A unified approach to the four vertex theorems I

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CONTENTS

Introduction. In 1932, Bose [Bo] established the following formula for a given non-circular simple closed convex plane curve γ

$$s^{\bullet} - t^{\bullet} = 2, \tag{0.1}$$

where $s^{\bullet}$ is the number of enclosing osculating circles and $t^{\bullet}$ is the number of triple tangent enclosed circles in γ. Haupt [Hu] (1969) extended it to simple closed curves in the category of Ordnungscharacteristiken(=OCh) mit der Grundzahl $k = 3$, which is defined in Haupt and Künneth [HK].

Roughly speaking, the formula for generic simple closed curves can be obtained by the following simple observations: Let γ be a generic C^{∞}-regular simple closed curve and D the domain bounded by γ. The *cut locus* K ($\subset D$) of γ is the closure of the set of points which have more than one minimizing line segments from γ. Then K has a structure of a tree and each boundary point corresponds to the center of an enclosed osculating circle. (See Thom [Tm1] and [Tm2].) Moreover, it can be observed that the branch points of K are the centers of triple tangent enclosed circles. Hence $s^{\bullet}$ is the number of the boundary points of K and $t^{\bullet}$ is the total branching number at the branch points. Since K is contractible, the formula $s^{\bullet} - t^{\bullet} = 2$ follows immediately. (This observation is justified for any C^2-regular simple closed curves with $s^{\bullet} < \infty$. See the last remark in §2.)

We give here a brief history of the four vertex theorems for simple closed curves. In 1909, Mukhopadhayaya [Mu1] proved it for convex closed curves. A. Kneser [A.K] (1912) extended it to simple closed curves. But a vertex (that is, a critical point of the curvature function) may not be a point where the osculating circle is completely inside and outside the curve. The inequality $s^\bullet \geq 2$ for simple closed curves was proved by H. Kneser [H.K] (1922-1923) who is a son of A. Kneser. The Bose formula and its generalization by Haupt [Hu] is a refinement of it. Jackson [J1] (1944) gave many other fundamental tools for the study of vertices on plane curves.

On the other hand, the four vertex theorem was extended to simple closed curves on closed convex surfaces by Mohrmann [Mo](1917) without details and its complete proof was given by Barner and Flohr [BF] in 1958. To generalize the four vertex theorem for simple closed convex space curves (that is, curves lying on the boundary of their convex hulls) with non-vanishing curvature, Romero-Fuster [R] proved a Bose type formula

$$s - t = 4 \tag{0.2}$$

for convexly generic convex curves γ in $\mathbf{R}^3$, where s is the number of supporting osculating planes and t is the number of tritangent supporting planes. (Various approaches for the same problem are found in [Bi], [RCN2] and [BR1-2].) After that, Sedykh [Sd2] showed that (0.2) is true for simple closed strictly convex space curves. (Moreover, he gave a generalization of (0.2) for strictly convex manifolds M^k in the Euclidean space $\mathbf{R}^n$ ($k < n-1$).) The four vertex theorem for simple closed convex space curves with non-vanishing curvature itself was proved in Sedykh [Sd1] by a different approach. Recently, Kazarian [Ka] established some formulas similar to (0.1) representing the Chern-Euler class of a circle bundle over a Riemann surface in terms of global singularities of restrictions of a generic function to the fibers.

There are interesting connections between vertices and integral geometry (e.g. [Bl2], [Hy],[Ba],[Gu1-2],[He5].) or contact geometry. The author was inspired by them, especially recent papers [A1-4],[GMO],[OT],[Ta1-3] in which several variations of the four vertex theorem are observed from the view of contact geometry or proved by using the technique of disconjugate operators on S^1.

The purpose of the paper is to give a unified treatment of the formulas (0.1) and (0.2). More precisely, we will introduce a notion "intrinsic circle system" as a certain multivalued function on the unit circle without referring to ambient spaces, which characterizes the cut loci of plane curves intrinsically and enables us to prove the formula (0.1) abstractly. Consequently, (0.1) or (0.2) is proved under much weaker assumptions for the following three cases:

(1) piecewise C^1-regular simple closed curves on the Euclidean or Minkowski plane, which bounds a domain whose internal angles are less than or equal to π,
(2) piecewise C^1-regular simple closed curves on an embedded surface with positive Gaussian curvature in $\mathbf{R}^3$, which bounds a domain whose internal angles are less than or equal to π,
(3) convex simple closed space curves in $\mathbf{R}^3$ with some additional conditions. (As an application, the Sedykth's 4-vertex theorem is obtained.)

The formula like as (0.1) will be shown for these three cases. (See Theorem 2.7 and Theorem 3.2.) However, the formula like as (0.2) requires C^2-regularity of curves. (See Corollary 3.3 and Theorem 4.14.) Haupt's proof partially covers the cases (1)-(2) but not (3). (In his paper, the existence of osculating circles is assumed.) Here the vertices on curves defined for the cases (1)-(2) include singular points of curves. This gives a new interpretation for the existence of the unique inscribed circle in a triangle. (In this case, $s^\bullet = 3$ is the number of vertices and $t^\bullet = 1$ is the number of inscribed circles and they satisfy the relation $s^\bullet - t^\bullet = 2$ trivially.) Though it is not directly concerned with the Bose-type formulas, several generalization of four vertex theorems without differentiability have been investigated by [LSc],[J2],[LSp],[Sp1-4] etc. It should also be remarked that vertices for polygons are studied by several authors. (See [Sa],[W2] and [Sd3].) But their definition of vertex is different from ours. (In our setting, the vertices of polygons have the usual meaning.)

Finally, we remark here that this paper is prepared for the ensuing paper Thorbergsson and Umehara [TU], in which we shall prove in the same axiomatic setting that *for any C^2-regular simple closed curve $\gamma : [a,b] \to \mathbf{R}^2$, there exist four points t_1, t_2, t_3, t_4 ($t_1 < t_2 < t_3 < t_4$) such that the osculating circles at t_1 and t_3 are enclosed in γ and the osculating circles at t_2 and t_4 enclose γ.* (Here the the order of the osculating circles is important. The corresponding version for convex simple closed space curves also holds.) The statement looks obvious at the first glance, but it is one of the deepest versions of the four vertex theorems, and provides many applications.

§1 Intrinsic circle systems.

We fix an oriented unit circle S^1. Let $\succ$ denote the order induced by the orientation on the complement of any interval in S^1. Any two distinct points $p, q \in S^1$ divide S^1 into two closed arcs $[p,q]$ and $[q,p]$ such that on $[p,q]$ we have $q \succ p$ and on $[q,p]$ we have $p \succ q$. We let (p,q) and (q,p) denotes the corresponding open arcs. We also use the notation $p \succeq q$, which means $p = q$ or $p \succ q$. Let A be a subset of S^1 and $p \in A$. We denote by $Z_p(A)$ the connected component of A containing p.

Definition 1.1. A family of non-empty closed subsets $F := (F_p)_{p\in S^1}$ of S^1 is called an *intrinsic circle system* on S^1 if it satisfies the following three conditions for any $p \in S^1$.

(I1) If $q \in F_p$, then $F_p = F_q$.

(I2) If $q \in S^1 \setminus F_p$, then $F_q \subset Z_q(S^1 \setminus F_p)$. (Or equivalently, if $p' \in F_p$, $q' \in F_q$ and $q \succeq p' \succeq q' \succeq p (\succeq q)$, then $F_p = F_q$ holds.)

(I3) Let $(p_n)_{n\in\mathbf{N}}$ and $(q_n)_{n\in\mathbf{N}}$ be two sequences in S^1 such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$ respectively. Suppose that $q_n \in F_{p_n}$ $(n = 1, 2, 3, ...)$. Then $q \in F_p$ holds.

Remark. Let γ be a piecewise C^1-regular simple closed curve in $\mathbf{R}^2$. Let $C_p^\bullet$ be the maximal circle which is contained in γ and tangent to γ at p. Then $F_p := \gamma \cap C_p^\bullet$ satisfies the above three conditions. (See Proposition 3.1.) The definition of the intrinsic circle system characterizes the properties of maximal circles of a curve without referring to an ambient space, which enable us to generalize the Bose type formula to convex simple closed space curves. This is the reason for the terminology "intrinsic circle system". By (I1), F induces an equivalence relation. Later (See the last remark in §3), we will show that the quotient topological space S^1/F is homeomorphic to the cut locus K of γ. In this sense, the intrinsic circle system can also be interpreted as an abstract characterization of the cut loci of plane curves. We give here two elementary examples.

Let $\gamma : x^2/a^2 + y^2/b^2 = 1$ $(a > b)$ be an ellipse in $\mathbf{R}^2$. Then the maximal circle $C_p^\bullet$ at each point $p = (x,y)$ on γ has two contact points at p and $\bar{p} = (x, -y)$ unless $y \neq 0$. So if we set $F_p := C_p^\bullet \cap \gamma$, then

$$F_p := \begin{cases} \{p, \bar{p}\} & \text{if } p \neq (\pm a, 0), \\ \{p\} & \text{if } p = (\pm a, 0). \end{cases}$$

One can easily verify that $(F_p)_{p\in\gamma}$ is an intrinsic circle system.

Another typical example is the triangle $\triangle abc$ as in Figure 1.1, which is invariant under the reflections α, β and γ. We consider the maximal circle $C_p^\bullet$ at each point on the triangle. Then $C_p^\bullet$ has two contact points to the triangle unless $p = a, b, c, x, y, z$, where $x := (a+b)/2$, $y := (b+c)/2$ and $z := (c+a)/2$. So if we set $F_p := C_p^\bullet \cap \gamma$, then

$$F_p := \begin{cases} \{p, \alpha(p)\} & \text{if } p \in \overline{ay} \cup \overline{az} \text{ and } p \neq a, y, z \\ \{p, \beta(p)\} & \text{if } p \in \overline{bz} \cup \overline{bx} \text{ and } p \neq b, z, x \\ \{p, \gamma(p)\} & \text{if } p \in \overline{cx} \cup \overline{cy} \text{ and } p \neq c, x, y \\ \{p\} & \text{if } p = a, b, c \\ \{x, y, z\} & \text{if } p = x, y, z. \end{cases}$$

One can also easily verify that $(F_p)_{p\in\triangle abc}$ is an intrinsic circle system. We will give further examples of intrinsic circle systems in §3 and §4.

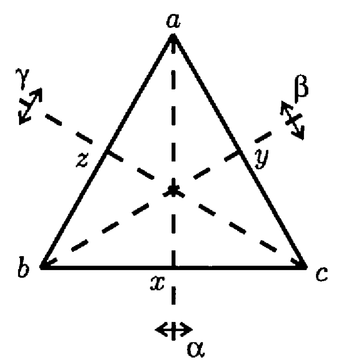

Figure 1.1

Let A be a subset of S^1. The number of connected components of A is called the *rank* of A and is denoted by $\operatorname{rank}(A)$. For a family of non-empty closed subsets $(F_p)_{p\in S^1}$, we set

$$\operatorname{rank}(p) := \operatorname{rank}(F_p).$$

The next lemma, which plays a fundamental role in this paper, is a generalization of the main argument in H. Kneser [K.H].

Lemma 1.1. *Let $(F_p)_{p\in S^1}$ be a family of non-empty closed subsets satisfying (I2). Let p, q be points on S^1 such that $q \in F_p$. Suppose that $(p,q) \not\subset F_p$. Then there exists a point $x \in (p,q)$ such that* $\operatorname{rank}(x) = 1$.

Proof. If necessary, taking a subarc in (p,q), we may assume that $F_p \cap (p,q)$ is empty. We fix a metric $d(\ ,\ )$ on S^1. Let x be the middle point of $[p,q]$ with respect to the distance function. If $\operatorname{rank}(x) = 1$, the proof is finished. So we may assume that $\operatorname{rank}(x) > 1$. By (I2), $F_x \subset (p,q)$. Since $S^1 \setminus F_x$ is an open subset, we can choose a connected component (p_1, q_1) of $S^1 \setminus F_x$ such that $(p_1, q_1) \subset [p,q]$. Then $p_1, q_1 \in F_x$. Instead of p and q, we apply the above argument for p_1 and q_1. Let x_1 be the middle point of the arc $[p_1, q_1]$. Then we find a subarc $[p_2, q_2]$ such that $p_2, q_2 \in F_{x_1}$ and

$(p_2, q_2) \subset S^1 \setminus F_{x_1}$. Continuing this argument, we get a sequence of arcs $\{[p_n, q_n]\}_{n\in\mathbf{N}}$ such that

$$d(p_n, q_n) < \frac{1}{2} d(p_{n-1}, q_{n-1}).$$

Thus, there exists a point $y \in (p, q)$ such that

$$y = \lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n.$$

If $\operatorname{rank}(y) \neq 1$, then there exists an element $z \in F_y$ different from y. Then $z \notin (p_n, q_n) = Z_y(S^1 \setminus F_{p_n})$ for a sufficiently large n. This contradicts (I2). Thus we have $\operatorname{rank}(y) = 1$. $\square$

Remark. Suppose that γ is a simple closed curve in $\mathbf{R}^2$. Let $C_p^\bullet$ be a maximal circle and $F_p = C_p^\bullet \cap \gamma$. Then the argument above was applied to show the existence of two distinct enclosed osculating circles in H. Kneser [H.K]. In fact, using Lemma 1.1, one can easily get the existence of two distinct maximal circles $C_x^\bullet$ and $C_y^\bullet$ ($x \neq y$), which are tangent to γ with only one connected component. If the curve γ is C^2-differentiable, then $C_x^\bullet$ and $C_y^\bullet$ must coincide with the osculating circles at $x, y \in \gamma$ respectively. (See Proposition A.5 in Appendix A.) We remark that Thorbergsson [Tr] generalized this argument for a certain class of simple closed curves in any complete Riemannian 2-manifold.

From now on, we fix an intrinsic circle system $F = (F_p)_{p\in S^1}$ on S^1.

Definition 1.2. $p \in S^1$ is called *regular* (resp. *weakly regular*) if $\operatorname{rank}(p) = 2$ (resp. $2 \leq \operatorname{rank}(p) \leq \infty$). A subarc I of S^1 whose elements are all regular (resp. weakly regular) is called a *regular arc* (resp. *weakly regular arc*).

The following lemma immediately follows from Lemma 1.1.

Corollary 1.2. *Let I be an open weakly regular arc. Then for each $p \in I$, the set*

$$Y_p := F_p \setminus Z_p(F_p)$$

is contained in $S^1 \setminus \overline{I}$. In particular, the closure $\overline{Y_p}$ lies in $S^1 \setminus I$.

Definition 1.3. Let I be a closed arc on S^1 and A be a subset in I. Then the points $\sup_I(A)$ and $\inf_I(A)$ which are called the least upper bound and the greatest lower bound of A, are defined as the smallest (resp. greatest) points satisfying

$$\sup_I(A) \succeq x \qquad (\text{for all } x \in A),$$
$$x \succeq \inf_I(A) \qquad (\text{for all } x \in A).$$

Definition 1.4. Let $I = (x_1, x_2)$ be a weakly regular arc. For any $p \in I$, we set

$$\mu_+(p) := \sup_{S^1\setminus I}(Y_p), \qquad \mu_-(p) := \inf_{S^1\setminus I}(Y_p),$$

where $Y_p := F_p \setminus Z_p(F_p)$. Moreover, we extend the definition of $\mu_\pm^\bullet$ to the boundary of I as follows. If x_j $(j = 1, 2)$ is weakly regular, we set

$$(1.1) \qquad \mu_+(x_j) := \sup_{S^1\setminus I}(Y_{x_j}), \qquad \mu_-(x_j) := \inf_{S^1\setminus I}(Y_{x_j}).$$

On the other hand, if x_j is of rank 1, we set

$$(1.2) \qquad \mu_+(x_j) := \sup_{S^1\setminus I}(F_{x_j}), \qquad \mu_-(x_j) := \inf_{S^1\setminus I}(F_{x_j}).$$

We will call $\mu_\pm$ antipodal maps. By definition, $\mu_\pm(\overline{I}) \subset S^1 \setminus I$ holds.

The following lemma is a simple consequence of the properties (I1) and (I2).

Lemma 1.3. *Let $I = (x_1, x_2)$ be an open weakly regular arc and $p, q \in \overline{I}$ two points such that $p \succ q$ on $\overline{I}$. Then the following relations hold.*

$$\mu_+(q) \succeq \mu_+(p), \qquad \mu_-(q) \succeq \mu_-(p) \qquad (\textit{on } S^1 \setminus I).$$

Moreover if $F_p \neq F_q$, then $\mu_-(q) \succ \mu_+(p)$ holds on $S^1 \setminus I$.

Proof. We only prove the first relation. (The second relation is obtained if one reverses the orientation of S^1 and replaces p by q.) Suppose that $\mu_+(p) \succ \mu_+(q)$ on $S^1 \setminus I$. Then we have

$$q \succeq x_1 \succeq \mu_+(p) \succ \mu_+(q) \succeq x_2 \succeq p \qquad \text{on } [p, q].$$

By (I2), we have $F_p = F_q$. Since I contains no points of rank 1, Lemma 1.1 yields that $Z_p(F_p) = Z_q(F_q)$. Hence $\mu_+(p) = \mu_+(q)$ but it is a contradiction. Thus we have $\mu_+(q) \succeq \mu_+(p)$.

Next we suppose that $\mu_+(p) \succeq \mu_-(q)$ holds. Then we have

$$\mu_+(q) \succeq \mu_+(p) \succeq \mu_-(q)(\succeq p).$$

Since F_p and F_q are closed subsets of S^1, we have $\mu_\pm(q) \in F_q$ and $\mu_+(p) \in F_p$. Thus (I2) yields that $F_p = F_q$, which proves the second assertion. □

Theorem 1.4. *Let $I = (x_1, x_2)$ be an open weakly regular arc. Then the following two formulas hold:*

$$\lim_{x \to p-0} \mu_+(x) = \mu_+(p) + 0 \qquad (\textit{for } p \in (x_1, x_2]),$$
$$\lim_{x \to p+0} \mu_-(x) = \mu_-(p) - 0 \qquad (\textit{for } p \in [x_1, x_2)).$$

Proof. We shall prove the first formula. The second formula follows by the same arguments. We take a sequence $(p_n)_{n\in\mathbf{N}}$ such that $p_n \to p - 0$. Since $p_n \to p - 0$, we may assume that $p_{n+1} \succ p_n$ for any $n \in \mathbf{N}$. Since S^1 is compact, $(\mu_+(p_n))_{n\in\mathbf{N}}$ contains a convergent subsequence. Thus, without loss of generality, we may assume that there exists a point $q \in S^1 \setminus I$ such that $\mu_+(p_n) \to q$. Since $p_{n+1} \succ p_n$, it holds that $\mu_+(p_n) \succeq \mu_+(p_{n+1})$ by Lemma 1.3. So we have $\mu_+(p_n) \to q + 0$. Then the proof of the formula follows from the following lemma. □

Lemma 1.5. *Let $(p_n)_{n\in\mathrm{N}}$ be a sequence in an open weakly regular arc $I = (x_1, x_2)$ such that $p_n \to p - 0$, where $p \in (x_1, x_2]$. Suppose there exists $q \in S^1 \setminus I$ such that $\mu_+(p_n) \to q + 0$. Then $q = \mu_+(p)$.*

Proof. First, we consider the case that $\mathrm{rank}(p) \geq 2$. By (I3), we have $p, q \in F_p$. Since $\mu_+(I) \subset S^1 \setminus I$, Lemma 1.3 yields

$$x_1 \succeq \mu_+(p_n) \succeq \mu_+(p) \qquad \text{on } S^1 \setminus I.$$

By taking the limit $\mu_+(p_n) \to q$, we have

$$x_1 \succeq q \succeq \mu_+(p) \qquad \text{on } S^1 \setminus I. \tag{1.3}$$

In particular $p \neq q$. Suppose that $q \in Z_p(F_p)$. Then $[q, p] \subset F_p$. Since $\mu_+(p_n) \to q + 0$, we have $p_n \in Z_p(F_p)$ and thus $\mu_+(p_n) = \mu_+(p)$ for sufficintly large n. Hence we have $q = \mu_+(p)$. So we may assume that $q \in \overline{Y_p}$. Since $\mu_+(p) = \sup_{S^1\setminus I}(Y_p)$, we have $q = \mu_+(p)$ by (1.3).

Next we consider the case that $\mathrm{rank}(p) = 1$. This case happens only if $p = x_2$. By (I3), we have $q \in F_{x_2}$. If $F_{x_2} = \{x_2\}$, then we have $q = x_2 = \mu_+(x_2)$. So we may assume that F_{x_2} consists of more than two points. Then F_{x_2} is written as

$$F_{x_2} = [x_2, y] \qquad (y \in S^1 \setminus \overline{I}).$$

Suppose that $q \in [x_2, y)$. Since $\mu_+(p_n) \to q + 0$, we have $\mu_+(p_n) \in (x_2, y)$. Then by (I1), $F_{p_n} = F_{\mu_+(p_n)} = F_{x_2}$. But this contradicts the fact $\mathrm{rank}(p_n) \geq 2$. Hence we have $q = y = \mu_+(x_2)$ because of $q \in F_{x_2}$. $\square$

Theorem 1.6. *Let $I = (x_1, x_2)$ be an open weakly regular arc. Then $\mu_-(x_1) \succ \mu_+(x_2)$ holds on the arc $S^1 \setminus I$. Moreover, for any $q \in (\mu_+(x_2), \mu_-(x_1))$, there exists a point $p \in I$ such that*

$$\mu_+(p) \succeq q \succeq \mu_-(p) \qquad (\textit{on } S^1 \setminus I). \tag{1.4}$$

Proof. We divide the proof into three steps.

(*Step 1*) First prove the relation $\mu_-(x_1) \succ \mu_+(x_2)$ on $S^1 \setminus I$. Suppose $F_{x_2} = F_{x_1}$. Then there is a point of rank 1 on I by Lemma 1.1. But this contradicts the weak regularity of I. So we have $F_{x_2} \neq F_{x_1}$. Then $\mu_-(x_1) \succ \mu_+(x_2)$ holds by Lemma 1.3.

(*Step 2*) Next we prove the second assertion. We set

$$p := \inf_{\overline{I}}(B_q),$$

where B_q is the set defined by

$$B_q := \{x \in \overline{I}\,;\, q \succeq \mu_+(y) \text{ for all } x_2 \succeq y \succeq x\}.$$

For any $z \in I$ which is sufficiently close to x_2, it holds that $q \succ \mu_+(z)$ by Theorem 1.4. This implies $z \in B_q$, and thus B_q is non-empty. Moreover, definition of p yields that

$$x_2 \succ z \succ p.$$

In particular $p \neq x_2$. Next we suppose that $p = x_1$. By Theorem 1.4, we have $\lim_{w \to x_1+0} \mu_-(w) = \mu_-(x_1)$. In particular, it holds that $\mu_-(w) \succ q$ for $w \in I$ sufficiently close to x_1. On the other hand, the definition of p yields $q \succeq \mu_+(w)$. Thus (1.4) holds for $p = w$.
(*Step 3*) So we may assume that $p \in I$. By Theorem 1.4, we have

$$\mu_+(p) = \lim_{x \to p-0} \mu_+(x), \tag{1.5}$$

$$\mu_-(p) = \lim_{x \to p+0} \mu_-(x). \tag{1.6}$$

Suppose that $q \succ \mu_+(p)$ on $S^1 \setminus I$. Then (1.5) implies that there exists $u(\prec p)$ such that $q \succ \mu_+(x)$ for $x \in (u,p)$. This means that $q \succeq \mu_+(x)$ holds for $x \in (u, x_2)$. Hence $u \in B_q$. But this contradicts that $p = \inf_{\bar{I}}(B_q)$. So we have $\mu_+(p) \succeq q$ on $S^1 \setminus I$. Next we suppose that $\mu_-(p) \succ q$ on $S^1 \setminus I$. Then (1.6) implies that there exists $v(\succ p)$ such that $\mu_-(v) \succ q$. Since $\mu_+(v) \succeq \mu_-(v)$, we have $\mu_+(v) \succ q$. On the other hand, since $v \succ p$, we have $v \in B_q$. This contradicts the relation $\mu_+(v) \succ q$. So we have $q \in [\mu_-(p), \mu_+(p)]$. □

If the arc I is regular, the following stronger assertion follows immediately.

Corollary 1.7. *Let $I = (x_1, x_2)$ be a regular arc. Then $\mu_-(x_1) \succ \mu_+(x_2)$ holds on the arc $S^1 \setminus I$. Moreover, for any $q \in (\mu_+(x_2), \mu_-(x_1))$, there exists a point $p \in I$ such that $F_p = F_q$. In particular, $(\mu_+(x_2), \mu_-(x_1))$ is also a regular arc.*

§2 A generalization of the Bose formula.

In this section, we fix an intrinsic circle system $F = (F_p)_{p \in S^1}$. We define a relation $\sim$ on S^1 as follows. For $p, q \in S^1$, we denote $p \sim q$ if $F_p = F_q$. Then by (I1), this is an equivalence relation on S^1. We denote by S^1/F the quotient space of S^1 by the relation. The equivalence class containing $p \in S^1$ is denoted by $[p]$. Then $\mathrm{rank}([p]) := \mathrm{rank}(p)$ is well defined on S^1/F by (I1).

Definition 2.1. We set

$$S(F) := \{[p] \in S^1/F\,;\, \mathrm{rank}([p]) = 1\},$$
$$T(F) := \{[p] \in S^1/F\,;\, \mathrm{rank}([p]) \geq 3\}.$$

The set $S(F)$ is called the single tangent set and $T(F)$ is called the tritangent set. Moreover, we set

$$s(F) := \text{the cardinality of the set } S(F),$$
$$t(F) := \sum_{[p] \in T(F)} (\mathrm{rank}(p) - 2).$$

Definition 2.2. The single tangent set $S(F)$ is said to be *supported by a continuous function* $\tau : S^1 \to \mathbf{R}$ if for each $[p] \in S(F)$, F_p is a connected component of the zero set of τ.

In §3, we will give several examples of intrinsic circle systems whose single tangent sets are supported by continuous functions. (See Remark of Theorem 3.2.)

Lemma 2.1. *Suppose that $3 \leq s(F) < \infty$. Let $p, q \in S^1$ be points such that* $\operatorname{rank}(p) = \operatorname{rank}(q) = 1$ *and $F_p \neq F_q$. Then there is a point $x \in (p,q)$ such that* $\operatorname{rank}(x) \geq 3$. *Moreover, if the single tangent set $S(F)$ is supported by a continuous function τ, the assumption $s(F) < \infty$ is not needed.*

Proof. Suppose that there are no points $x \in (p,q)$ such that $\operatorname{rank}(x) \geq 3$. Since $s(F) < \infty$, we may assume that there are no points of rank $= 1$ on (p,q). Then (p,q) is a regular arc. By Corollary 1.7, the open arc $(\mu_+(q), \mu_-(p))$ is also a regular arc. On the other hand, we have $\mu_+(p) = p$ and $\mu_-(q) = q$ by (1.2). So all the elements in $[\mu_-(p), p] \cup [q, \mu_+(p)]$ are of rank one. Since γ is expressed as

$$\gamma = (p,q) \cup [q, \mu_+(p)] \cup (\mu_+(q), \mu_-(p)) \cup [\mu_-(p), p],$$

there are no elements of $\operatorname{rank}(\geq 3)$ and $s(F) = 2$. But this contradicts $s(F) \geq 3$. This proves the first assertion. When $S(F)$ is supported by τ, we do not need the assumption $s(F) < \infty$. In fact, we get the same contradiction if we can take an open subarc (p', q') of (p,q) satisfying the following three properties;

(1) $[p'], [q'] \in S(F)$,
(2) $F_{p'} \neq F_{q'}$,
(3) (p', q') is a regular arc.

If there are no such p' and q', then the subset

$$\{x \in (p,q)\,;\, [x] \in S(F)\}$$

is dense in (p,q). This implies that the function τ vanishes identically on (p,q) and thus $F_p = F_q$, which is a contradiction. $\square$

Theorem 2.2. *If $s(F) < \infty$ then $t(F) < \infty$. The converse is also true if the single tangent set $S(F)$ is supported by a continuous function $\tau : S^1 \to \mathbf{R}$.*

Remark. In general, $t(F) < \infty$ does not imply $s(F) < \infty$. For example, we set $F_p := \{p\}$ $(p \in S^1)$. Then $F = (F_p)_{p \in S^1}$ is an intrinsic circle, which satisfies $s(F) = \infty$ but $t(F) = 0$.

The theorem follows from the following three lemmas.

Lemma 2.3. *If there exists a point $p \in S^1$ such that* $\operatorname{rank}(p) = \infty$, *then $s(F) = \infty$.*

Proof. Let O be the open subset of S^1 given by $O := S^1 \setminus F_p$. We take a sequence $(x_n)_{n \in \mathbf{N}}$ in O such that x_i and x_j are in mutually different components of O unless $i = j$. Let (p_n, q_n) be the maximal open interval in O containing x_n. Then $p_n, q_n \in F_p$. By Lemma 1.1, there exists $[y_n] \in S(F)$ on (p_n, q_n). By (I2), we have $F_{y_n} \subset (p_n, q_n)$. Thus $(F_{y_n})_{n \in \mathbf{N}}$ are all disjoint. Hence $s(F) = \infty$. $\square$

Lemma 2.4. *Suppose that $S(F)$ is supported by a continuous function $\tau : S^1 \to \mathbf{R}$. If $s(F) = \infty$, then $t(F) = \infty$.*

Proof. Let $n \geq 3$ be a fixed integer. We assume that $s(F) = \infty$. Then there exists a mutually distinct equivalence classes $[x_1], \cdots, [x_n] \in S(F)$. We set

$$M := \bigcup_{j=1}^{n} F_{x_j}.$$

Then $S^1 \setminus M$ is a union of disjoint open subsets $\{(p_j, q_j)\}_{j=1,...,n}$. By Lemma 2.1, there exists a point y_j ($j = 1, ..., n$) on (p_j, q_j) such that $\operatorname{rank}(y_j) \geq 3$. This implies that $t(F) \geq n$. Since n is an arbitrary integer, we have $t(F) = \infty$. $\square$

Definition 2.3. Let Δ be a subset of $T(F)$ such that $\operatorname{rank}([x]) < \infty$ for all $[x] \in \Delta$. Then for each $x \in \Delta$, $S^1 \setminus F_x$ is a union of disjoint open arcs $I_x^1, ..., I_x^{r_x}$, where $r_x := \operatorname{rank}(x)$. Such an open arc I_x^ℓ is called a *primitive arc* with respect to the subset Δ if $I_x^\ell \cap F_y$ is empty for all $[y] \in \Delta$. If Δ is a finite subset and given by $\Delta := \{[x_1], ..., [x_n]\}$, then we set

(2.1)

$$N(\Delta) := \sharp\{I_{x_j}^{\ell_j} \; ; \; 1 \leq j \leq n, \, 1 \leq \ell_j \leq r_{x_j} \text{ and } I_{x_j}^{\ell_j} \cap F_{x_k} = \varnothing \text{ for all } k = 1, ..., n\},$$

that is $N(\Delta)$ is the total number of primitive arcs with respect to Δ among $\{I_{x_j}^{\ell_j}\}$.

We give an example which will be helpful for the arguments below.

Example. Let γ be the smooth curve as shown in Figure 2.1 and $C_p^\bullet$ the maximal circle $C_p^\bullet$ at each point $p \in \gamma$. We set $F_p := C_p^\bullet \cap \gamma$. Then it can be easily checked that $(F_p)_{p \in \gamma}$ is an intrinsic circle system. The points $a_1, ..., a_{12}$ are of rank one and the points b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3, f_1, f_2, f_3, g_1, g_2, g_3 are of rank three. Finally, h_1, h_2, h_3, ℓ_4 and i_1, i_2, i_3, i_4 are of rank four. Other points of γ are all regular. In this case,

$$S(F) := \{[a_1], ..., [a_{12}]\},$$
$$T(F) := \{[b_1], [c_1], [d_1], [e_1], [f_1], [g_1], [h_1], [i_1]\}.$$

For example, $\gamma \setminus F_{b_1}$ has three components $J_1 := \gamma_{(b_1, b_2)}$, $J_2 := \gamma_{(b_2, b_3)}$ and $J_3 := \gamma_{(b_3, b_1)}$. In this case J_1 and J_2 are primitive with respect to $T(F)$, but J_3 is not.

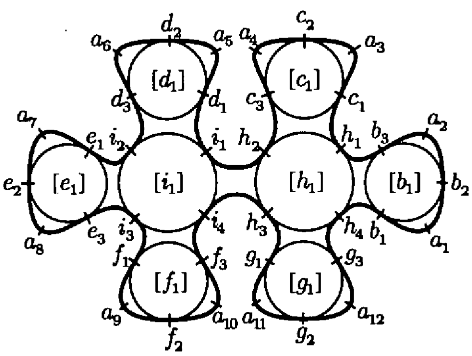

Figure 2.1

Definition 2.4. Let Δ be a subset of $T(F)$. An element $[x] \in \Delta$ ($x \in S^1$) is called *totally primitive* if there exists a non-primitive arc I_x^ℓ such that all the other arcs

$$I_x^i \, (\subset S^1 \setminus F_x) \qquad (i \neq \ell, 1 \leq i \leq r_x)$$

are primitive with respect to Δ.

Let γ be the curve as in Figure 2.1 and F the intrinsic circle system defined in Example. Then $[b_1], [c_1], [d_1], [e_1], [f_1], [g_1]$ are totally primitive with respect to $T(F)$, but $[h_1]$, $[i_1]$ are not.

Lemma 2.5. *If $s(F) < \infty$, then $t(F) < \infty$.*

Proof. We prove the lemma by induction. If $t(F) \geq 1$, then by Lemma 1.1, we have $s(F) \geq 3$. Thus the lemma holds for $s(F) \leq 2$. So we assume that $t(F) < \infty$ holds if $s(F) < n$ $(n \geq 3)$ and prove the assertion in the case $s(F) = n$. We suppose that $t(F) = \infty$. Although the set $T(F)$ need not to be finite, but the rank of each element is finite by Lemma 2.3.

(*Step 1*) Suppose that there is a totally primitive element $[x] \in T(F)$ with respect to $T(F)$. Without loss of generality, we may assume that I_x^1 is not a primitive arc and the other arcs $I_x^2, ..., I_x^{r_x}$ are all primitive. We consider the quotient topological space $S^1/(S^1 \setminus I_x^1)$ and $\pi : S^1 \to S^1/(S^1 \setminus I_x^1)$ by the canonical projection. Then $S^1/(S^1 \setminus I_x^1)$ is also homeomorphic to S^1. For each $p \in S^1$, we set

$$\hat{F}_{\pi(p)} := \begin{cases} \pi(F_p) & \text{if } p \in I_x^1, \\ \pi(F_x) & \text{if } p \notin I_x^1. \end{cases}$$

Then it can be easily checked that $\hat{F}$ is an intrinsic circle system on $S^1/(S^1 \setminus I_x^1)$. By Lemma 1.1, each I_x^ℓ $(\ell \neq 1)$ contains at least one components of rank one points. On the other hand, I_x^ℓ has at most one component of rank one points by Lemma 2.1. Thus each I_x^ℓ $(\ell \neq 1)$ contains exactly one component of rank one points. Thus, we have

$$s(\hat{F}) = s(F) - (\mathrm{rank}(x) - 2), \tag{2.2}$$

$$t(\hat{F}) = t(F) - (\mathrm{rank}(x) - 2). \tag{2.3}$$

Since $s(\hat{F}) < n$, we have $t(\hat{F}) < \infty$. So $t(F)$ is also finite by (2.3).

(*Step 2*) Next we consider the case that there are no totally primitive elements in $T(F)$. Assume that $t(F) = \infty$. We take two mutually different elements $[x_1]$ and $[x_2]$. Without loss of generality, we may assume that $F_{x_1} \subset I_{x_2}^1$. Since $[x_2]$ is not totally primitive, there exists an element x_3 $(x_3 \neq x_1, x_2)$ such that F_{x_3} is contained in $I_{x_2}^k$ for some $k \neq 1$. By (I2), F_{x_2} is contained in one of $(I_{x_3}^\ell)_{\ell=1,...,r_{x_3}}$, here we may assume $F_{x_2} \subset I_{x_3}^1$. Then we also have $F_{x_1} \subset I_{x_3}^1$ by (I2). Since $[x_3]$ is not totally primitive, there exists an element x_4 $(x_4 \neq x_1, x_2, x_3)$ such that F_{x_4} is contained in $I_{x_3}^k$ for some $k \neq 1$. Repeating this argument inductively, we can find a sequence $([x_n])_{n\in\mathbf{N}}$ such that

$$\begin{aligned} &F_{x_j} \subset I_{x_n}^1 && (j = 1, ..., n-1), \\ &F_{x_{n+1}} \subset I_{x_n}^k && \text{for some } k \ (1 < k \leq r_{x_n}). \end{aligned} \tag{2.4}$$

By Lemma 1.1, we have

$$s(F) \geq N(\{[x_1], ..., [x_k]\}). \tag{2.5}$$

On the other hand, by (2.4), we have

$$N(\{[x_1], ..., [x_k], [x_{k+1}]\}) = N(\{[x_1], ..., [x_k]\}) + (\mathrm{rank}([x_{k+1}]) - 2). \tag{2.6}$$

Thus $N(\{[x_{i_1}], ..., [x_{i_k}]\}) \to \infty$ if $k \to \infty$. Hence $s(F) = \infty$, a contradiction. So $t(F)$ is finite. $\square$

Corollary 2.6. *Suppose that $s(F) < \infty$. Then the set of all regular (resp. weakly regular) points is an open subset of S^1.*

Proof. Since $s(F) < \infty$, $t(F) < \infty$ holds by Lemma 2.5. Thus there exists finitely many points $p_1, ..., p_n$ such that $S^1 \setminus (F_{p_1} \cup \cdots \cup F_{p_n})$ is the set of all regular (resp. weakly regular) points. Since each F_{p_j} $(j = 1, ..., n)$ is closed, the set is an open subset. $\square$

We now prove the following theorem which is a generalization of the Bose formula.

Theorem 2.7. *Let $F := (F_p)_{p \in S^1}$ be an intrinsic circle system. Suppose that $s(F) < \infty$ and there exists a point $p \in S^1$ such that $[p] \notin S(F)$. Then $t(F) < \infty$ and*

$$s(F) - t(F) = 2$$

holds.

Proof. Assume that $s(F) < \infty$. If $s(F) = 0$, then this contradicts Lemma 1.1. If $s(F) = 1$, we can conclude that $[p] \in S(F)$ for all $p \in S^1$ by Lemma 1.1. Next we suppose that $s(F) = 2$ and $t(F) \geq 1$. Then by (2.5), we have

$$2 = s(F) \geq N(\{[x_1]\}) = \operatorname{rank}(x_1) \geq 3$$

for any $x_1 \in T(F)$, which yields a contradiction. Thus $t(F) = 0$. So we may assume $s(F) \geq 3$. Then Lemma 2.1 implies $T(F)$ is a non-empty set. Let $[x_1], ..., [x_{t(F)}]$ be all of the elements of $T(F)$. To we complete the proof of the theorem, we need the following lemma.

Lemma 2.8. *Suppose that $3 \leq s(F) < \infty$ There exists an integer j $(1 \leq j \leq s(F))$ such that $[x_j]$ is totally primitive with respect to $T(F)$.*

Proof. If $[x_1]$ is totally primitive, the prove is finished. If not, we fix a non-primitive arc $I_{x_1}^{\ell_1}$. Then by (I2), we may suppose that F_{x_2} lies in $I_{x_1}^{\ell_1}$. (If not, we can exchange $[x_2]$ for a suitable $[x_k]$ $(k > 2)$.) If $[x_2]$ is totally primitive, the proof is finished. If not, we fix a non-primitive arc $I_{x_2}^{\ell_2}$ contained in $I_{x_1}^{\ell_1}$. Then we may assume that F_{x_3} lies in $I_{x_2}^{\ell_2}$. (If not, we can exchange $[x_3]$ for a suitable $[x_k]$ $(k > 3)$.) Continuing this argument, we find a totally primitive $[x_j]$ since $t(F)$ is finite. $\square$

(*Proof of Theorem 2.7 continued.*) We will prove the formula by induction on the number $s(F)$. We have already seen that the formula is true whenever $s(F) \leq 2$. So we assume that the formula holds if $s(F) < n$ $(n \geq 3)$ and prove the assertion in the case $s(F) = n$. By Lemma 2.8, there is a totally primitive element $[x]$ in $T(F)$. Then as shown in the proof of Lemma 2.5, the induced intrinsic circle system $\hat{F}$ on $S^1/(S^1 \setminus I_x^1)$ satisfies (2.2) and (2.3). Since $s(\hat{F}) < n$, we have $s(\hat{F}) - t(\hat{F}) = 2$, which yields the formula $s(F) - t(F) = 2$. $\square$

Remark. Let $\gamma : S^1 \to \mathbf{R}^2$ be a C^2-regular simple closed curve with positive orientation and $C_p^\bullet$ a maximal circle of γ at $p \in \gamma$. Then $F_p := \gamma \cap C_p^\bullet$ is a typical example of intrinsic circle system. (See §3.) We define a map $\Phi : S^1 \to \mathbf{R}^2$ by $\Phi(p) = c_p$, where c_p is the center of the circle $C_p^\bullet$. Suppose that $s(F) < \infty$. As will seen in Appendix B, the map Φ is continuous by the C^2-regularity of the curve. Then Φ induces an injective continuous map $\varphi : S^1/F \to \mathbf{R}^2$. Since S^1/F is compact, S^1/F is homeomorphic to

$\Phi(S^1)$. Let $K_0(\subset D^\bullet(\gamma))$ be the set of points which have more than one minimizing normal geodesics from γ. The cut locus K of γ defined in introduction is the closure of K_0. Then obviously $K_0 \subset \Phi(S^1)$. Since $\Phi(S^1)$ is closed, we have $K \subset \Phi(S^1)$. On the other hand, we set

$$R := \{p \in \gamma \,:\, F_p = \{p\}\}.$$

Since $s(F) < \infty$, R is a finite subset in S^1. Moreover $\Phi(S^1 \setminus R) \subset K_0$ by the definition of K_0. By the continuity of Φ, we have $\Phi(S^1) \subset K$, which implies $\Phi(S^1) = K$. Thus S^1/F is homeomorphic to K. So we can identify S^1/F with K of the cut locus of γ. We have thus seen that the concept of the intrinsic circle system characterizes the cut locus of a simple closed curve abstractly. Since S^1/F has the structure of tree by Theorem 2.7, the observation in the introduction is justified for any C^2-regular simple closed curves with $s(F) < \infty$.

§3 Application to plane curves.

As an application of the results of §1-2, we give a general framework to discuss the number of vertices on a curve, which is similar to (but more elementary than) that of Och mit Grundzahl $k = 3$ (cf. Haupt and Künneth [HK]).

Let X be a topological space homeomorphic to S^2 with fixed orientation. We denote by $J(X)$ the set of all oriented simple closed curves. Each $\gamma \in J(X)$ separates X by two domains D_1 and D_2. We assume that D_1 is the left-hand domain bounded by γ and we set

$$D^\bullet(\gamma) := \overline{D_1}, \qquad D^\circ(\gamma) = \overline{D_2}. \tag{3.1}$$

We call $D^\bullet(\gamma)$ the *internal domain* and $D^\circ(\gamma)$ the *external domain.*

For the sake of simplicity, we use the following notations: Let $\gamma \in J(X)$ and p, q different points on γ. Then we denote by

$$\gamma|_{[p,q]} := \{x \in \gamma\,;\, q \succeq x \succeq p\}, \quad \gamma|_{(p,q)} := \{x \in \gamma\,;\, q \succ x \succ p\}.$$

Definition 3.1. Let $\gamma \in J(X)$. If a sequence $(\gamma_n)_{n\in\mathrm{N}}$ satisfies the following two properties, we write $\gamma_n \to \gamma$.

1. (1) Let $(p_n)_{n\in\mathrm{N}}$ be a sequence in X converging to $p \in X$. If $p_n \in D^\bullet(\gamma_n)$ for all $n \in \mathrm{N}$, then $p \in D^\bullet(\gamma)$.
2. (2) Let $(p_n)_{n\in\mathrm{N}}$ be a sequence in X converges to $p \in X$. If $p_n \in D^\circ(\gamma_n)$ for all $n \in \mathrm{N}$, then $p \in D^\circ(\gamma)$.

Remark. This convergence properly coincides with the compact open topology on $J(X)$ or equivalently compatible with the uniform distance on $J(X)$ induced from an arbitrary distance function $d(\,,\,)$ on X. (See Greenberg and Harper [GH;§7]. Here $d(\,,\,)$ is assumed to be compatible with the topology of X.) In fact, assume $\gamma_n \to \gamma$. Let $d(\,,\,)$ be the uniform distance on $J(X)$ induced by a distance function of X. Suppose that $d(\gamma_n, \gamma) \not\to 0$. Then there is a sequence $(p_n)_{n\in\mathrm{N}}$ such that $p_n \in \gamma_n$ and $d(p_n, \gamma) > \varepsilon > 0$. Since X is compact, there is a subsequence $(p_{j_n})_{n\in\mathrm{N}}$ converging to q. Then $q \in \gamma$ since $\gamma_n \to \gamma$. But this contradicts the fact $d(p_{j_n}, \gamma) > \varepsilon > 0$.

On the other hand, assume that $(\gamma_n)_{n\in\mathrm{N}}$ converges to γ with respect to the compact open topology. Let $d(\,,\,)$ be the canonical distance function on $X = S^2(1)$. Then we have $d(\gamma, \gamma_n) \to 0$. Let $(p_n)_{n\in\mathbf{N}}$ be a sequence in X converging to $p \in X$. Suppose that $p_n \in D^\bullet(\gamma)$ and $p \in D^\circ(\gamma) \setminus \gamma$. Let $\overline{p_n p}$ be the geodesic segment in X. Then there exists a point q_n on $\gamma \cap \overline{p_n p}$. Then we have

$$d(p_n, \gamma) \le d(p_n, q_n) \le d(p_n, p).$$

Since $d(p_n, p) \to 0$, we have $d(p, \gamma) = 0$, which is a contradiction. Hence $\gamma_n \to \gamma$ in the sense of the above definition.

Let $q \in X$ be a point. We interpret q as collapsing of simple closed curves. We consider two orientations of q. The point q is said to be *positively oriented* if we regard it as

$$D^\bullet(q) = q, \qquad D^\circ(q) = X \setminus \{q\} \tag{3.2}$$

and q is said to be *negatively oriented* if we regard it as

$$D^\circ(q) = q, \qquad D^\bullet(q) = X \setminus \{q\}. \tag{3.3}$$

In the first case, we denote q by $q^\bullet$ and in the second case q°. Then the notations $\gamma_n \to q^\bullet$ or $\gamma_n \to q^\circ$ make sense. We denote by $\partial J(X)$ the set of all oriented points on X, that is

$$\partial J(X) := \{q^\bullet, q^\circ\}_{q \in X}. \tag{3.4}$$

Now we define a notion "circle system" which will produce typical examples of intrinsic circle system defined in §1.

(*Definition of a "circle system".*) A subset Γ of $J(X)$ is called a *circle system* if the following three conditions are satisfied: (We set $\hat{\Gamma} = \Gamma \cup \partial J(X)$.)

- (C1) Any distinct curves $C, C' \in \Gamma$ have at most two common points. Moreover, if $D^\bullet(C) \subset D^\bullet(C')$ then they have at most one common point.
- (C2) Let $(p_n)_{n\in\mathbf{N}}$ be a sequence in X which converges to a point $p \in X$. Let $(C_n)_{n\in\mathbf{N}}$ be a sequence in $\hat{\Gamma}$ such that $C_n \ni p_n$. Then $(C_n)_{n\in\mathbf{N}}$ has a subsequence converging to an element in $\hat{\Gamma}$.
- (C3) Let p be a point on X and A a subset of Γ such that any two elements of A have only one common point p. Then there exist $C_A^\bullet, C_A^\circ \in \hat{\Gamma}$ such that

  (1) $D^\bullet(C_A^\bullet) \subset D^\bullet(C)$ and $D^\bullet(C) \subset D^\bullet(C_A^\circ)$ for all $C \in \Gamma$.

  (2) There exist sequences $(C_n)_{n\in A}$ and $(C'_n)_{n\in A}$ such that $C_n \to C_A^\bullet$ and $C'_n \to C_A^\circ$ respectively.

An element of Γ is called a *circle*. The followings are examples of circle systems.

Example 1 (The Möbius plane). Let $X_1 = \mathbf{R}^2 \cup \{\infty\}$ and Γ_1 be the set of oriented circles and lines. (Since the circles are invariant under the Möbius transformations, it is natural to compactify the Euclidean plane by attaching the infinity.) Then the pair (X_1, Γ_1) satisfies the conditions of a circle system. Via the stereographic projection from the north pole of the unit sphere $S^2(1)$ in $\mathbf{R}^3$, this model is equivalent to the following one

$$X_1 := S^2(1),$$
$$\hat{\Gamma}_1 := \text{the oriented intersections between } S^2(1) \text{ and planes.}$$

Example 2 (Closed strictly convex surfaces). As a canonical generalization of Example 1, the following model also satisfies the above conditions:

$$X_2 := \text{A closed } C^2\text{-embedded surface in } \mathbf{R}^3 \text{ with positive Gaussian curvature,}$$
$$\hat{\Gamma}_2 := \text{the oriented intersections between } X_2 \text{ and planes.}$$

Example 3 (The Minkowski plane). Let $\mathcal{I}$ be a fixed C^2-regular simple closed curve with positive curvature in $\mathbf{R}^2$ enclosing the origin. We call $\mathcal{I}$ an *indicatrix*. The *Minkowski distance* $d_{\mathcal{I}}(x,y)$ associated with the indicatrix $\mathcal{I}$ is defined by

$$d_{\mathcal{I}}(x,y) := \inf\{t > 0\,;\, \frac{1}{t}(y-x) \in D^\bullet(\mathcal{I})\}.$$

It satisfies the usual properties of a distance function except for the symmetry property $d_{\mathcal{I}}(x,y) = d_{\mathcal{I}}(y,x)$. The Minkowski geometry is the geometry with respect to this distance function. The indicatrix $\mathcal{I}$ is characterized as the level set

$$\mathcal{I} = \{x \in \mathbf{R}^2\,;\, d_{\mathcal{I}}(0,x) = 1\}.$$

When $\mathcal{I}$ is the unit circle, $d_{\mathcal{I}}$ coincides with the usual Euclidean distance. A Minkowski circle C is the image of the indicatrix $\mathcal{I}$ under a translation and a homothety with a positive ratio. The point in C corresponding to the origin in $D^\bullet(\mathcal{I})$ is called the *center* of C and the magnification of C with respect to $\mathcal{I}$ is called the *Minkowski radius*. We set $X_3 := \mathbf{R}^2 \cup \{\infty\}$ as a stereographic image of the unit sphere. Let Γ_3 be the set of Minkowski circles and straight lines. Then (X_3, Γ_3) satisfies condition (C1) obviously. Condition (C3) is also easily checked. (In this setting, two different lines meet only at infinity if they are parallel. So condition (C3) with $p = \infty$ is also easily checked.) Condition (C2) is verified as follows:

(*Case 1*) First we consider the case $p \neq \infty$. Let $(p_n)_{n\in\mathbf{R}^2}$ be a sequence converging to $p \neq \infty$ and $(C_n)_{n\in\mathbf{N}}$ a sequence in Γ_3 such that $p_n \in C_n$. If $(C_n)_{n\in\mathbf{N}}$ contains either infinitely many straight lines or infinitely many oriented points, then such a subsequence of lines has a subsequence converging a line through p obviously. So we may assume that $(C_n)_{n\in\mathbf{N}}$ does not contain neither straight lines nor oriented points. If necessary by taking a subsequence, we may assume that $(C_n)_{n\in\mathbf{N}}$ have the same orientation. Moreover, by reversing the orientation of $(C_n)_{n\in\mathbf{N}}$ simultaneously, we may assume that $(C_n)_{n\in\mathbf{N}}$ are all positively oriented, that is, $\big(D^\bullet(C_n)\big)_{n\in\mathbf{N}}$ are all bounded in $\mathbf{R}^2$. Let r_n be the Minkowski radius of C_n. If $(r_n)_{n\in\mathbf{N}}$ is bounded, (C2) is easily checked. So we may assume that $r_n \to \infty$. Let L_n be the line which is tangent to C_n at p_n. Then $(L_n)_{n\in\mathbf{N}}$ contains a subsequence converging to a line L passing through p. So we may assume that $(L_n)_{n\in\mathbf{N}}$ converges to L. One can easily prove the following two assertions.

(1) There exists $\varepsilon > 0$ such that the Euclidean circle with radius ε which is tangent to $\mathcal{I}$ at p from the same direction, lies in $D^\bullet(\mathcal{I})$ for each point $p \in \mathcal{I}$.

(2) Suppose that $(s_n)_{n\in\mathbf{R}}$ is a sequence of positive real numbers such that $s_n \to \infty$. Then $E_n(s_n) \to L$, where $E_n(s_n)$ is the Euclidean circle with radius s_n which is tangent to C_n at p_n from the same direction.

By (1), we have

$$D^\bullet\left(E_n(\varepsilon r_n)\right) \subset D^\bullet(C_n) \subset D^\bullet(L_n). \tag{3.5}$$

By (2), we have $E_n(\varepsilon r_n) \to L$. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in $D^\bullet(C_n)$ (resp. $D^\circ(C_n)$) converging to $x \in X_3$. Then by (3.5), we have $x_n \in D^\bullet(L_n)$ (resp. $x_n \in D^\circ(E_n(\varepsilon r_n))$). Since $L_n \to L$ (resp. $E_n(\varepsilon r_n) \to L$), we have $x \in D^\bullet(L)$ (resp. $x \in D^\circ(L)$). This proves $C_n \to L$.

(*Case 2*) Next we consider the case $p = \infty$. Let $(p_n)_{n\in\mathbf{R}^2}$ be a sequence converging to ∞ and $(C_n)_{n\in\mathbf{N}}$ a sequence in Γ such that $p_n \in C_n$. Without loss of generality, we may assume that C_n is positively oriented. Suppose that $q_n \to \infty$ holds for any sequence $(q_n)_{n\in\mathbf{N}}$ such that $q_n \in C_n$. Let $x_n \in C_n$ be the point which attains the minimum of the distance function of C_n from the origin. Then we have $x_n \to \infty$, which implies $C_n \to \infty^\circ$. Thus we may assume that there exists a sequence $(q_n)_{n\in\mathbf{N}}$ such that $q_n \in C_n$ and $q_n \to q \neq \infty$. Then it reduces to Case 1.

Hence (X_3, Γ_3) satisfies the conditions of a circle system. The vertices on curves in the Minkowski plane have been investigated by many geometers (See [Su], [He2-5], [Gu1].) Here the vertex is regarded as a point where the osculating circle has the third order tangency with the curve. Later in this section, we define clean maximal (resp. minimal) vertices. Maximal (resp. minimal) vertices are defined in Appendix A. If a closed curve in the Minkowski plane is C^3-regular, these vertices are all vertices in this sense. For the relationship between Minkowski vertices and contact geometry, see Tabachnikov [Ta2].

Example 4. Let $\varphi : X_i \to X_i$ be a homeomorphism of X_i. Then $(X_i, \varphi(\Gamma_i))$ $(i = 1, 2, 3)$ also satisfies conditions (C1)-(C3).

Definition 3.2. Let $\gamma \in J(X)$. For each $p \in \gamma$, we set

$$\begin{aligned} \mathcal{A}_p^\bullet &:= \{C \in \hat{\Gamma}\,;\, C \ni p,\ C \subset D^\bullet(\gamma)\}, \\ \mathcal{A}_p^\circ &:= \{C \in \hat{\Gamma}\,;\, C \ni p,\ C \subset D^\circ(\gamma)\}. \end{aligned} \tag{3.6}$$

A point p on γ is called $\bullet$-*admissible* if $\mathcal{A}_p^\bullet = \{q^\bullet\}$ or if any two distinct elements in $\mathcal{A}_p^\bullet \setminus \{q^\bullet\}$ meets only at p. (A $\circ$-admissible point is defined similarly.)

Definition 3.3. For a $\bullet$-admissible (resp. $\circ$-admissible) point p, we set

$$C_p^\bullet := C_{\mathcal{A}_p^\bullet}^\bullet \qquad (\text{resp. } C_p^\circ := C_{\mathcal{A}_p^\circ}^\circ). \tag{3.7}$$

$C_p^\bullet$ (resp. C_p°) is called the *maximal* (resp. *minimal*) *circle* at p. (Such circles exist by condition (C3).) A curve $\gamma \in J(X)$ is called $\bullet$-*admissible* (resp. $\circ$-*admissible*) if all points on it are $\bullet$-admissible (resp. $\circ$-admissible).

If $(X, \Gamma) = (X_i, \Gamma_i)$ $(i = 1, 2, 3)$, then every piecewise C^1-regular curve in $J(X)$ whose internal angles with respect to $D^\bullet(\gamma)$ are less than or equal to π is $\bullet$-admissible. (See Proposition A.1 in Appendix A.) For example, the triangle figure as in Figure 1.1 with positive orientation is $\bullet$-admissible, but not $\circ$-admissible because the three vertices of the triangle are not $\circ$-admissible points.

Definition 3.4. Let γ be a $\bullet$-admissible (resp. $\circ$-admissible) curve. We set

$$F_p^\bullet := C_p^\bullet \cap \gamma \qquad (\text{resp. } F_p^\circ := C_p^\circ \cap \gamma).$$

Proposition 3.1. *Let $\gamma \in J(X)$ be a $\bullet$-admissible (resp. $\circ$-admissible) curve. Then $(F_p^\bullet)_{p\in\gamma}$ (resp. $(F_p^\circ)_{p\in\gamma}$) is an intrinsic circle system on $S^1 = \gamma$.*

Proof. The condition (I1) obviously follows from the definition of $C_p^\bullet$. The condition (I2) follows from (C1). Finally, we prove that $F^\bullet$ satisfies (I3). Let $(p_n)_{n\in\mathbf{N}}$ and $(q_n)_{n\in\mathbf{N}}$ be two sequences in S^1 such that $\lim_{n\to\infty} p_n = p$, $\lim_{n\to\infty} q_n = q$ and $q_n \in F_{p_n}^\bullet$. By (C2), $C_{p_n}^\bullet$ contains a convergent subsequence. So we may assume that $C_{p_n}^\bullet \to C \in \hat{\Gamma}$. If $p = q$, then $q \in F_p$ is obvious. So we may assume $p \neq q$. Since $C_{p_n}^\bullet \to C$ and $C_{p_n}^\bullet \subset D^\bullet(\gamma)$, we have $C \subset D^\bullet(\gamma)$. On the other hand, we have

$$p, q \in D^\bullet(C) \cap D^\circ(C) = C.$$

(In fact, it follows from $p_n, \mu_+^\bullet(p_n) \in D^\bullet(C_{p_n}^\bullet) \cap D^\circ(C_{p_n}^\bullet)$ because of $C_{p_n}^\bullet \to C$.) Since $p \neq q$, we have $C_p^\bullet = C$ by the definition of $C_p^\bullet$. □

Let $\gamma \in J(X)$ be a $\bullet$-admissible (resp. $\circ$-admissible) curve. Then we set

$$\operatorname{rank}^\bullet(p) := \operatorname{rank}(F_p^\bullet) \quad (\text{resp. } \operatorname{rank}^\circ(p) := \operatorname{rank}(F_p^\circ)).$$

Namely, $\operatorname{rank}^\bullet(p)$ is the number of connected components of $C_p^\bullet \cap \gamma$.

Definition 3.5. Let γ be a $\bullet$-admissible (resp. $\circ$-admissible) curve. A point p on γ is called a *clean maximal vertex* (resp. *clean minimal vertex*) if $\operatorname{rank}^\bullet(p) = 1$ (resp. $\operatorname{rank}^\circ(p) = 1$). A point p on γ is called *$\bullet$-regular* (resp. *$\circ$-regular*) if $\operatorname{rank}^\bullet(p) = 2$ (resp. $\operatorname{rank}^\circ(p) = 2$). A point p on γ is called *weakly $\bullet$-regular* (resp. *weakly $\circ$-regular*) if $2 \leq \operatorname{rank}^\bullet(p) \leq \infty$ (resp. $2 \leq \operatorname{rank}^\circ(p) \leq \infty$). An open arc I of γ is called *$\bullet$-regular* (resp. *weakly $\bullet$-regular*) if all points on I are $\bullet$-regular (resp. weakly $\bullet$-regular). Similarly *$\circ$-regular* (resp. *weakly $\circ$-regular*) arc is also defined.

By definition, I is (weakly) $\bullet$-regular (resp. $\circ$-regular) if it is a (weakly) regular arc with respect to the intrinsic circle system $F^\bullet$ (resp. F°). (See Definition 1.2.)

We set

$$S^\bullet(\gamma) := S(F^\bullet) \qquad (\text{resp. } S^\circ(\gamma) := S(F^\circ)),$$
$$T^\bullet(\gamma) := T(F^\bullet) \qquad (\text{resp. } T^\circ(\gamma) := T(F^\circ)).$$

Then $S^\bullet(\gamma)$ (resp. $S^\circ(\gamma)$) is the set of connected components of clean maximal (resp. minimal) vertices on γ. Moreover, we set

$$s^\bullet(\gamma) := \sharp\{S^\bullet(\gamma)\},$$
$$t^\bullet(\gamma) := \sum_{[p]\in T^\bullet(\gamma)} (\operatorname{rank}^\bullet(p) - 2).$$

Similarly, $s^\circ(\gamma)$ and $t^\circ(\gamma)$ are also defined. Then Theorem 2.7 yields the following generalization of Bose's formula (I.1).

Theorem 3.2. *Let γ be a $\bullet$-admissible (resp. $\circ$-admissible) simple closed curve, which is not a circle. Suppose that $s^\bullet(\gamma) < \infty$ (resp. $s^\circ(\gamma) < \infty$). Then $t^\bullet(\gamma) < \infty$ (resp. $t^\circ(\gamma) < \infty$) and*

$$s^\bullet(\gamma) - t^\bullet(\gamma) = 2 \qquad (\text{resp. } s^\circ(\gamma) - t^\circ(\gamma) = 2).$$

Remark. If $(X, \Gamma) = (X_1, \Gamma_1)$ as in Example 1 and γ is a C^3-regular curve, then $S^\bullet(\gamma)$ (resp. $S^\circ(\gamma)$) is supported by the derivative of the curvature function. Similarly, if $(X, \Gamma) = (X_2, \Gamma_2)$ as in Example 2 and γ a C^3-regular curve as a space curve, then $S^\bullet(\gamma)$ (resp. $S^\circ(\gamma)$) is supported by the torsion function of γ as a space curve in $\mathbf{R}^3$. Thus in these two cases, $t^\bullet(\gamma) < \infty$ (resp. $t^\circ(\gamma) < \infty$) is equivalent to the condition $s^\bullet(\gamma) < \infty$ (resp. $s^\circ(\gamma) < \infty$).

In our general settings, a clean minimal vertex might be a clean maximal vertex. If X has C^2-differentiable structure and Γ satisfies the additional condition (C4) in Appendix A. Then any C^2-regular simple closed curves γ are $\bullet$-admissible and also $\circ$-admissible by Proposition A.1 in Appendix A. Moreover, a clean maximal vertex never be a clean minimal vertex by Proposition A.5. Thus the number $s(\gamma)$ of connected component of clean (maximal or minimal) vertices is equal to $s^\bullet(\gamma) + s^\circ(\gamma)$. Thus we get the following corollary.

Corollary 3.3. *Let X be a C^2-differentiable sphere and Γ a circle system on X satisfying the additional condition (C4) in Appendix A. Let γ be a C^2-regular curve on X. Suppose that the number $s(\gamma)$ of connected components of clean vertices is finite. Then*

$$s(\gamma) - t(\gamma) = 4$$

holds, where $t(\gamma) := t^\bullet(\gamma) + t^\circ(\gamma)$.

§4 Application to space curves.

In this section, we apply Theorem 2.7 to convex simple closed space curves. An immersed closed C^1-curve $\gamma : S^1 \to \mathbf{R}^3$ is called *convex* if it lies on the boundary ∂H of its convex hull H. We fix a convex simple closed curve γ and assume that it is not planar. We fix an interior point o of the convex hull and consider the unit sphere S^2_o centered at o. We denote by $\pi : \partial H \to S^2_o$ the canonical projection. Then π is a bijective continuous map. Since ∂H is compact, π is a homeomorphism. In particular, the boundary ∂H of the convex hull is homeomorphic to a sphere and γ divides ∂H into two domains. Let $\partial H^\bullet$ (resp. ∂H°) be the left-hand (right-hand) closed domain of γ in ∂H. Moreover,

$$\tilde{\gamma} := \pi \circ \gamma : S^1 \to S^2_o$$

is an embedded curve. By the projection π, the left-hand (resp. right-hand) domain of $\tilde{\gamma}$ corresponds to $\partial H^\bullet$ (resp. ∂H°). Now we fix a point p on γ arbitrarily. A plane U is called *tangent plane* if it contains the tangent line L_p at p. Let $\mathcal{P}_p$ be the pencil of oriented planes which is tangent to γ at p. Then $\mathcal{P}_p$ is identified with a circle.

We denote by $V_x \in \mathcal{P}_p$ the oriented plane passing through $x \in \mathbf{R}^3 \setminus L_p$, where the orientation of V_x is chosen so that the line segment $\overline{px}$ lie in a upper half plane on V_x. A plane $V_x(\neq V_o)$ is said to be *upper* (resp. *lower*) than V_o if $\overline{px}$ lies in the closed upper (resp. lower) half region bounded by V_o. We give an orientation of $\mathcal{P}_p$ such that any tangent plane at p upper than V_o is greater than V_o. The orientation is independent of the choice of the interior point o, and thus it induces a canonical cyclic order of $\mathcal{P}_p$.

An oriented plane U is called a *supporting plane* of γ at p if $p \in U$ and the curve lies entirely in the positive closed half-spaces bounded by U. Let $\mathcal{S}_p$ be the set of supporting plane at p which does not contain any points in $\gamma \setminus L_p$. Then by definition, $\mathcal{S}_p$ is a subset of $\mathcal{P}_p$ and the set of supporting plane is just the closure $\overline{\mathcal{S}_p}$ of $\mathcal{S}_p$. Since γ is a convex simple closed curve, there is at least one supporting plane passing through p. Hence $\overline{\mathcal{S}_p}$ is non-empty. One can easily see that $\mathcal{S}_p$ is connected, that is, there exists $U_p^\bullet, U_p^\circ \in \mathcal{P}$ such that one of the following four possibilities occur;

$$(1)\quad \mathcal{S}_p = (U_p^\circ, U_p^\bullet) \qquad (2)\quad \mathcal{S}_p = [U_p^\circ, U_p^\bullet)$$
$$(3)\quad \mathcal{S}_p = (U_p^\circ, U_p^\bullet] \qquad (4)\quad \mathcal{S}_p = [U_p^\circ, U_p^\bullet]$$

The plane $U_p^\bullet$ (resp. U_p°) is called *maximal* (resp. *minimal*) supporting plane at p. (It may possible to be $U_p^\bullet = U_p^\circ$.) Later, we will need the following lemma. (Except for the lemma, we do not need C^2-regularity of curves until Proposition 4.9.)

Lemma 4.1. *Let γ be a C^2-convex simple closed space curve and $p \in \gamma$ has non-vanishing curvature. Suppose that $L_p \cap \gamma = \{p\}$. Then case (4) never occurs. Moreover, if case (2) (resp. case (3)) occurs, then $U_p^\bullet$ (resp. U_p°) is the osculating plane at p.*

The lemma is well known (cf. Lemma 1 of [Sd1]) and can be proved with the standard method. So we omit the proof.

Definition 4.1. We set

$$F_p^\bullet := \{q \in \gamma\,;\, \overline{pq} \subset \partial H^\bullet\}, \quad (\text{resp. } F_p^\circ := \{q \in \gamma\,;\, \overline{pq} \subset \partial H^\circ\}).$$

Now we prepare lemmas to give some sufficient conditions that $F^\bullet$ and F° are intrinsic circle systems.

Lemma 4.2. *Let γ be a convex simple closed space curve. Then for each $p \in \gamma$, the following inclusions hold*

$$F_p^\bullet \subset U_p^\bullet, \qquad F_p^\circ \subset U_p^\circ.$$

Proof. We fix $q \in F_p^\bullet$ and will show that $q \in U_p^\bullet$. Since L_p is contained in $U_p^\bullet$, we may assume that q does not lie in L_p. First, we show that either $V_q = U_p^\bullet$ or $V_q = U_p^\circ$ holds. In fact, we take the middle point m on the line segment $\overline{pq}$. Since $m \in \partial H^\bullet$, there exists a plane U passing through m such that H lies in the upper or the lower half region of U. Then $\overline{pq} \in U$ holds, and consequently U is a supportinf plane at p. Hence we have $V_q = U$, and thus $V_q = U_p^\bullet$ or $V_q = U_p^\circ$ holds.

Let $U_+^\bullet$ (resp. U_-°) be the upper (resp. lower) half plane of $U_p^\bullet$ (resp. U_p°). Then γ lies in the region D bounded by $U_+^\bullet$ and U_-°. We have seen that $V_q = U_p^\bullet$ or $V_q = U_p^\circ$ holds. Since $\pi(\overline{pq})$ lies in a left hand side of $\tilde{\gamma}$ at p, $\overline{pq}$ lies in the closed upper half domain bounded by V_o. Thus we have $q \in U_p^\bullet$. $\square$

Lemma 4.3. *Let γ be a convex simple closed space curve and L_p the tangent line of γ at p. Suppose that there exists $q(\neq p)$ such that $q \in L_p \cap \gamma$ and the tangent line L_q at q does not coincide with L_p. Then there exists a unique supporting plane U at p. Moreover U contains the lines L_p and L_q.*

Proof. Since γ is a convex curve, there exists at least one supporting plane U at p. Obviously U contains L_p. If U does not contain L_q, it is transversal to γ at q, which is impossible. Thus U contains also L_q. Since $L_p \neq L_q$, U is uniquely determined. $\square$

Lemma 4.4.. *Let γ be a convex simple closed curve which has no planar open subarcs. Suppose that U is a supporting plane at $p \in \gamma$ and $p, x, y \in \gamma \cap U$ are not collinear. Then the triangle $\triangle pxy$ is contained in $\partial H^\bullet$ or ∂H°.*

Proof. Obviously, the triangle $\triangle pxy$ on U lies in ∂H. Suppose that the triangle $\triangle pxy$ contains a point q of γ in its interior. Then $\pi(q)$ lies in the interior of $\pi(\triangle pxy)$ in S^2_o. Thus a sufficiently small open arc of $\tilde{\gamma}$ containing q also lies in its interior. Hence the corresponding arc of γ containing q lies in $\triangle pxy$. But this contradicts that γ has no planar subarcs. Thus $\triangle xqp \subset \partial H^\bullet$ or $\triangle xqp \subset \partial H^\circ$ holds. $\square$

Proposition 4.5. *Let γ be a convex simple closed curve which has no planar open subarcs and p a point on γ. Suppose that $U^\bullet_p$ satisfies the following two conditions*

(1) *the set $U^\bullet_p \cap \gamma$ does not lie in any line passing through p,*
(2) *$F^\bullet_p \neq \{p\}$ (resp. $F^\circ_p \neq \{p\}$).*

Then it holds that $F^\bullet_p = U^\bullet_p \cap \gamma$ (resp. *$F^\circ_p = U^\circ_p \cap \gamma$*).

Proof. We prove the assertion for $F^\bullet$. By Lemma 4.2, we have $F^\bullet \subset U^\bullet_p \cap \gamma$. It is sufficient to show that $U^\bullet_p \cap \gamma \subset F^\bullet$. By condition (1), there are points $q, x \in U^\bullet_p \cap \gamma$ such that p, q, q' are not collinear. To prove it, we divide the proof into the the following two cases. Let $x \in U^\bullet_p \cap \gamma$ be an arbitrary point.
(*Case 1*) Suppose that $p, q, x \in \gamma \cap U^\bullet_p$ are not collinear. Then by Lemma 4.4, either $\triangle xpq \subset \partial H^\bullet$ or $\triangle xpq \subset \partial H^\circ$ holds. But in the latter case, we have

$$\overline{pq} \subset \partial H^\bullet \cap \partial H^\circ = \gamma,$$

which contradicts the fact that γ has no planar subarcs. Thus we have $\triangle xpq \subset \partial H^\bullet$. In particular, we have $\overline{px} \subset \partial H^\bullet$, which implies $x \in F^\bullet_p$.
(*Case 2*) Next we consider the case that $p, q, x \in \gamma \cap U^\bullet_p$ lie on a line L. Since p, q, q' is not collinear, we have $q' \notin L$. Suppose that $\overline{px} \not\subset \partial H^\bullet$. Then by Lemma 4.4, we have $\triangle pq'x \subset \partial H^\circ$. In particular $\overline{pq'} \in \partial H^\circ$. On the other hand, $\overline{pq} \subset \partial H^\bullet$ yields that $\triangle pqq' \subset \partial H^\bullet$ by Lemma 4.2 In particular,

$$\overline{pq'} \in \partial H^\circ \cap \partial H^\bullet = \gamma,$$

which is a contradiction. Hence we have $\overline{px} \subset \partial H^\bullet$. So $x \in F^\bullet_p$. $\square$

Lemma 4.6. *Let γ be a convex simple closed space curve. Suppose that for each $p \in \gamma$ there exists a supporting plane U such that $U \cap \gamma = \{p\}$. Moreover, if U_0 is a supporting plane of γ such that $U_0 \cap \gamma$ contains three distinct points $x, y, z \in \gamma$, then these three points are not collinear.*

Proof. By the assumption, we can easily see that

$$L_z \cap \gamma = \{z\} \qquad (z \in \gamma). \tag{4.1}$$

Suppose that $x, y, z \in U_0 \cap \gamma$ lie in a line L with this order. If $L = L_y$, this contradicts $L_y \cap \gamma = \{y\}$. So $L \neq L_y$. Then U_0 must be a unique supporting plane passing through y by Lemma 4.3. This contradicts the fact that there exists a supporting plane U such that $U \cap \gamma = \{y\}$. $\square$

Proposition 4.7. *Let γ be a convex simple closed space curve. Suppose that for each $p \in \gamma$ there exists a supporting plane U such that $U \cap \gamma = \{p\}$. Then for each $p \in \gamma$, it holds that*

$$F_p^\bullet = U_p^\bullet \cap \gamma, \qquad F_p^\circ = U_p^\circ \cap \gamma. \tag{4.2}$$

In particular, $U_p^\bullet \neq U_p^\circ$ holds.

Proof. We prove the first equality. (The second equality is obtained by the same manner.) If $F_p^\bullet = \{p\}$, then (4.2) is obvious. So we may assume that there exists a point $q \in F_p^\bullet$ such that $q \neq p$. By Lemma 4.2, we have $q \in U_p^\bullet$. If $\sharp(U_p^\bullet \cap \gamma) = 2$, (4.2) is obvious. So we may assume that $\sharp(U_p^\bullet \cap \gamma) > 2$. We fix a point $x \in U_p^\bullet \cap \gamma$ such that $x \neq p, q$. By Lemma 4.6, p, q, x are not collinear and thus the triangle $\triangle pqx$ is considered. Suppose that there exists a point $y \in \gamma$ in the triangle. Then the tangent line L_y separates one of three points p, q, x with the other two in the plane $U_p^\bullet$. Hence $U_p^\bullet$ must be a unique supporting plane passing through y. This contradicts the fact that there exists a supporting plane U such that $U \cap \gamma = \{y\}$. So there is no points on γ inside the triangle. In particular, $\triangle pqx \subset \partial H^\bullet$ or $\triangle pqx \subset \partial H^\circ$ holds. But if $\triangle pqx \subset \partial H^\circ$, then

$$\overline{pq} \subset \partial H^\bullet \cap \partial H^\circ = \gamma.$$

This contradicts (4.1). So $\triangle pqx \subset \partial H^\bullet$. In particular $x \in F_p^\bullet$. Thus we have $U_p^\bullet \cap \gamma \subset F_p^\bullet$. The opposite inclusion follows from Lemma 4.2. $\square$

Theorem 4.8. *Let γ be a convex simple closed space curve satisfying the one of the following two conditions;*

(a) *for each $p \in \gamma$, there exists a supporting plane U such that $U \cap \gamma = \{p\}$,*
(b) *γ has no planar open subarcs.*

Then $(F_p^\bullet)_{p\in\gamma}$ (resp. $(F_p^\circ)_{p\in\gamma}$) is an intrinsic circle system on $S^1 = \gamma$.

Proof. We divide the proof into three steps. (We prove the assertion for $F^\bullet$.)
(Step 1) We check the property (I1). By Proposition 4.7, this is obvious for case (a). So we prove the assertion only for case (b). Let $q \in F_p^\bullet$. It is sufficient to show that $F_p^\bullet \subset F_q^\bullet$. (Opposite inclusion is obtained by interchanging the role of p and q.) If $p = q$, then the property (I1) is obvious. So we may assume that $q \neq p$.
(*Case 1*) First we consider the case that $U_p^\bullet \cap \gamma$ does not lie in any line passing through p. If $F_p^\bullet = \{p\}$, the statement is obvious. If $F_p^\bullet \neq \{p\}$, we have the assertion by Proposition 4.5.
(*Case 2*) So we may assume that $U_p^\bullet \cap \gamma$ lies on a line L passing through p. Let $x \in F_p^\bullet$. Then $\overline{xp}$ and $\overline{qp}$ both lie in $L \cap \partial H^\bullet$. In particular so does $\overline{qx}$, and hence $x \in F_q^\bullet$. Thus we have $F_p^\bullet \subset F_q^\bullet$.
(*Step 2*) We show (I2). Suppose that there exist $p' \in F_p^\bullet \setminus \{p\}$ and $q' \in F_q^\bullet \setminus \{q\}$ such that $F_p^\bullet \neq F_q^\bullet$ and

$$q \succeq p' \succeq q' \succeq p \qquad \text{on } [p, q]. \tag{4.3}$$

Then $\overline{pp'}, \overline{qq'} \subset \partial H^\bullet$. Since $\overline{pp'}$ separates $\partial H^\bullet$ into two domains, $\overline{pp'} \cap \overline{qq'}$ is not empty by (4.3). Let $z \in \overline{pp'} \cap \overline{qq'}$. Then $z \neq p, p', q, q'$. (For example, if $z = p$ or $z = p'$, then $q \in F^\bullet_z = F^\bullet_p \neq F^\bullet_q$ by Step 1, which is a contradiction.) In particular, $\overline{pp'}$ and $\overline{qq'}$ can not lie in a common line. This implies that they are transversal at a point z. By Lemma 4.2, these four points p, p', q, q' lie in $U^\bullet_p$. In particular, $U^\bullet_p \cap \gamma$ does not lie in any line passing through p. By Proposition 4.5 or Proposition 4.7, $F^\bullet_p = U^\bullet_p \cap \gamma \ni q$. This is a contradiction.

(*Step 3*) Finally, we show the property (I3). Let $(p_n)_{n\in\mathbf{N}}$ and $(q_n)_{n\in\mathbf{N}}$ be two sequences in γ such that $q_n \in F^\bullet_{p_n}$, $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} q_n = q$. Since $\overline{p_n q_n} \in \partial H^\bullet$, we have $\overline{pq} \in \partial H^\bullet$. Thus $F^\bullet_p \ni q$. $\square$

Let γ be a convex simple closed space curve as above. We denote by $\operatorname{rank}^\bullet(p)$ (resp. $\operatorname{rank}^\circ(p)$) the rank of $p \in \gamma$ with respect to $F^\bullet$ (resp. F°). By Theorem 2.7, we can get a Bose type formula for γ satisfying the assumptions of Theorem 4.8. But unfortunately, in such a general setting, the points of rank one with respect to $F^\bullet$ or F° may not be neither clean vertices nor clear vertices defined below.

Definition 4.2. Let γ be a C^2-convex simple closed space curve. Then a *clear maximal* (resp. *minimal*) *vertex* is a the point with non-vanishing curvature, which is a maximum (resp. minimum) of the height function with respect to the bi-normal vector. Moreover, if the maximum (resp. minimum) level set of the height function is connected, it is called a *clean maximal* (resp. *minimal*) *vertex.*

We remark that $p \in \gamma$ is a clear vertex (namely clear maximal or clear minimal vertex) if and only if the osculating plane U at p is a supporting plane. Moreover it is clean vertex if and only if $U \cap \gamma$ is connected.

If γ lies in X_2 as in §3-Example 2, this definition of clean vertices has the same meaning as the one in §3. In other words, a point p of $\operatorname{rank}^\bullet(p) = 1$ or $\operatorname{rank}^\circ(p) = 1$ is a clean vertex in the above sense. Our next goal is to give much weaker sufficient conditions for convex simple closed space curves that $\operatorname{rank}^\bullet(p) = 1$ (resp. $\operatorname{rank}^\circ(p) = 1$) implies a clean or clear maximal (resp. minimal) vertex.

Proposition 4.9. *Let γ be a C^2-convex simple closed space curve and $p \in \gamma$ a point with non-vanishing curvature. Suppose that there exists a supporting plane U at p passing through a point $q(\neq p)$ on γ. Then there exists $x \in U \cap \gamma$ $(x \neq p)$ satisfying the following two properties*

(1) $x \in \overline{pq}$,
(2) $x \in F^\bullet_p$ *or* $x \in F^\circ_p$.

Proof. $\overline{pq} \cap \gamma$ is a closed subset of $\overline{pq}$. Suppose that there is no such $x \in \overline{pq}$. Then we can take a sequence $(q_n)_{n\in\mathbf{N}}$ consisting of mutually different points in $\overline{pq} \cap \gamma$ such that $\lim_{n\to\infty} q_n = p$. Since the unit vectors $(q_n - p)/|q_n - p|$ converge to the unit tangent vector at p of γ, $\overline{pq}$ lies in the tangent line L_p at p. Thus $q_n \in L_p$ for all n. But this contradicts the fact that the curvature function of γ does not vanish at p. $\square$

Definition 4.3. A convex simple closed space curve γ is called *tame* if $L_p \cap \gamma = \{p\}$ for any $p \in \gamma$.

Remark. In Ballestero and Romero-Fuster [BR2], such a curve is called strictly convex. But there is another definition of strictly convexity. (The strictly convexity defined in

Sedykh [Sd2] is stronger than that in [BR2].) So we use here the term "tame" to avoid confusions.

By Propositions 4.9, the following is obvious.

Lemma 4.10. *Let γ be a C^2-convex simple closed space curve satisfying (a) or (b) as in Theorem 4.7. Suppose that $p \in \gamma$ has non-vanishing curvature and $L_p \cap \gamma = \{p\}$. Then* $\operatorname{rank}^\bullet(p) = 1$ *(resp.* $\operatorname{rank}^\circ(p) = 1$*) if and only if p is a clean maximal (resp. minimal) vertex.*

It should be remarked that If γ satisfies (a), then $L_p \cap \gamma = \{p\}$ is automatically satisfied by (4.1). Since the clean maximal vertex is not a clean minimal vertex by definition, we get the following

Corollary 4.11. *Let γ be a C^2-convex simple closed space curve with non-vanishing curvature satisfying the one of the following two conditions.*

(1) *For each $p \in \gamma$, there exists a supporting plane U such that $U \cap \gamma = \{p\}$.*
(2) *γ is tame and has no-planar open subarcs.*

Then the number $s(\gamma)$ of connected components of clean vertices is given by $s(\gamma) = s(F^\bullet) + s(F^\circ)$.

Let γ be a convex simple closed space curve. A plane U is called a *tangent plane* of γ if it contains the tangent vector of c at some point. We denote by $\operatorname{rank}(U \cap \gamma)$ the number of the connected components in $U \cap \gamma$. A tangent plane U is called *tritangent plane* if $\operatorname{rank}(U \cap \gamma) \geq 3$.

Definition 4.4. Let $T(\gamma)$ be the set of tritangent supporting planes of γ. We set

$$t(\gamma) := \sum_{U \in T(\gamma)} (\operatorname{rank}(U \cap \gamma) - 2).$$

We call $t(\gamma)$ the total order of tritangent supporting planes.

Lemma 4.12. *Let γ be a C^2-convex simple closed space curve and U a tritangent plane. Suppose that γ is tame. Then $U \cap \gamma$ does not lie in a line.*

Proof. Suppose that U lies in a line L. Then there are three distinct points $x, y, z \in U \cap L$. Without loss of generality, we may assume that y is an intermediate point between $\overline{xz}$. Since γ is convex, we have $L = L_y$, which contradicts that γ is tame.

Proposition 4.13. *Let γ be a C^2-convex simple closed space curve with non-vanishing curvature satisfying the one of the following conditions;*

(1) *for each $p \in \gamma$, there exists a supporting plane U such that $U \cap \gamma = \{p\}$,*
(2) *γ is tame and has no-planar open subarcs.*

Then the following identity holds $t(\gamma) = t(F^\bullet) + t(F^\circ)$.

Proof. If γ satisfies (1), then the assertion follows immediately from Proposition 4.7. So we consider the second case. Let U be a tritangent supporting plane of γ which is tangent at p. By Lemma 4.12, we may assume that $U \cap \gamma$ does not lie in any line. Since γ has non-vanishing curvature function, by Proposition 4.9, $F_p^\bullet \neq \{p\}$ or $F_p^\circ \neq \{p\}$. Hence by Proposition 4.5, either $U \cap \gamma = F_p^\bullet$ or $U \cap \gamma = F_p^\circ$ holds. Thus we have

$$t(\gamma) \leq t(F^\bullet) + t(F^\circ).$$

On the other hand, suppose that the number of connected components of $F_p^\bullet$ (resp. F_p°) is greater than 2. By Lemma 4.2, $U_p^\bullet$ (resp. U_p°) is a tritangent plane. Since γ is tame, we have $L_p \cap \gamma = \{p\}$. So there is an element in $q \in U_p^\bullet$ (resp. $q \in U_p^\circ$) such that $\overline{pq} \subset \partial H^\bullet$ (resp. $\overline{pq} \subset \partial H^\circ$). By Lemma 4.12, $U_p^\bullet$ (resp. U_p°) does not lie in any line. By Proposition 4.5, we have $F_p^\bullet = U \cap \gamma$ (resp. $F_p^\circ = U \cap \gamma$). Hence we have

$$t(\gamma) \geq t(F^\bullet) + t(F^\circ).$$

□

Theorem 4.14. *Let γ be a C^2-convex simple closed space curve with non-vanishing curvature satisfying the one of the following conditions.*

(1) *For each $p \in \gamma$, there exists a supporting plane U such that $U \cap \gamma = \{p\}$.*
(2) *γ is tame and has no-planar open subarcs.*

Suppose the number $s(\gamma)$ of connected components of clean vertices is finite. Then the total order $t(\gamma)$ of tritangent supporting plane is also finite and the following formula holds

$$s(\gamma) - t(\gamma) = 4.$$

The theorem follows immediately from Theorem 2.7, Corollary 4.11 and Proposition 4.13. If γ is C^3-differentiable, then $s(F^\bullet)$ and $s(F^\circ)$ are supported by the torsion function. Thus $t(\gamma) < \infty$ is equivalent to $s(\gamma) < \infty$.

Remark 1. The formula is a generalization of the one obtained by Romero-Fuster [R] in the convexly generic case and by Sedykh [Sd2] in the strictly convex case. In fact, condition (1) is weaker than strictly convexity of curves in the sense of Sedykh [Sd2], and (2) is weaker than the convexily generic assumption as in [R]. When γ is convexily generic in the sense of [R], the disjoint union of quotients $(S^1/F^\bullet) \cup (S^1/F^\circ)$ is identified with the Maxwell graph of γ. (See [R] for definition.)

Remark 2. If γ is a C^2-regular curve on X_2 as in §3-Example 2, then γ satisfies (1) obviously. In this case, the assertion follows from Corollary 3.3 directly.

Next we consider convex simple closed space curves which may not satisfy the assumption of Theorem 4.14.

Proposition 4.15. *Let γ be a C^2-convex simple closed space curve, which has no planar open subarcs and has at most finitely many zeros of the curvature function. Suppose that every element in the set*

$$M_\gamma := \{x \in \gamma\,;\, L_x \cap \gamma \neq \{x\},\ \ \kappa(L_x \cap \gamma) \not\ni 0\}$$

is isolated, where κ is the curvature function. Then any point p on γ satisfying $\operatorname{rank}^\bullet(p) = 1$ (*resp.* $\operatorname{rank}^\circ(p) = 1$) *is a zero of curvature function or a clear maximal* (*resp. minimal*) *vertex.*

Remark. If γ has non-vanishing curvature, we have a simple expression $M_\gamma = \{x \in \gamma\,;\, L_x \cap \gamma \neq \{x\}\,\}$. In this case, every element in M_γ is isolated if and only if M_γ is finite. In fact, if an accumulation point $p \in \gamma$ of M_γ exists, one can easily verify that $p \in M_\gamma$ using the property $\kappa(p) \neq 0$.

To prove it, we prepare the following two lemmas.

Lemma 4.16. *Let γ be a C^2-convex simple closed space curve, which has no planar open subarcs and has at most finitely many zeros of the curvature function. Let p be a point on γ with non-vanishing curvature and* $\mathrm{rank}^\bullet(p) = 1$ *(resp.* $\mathrm{rank}^\circ(p) = 1$*). Suppose that p is an isolated point in the set*

$$\{x \in \gamma\,;\, \mathrm{rank}^\bullet(x) = 1\} \qquad (\textit{resp. } \{x \in \gamma\,;\, \mathrm{rank}^\circ(x) = 1\}).$$

Then p is a clear maximal (resp. minimal) vertex.

Proof. By assumption, there is an open arc I containing p such that all points on $I \setminus \{p\}$ is weakly regular with respect to $F^\bullet$ (resp. F°). We take a sequence $(p_n)_{n\in\mathbf{N}}$ on $I \setminus \{p\}$ such that $p_n \to p - 0$. Then by Theorem 1.4, we have $\mu_+(p_n) \to p + 0$. On the other hand, there exists a supporting plane U_n of γ containing p_n and $\mu_+(p_n)$. Then U_n converges to the osculating plane U at p. In particular U is also a supporting plane, that is p is a clear vertex. □

Lemma 4.17 (Romero-Fuster and Sedykh [RS; Proposition 1]). *Let $\sigma : (a, b) \to \mathbf{R}^3$ be a C^2-regular curve with non-vanishing curvature, which may not be closed. Let p be a point of σ and $q(\neq p)$ a point in $\mathbf{R}^3$. Then there is an open arc I containing p such that $q \notin L_x \cap \gamma$ for all $x \in I \setminus \{p\}$.*

As mentioned in [RS], the lemma is a simple exercise.

(*Proof of Proposition 4.15.*) Let $p \in \gamma$ be a point satisfying $\mathrm{rank}^\bullet(p) = 1$. Assume that p has non-vanishing curvature. If $L_p \cap \gamma = \{p\}$, then p is a clean vertex by Lemma 4.10. So we may assume that $L_p \cap \gamma \neq \{p\}$. Consider the subset

$$K = \{x \in \gamma\,;\, \mathrm{rank}^\bullet(x) = 1\}.$$

If p is isolated in K, then it is a clear vertex by Lemma 4.16. So we may assume that there is a sequence $(p_n)_{n\in\mathbf{N}}$ in K which converges to p. Since $\kappa(p) \neq 0$, there exists a neighborhood I of p such that $(L_q)_{q\in I}$ are mutually distinct. Thus there exists a positive integer n_0 such that

$$0 \notin \kappa(L_{p_n} \cap \gamma) \qquad (\text{for } n > n_0). \tag{4.4}$$

(In fact, if (4.4) fails, there is a point $q \in \gamma$ such that $q \in L_{p_n} \cap \gamma$ for infinitely many n. But this contradicts Lemma 4.17.)

We fix p_n $(n > n_0)$ arbitrarily. It is sufficient to show that each p_n is a clear maximal vertex. (Then the limit point p is also a clear maximal vertex.) If $L_{p_n} \cap \gamma = \{p_n\}$, then p_n is a clean maximal vertex by Lemma 4.10. So we may assume that $p_n \in M_\gamma$.
(*Case 1*) Suppose that each p_n is isolated in K. By Lemma 4.17, p_n is a clear maximal vertex.
(*Case 2*) Next we suppose that p_n is an accumulation point of the set $K = \{x \in \gamma\,;\, \mathrm{rank}^\bullet(x) = 1\}$. Then there is a sequence $(q_m)_{m\in\mathbf{N}}$ in K converging to p_n. By assumption, every sufficiently large q_m is not contained in M_γ. Thus q_m is a clean vertex by Lemma 4.10. Thus the limit point p_n is a clear vertex. □

For the following applications, we recall important two facts from [Sd1].

Lemma 4.18([Sd1: Proposition 4]). *Let γ be a C^3-convex simple closed space curve and $p, q \in \gamma$ be points with non-vanishing curvature and torsion. Then the straight line pq is tangent to γ at p if and only if it is tangent to the curve at q.*

Lemma 4.19 ([Sd1: Proposition 7]). *Let γ be a C^2-convex simple closed space curve and let p be a point such that $0 \notin \kappa(L_p \cap \gamma)$. Then there exists an open arc I containing p such that the tangent line L_q at each $q \in I \setminus \{p\}$ is not tangent to the curve at any other points.*

Remark. The statement of the lemma is slightly modified as in [RS: Proposition 4]. As explained in [RS], the proof is essentially the same as that of [Sd1: Proposition 7].

Above two lemmas yield the following

Lemma 4.20. *Let γ be a C^3-convex simple closed space curve whose curvature function and torsion function have only finitely many zeros. Then every element in the set M_γ is isolated.*

Proof. Suppose that there exists a point $p \in M_\gamma$ such that a sequence $(p_n)_{n\in\mathrm{N}}$ in $M_\gamma \setminus \{p\}$ exists and converges to p. For each p_n, we can choose $q_n \in L_q \cap \gamma$ such that $q_n \neq p_n$. By Lemma 4.19, L_{p_n} is not tangent to γ at q_n. Then by Lemma 4.18, the torsion function vanishes at p_n or q_n. Since the number of zeros of the torsion function is finite, there exists a positive number $n_0 > 0$ such that $q_n = q_0$ for all $n \geq n_0$. But this contradicts Lemma 4.17. □

By Proposition 4.15 and Lemma 4.20, we get the following two corollaries.

Corollary 4.21. ([RS]) *Let γ be a C^3-convex simple closed space curve. Then*

$$v(\gamma) + 2c(\gamma) \geq 4,$$

where $v(\gamma)$ is the number of zeros of the torsion function and $c(\gamma)$ is the number of zeros of the curvature function.

Corollary 4.22. ([Sd1]) *Let γ be a C^3-convex simple closed space curve with non-vanishing curvature function. Then*

$$v(\gamma) \geq 4.$$

Further generalizations of four vertex theorem for space curves will be found in Thorbergsson-Umehara [TU]. The inequality $v(\gamma) \geq 4$ does not hold if the curvature function of γ has zeros. (According to Barner [Ba;p210], Flohr pointed out it in the 1950s.) The explicit examples of $(v, c) = (1, 1)$ or $(0, 2)$ are found in [Sd1] and [RS].

Appendix A. Vertices on C^2-regular plane curves

As written in introduction, the four vertex theorem for simple closed Euclidean plane curves has been extended for various umbient spaces. On the other hand, there are many other known results for vertices on Euclidean plane curves with self-intersections, but it is still unclear that such a generalization works for these results or not. In this appendix, we give an abstract approach for the study of vertices on C^2-plane curves which may have self-intersections, and show that several known results are generlized for Minkowski plane curves and for curves on a convex surfaces with positive Gaussian curvature.

Let X be a differentiable sphere and Γ a subset of C^2-regular simple closed curves satisfying the axioms of circle system. Assume that Γ satisfies the following additional condition, which asserts the existence and the uniqueness of the osculating circles.

(C4) For any $p \in X$ and a C^2-regular curve γ passing through p, there exists a unique circle $C_p \in \Gamma$ which has second order tangency with γ at p.

Such a circle C_p is called the *osculating circle* of γ at p. Example 1-3 in §3 satisfy this condition.

Proposition A.1. *Let γ be a piecewise C^1-regular simple closed curve. Suppose that all internal angles of $\partial D^\bullet(\gamma)$ (resp. $\partial D^\circ(\gamma)$) are less than or equal to π. Then γ is $\bullet$-admissible (resp. $\circ$-admissible).*

Proof. We prove for $\partial D^\bullet(\gamma)$. (The corresponding assertion for $\partial D^\circ(\gamma)$ is obtained if one reverses the direction of the curve.) $A_p^\bullet$ is not empty, since $p \in A_p^\bullet$. If $A_p^\bullet = \{p\}$, p is an admissible point by definition. (See Definition 3.2.) So we may assume that $A_p^\bullet \neq \{p\}$. If p is a singular point of γ, $A_p^\bullet = \{p\}$ holds, because the internal angle at p is less than π. Thus we may also assume that γ is C^1-regular at p. Then each element of $A_p^\bullet \setminus \{p\}$ is tangent to γ at p. Then the $\bullet$-admissibility of γ follows from the following lemma. $\square$

Lemma A.2. *Let C_1 and C_2 be two distinct circles which are tangent at $p \in X$. Then they meet only at p.*

Proof. By (C4), the 2-jets of C_1 and C_2 at p are mutually different. Thus there exists a sufficiently small neighborhood W of p in X such that $C_2 \cap W$ is contained in $D^\bullet(C_1)$ or $D^\circ(C_1)$. If necessary, by interchanging C_1 and C_2, we may assume that $C_2 \cap W \subset D^\bullet(C_1)$ holds. If $D^\bullet(C_2) \not\subset D^\bullet(C_1)$, C_2 must meet C_1 at least three points. By (C1), it is impossible. Thus we have $D^\bullet(C_2) \subset D^\bullet(C_1)$. Then again by (C1), we have $C_1 \cap C_2 = \{p\}$. $\square$

Lemma A.3. *Let γ be a C^2-regular simple closed curve. Then for each point $p \in \gamma$, the osculating circle C_p at p satisfies the following relation*

$$D^\bullet(C_p^\bullet) \subset D^\bullet(C_p) \subset D^\bullet(C_p^\circ).$$

Proof. Let Γ_p be the subset of circles which are tangent to γ at p. The set $\mathcal{A}_p^\bullet$ defined in Definition 3.3 can be written as

$$\mathcal{A}_p^\bullet = \{C \in \Gamma_p \cup \{p^\bullet\}\,;\, C \subset D^\bullet(\gamma)\}.$$

Let C' be a circle satisfying the relation $D^\bullet(C_p) \subsetneq D^\bullet(C')$. Then the 2-jet of C' at p is different from C_p by (C4). Since γ has the second order tangency with C_p at p, any points on γ close to p are contained in $D^\bullet(C')$. This implies $C' \notin \mathcal{A}_p^\bullet$. Thus $D^\bullet(C_p^\bullet) \subset D^\bullet(C_p)$ holds. Similarly, $D^\bullet(C_p) \subset D^\bullet(C_p^\circ)$ can be also proved. $\square$

Lemma A.4. *Let γ be an embedded C^2-regular curve on X. Then for each p, $C_p^\bullet$ and C_p° are not collapsed into points, namely $C_p^\bullet, C_p^\circ \in \Gamma$.*

Proof. We prove for $C_p^\bullet$. Let Γ_p be the subset of circles which are tangent to γ at p. Suppose that $C_p^\bullet = p^\bullet$. Then there exists a sequence $(C_n)_{n\in\mathbf{N}}$ in $\Gamma_p \setminus \{p^\bullet\}$ such that $C_n \to p^\bullet$ and $D^\bullet(C_n) \not\subset D^\bullet(\gamma)$. By Lemma A.2 and (C1) in §3, either $D^\bullet(C_{n+1}) \subset D^\bullet(C_n)$ or $D^\bullet(C_n) \subset D^\bullet(C_{n+1})$ holds. Since $C_n \to p^\bullet$, without loss of generality, we may assume that

$$D^\bullet(C_{n+1}) \subset D^\bullet(C_n) \subsetneq D^\bullet(C_p) \qquad (n = 1, 2, 3, \ldots). \tag{A.1}$$

Since $D^\bullet(C_n) \not\subset D^\bullet(\gamma)$, there exists a point $q_n \in C_n \cap \gamma$ such that $q_n \neq p$ for each $n \in \mathbf{N}$. Since $C_n \to p^\bullet$, we have $q_n \to p$. On the other hand, since $D^\bullet(C_1) \subsetneq D^\bullet(C_p)$, the 2-jets of C_1 and C_p at p are distinct. So there exists an open subarc I of γ containing p such that $D^\bullet(C_1) \cap I = \{p\}$ and $I \subset D^\circ(C_1)$. By (A.1), we have

$$D^\bullet(C_n) \cap I = \{p\}. \tag{A.2}$$

Since $q_n \in \gamma$ and $q_n \to p$, we have $q_n \in I$ for any sufficiently large n. But this contradicts (A.2). Thus $C_p^\bullet \neq p^\bullet$, that is $C_p^\bullet \in \Gamma$. $\square$

For simple closed curves, we defined clean vertices in §3, but for curves with self-intersections, they cannot be defined. Instead of clean vertices, we define maximal and minimal vertices on C^2-regular curves as follows:

Definition A.1. A point p on γ is called a *maximal vertex* (resp. *minimal vertex*) if there exists an open subarc I of γ containing p such that $I \subset D^\circ(C_p)$ (resp. $I \subset D^\bullet(C_p)$). (In particular, all points on a circle are maximal and minimal vertices at the same time.)

In this appendix, the term "*honest vertex*" refers to a maximal or a minimal vertex unless otherwise stated.

Remark. This abstract definition of an honest vertex is slightly different from the original concept in Euclidean plane curves. When γ is a Euclidean plane curve, an honest vertex should be defined as an extremal point of the curvature function. But in our general setting, we can not define a curvature function. The honest vertices in the sense of the above definition and the extremal points of the curvature function coincide whenever the number of honest vertices is finite. On the other hand, if the number of honest vertices is infinite, honest vertices are divided into the following two cases

(1) extremal points of the curvature function,
(2) an accumulate point of extremal points of the curvature function.

(This observation is due to H. Kneser [H.K].) The example of the graph of $t \to t^4 sin(1/t)$ at $t = 0$ demonstrates this phenomenon, which was suggested by Dombrowski. Since we never use the curvature function in the following discussion, our definition of an honest vertex will makes no confusions even when the curve has infinitely many honest vertices.

Proposition A.5. *Let γ be a C^2-regular simple closed curve. If p is a clean maximal (resp. minimal) vertex, then p is a maximal (resp. minimal) vertex. Furthermore, $C_p^\bullet = C_p$ (resp. $C_p^\circ = C_p$) holds.*

Proof. Let p be a clean maximal vertex and Γ_p the set of circles which are tangent to γ at p. It is sufficient to show that $C_p^\bullet = C_p$. (If one reverse the orientation of the curve, the corresponding assertion for minimal vertex is obtained.) Suppose that $C_p^\bullet \neq C_p$. Then by Lemma A.3, we have

$$D^\bullet(C_p^\bullet) \subsetneq D^\bullet(C_p).$$

Since $C_p^\bullet \neq C_p$, the second derivative of C_p and $C_p^\bullet$ at p are mutually different by (C4). Moreover, by the existence of circles with given 2-jets as in (C4), there exists a sequence $(C_n)_{n\in\mathbf{N}}$ in Γ_p such that $C_n \to C_p^\bullet$ and

$$D^\bullet(C_p^\bullet) \subsetneq D^\bullet(C_n) \subsetneq D^\bullet(C_p) \qquad (n = 1, 2, 3, \ldots).$$

Here we also used the fact that any two elements in Γ_p meet only at p by Lemma A.2. Without loss of generality, we may assume that

$$D^\bullet(C_{n+1}) \subset D^\bullet(C_n) \qquad (n = 1, 2, 3, \ldots). \tag{A.3}$$

Since C_1 and C_p have the distinct 2-jets and γ is approximated by C_p at p in C^2-topology, there exists an open subarc I containing p such that $I \setminus \{p\}$ lies in the interior of $D^\circ(C_1)$. We fix an arbitrary distance function $d(\ ,\ )$ on X compatible with the topology. Since $C_p^\bullet$ and $\gamma \setminus I$ are disjoint closed subsets, the uniform distance $d(C_p^\bullet, \gamma \setminus I)$ is positive. As remarked in §3, the convergence $C_n \to C_p^\bullet$ is the same as that of the induced uniform distance of $J(X)$. Thus for a sufficiently large n, $d(C_n, \gamma \setminus I) > 0$. On the other hand, since $D^\bullet(C_n) \subsetneq D^\bullet(C_1)$, we have $C_n \cap I = \{p\}$. Thus C_n is a circle contained in $D^\bullet(\gamma)$. But this contradicts the maximality of $C_p^\bullet$. $\square$

Definition A.2. A C^2-regular curve $\sigma : [a, b] \to X$ is called a *shell* at p if $p = \sigma(a) = \sigma(b)$ and $\sigma|_{(a,b)}$ has no self-intersection. A shell is said to be *positive* (resp. *negative*) if the velocity vector $\sigma'(a)$ coincides with $\sigma'(b)$ or it points to the left (resp. right) of $\sigma'(b)$. The point p is called the *node* of the shell.

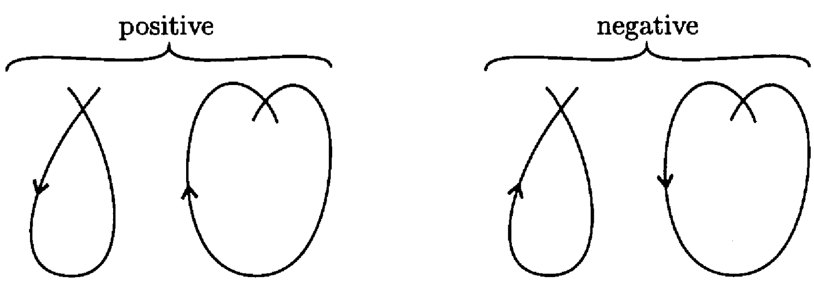

Figure A.1.

Lemma A.6. *Let $\gamma : [a,b] \to X$ be a positive (resp. negative) shell. Then there exists $c \in (a,b)$, such that $C_{\gamma(c)} = C^{\bullet}_{\gamma(c)}$ and $C_{\gamma(c)} \neq C_{\gamma(a)}, C_{\gamma(b)}$.*

Proof. By changing the orientation of the curve, we may assume that the shell is positive. (The maximal vertices and minimal vertices are exchanged if the direction of curves is reversed.) A positive shell is a $\bullet$-admissible simple closed curve by Proposition A.1. Thus by Theorem 3.2, there are at least two distinct maximal circles $C^{\bullet}_p$ and $C^{\bullet}_q$. We may assume that one of them, say q is not the node of the shell. If $C^{\bullet}_q = C_{\gamma(a)}$, then $C_{\gamma(a)} = C^{\bullet}_{\gamma(a)} = C^{\bullet}_p$, but it contradicts to $C_p \neq C_q$. Thus $C^{\bullet}_q \neq C_{\gamma(a)}$. Similarly we also have $C^{\bullet}_q \neq C_{\gamma(b)}$. By Proposition A.5, we have $C_q = C^{\bullet}_q$. Hence the the point $c \in (a,b)$ such that $\gamma(c) = q$ is the desired one. □

The following corollary is an abstract version of Jackson [J;Lemma4.3].

Corollary A.7. *A positive (resp. negative) shell $\gamma : [a,b] \to X$ has at least one maximal (resp. minimal) vertex in (a,b).*

Proposition A.8. *Let $\gamma : [a,b] \to X$ be a curve which contains neither a maximal vertex nor a minimal vertex on (a,b). Then the one of the following two assertions are true;*

(1) *$\gamma|_{(a,b]}$ lies in $\mathcal{D}_a$,*
(2) *$\gamma|_{(a,b]}$ lies in $\mathcal{D}_a$,*

where $\mathcal{D}_a$ is the interior of $D^{\bullet}(C_{\gamma(a)})$ (resp. $D^{\circ}(C_{\gamma(a)})$).

Proof. Suppose that $\gamma|_{(a,b]}$ intersects $C_{\gamma(a)}$ firstly at p. Then composing γ with $C_{\gamma(a)}$ at $\gamma(a)$, we get a no-vertex shell at p. But the shell does not satisfy the conclusion of Lemma A.6. □

Definition A.3. Let $\gamma : [a,b] \to X$ be a curve which contains maximal vertices nor minimal vertices on (a,b). Then γ is called a *positive scroll* (resp. *negative scroll*) if (1) (resp. (2)) of Proposition A.8 occurs.

By definition, positivity or negativity of scrolls does not depend on the choice of orientation of the scrolls. Lemma A.6 yields the following abstract version of Kneser's theorem [K.A].

Theorem A.9. *Let $\gamma : [a,b] \to X$ be a positive scroll (resp. negative scroll). Then the osculating circle $C_{\gamma(b)}$ lies in $\mathcal{D}_a$ (resp. $\mathcal{D}_b$).*

Proof. Suppose that two osculating circles intersect. Then we can use arcs of $C_{\gamma(a)}$, γ and $C_{\gamma(b)}$ to find a shell at the one of intersection points of two circles $C_{\gamma(a)}$ and $C_{\gamma(b)}$. This contradicts to Lemma A.6, since γ has no honest vertex. Thus $C_{\gamma(a)} \cap C_{\gamma(b)}$ is empty. Since $\gamma(b)$ lies in $\mathcal{D}_a$ (resp. D_b) by Proposition A.8, we have $C_{\gamma(b)} \subset \mathcal{D}_a$. □

Corollary A.10. *Let γ be a C^2-regular closed curve with finitely many maximal vertices. Then the number of maximal vertices is equal to the number of minimal vertices. More precisely, for any two different maximal vertices p, q on γ, there is a minimal vertex on $\gamma|_{(p,q)}$.*

Proof. Suppose that there is no minimal vertex between p and q. Without loss of generality, we may assume that $\gamma|_{(p,q)}$ is vertex-free. Since p is a maximal vertex,

$\gamma|_{[p,q]}$ is a negative scroll. On the other hand, Since q is also a maximal vertex, $\gamma|_{[p,q]}$ is a positive scroll. This is a contradiction. $\square$

As an application, we give the following $2n$-vertex theorem which is a generalization of Jackson [J]. (For convex curves, it was proved by Blaschke [Bl1]. Similar axiomatic treatment of $2n$-vertex theorem are found in Haupt and Künneth [HK2-3].)

Theorem A.11. *Let γ be a C^2-regular simple closed curve on (X, Γ) such that a circle $C \in \Gamma$ meets γ transversally at $p_1, q_1, ..., p_n, q_n \in \gamma \cap C$. Suppose that the rotational order of the crossings $p_1, q_1, ..., p_n, q_n$ of γ is the same as that of C. Then γ has at least $2n$ different honest vertices.*

The outline of the proof is the essentially same as in [J; Theorem 7.1]. But in our general setting, we can not apply Jackson [J; Lemma 3.1]. The following lemma will replace Jackson's lemma.

Lemma A.12. *Let C be a circle and γ_j $(j = 1, 2)$ two C^2-regular curves with finitely many honest vertices transversally intersecting C at two points p_j, q_j $(j = 1, 2)$. Suppose that $\gamma_1|_{[p_1,q_1]}$ and $\gamma_2|_{[p_2,q_2]}$ lie in $D^\bullet(C)$ and have no intersections with each other. (See Figure A.2.) Then there is a circle C' which lies in $D^\bullet(C)$ such that it is tangent to the three arcs $\gamma_1|_{[p_1,q_1]}$, $C|_{[q_1,p_2]}$ and $\gamma_2|_{[p_2,q_2]}$.*

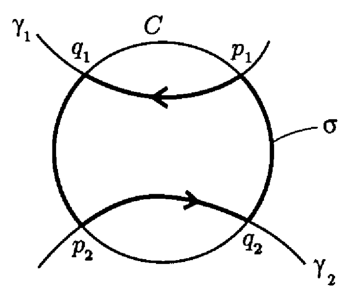

Figure A.2.

Proof. Let σ be a piecewise C^1-regular curve consisting of the four arcs $\gamma_1|_{[p_1,q_1]}$, $C|_{[q_1,p_2]}$, $\gamma_2|_{[p_2,q_2]}$ and $C|_{[q_2,p_1]}$. Since each interior angle of $\partial D^\bullet(\sigma)$ is less than π, σ is a $\bullet$-admissible curve by Proposition A.1. The four points p_1, q_1, p_2, q_2 are clean maximal vertices on σ. Thus the set

$$T := \{x \in C|_{[q_1,p_2]} \; : \; \operatorname{rank}^\bullet(x) \geq 3\}$$

is not empty by Lemma 2.1. Let $x \in T$. By (C1), $C_x^\bullet \cap C|_{[q_2,p_1]} = \varnothing$. Suppose there is no such circle C' as stated in the theorem. Then it holds either $F_x^\bullet \cap \gamma_1|_{[p_1,q_1]} = \varnothing$ or $F_x^\bullet \cap \gamma_2|_{[p_2,q_2]} = \varnothing$. But $F_x^\bullet \cap \gamma_j|_{[p_1,q_1]} = \varnothing$ $(j = 1, 2)$ never hold at the same time. (In fact, if so, the circle $C_x^\bullet$ coincides with C by the same arguments as in the proof of Proposition A.5, which is a contradiction.) Thus the set T is a disjoint union of the following two subsets

$$T^- := \{x \in T \; : \; F_x^\bullet \cap \gamma_2|_{[p_2,q_2]} = \varnothing\},$$
$$T^+ := \{x \in T \; : \; F_x^\bullet \cap \gamma_1|_{[p_1,q_1]} = \varnothing\}.$$

We show $T^- \neq \varnothing$. (Similarly $T^+ \neq \varnothing$ is also verified.) Since γ_1 and γ_2 have only finitely many honest vertices, so does σ. In particular, $s^\bullet(\sigma) < \infty$ and so $t^\bullet(\sigma) < \infty$ by Theorem 3.2. By Lemma A.4, we have $F^\bullet_{q_1} = \{q_1\}$ because of $C^\bullet_{q_1} = q^\bullet_1$. So any point x on $C|_{[q_1,p_2]}$ sufficiently close to q_1 is $\bullet$-regular and $\mu^\bullet_-(x) \in \gamma_1|_{(p_1,q_1)}$, because $\lim_{x\to q_1+0}\mu^\bullet_-(x) = q_1 - 0$ by Theorem 1.4. Since x is of rank$^\bullet$ 2, we have $F^\bullet_x \cap \gamma_2|_{[p_2,q_2]} = \varnothing$. Hence $x \in T^-$, and T^- is non-empty. We set

$$y^- := \sup(T^-), \qquad y^+ := \inf(T^+),$$

where the lowest upper bound and the greatest lower bound are taken with respect to the canonical order of the arc $C|_{[p_1,p_2]}$. Since $(F^\bullet_p)_{p\in\sigma}$ is an intrinsic circle system, by (I2), we have $y^+ \succeq y^-$. On the other hand, $y = y^+ = y^-$ does not occur since T^+ and T^- are disjoint. Thus we have $y^+ \succ y^-$. Consequently, $C|_{(y^-,y^+)}$ is a $\bullet$-regular arc on σ. By Corollary 1.7, $\sigma|_{(\mu^\bullet(y^+),\mu^\bullet(y^-))}$ is also $\bullet$-regular. On the other hand, $\sigma|_{(\mu^\bullet(y^+),\mu^\bullet(y^-))}$ contains two clean maximal vertices q_1 and q_2. This is a contradiction. $\square$

(*Proof of Theorem A.11*) We set $I_k := \gamma|_{[p_k,q_k]}$. Without loss of generality, we may assume that $\gamma \cap D^\bullet(C) = I_1 \cup \cdots \cup I_n$. We set

$$J_1 := C|_{[q_n,p_1]}, \quad J_2 := C|_{[q_1,p_2]}, \quad \cdots, \quad J_n := C|_{[q_{n-1},p_n]}.$$

By Lemma A.11, there exists a circle C'_k $(k = 1, ..., n)$ which is tangent to I_k, J_k and I_{k+1} respectively. Let x_k (resp. y_k) be a tangent point between C'_k and I_k (resp. I_{k+1}). Then there is a maximal vertex on $\gamma|_{(x_k,y_k)}$ by Lemma 1.1. (It is a clean vertex of the simple closed curve obtained by joining $\gamma|_{(x_k,y_k)}$ and C'_k, but not a clean vertex of γ in general.) Moreover, by (C1) in §3, we have

$$(y_1 \succ) y_n \succ x_n \succ \cdots \succ x_1 \succ y_1,$$

where $\succ$ is the rotational order of γ. (See Figure A.3.) Thus γ has n clean maximal vertices. By Corollary A.10, γ has n clean minimal vertices between them. $\square$

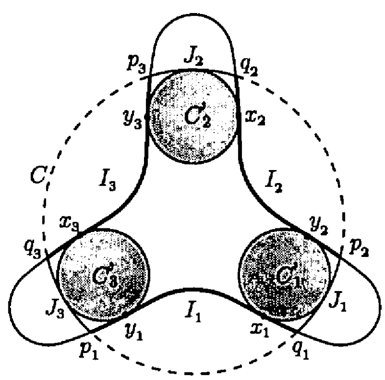

Figure A.3

The the following lemma is a refinement of Corollary A.7: (The proof below is the a slight modification of the original one in Kobayashi-Umehara [KU].)

Lemma A.13. (The abstract version of [KU; Lemma 3.1])

Let $\gamma : [a,b] \to \mathbf{R}^2$ be a positive shell at $p = \gamma(a) = \gamma(b)$.

(1) *If γ has only one (necessary maximal) vertex, then $\gamma \setminus \{p\} \subset \mathcal{D}_a \cap \mathcal{D}_b$, where $\mathcal{D}_a$ (resp. $\mathcal{D}_b$) is the interior of the closed domain $D^\bullet(C_{\gamma(a)})$ (resp. $D^\bullet(C_{\gamma(b)})$).*

(2) *If γ has exactly two honest vertices, maximal at $t_1 \in (a,b)$ and minimal at $t_2 \in (a,b)$, then $\gamma \setminus \{p\} \subset \mathcal{D}_a$ if $t_1 < t_2$ and $\gamma \setminus \{p\} \subset \mathcal{D}_b$ if $t_2 < t_1$.*

(3) *If γ has exactly three honest vertices, two of which are maximal and the other is minimal, then either $\gamma \setminus \{p\} \subset \mathcal{D}_a$ or $\gamma \setminus \{p\} \subset \mathcal{D}_b$.*

Proof. By Proposition A.1, γ is a $\bullet$-admissible curve. First we prove (1). Let q be the maximal vertex and $x \in \gamma|_{(p,q]}$. Then by Theorem A.9, we have

$$x \in \mathcal{D}_a. \tag{A.4}$$

On the other hand, let $y \in \gamma|_{(q,p)}$. Since y is not a maximal vertex, $C^\bullet_y$ meets another point $z \in \gamma|_{(p,q)}$ by Corollary 1.2. Thus we have

$$y \in D^\bullet(C_{\gamma(z)}) \subset \mathcal{D}_a. \tag{A.5}$$

By (A.4) and (A.5), we have $\gamma \subset \mathcal{D}_a$. Similarly, we can also show $\gamma \subset \mathcal{D}_b$.

Next we prove (2). Assume that γ has exactly two honest vertices, maximal at $t_1 \in (a,b)$ and minimal at $t_2 \in (a,b)$ and $t_1 < t_2$. Then by the same argument as in the proof of (1), (A.4) holds for $x \in \gamma|_{(p,\gamma(t_1)]}$ and (A.5) holds for $y \in \gamma|_{(\gamma(t_1),p)}$. Thus we have $\gamma \subset \mathcal{D}_a$.

Finally, we prove (3). Let $q_1 = \gamma(t_1)$ and $q_3 = \gamma(t_3)$ be maximal vertices and $q_2 = \gamma(t_2)$ a minimal vertex. We may assume that $t_1 < t_2 < t_3$. By Proposition A.8, we have $\gamma_{[p,q_1]} \subset \mathcal{D}_a$ and $\gamma_{[q_3,p]} \subset \mathcal{D}_b$. On the other hand, for an arbitrary $x \in \gamma|_{(q_1,q_3)}$, there exists $y \in \gamma|_{(a,q_1)} \cup \gamma|_{(q_3,b)}$ such that $C^\bullet_x \ni y$. Thus we have $x \in C^\bullet_x \subset \mathcal{D}_y \subset \mathcal{D}_a($ or $\mathcal{D}_b)$. Hence we have shown that $\gamma \setminus \{p\} \subset \mathcal{D}_a \cup \mathcal{D}_b$. We set

$$\begin{aligned} s_a &:= \inf\{s \in (a,b)\,;\, \gamma(t) \in \mathcal{D}_a \text{ if } t \in (s,b)\},\\ s_b &:= \sup\{s \in (a,b)\,;\, \gamma(t) \in \mathcal{D}_b \text{ if } t \in (a,s)\}. \end{aligned}$$

Then it holds that $a < s_a < s_b < b$. Now we suppose that neither $\gamma \setminus \{p\} \not\subset \mathcal{D}_a$ nor $\gamma \setminus \{p\} \not\subset \mathcal{D}_b$ holds. We can extend γ to $\tilde{\gamma} : [a, b+c] \to X$ such that $\tilde{\gamma}|_{[a,b]} = \gamma$, $\tilde{\gamma}|_{[b,b+c]} = C_b|_{[\gamma(b),\gamma(s_b)]}$. Then $\tilde{\gamma}|_{[s_a,b+c]}$ is a negative shell at $\tilde{\gamma}(\tilde{b})$. By Lemma A.6, there is a minimal vertex on (s_b, b). Similarly, we can find another minimal vertex on (a, s_a). This is a contradiction. $\square$

Using Lemma A.13, the following theorem can be proved by the same arguments as in [KU; Theorem 3.5].

Theorem A.14 (The abstract version of [KU; Theorem 3.5]). *If a closed curve contains three positive shells or three negative shells, then it has at least six honest vertices.*

The above 6-vertex theorem is stronger than the following 6-vertex theorem:

Corollary A.15 (The abstract version of [CMO] and [U1]). *A closed curve has at least six honest vertices if it bounds an immersed surface other than the disc.*

Proof. It is sufficient to show that *any closed curve γ which bounds immersed surface with positive genus has three negative shells.* (The immersed surface is assumed to lie on the left hand side of γ.) If γ has a positive shell then by the proof in [KU: Corollary 3.7], we found three negative shells. Hence, we may assume that γ is not embedded and γ has no positive shell. Suppose that γ has at most two negative shells. Let x is a self-intersection of γ. Then γ can be expressed as a union of two distinct loops γ_1 and γ_2 at x. Each loop γ_i contains at least one shell S_i, which must be negative because γ has no positive shells. We take points $q_j \in S_j \setminus \{p_j\}$ $(j = 1, 2)$ respectively, where p_j is the node of the shell S_j. Then γ can be divided into two arcs $\gamma|_{[q_1,q_2]}$ and $\gamma|_{[q_2,q_1]}$. Moreover these two closed arcs $\gamma|_{[q_1,q_2]}$ and $\gamma|_{[q_2,q_1]}$ are both embedded. (In fact, for example, if $\gamma|_{[q_1,q_2]}$ is not embedded, then we find third shell S on $\gamma|_{[q_1,q_2]}$, which must be negative. This is a contradiction.) Then by [U1;Theorem 3.1], γ only bounds a disc, which is a contradiction. $\square$

The Corollary A.15 for Euclidean plane curves was first proved for normal curves in [CMO] and extended to the general case in [U1]. It should be remarked that Corollary A.15 itself is obtained by Corollary A.7 using purely topological arguments. The following related result can be proved by the method in [Pe] using Corollary A.7.

Theorem A.16 (The abstract version of [Pe; Theorem 4]). *A closed curve has at least $(4g + 2)$-vertices if it bounds an immersed surface of genus g, provided that the number of self-intersections does not exceed $2g + 2$.*

In the rest of this appendix, we consider an intersection sequence of a positive scroll and a negative scroll, which is an abstract version of [KU;§4]. As an application, a structure theorem for 2-vertex curve is obtained. In [KU;§4], we use corner rounding technique on curves. But this method is not valid in our general setting. So the following is the modified version of [KU;§4].

Let γ^- and γ^+ be positive and negative scrolls respectively satisfying the following two properties:

(a) All intersections of γ^- to γ^+ are transversal.
(b) The first crossing of γ^+ is the last crossing of γ^-.

A crossing of γ is called *positive* (resp. *negative*) if γ^+ crosses γ^- from the left (resp. right). We use small letters for positive crossings. For the sake of simplicity, we use the following notations: Let γ be an open arc and p a point on γ. Then we denote by $\gamma|_{>p}$ (resp. $\gamma|_{<p}$) the future part (resp. the past part) from p.

Definition A.5 (The $*$-pairing). Let a be a positive crossing. If a crossing is the first one at which $\gamma^-|_{>a}$ meets $\gamma^+|_{<a}$, then it is expressed by a^*.

Lemma A.17. *Let γ^+ and γ^- be positive and negative scrolls satisfying (a) and (b). If there exists a crossing a^* for a positive crossing a, then a^* is a negative crossing.*

Proof. Suppose that a^* is a positive crossing.

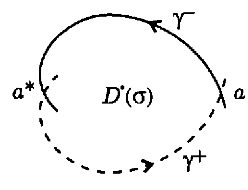

Figure A.4a.

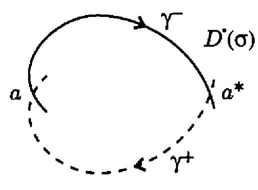

Figure A.4b.

Let σ be a simple closed curve defined as a union of two arcs $\sigma := \gamma^-|_{[a,a^*]} \cup \gamma^+|_{[a^*,a]}$. Let $D^\bullet(\sigma)$ be the left-hand closed domain with respect to σ as in Figure A.4a or A.4b. The angle at a^* of the domain is greater than π. We consider a sufficiently small circle C , which is tangent to γ^- at a^* and lies in $D^\bullet(\sigma)$. Expand C continuously. Let $x \neq a^*$ be the first attachment of C to the heart figured domain. Then $x \neq a$, and C is tangent to γ^- or γ^+ at x. If $x \in \gamma^-$, we have $C \subset \overline{\mathcal{D}_x}$, where $\mathcal{D}_x$ is the left open domain of the osculating circle C_x. Since γ^- is a negative arc, we have $\overline{\mathcal{D}_x} \subset \mathcal{D}_{a^*}$ by Theorem A.9. Hence C can not meet C_{a^*}, which is a contradiction because of $a^* \in C \cap C_{a^*}$. $\square$

Lemma A.18. *Let γ^+ and γ^- be positive and negative scrolls satisfying (a) and (b). Suppose that there exists a crossing a^* for a positive crossing a. If $\gamma^+|_{>a}$ (resp. $\gamma^-|_{<a}$) meets γ^- (resp. γ^+) at q firstly, then q lies on $\gamma^-|_{>a}$ (resp. $\gamma^+|_{<a}$).*

Proof. Let $\sigma = \gamma^-_{[a,a^*]} \cup \gamma^+_{[a^*,a]}$ be a simple closed curve. Let C_a^- (resp. C_a^+) be the osculating circle at a with respect to γ^- (resp. γ^+). By Proposition A.8 and Definition A.3, we have $(D^\bullet(C_a^-))^c \supset \gamma^-|_{>a}$ and $(D^\bullet(C_a^+))^c \supset \gamma^+|_{<a}$, where $(D^\bullet(C_a^\pm))^c$ are the complements of $D^\bullet(C_a^\pm)$. Thus

$$(D^\bullet(C_a^-) \cap D^\bullet(C_a^+))^c = (D^\bullet(C_a^-))^c \cup (D^\bullet(C_a^+))^c \supset \sigma.$$

This implies that

$$D^\bullet(C_a^-) \cap D^\bullet(C_a^+) \subset D^\bullet(\sigma). \tag{A.6}$$

On the other hand, $\gamma^+|_{>a} \subset D^\bullet(C_a^+)$ and $\gamma^-|_{<a} \subset D^\bullet(C_a^-)$. Suppose that $\gamma^+|_{>a}$ meets $\gamma^-|_{<a}$ at some point x. Then $x \in D^\bullet(C_a^-) \cap D^\bullet(C_a^+)$, so $x \in D^\bullet(\sigma)$ by (A.6). This means that $\gamma^+|_{>a}$ (resp. $\gamma^-|_{<a}$) meets $\gamma^-|_{[a,a^*]}$ (resp. $\gamma^+|_{[a^*,a]}$) before x (resp. after x). $\square$

Lemma A.19. *Let a be a positive crossing.*

(1) *a^* coincides with the first crossing at which the past part of γ^+ from a meets the future part of γ^- from a.*
(2) *If $a^* = b^*$, then $a = b$.*

Proof. We prove the first assertion. Suppose $p \neq a^*$ is the first crossing at which $\gamma^+|_{<a}$ meets $\gamma^-|_{>a}$. Then p lies on $\gamma^+|_{[a^*,a]}$.(See Figure A.5a.) Consequently, p is a positive crossing. Then $p = a^\diamond$, where $a^\diamond$ is the $*$-paring between the negative scroll γ^- and the positive scroll $\gamma^+|_{>p}$. On the other hand $p = a^\diamond$ is a negative crossing by Lemma A.17. This is a contradiction.

Next we prove (2). Suppose that $a \neq b$. Without loss of generality, we may assume that γ^- meets γ^+ firstly at a, next at b and finally at $c = a^* = b^*$. Since b is a positive crossing, there is a negative crossing x on $\gamma^-|_{[a,b]}$ at which $\gamma^+|_{>a}$ meets $\gamma^-|_{[a,b]}$ firstly. (See Figure A.5b.) Now we reverse the orientation of γ^-, which is denoted by $\langle -\gamma^- \rangle$. We denote by $\sharp$ the $*$-pairing between the negative scroll $\langle -\gamma^- \rangle$ and the positive scroll γ^+. Then the signs of crossings are all reversed. We have $a = x^\sharp$. But $\gamma^+|_{>x}$ meets $\langle -\gamma^- \rangle$ at b, which contradicts Lemma A.18. $\square$

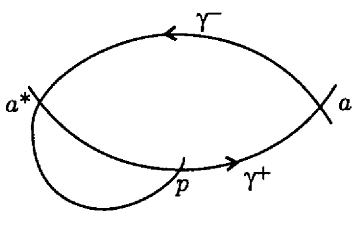

Figure A.5a.

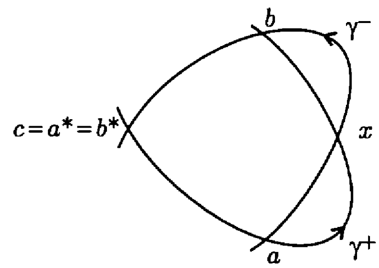

Figure A.5b.

Definition A.8. If a negative crossing does not have a $*$-pairing, then it is called a *solitary negative crossing* and is denoted by a capital letter.

The remaining discussions in [KU;§4] can be easily translated to our abstract setting. In particular, the intersection sequence of γ^- consists of the following three type of words:

$$\begin{aligned} \text{Type T}: &\quad A_1 A_2 \ldots A_n, \\ \text{Type D}: &\quad [a_1 a_2 \ldots a_n] := a_1 \ldots a_n a_n^* \ldots a_1^*, \\ \text{Type S}: &\quad [a_1 a_2 \ldots a_n : B] := a_1 \ldots a_n B a_n^* \ldots a_1^*. \end{aligned}$$

We define the length of the each type of words by

$$\big|A_1 A_2 \ldots A_n\big| := n, \quad \big|[a_1 a_2 \ldots a_n]\big| := n, \quad \big|[a_1 a_2 \ldots a_n : B]\big| := n+1.$$

The following theorem holds by exactly the same argument in [KU;§4].

Theorem A.20. *Let γ^+ and γ^- be positive and negative scrolls satisfying* (a) *and* (b). *Then the intersection sequences W^- of γ^- is of the form $W^- = W_1 W_2 \cdots W_n$, where W_i $(i = 1, \ldots, n)$ is of type T, D or S and the intersection sequence of γ^+ is obtained by the head picking rule as in* [KU]. *Moreover W^- satisfies the following grammar:*

(1) *If W_i is of type D, then W_j $(j < i)$ is of type T or D.*
(2) *If W_i is of type T and W_{i+1} is of type D, then $|W_i| \le |W_{i+1}|$. Moreover if W_{i-1} is of type D, then $|W_i| + |W_{i-1}| \le |W_{i+1}|$ holds.*
(3) *If W_i is of type T and W_{i-1}, W_{i+1} is of type S, then $|W_i| + |W_{i-1}| \ge |W_{i+1}|$.*

An immersed curve is called *normal* if all crossings are transversal and there are only double points. The following theorem is obtained by exactly the same argument as in the proof of [KU; Theorem 4.8 and 4.9].

Theorem A.21 (A structure theorem of 2-vertex curves). *Let γ be a closed normal 2-vertex curve divided by negative and positive scrolls $\gamma = \gamma^- \cup \gamma^+$. Then the intersection* *sequences of γ^- and γ^+ are translated mutually by the head picking rule as in* [KU]. *Moreover, the grammar of the intersection sequence of γ^- is given as follows.*

(1) *The intersection sequence consists of words of type T and type S and written in the form $T_0S_1T_1S_2T_2\cdots S_kT_k$. Each T_i $(i=1,...,k)$ may possibly be empty.*
(2) $|T_0|>0$, $|T_0|\geq|S_1|$ *and* $|S_i|+|T_i|\geq|S_{i+1}|$ $(i=1,...,k)$.

When $X=\mathbf{R}^2\cup\{\infty\}$ and Γ is the set of circles in the Möbius plane (cf. §3-Example 1), the converse assertions of Theorem A.20 and Theorem A.21 are true. (See [KU].) Moreover, in [KU], the intersection sequences of two scrolls of the same kind are also characterized in a similar manner.

For a plane curve γ, there exists an interesting invariant $J^+(\gamma)\in\mathbf{Z}$, which is related to the linking number of the corresponding Legendrian knot in the unit sphere bundle on $\mathbf{R}^2$. (See [A1],[A2] and [A3]. Selwat [Sl] is also a nice reference.). Since $J^+(\gamma)$ is not invariant under the diffeomorphism of $S^2=\mathbf{R}^2\cup\{\infty\}$, it is convenient to define a modified invariant

$$SJ^+(\gamma):=J^+(\gamma)+\frac{i_\gamma^2}{2},$$

where i_γ is the rotation number of γ as a plane curve. As an application of Theorem A.21, we can get the following by the same method as in [U2]. (See also Remark in [U2:§1].)

Theorem A.22 (The abstract version of [U2].). *Let γ be a normal closed curve in X. Suppose that $SJ^+(\gamma)>0$, then γ has at least four honest vertices.*

Two closed normal curves $\gamma_1,\gamma_2:S^1\to S^2$ are called geotopic if there is a diffeomorphism φ on S^2 such that $\varphi(\mathrm{Im}(\gamma_1))=\mathrm{Im}(\gamma_1)$. It is an interesting problem to determine the minimum number of honest vertices that a closed normal curve with given geotopy type can have. Minimizing numbers for normal curves are determined by Heil [He1] for crossings(≤ 3) and in [KU] and Kobayashi [Ko] for crossings(≤ 5).

Appendix B. The continuity of the maximal circles

In this Appendix, we shall prove the continuity of the center of maximal circles of a simple closed curve in the Euclidean plane. This was used in the last remark in §2. First, we prove the following general statement.

Theorem B.1. *Let X be a differentiable sphere and Γ a subset of C^2-regular simple closed curves satisfying the axioms of a circle system. Let γ be a C^2-regular simple closed curve satisfying $s^\bullet(\gamma)<\infty$ and $c^\bullet:X\to\Gamma$ a map defined by $c^\bullet(p):=C_p^\bullet$. Then $c^\bullet$ is a continuous mapping with respect to the compact open topology on Γ.*

Proof. If γ is a circle, the statement is obvious. So we assume γ is not a circle. It is sufficient to show that $C_{p_n}^\bullet\to C_p^\bullet$ if $p_n\to p$ holds for any $p\in\gamma$ and a sequence $(p_n)_{n\in\mathbf{N}}$ converging to p. By (C2), the sequence $(C_{p_n}^\bullet)_{n\in\mathbf{N}}$ has a convergent subsequence which converges to a circle C. To prove $C=C_p^\bullet$, we may assume that $(C_{p_n}^\bullet)_{n\in\mathbf{N}}$ itself is a convergent sequence. Obviously $C\subset D^\bullet(\gamma)$. Since $p_n\in C_{p_n}^\bullet$, we have $p\in C$. Hence C is a circle contained in $D^\bullet(\gamma)$ which is tangent at p. Suppose that $C\neq C_p^\bullet$. Then $C\cap\gamma=\{p\}$ holds. Thus there exists an integer $n_0>0$ and a sufficiently small open arc J containing p such that

$$\text{(B.1)}\qquad C_{p_n}^\bullet\cap\gamma\subset J\qquad(\text{ for all } n\geq n_0).$$

First, we consider the case $\operatorname{rank}^\bullet(p) \geq 2$. In this case, J can be taken to be weakly $\bullet$−regular. (See Corollary 2.6.) Then by Corollary 1.2, $(B.1)$ implies that $C_n \cap \gamma$ consists of only one component. But it is impossible because $\operatorname{rank}^\bullet(p_n) \geq 2$ in this case. Thus we have $C = C_p^\bullet$. Next, we consider the case $\operatorname{rank}^\bullet(p) = 1$. If $p \neq \mu_-^\bullet(p)$, then $p, \mu_-^\bullet(p) \in C$, which contradicts $C \cap \gamma = \{p\}$. Thus $p = \mu_-^\bullet(p)$, which implies $F_p = \{p\}$. Since $p_n \to p - 0$ and $\lim_{n\to\infty} \mu_-^\bullet(p_n) \to p + 0$, the C^2-differentiability of γ yields that C is the osculating circle at p. By Lemma A.1, we have $C = C_p^\bullet$. $\square$

Let $\gamma : S^1 \to \mathbf{R}^2$ be a C^2-regular simple closed curve in the Euclidean plane. Assume that γ is oriented so that $D^\bullet(\gamma)$ is a bounded domain in $\mathbf{R}^2$. For each point $p \in \gamma$, let c_p be the center of the maximal circle $C_p^\bullet$. Then we have the following

Corollary B.2. *Suppose that $s^\bullet(\gamma) < \infty$. Then the map $\Phi : \gamma \to \mathbf{R}^2$ defined by $\Phi(p) = c_p$ is continuous.*

Proof. Let Γ_1 be the set of circles in the Euclidean plane X_1. It is not so hard to see that the map $\psi : \Gamma_1 \to \mathbf{R}^2 \times (0, \infty)$ defined by $\psi(C) = (z(C), r(C))$ is homeomorphism, where $z(C)$ and $r(C)$ are the center and the radius of C respectively. Since $\psi \circ c^\bullet$ is continuous by Theorem B.1, we have the conclusion. $\square$

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