_1})} (A_{\mathcal{F}_1})_{x_j}$, one has a direct sum decomposition

$$Hom_S(\mathcal{F}_1, A_{\mathcal{F}_1}) = \bigoplus_{x_j \in Supp(A_{\mathcal{F}_1})} Hom_S(\mathcal{F}_1, (A_{\mathcal{F}_1})_{x_j}).$$

Recall that $\mu_j = mult_{x_j}(A_{\mathcal{F}_1})$. Suppose that $\mathcal{F}_1$ is generic in the sense that it has a local description given before this lemma. It is easy to show using this local description that $T_{\mathcal{F}_1}(L^{\mu,s})$ corresponds precisely to the subspace

$$\bigoplus_{x_j \in Supp(A_{\mathcal{F}_1})} Hom_S(\mathcal{F}_1/\mathcal{E}(-\mu_j C), (A_{\mathcal{F}_1})_{x_j}) \subset Hom_S(\mathcal{F}_1, A_{\mathcal{F}_1}).$$

Similar computation applies to $\mathcal{F}_2 \in L^{\lambda,s}$.

Step 2. Recall that any tangent vector $v \in T_{\mathcal{F}_1}(M^G(r, s+|\mu|)) \subset Ext^1_S(\mathcal{F}_1, \mathcal{F}_1)$ corresponds to an extension $0 \to \mathcal{F}_1 \to \mathcal{G}_1 \to \mathcal{F}_1 \to 0$ (i.e. deformation with base $\mathbb{C}[\epsilon]/\epsilon^2$).

The conditions *Supp* $(\mathcal{F}_1/\mathcal{F}_2) = x \in C$ and $length(\mathcal{F}_1/\mathcal{F}_2) = i$ imply that $\mathcal{F}_1(-iC) \subset \mathcal{F}_2$. Moreover, the following diagram

$$\begin{tikzcd} \mathcal{F}_1(-iC) \arrow[r] \arrow[d, equal] & \mathcal{F}_2 \arrow[d] \\ \mathcal{F}_1(-iC) \arrow[r, "a"] & \mathcal{F}_1 \end{tikzcd} \tag{3}$$

commutes. Here the map a is the muplitplication by i-th power of the local equation for C.

The diagram (3) should be preserved under infinitesimal deformation of the point $(\mathcal{F}_1, \mathcal{F}_2)$ within $P_C[-i]$. This means that

(a) There exists a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0
\end{array}$$

It is a standard fact of homological algebra that this is equivalent to requiring that the images of v and w under the natural maps

$$Ext^1_S(\mathcal{F}_1, \mathcal{F}_1) \to Ext^1_S(\mathcal{F}_2, \mathcal{F}_1) \quad \text{and} \quad Ext^1_S(\mathcal{F}_2, \mathcal{F}_2) \to Ext^1_s(\mathcal{F}_2, \mathcal{F}_1)$$

coincide.

(b) There exists a similar diagram of extensions corresponding to the embedding $\mathcal{F}_1(-iC) \subset \mathcal{F}_2$. This is can be expressed in terms of Ext groups in a similar way.

(c) The diagram of middile terms

$$\begin{array}{ccc}
\mathcal{G}_1(-iC) & \longrightarrow & \mathcal{G}_2 \\
\| & & \downarrow \\
\mathcal{G}_1(-iC) & \xrightarrow{a} & \mathcal{G}_1
\end{array}$$

commutes. This condition can be expressed as vanishing of some homomorphism from the quotient copy of $\mathcal{F}_1(-iC)$ to the subsheaf copy of $\mathcal{F}_1$ (i.e. the embedding of sheaves a should not be deformed). We will not write this down explicitly as we need this condition only in a special case (see below).

Step 3. We will prove that if $(v, w) \in U$ and $v \in T_{\mathcal{F}_1}(L^{\mu,s})$ then $w \in T_{\mathcal{F}_2}(L^{\lambda,s})$ and $w = 0$ implies $v = 0$. The condition $w \in T_{\mathcal{F}_2}(L^{\lambda,s})$ will follow from (a) above while the implication $(w = 0) \Rightarrow (v = 0)$ is a consequence of (c).

To use (a), consider the diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{E} & \longrightarrow & A_{\mathcal{F}_1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{E} & \longrightarrow & A_{\mathcal{F}_2} & \longrightarrow & 0
\end{array} \qquad (4)$$

and the induced commutative digram of Ext-groups:

$$\begin{array}{ccccc}
Hom_S(\mathcal{F}_1, A_{\mathcal{F}_1}) & \longrightarrow & Ext^1_S(\mathcal{F}_1, \mathcal{F}_1) & \longrightarrow & Ext^1_S(\mathcal{F}_1, \mathcal{E}) \\
\downarrow & & \downarrow & & \downarrow \\
Hom_S(\mathcal{F}_2, A_{\mathcal{F}_1}) & \longrightarrow & Ext^1_S(\mathcal{F}_2, \mathcal{F}_1) & \longrightarrow & Ext^1_S(\mathcal{F}_2, \mathcal{E}) \\
\uparrow & & \uparrow & & \| \\
Hom_S(\mathcal{F}_2, A_{\mathcal{F}_2}) & \longrightarrow & Ext^1_S(\mathcal{F}_2, \mathcal{F}_2) & \longrightarrow & Ext^1_S(\mathcal{F}_2, \mathcal{E})
\end{array}$$

Here the lower two rows are obtained by applying $Hom_S(\mathcal{F}_2, \,\cdot\,)$ to (4) and the upper row comes brom applying $Hom_S(\mathcal{F}_2, \,\cdot\,)$ to the upper row of (4). Note that by stability $Hom_S(\mathcal{E}, \mathcal{E}) = Hom_S(\mathcal{F}_i, \mathcal{F}_i) = Hom_S(\mathcal{F}_i, \mathcal{E}) = \mathbb{C}$ and hence the first arrow in each row is injective. Simple diagram chase shows that if

$$v \in \bigoplus_{x_j \in Supp(A_{\mathcal{F}_1})} Hom_S(\mathcal{F}_1/\mathcal{E}(-\mu_j C), (A_{\mathcal{F}_1})_{x_j}) \subset Hom_S(\mathcal{F}_1, A_{\mathcal{F}_1})$$

then by the condition (b) we have

$$w \in \bigoplus_{x_j \in Supp(A_{\mathcal{F}_2})} Hom_S(\mathcal{F}_2/\mathcal{E}(-\lambda_j C), (A_{\mathcal{F}_2})_{x_j}) \subset Hom_S(\mathcal{F}_2, A_{\mathcal{F}_2}).$$

To prove injectivity, recall that if $v \in Hom_S(\mathcal{F}_1, A_{\mathcal{F}_1}) \subset Ext^1(\mathcal{F}_1, \mathcal{F}_1)$ then the extension

$$0 \to \mathcal{F}_1 \to \mathcal{G}_1 \to \mathcal{F}_1 \to 0$$

can be recovered as a kernel of the map of extensions

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \oplus \mathcal{F}_1 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & 0 \\
& & {\scriptstyle b}\downarrow & & {\scriptstyle (b\oplus v)}\downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{\mathcal{F}_1} & = & A_{\mathcal{F}_1} & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where b is the cokernel of the natural embedding $\mathcal{F}_1 \subset \mathcal{E}$. Similar statement is true for w and the corresponding extension $0 \to \mathcal{F}_2 \to \mathcal{G}_2 \to \mathcal{F}_2 \to 0$. One shows that in our case the condition (c) translates into commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{F}_1(-iC) & \xrightarrow{v} & A_{\mathcal{F}_1}(-iC) \\
\downarrow & & \downarrow \\
\mathcal{F}_2 & \xrightarrow{w} & A_{\mathcal{F}_2}
\end{array}$$

Since both vertical arrows in this diagram are injective the condition $w = 0$ implies $v = 0$.

This completes the proof of the lemma and hence the proof of Theorem 4.6. $\square$

6 Computation of the intersection number.

The main result of this section is a computation of the intersection number of the fundamental classes of cycles $Quot^n_C(\mathcal{E})$ and $Quot^n_{C'}(s)$ in $M^G(r, n+s)$. In the case of Hilbert schemes this intersection is trasversal and the intersection number can be found by a simple set-theoretic argument (recall that we assume that C and C' are transversal). However, for higher ranks the set-theoretic intersection is not transversal any more and to compute the intersection index we have to apply the excess intersection formula (cf. [Fu]).

By definition $Quot^n_C(\mathcal{E}) \subset M^G(r, n+s)$ parametrizes all short exact sequences $0 \to \mathcal{E}_1 \to \mathcal{E}|_C \to A \to 0$ of sheaves on C, where A is a length n Artin sheaf. Any such sequence defines a sheaf $\mathcal{F} \in M^G(r, n+s)$ on S, namely the kernel of the composition $\mathcal{E} \to \mathcal{E}|_C \to A$. Note that $Quot^n_C(\mathcal{E})$ is *smooth* since the tangent space to it at a point represented by $0 \to \mathcal{E}_1 \to \mathcal{E}|_C \to A \to 0$ is $Hom_{\mathcal{O}_C}(\mathcal{E}_1, A)$. Since $\mathcal{E}_1$ is a rank r vector bundle on C this space is of constant dimension rn.

Similarly, $Quot^n_{C'}(s) \subset M^G(r, n+s)$ has open subset $(Quot^n_{C'}(s))^\circ$ isomorphic to a fiber bundle over the subset $(M^G(r,s))^\circ$ of locally free sheaves, with fiber $Quot^n_{\mathbb{C}'}(\mathcal{E}')$ over $\mathcal{E}' \in (M^G(r,s))^\circ$. Hence $(Quot^n_{C'}(s))^\circ$ is smooth.

Note that $Quot^n_{C'}(s) \setminus (Quot^n_{C'}(s))^\circ$ does not intersect $Quot^n_C(\mathcal{E})$ since all $\mathcal{F} \in Quot^n_C(\mathcal{E}) \subset M^G(r, n+s)$ satisfy $length(\mathcal{F}^{**}/\mathcal{F}) = n$ and for sheaves in $Quot^n_{C'}(s) \setminus (Quot^n_{C'}(s))^\circ$ this length is at least $(n+1)$.

Assume for the sake of simplicity that $S = \mathbb{P}^2$ and C, C' are two distinct lines intersecting at a point $x \in S$. The general case follows from our proof by a simple combinatorial argument.

Let $\mathcal{F} \in Quot^n_{C'}(s) \bigcap Quot^n_C(\mathcal{E})$. Then $\mathcal{F}^{**} = \mathcal{E}$, $\mathcal{F}$ is a kernel of $\mathcal{E} \to \mathcal{E}|_C \to A$ and also a kernel of $\mathcal{E} \to \mathcal{E}|_{C'} \to A'$. Hence $\mathcal{F}$ contains $\mathcal{E}(-C) + \mathcal{E}(-C') = \mathcal{E} \otimes \mathcal{J}_x$ as a subsheaf and $A \simeq A' \simeq (\mathbb{C}_x)^n$, where $\mathcal{J}_x$ is the ideal sheaf of $x \in S$ and $\mathbb{C}_x \simeq \mathcal{O}_S/\mathcal{J}_x$. This means that every sheaf F in our intersection can be obtained as a kernel of $\mathcal{E} \to \mathcal{E}_x \to A$ where A

is an n-dimensional quotient of an r-dimensional vector space $\mathcal{E}_x$. Hence $Quot^n_{C'}(s) \bigcap Quot^n_C(\mathcal{E})$ is nothing but the Grassman variety $Gr(\mathcal{E}_x, n)$ of all n-dimensional quotients of $\mathcal{E}_x$. In particular, $n \le r$.

Since $Quot^n_{C'}(s)$ and $Quot^n_C(\mathcal{E})$ are of complementary dimensions in the ambient $M^G(r, n+s)$ we see that for $r \ge 2$ their set-theoretic intersection has abnormally high dimension. However, it is true that $Quot^n_{C'}(s)$ and $Quot^n_C(\mathcal{E})$ are smooth along the points of $Gr(\mathcal{E}_x, n)$ and, moreover, for any $\mathcal{F} \in Gr(\mathcal{E}_x, n)$

$$T_{\mathcal{F}}\, Gr(\mathcal{E}_x, n) = T_{\mathcal{F}}\, Quot^n_{C'}(s) \bigcap T_{\mathcal{F}}\, Quot^n_C(\mathcal{E})$$

(this can be checked using methods of the previous section).

Therefore, we can apply excess intersection formula (cf. [Fu, Example 6.1.7]): let V be a rank $(r-n)n$ vector bundle on $Gr(\mathcal{E}_x, n)$ arising form the following exact sequence:

$$0 \to T_{Gr(r,n)} \to T_{Quot^n_C(\mathcal{E})} \oplus T_{Quot^n_{C'}(s)} \to T_{M^G(r,n+s)} \to V \to 0. \qquad (5)$$

Then the intersection number $[Quot^n_C(\mathcal{E})] \cdot [Quot^n_{C'}(s)]$ in $M^G(r, n+s)$ is equal to the top Chern class $c_{(r-n)n}(V)$.

To compute this Chern class we will need to consider certain sheaves on $M^G(r, n+s) \times S$. For any pair of sheaves $\mathcal{G}_1$, $\mathcal{G}_2$ on $M^G(r, n+s) \times S$ let $\mathcal{E}xt^i_{p_1}(\mathcal{G}_1, \mathcal{G}_2)$ be the relative Ext-sheaf on $M^G(r, n+s)$ with respect to the first projection $p_1 : M^G(r, n+s) \times S \to M^G(r, n+s)$. We will only deal with the cases when for all $x \in M^G(r, n+s)$ the global Ext group $Ext^i(p_1^{-1}(x); \mathcal{G}_1, \mathcal{G}_2)$ on the fiber $p_1^{-1}(x)$ is of constant dimension. Hence by [Kl] $\mathcal{E}xt^i_{p_1}(\mathcal{G}_1, \mathcal{G}_2)$ is a vector bundle on $M^G(r, n+s)$ of the same rank. Similar remarks apply to any closed subspace of $M^G(r, n+s)$ (abusing notation we will denote a sheaf on $M^G(r, n+s)$ and its restriction to a closed subspace by the same letter).

Consider a larger moduli space $\widetilde{M}^G(r, n+s)$ of H-stable sheaves with $c_1(\mathcal{F}) = c_1(L)$ and $c_2(\mathcal{F}) = n+s$ (i.e. the determinant is not fixed but is allowed to differ from L by any line bundle in $Pic^0(S)$). Since $gcd(r, c_1(H) \cdot c_1(L)) = 1$, there exists a universal sheaf $\mathcal{G}$ on $\widetilde{M}^G(r, n+s) \times S$ (cf. [HL]). It is known (cf. [Ma]) that the tangent bundle $T\, \widetilde{M}^G(r, n+s)$ is isomorphic to $\mathcal{E}xt^1_{p_1}(\mathcal{G}, \mathcal{G})$. Since $M^G(r, n+s)$ is a fiber of a smooth map $det : \widetilde{M}^G(r, n+$

$s) \to Pic^0(S)$, the tangent bundle to $M^G(r, n+s)$ has the same Chern classes as $\mathcal{E}xt^1_{p_1}(\mathcal{G}, \mathcal{G})$.

To compute the full Chern class of $\mathcal{E}xt^1_{p_1}(\mathcal{G}, \mathcal{G})|_{Gr(\mathcal{E}_x, n)}$ we use a short exact sequence of sheaves on $Gr(\mathcal{E}_x, n) \times S$:

$$0 \to \mathcal{G} \to p_2^*\mathcal{E} \to \mathcal{A} \to 0, \tag{6}$$

where p_1, p_2 are the projections, $\mathcal{A} = p_1^*Q \otimes p_2^*\mathbb{C}_x$ and Q is the universal quotient bundle on the Grassmanian.

First of all, applying $R\mathcal{H}om_{p_1}(\cdot, \mathcal{G})$ to (6) one obtains a long exact sequence of sheaves on the Grassmanian:

$$0 \to \mathcal{H}om_{p_1}(\mathcal{G}, \mathcal{G}) \to \mathcal{E}xt^1_{p_1}(\mathcal{A}, \mathcal{G}) \to \mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E}, \mathcal{G}) \to$$

$$\to \mathcal{E}xt^1_{p_1}(\mathcal{G}, \mathcal{G}) \to \mathcal{E}xt^2_{p_1}(\mathcal{A}, \mathcal{G}) \to 0$$

Note that $\mathcal{H}om_{p_1}(p_2^*\mathcal{E}, \mathcal{G})$ vanishes since its fiber over $\mathcal{F} \in Gr(\mathcal{E}_x, n)$ is $Hom_{\mathcal{O}_S}(\mathcal{E}, \mathcal{F})$ which is zero ($\mathcal{E}$ and $\mathcal{F}$ are stable and $c_2(\mathcal{F}) > c_2(\mathcal{E})$). Moreover, the fiber of $\mathcal{H}om_{p_1}(\mathcal{G}, \mathcal{G})$ over $\mathcal{F}$ is $Hom_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F})$. Since stable sheaves have only scalar automorphisms, $\mathcal{H}om_{p_1}(\mathcal{G}, \mathcal{G})$ is a trivial line bundle. One can also check that either both $\mathcal{E}xt^2_{p_1}(p_2^*\mathcal{E}, \mathcal{G})$ and $\mathcal{E}xt^2_{p_1}(\mathcal{G}, \mathcal{G})$ are zero (when $c_1(K) \cdot c_1(H) < 0$) or the map between them is an isomorphism (when $K \simeq \mathcal{O}$).

Thus, the full Chern class of $T_{M^G(r,n+s)}$ is equal to:

$$c(T_{M^G(r,n+s)}) = c(\mathcal{E}xt^1_{p_1}(\mathcal{G}, \mathcal{G})) = \frac{c(\mathcal{E}xt^2_{p_1}(\mathcal{A}, \mathcal{G}))c(\mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E}, \mathcal{G}))}{c(\mathcal{E}xt^1_{p_1}(\mathcal{A}, \mathcal{G}))}$$

As a second step we apply $R\mathcal{H}om_{p_1}(p_2^*\mathcal{E}, \cdot)$ to (6) and get

$$0 \to \mathcal{H}om_{p_1}(p_2^*\mathcal{E}, p_2^*\mathcal{E}) \to \mathcal{H}om_{p_1}(p_2^*\mathcal{E}, \mathcal{A}) \to$$

$$\to \mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E}, \mathcal{G}) \to \mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E}, p_2^*\mathcal{E}) \to 0$$

Here, the sheaf $\mathcal{H}om_{p_1}(p_2^*\mathcal{E}, \mathcal{G})$ vanishes as before and $\mathcal{E}xt^1(p_2^*\mathcal{E}, \mathcal{A})$ is zero since its fiber over a point $\mathcal{F}$ is $Ext^1_{\mathcal{O}_S}(\mathcal{E}, (\mathbb{C}_x)^n) = H^1(S, \mathcal{E}^* \otimes (\mathbb{C}_x)^n) = 0$. The bundles $\mathcal{H}om_{p_1}(p_2^*\mathcal{E}, p_2^*\mathcal{E})$ and $\mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E}, p_2^*\mathcal{E})$ are trivial on $Gr(\mathcal{E}_x, n)$ hence

$$c(\mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E},\mathcal{G})) = c(\mathcal{H}om_{p_1}(p_2^*\mathcal{E},\mathcal{A})) = c(Q\otimes\mathbb{C}^r) = (c(Q))^r.$$

Finally, we apply $R\mathcal{H}om_{p_1}(\mathcal{A},\cdot)$ to (6) and get

$$0\to\mathcal{H}om_{p_1}(\mathcal{A},\mathcal{A})\to Ext^1_{p_1}(\mathcal{A},\mathcal{G})\to 0\to Ext^1_{p_1}(\mathcal{A},\mathcal{A})\to$$
$$\to Ext^2_{p_1}(\mathcal{A},\mathcal{G})\to Ext^2_{p_1}(\mathcal{A},p_2^*\mathcal{E})\to Ext^2_{p_1}(\mathcal{A},\mathcal{A})\to 0$$

where $\mathcal{E}xt^1_{p_1}(\mathcal{A},p_2^*\mathcal{E})$ vanishes since it is dual to $\mathcal{E}xt^1_{p_1}(p_2^*\mathcal{E},\mathcal{A}\otimes p_2^*K_S)$ (cf. [Kl]). Since $Hom_{\mathcal{O}_S}(\mathbb{C}_x,\mathbb{C}_x) = Ext^2_{\mathcal{O}_S}(\mathbb{C}_x,\mathbb{C}_x) = \mathbb{C}$ and $Ext^1_{\mathcal{O}_S}(\mathbb{C}_x,\mathbb{C}_x) = \mathbb{C}^2$, we have

$$\mathcal{H}om_{p_1}(\mathcal{A},\mathcal{A})\simeq Ext^2_{p_1}(\mathcal{A},\mathcal{A})\simeq Q\otimes Q^*,\quad Ext^1_{p_1}(\mathcal{A},\mathcal{A}) = (Q\otimes Q^*)^{\oplus 2}.$$

To compute $\mathcal{E}xt^2_{p_1}(\mathcal{A},p_2^*\mathcal{E})$ we again apply a relative version of Serre's duality (*loc. cit.*) which gives

$$\mathcal{E}xt^2_{p_1}(\mathcal{A},p_2^*\mathcal{E})\simeq (Q^*)^{\oplus r}.$$

Summing up the results of our computation, we obtain

$$c(T_{M^G(r,n+s)}) = \frac{(c(Q^*))^r(c(Q))^r(c(Q\otimes Q^*))^2}{c(Q\otimes Q^*)c(Q\otimes Q^*)} = (c(Q)c(Q^*))^r.$$

A similar approach can be used with $T_{Quot^n_C(\mathcal{E})}$ and $T_{Quot^n_{C'}(s)}$ (one has to consider sheaves on $Gr(\mathcal{E}_x,n)\times C$ and $Gr(\mathcal{E}_x,n)\times C'$). One checks that

$$c(T_{Quot^n_C(\mathcal{E})}) = c(T_{Quot^n_{C'}}(s)) = (c(Q))^r.$$

Let S be the universal subbundle on $Gr(\mathcal{E}_x,n)$. Using the exact sequence (5) and the following short exact sequence:

$$0\to Q\otimes Q^*\to Q^{\oplus r}\to S^*\otimes Q\to 0$$

(recall that $T_{Gr(\mathcal{E}_x,n)}\simeq S^*\otimes Q$), we get

$$c(V) = \frac{c(T_{M^G(r,n+s)})c(T_{Gr(\mathcal{E}_x,n)})}{c(T_{Quot^n_C(\mathcal{E})})c(T_{Quot^n_{C'}(s)})} = \frac{(c(Q)c(Q^*))^r(c(S^*\otimes Q))}{(c(Q))^r(c(Q))^r} =$$

$$= \frac{(c(Q^*))^r(c(Q))^r}{(c(Q))^r c(Q\otimes Q^*)} = \frac{(c(Q^*))^r}{c(Q\otimes Q^*)} = c(Q^*\otimes S).$$

Hence the full Chern class $c(V)$ of the excess normal bundle V is equal to the full Chern class $c(Q^*\otimes S)$ of the cotangent bundle to $Gr(\mathcal{E}_x, n)$ (which is isomorphic to $Q^*\otimes S$). Therefore

$$c_{(r-n)n}(V) = (-1)^{rn-n^2}\binom{r}{n} = (-1)^{(r-1)n}\binom{r}{n}.$$

Recall that the top Chern class above is equal to the intersection number of $[Quot^n_C(\mathcal{E})]$ and $[Quot^n_{C'}(s)]$. Now Theorem 4.7 follows from the binomial formula. $\square$

7 Appendix: the punctual Quot scheme.

In this appendix we give a proof of Theorem 2.2 saying that the scheme $Quot(r,n)$ is irreducible of dimension $(rn-1)$.

Our strategy is to find a dense irreducible open subset $W\subset Quot(r,n)$ of dimension $(rn-1)$. This subset will turn out to be a vector bundle over the punctual Hilber scheme $Hilb^n := Quot(1,n)$ considered by Briançon and Iarrobino.

We define W as the set of all quotients $\mathcal{O}^{\oplus r}\xrightarrow{\phi} A$, $\phi=(\phi_1+\phi_2+\ldots+\phi_r)$ such that the first component $\phi_1:\mathcal{O}\to A$ is surjective (this is clearly an open condition). Such a ϕ_1 corresponds to a point in $Hilb^n$. Once ϕ_1 is chosen, the other components $(\phi_2,\ldots,\phi_r)$ are given by an arbitrary element of $Hom(\mathcal{O}^{\oplus(r-1)}, A) = \mathbb{C}^{(r-1)n}$. Therefore W is a rank $(r-1)n$ vector bundle over $Hilb^n$. By [Br] or [Ia] the subset W is irreducible of dimension $(rd-1)$.

Now we want to show that W is dense in $Quot(r,n)$. In fact, for any point $x\in Quot(r,n)$ we will find an irreducible rational curve $C\subset Quot(r,n)$ connecting it with some point in W.

To that end, we generalize Nakajima's construction (cf. [Na, Chapter 2]) of the global Hilbert scheme $Hilb^n(\mathbb{C}^2)$ to the Quot scheme. Once we do that, the existence of the irreducible curve will amount to an exercise in linear algebra (cf. Lemma 7.3).

Fix a complex vector space V of dimension n, and N_n let be the space of pairs of commuting nilpotent operators on V. The space N_n is naturally a closed affine subvariety of End(V) $\oplus$ End(V).

Consider a subspace U_r of $N_n \times V^{\oplus r}$ formed by all $(B_1, B_2, v_1, \ldots, v_r)$, such that there is no proper subspace of V which is invariant under B_1, B_2 and contains $v_1, \ldots, v_r$. Then $U_1 \times V^{\oplus(r-1)} \subset U_2 \times V^{\oplus(r-2)} \subset \ldots \subset U_r$ is a chain of open subsets in $N_n \times V^{\oplus r}$ (each of them is given by a condition saying that some system of vectors in V has maximal rank).

Ona has a natural $GL(V)$-action on V_r and it is easy to prove that U_r is $GL(V)$-stable.

Lemma 7.1 *$GL(V)$ acts freely on U_r.*

Proof. Suppose $g \in GL(V)$ stabilizes $(B_1, B_2, v_1, \ldots, v_r) \in U_r$. Then $\mathrm{Ker}(1-\mathrm{g})$ contains $v_1, \ldots, v_r$. Since it is also preserved by B_1, B_2 , we have $Ker(1-g) = V$ and therefore $g = 1$. □

The following lemma gives an explicit construction of the punctual Quot scheme:

Lemma 7.2 *There exists a morphism $\pi : U_r \to Quot(r, n)$ such that*
(i) π is surjective;
(ii) the fibers of π are precisely the orbits of $GL(V)$ action on U_r;
(iii) $\pi^{-1}(W) = U_1 \times V^{\oplus(r-1)}$.

Proof. Note that the punctual Quot scheme does not depend on the surface since the n-th power of the maximal ideal $\mathfrak{m}_x$ acts by zero on any length n sheaf supported at x. Hence we can assume that $S = \mathbb{C}^2 = Spec\, \mathbb{C}[x_1, x_2]$ and that all the quotients are supported at $x = 0 \in \mathbb{C}^2$.

To construct π suppose that $(B_1, B_2, v_1, \ldots, v_r)$ is a point in U_r and consider a $\mathbb{C}[x_1, x_2]$-module structure on V in which x_1 acts by B_1 and x_2 acts by B_2. We can view V as a quotient of a free $\mathbb{C}[x_1, x_2]$-module with generators $v_1, \ldots, v_r$. Since B_1 and B_2 are nilpotent $\sqrt{\mathrm{Ann(V)}} = (x_1, x_2)$. Therefore a coherent sheaf A on $\mathbb{C}^2$ associated with V is a quotient of $\mathcal{O}^{\oplus r}$ supported at s. Moreover, $V \simeq H^0(S, A)$ as vector spaces.

A different point in the same $GL(V)$-orbit defines an isomorphic quotient, hence the fibers of $\pi : U_r/\mathrm{GL(V)} \to \mathrm{Quot(r, n)}$ are $GL(V)$-invariant. Moreover, suppose that two points u_1, u_2 of U_r give rise to isomorphic quotients A_1, A_2. Then the induced isomorphism between $H^0(S, A_1)$ and $H^0(S, A_2)$ defines an element of $GL(V)$ taking u_1 to u_2. Therefore, each fiber of π is precisely one $GL(V)$-orbit. This proves (ii).

To prove (i), suppose we have a quotient $\mathcal{O}^{\oplus r} \to A \to 0$ of length d supported at zero. Multiplication by x_1 and x_2 induces a pair of commuting nilpotent operators on $H^0(S, A)$. Choose a $\mathbb{C}$-linear isomorphism

$H^0(S,A) \simeq V$. The generators of the free $\mathbb{C}[x_1,x_2]$-module $H^0(S,\mathcal{O}^{\oplus r})$ project to some vectors $v_1,\dots,v_r$ in V. Since $v_1,\dots,v_r$ generate V as a $\mathbb{C}[x_1,x_2]$-module, $(x_1,x_2,v_1,\dots v_r)$ is a point of U_r. Thus (i) is proved.

Finally, (iii) follows from definitions of W and U_1. □

Now we want to show that any point in U_r can be deformed to a point in the preimage of W. The above construction will allow us to construct this deformation using the following lemma

Lemma 7.3 *Let B_1, B_2 be two commuting nilpotent operators on a vector space V. There exists a third nilpotent operatov B_2' and a vector $w \in V$ such that*

(i) B_2' commutes with B_1;

(ii) any linear combination $\alpha B_2 + \beta B_2'$ is nilpotent;

(iii) $(B_1, B_2', w) \in U_1$, i.e. w is a cyclic vector for the pair of operators (B_1, B_2').

This lemma will be proved later. Now we will show how it can be used to give an

End of Proof of Theorem 2.2:

Let x be a point of $Quot(r,n)$ and $u_1 = (B_1, B_2, v_1, \dots, v_r)$ be any point of $\pi^{-1}(x) \subset U_r$. Choose a nilpotent operator B_2' and a vector $w \in V$ as in the lemma above. Connect the points u_1 and $u_2 = (B_1, B_2', w, v_2, \dots, v_r)$ with a straight line $\Phi(t)$, $t \in \mathbb{C}$ such that $\Phi(1) = u_1$ and $\Phi(0) = u_2$. This $\Phi(t)$ is given by equation:

$$\Phi(t) = (B_1, tB_2' + (1-t)B_2, tw + (1-t)v_1, v_2, \dots, v_r)$$

Note that for all $t \in C$, $B_2(t) = tB_2' + (1-t)B_2$ is nilpotent and commutes with B_1. Therefore the image of $\Phi(t)$ is a subset of $N_n \times V^{\oplus r}$. Since U_r is open in $N_n \times V^{\oplus r}$, there is a dense open subset $C \subset \mathbb{C}$ such that $\Phi(C) \subset U_r$. Similarly, there exists a dense open subset $C_1 \subset C$ such that $\Phi(C_1) \subset U_1 \times V^{\oplus(r-1)}$.

Hence the image $\pi(\Phi(C)) \subset Quot(r,n)$ is an irreducible rational curve connecting $x = \pi(u_1)$ with $\pi(u_2) \in W$. Note that $\pi(\Phi(C_1)) \subset W$. Therefore x belongs to the closure of W. Since by Theorem 1.1 W is irreducible of dimension $(rd-1)$, the scheme $Quot(r,n)$ is also irreducible of dimension $(rd-1)$. Theorem 2.2 is proved. □

Proof of Lemma 7.3:

Step 1. We will find a basis $e_{i,j}$ of V, where $1 \le i \le k$ and $1 \le j \le \mu_i$ such that

(a) $B_1^{j-1}(e_{i,1}) = e_{i,j}$ for $j \le \mu_i$ and $B_1^{\mu_j}(e_{i,1}) = 0$ (i.e. B_1 has Jordan canonical form in the basis $e_{i,j}$);

(b) $B_2(e_{i,1}) \in \left(\bigoplus_{k \ge i+1} \mathbb{C} \cdot e_{k,1}\right) \oplus B_1 \cdot V$.

To that end, recall one way to construct a Jordan basis for B_1. Let $d = \dim V$ and $V_i = Ker(B_1^{d-i})$. The subspaces V_i form a decreasing filtration $V = V_0 \supset V_1 \supset V_2 \dots$. Moreover, $B_1 \cdot V_i \subset V_{i+1}$. Firstly, we choose a basis $(w_1, \dots, w_{a_1})$ of $W_1 := V_0/V_1$. Lift this basis to some vectors $e_{1,1}, e_{2,1}, \dots, e_{a_1,1}$ in V_0 and set all $\mu_1, \dots, \mu_{a_1}$ equal to d. Secondly, choose a basis $(w_{a_1+1}, \dots, w_{a_2})$ of $W_2 := V_1/(B_1 \cdot V_0 + V_2)$. Lift this basis to some vectors $e_{a_1+1,1}, e_{2,1}, \dots, e_{a_2,1}$ in V_1 and set all $\mu_{a_1+1}, \dots, \mu_{a_2}$ equal to $d-1$. Continue in this manner by choosing bases of the spaces $W_{i+1} = V_i/(B_1 \cdot V_{i-1} + V_{i+1})$ and lifting them to V_i. This procedure gives us vectors $e_{1,1}, e_{2,1}, \dots, e_{k,1}$ and the formula (a) tells us how to define $e_{i,j}$ for $j \ge 2$. It is easy to check that the system of vectors $\{e_{i,j}\}$ is in fact a basis of V.

If we want to have property (b) we should be more careful with the choice of w_i. Note that all the subspaces V_i and $B_1 \cdot V_i$ are B_2-invariant. Therefore we have an induced action of B_2 on each W_i. We can choose our basis $(w_{a_{i-1}+1}, \dots, w_{a_i})$ of W_i in such a way that $B_2(w_i) \in \bigoplus_{s=i+1}^{a_i} \mathbb{C} \cdot w_s$ for all $i \in \{a_{i-1}+1, \dots, a_i\}$. This ensures that (b) holds as well.

Step 2. Define B_2' by $B_2'(e_{i,j}) = e_{i+1,j}$ if $j \le \mu_{i+1}$ and 0 otherwise. It is immediate that B_2' is nilpotent and that $[B_1, B_2'] = 0$. Let $w = e_{1,1}$. Then $e_{i,j} = B_1^{j-1}(B_2^{i-1}(w))$ hence $(B_1, B_2', w) \in U_1$.

Step 3. Note that both B_2 and B_2' are lower-triangular with zeros on the diagonal in the basis of V given by

$$e_{1,1}, e_{2,1}, \dots, e_{k,1}, e_{1,2}, e_{2,2}, \dots e_{k,2}, \dots$$

Hence any linear combination of B_2 and B_2' is also lower-triangular and has zeros on the diagonal. Therefore $\alpha B_2 + \beta B_2'$ is nilpotent for any complex α and β. The Lemma is proved. $\square$

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