On the volumes of hyperbolic 5 - orthoschemes and the Trilogarithm

R. Kellerhals

Mathematisches Institut der Universität
Wegelerstraße 10
D-5300 Bonn 1

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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Ruth Kellerhals *

Max-Planck-Institut für Mathematik, Gottfried-Claren-Strasse 26, W-5300 Bonn 3, FRG

and

Mathematisches Institut der Universität Bonn, Wegelerstrasse 10, W-5300 Bonn 1, FRG

Introduction

The purpose of this paper is to calculate volumes of five-dimensional hyperbolic orthoschemes. Orthoschemes in a space X of constant curvature are simplexes whose vertices $P_0, \ldots, P_n$ are such that

$$\operatorname{span}(P_0, \ldots, P_k) \perp \operatorname{span}(P_k, \ldots, P_n) \quad \text{for} \quad 1 \leq k \leq n-1 . \tag{1}$$

These are the most basic objects in polyhedral geometry: They generate the scissors congruence groups $\mathcal{P}(X)$ of polytopes in X (see **1.4**). In addition, orthoschemes are characterized by nice metrical properties, e.g., they have at most n non-right dihedral angles $\alpha_1, \ldots, \alpha_n$, and all their faces and vertex figures are orthoschemes. It is therefore natural to restrict the volume problem to orthoschemes. In doing so, Lobachevsky found a volume formula for hyperbolic 3-orthoschemes (see **2.2**), which, for a 2-asymptotic orthoscheme $R(\alpha)$ with angles $\alpha_1 = \frac{\pi}{2} - \alpha_2 = \alpha_3 =: \alpha$, reduces to

$$\operatorname{vol}_3(R(\alpha)) = \frac{1}{2}\,Л(\alpha) \quad . \tag{2}$$

Here, $Л(\alpha)$ denotes the classical Lobachevsky function related to Euler's Dilogarithm $\operatorname{Li}_2(z) = \sum\limits_{r=1}^{\infty} \frac{z^r}{r^2}$, $z \in \mathbf{C}$, $|z| \leq 1$, by

$$Л(\alpha) = \frac{1}{2}\operatorname{Im}(\operatorname{Li}_2(e^{2i\alpha})) \quad .$$

Since, for even-dimensional orthoschemes, volumes are expressible in terms of those of certain lower (odd) dimensional orthoschemes (see [K, §14.2.2]), the next step is to look for a volume formula for hyperbolic orthoschemes of dimension five. In this context, Dehn

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[B, p.308] raised the question whether this can still be done by means of a function in one variable. This problem was solved affirmatively by Böhm [B] resp. Paul Müller [M] using different approaches; they showed that - apart from logarithms of lower orders - the Trilogarithm $\mathrm{Li}_3(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^3}$ is sufficient to express the volume of a compact resp. 1-asymptotic 5-orthoscheme. However, their volume formulae are very difficult to survey involving dozens of Trilogarithms with rational arguments in trigonometrical expressions of the dihedral angles.

By results of Dupont and Sah (see **1.4**), the hyperbolic scissors congruence groups of dimensions ≥ 2 are isomorphic to the scissors congruence groups of polytopes in extended hyperbolic space which, for odd dimensions, are generated by the 2-asymptotic orthoschemes (i.e., P_0 , P_n are points at infinity). Focussing on 2-asymptotic orthoschemes, we can derive a comparatively simple volume formula for a certain subclass among them. Let $R(\alpha,\beta,\gamma)$ denote a 5-orthoscheme with angles $\alpha_1 = \alpha_4 =: \alpha$, $\alpha_2 = \alpha_5 =: \beta$, $\alpha_3 =: \gamma$ satisfying

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad . \tag{3}$$

Then, $R(\alpha,\beta,\gamma)$ is 2-asymptotic, and its volume is given by

$$\begin{aligned} \mathrm{vol}_5(R(\alpha,\beta,\gamma)) =& \frac{1}{4}\{Л_3(\alpha) + Л_3(\beta) - \frac{1}{2}Л_3(\frac{\pi}{2}-\gamma)\} - \\ & -\frac{1}{16}\{Л_3(\frac{\pi}{2}+\alpha+\beta) + Л_3(\frac{\pi}{2}-\alpha+\beta)\} + \frac{3}{64}\zeta(3) \quad , \end{aligned} \tag{4}$$

where $Л_3(\omega)$ denotes the Lobachevsky function of order three (see **2.**) related to the Trilogarithm by

$$Л_3(\omega) = \frac{1}{4}\,\mathrm{Re}(\mathrm{Li}_3(e^{2i\omega})) \quad , \quad \omega \in \mathbf{R} \quad .$$

The proof of formula (4) is based on Schläfli's theorem about the volume differential (see **3.1**) and the results of Lobachevsky in dimension three (see **2.2**).

Together with some dissection properties for regular crosspolytopes (see **1.5**), equation (4) enables us to compute, among other things, the volumes of the three Coxeter orthoschemes (i.e., all dihedral angles are submultiples of π) of dimension five (cf. **3.2**). It turns out that the corresponding reflection groups have commensurable covolumes being rational multiples of $\zeta(3)$. Hence, by passing over to torsionfree subgroups, we obtain examples of hyperbolic cusped 5-manifolds whose volumes are rational multiples of $\zeta(3)$. This result gives a first glimpse into the structure of the volume spectrum for hyperbolic 5-space forms which, by a theorem of Wang [W], forms a discrete subset of $\mathbf{R}_+$.

1. Orthoschemes in hyperbolic space

1.1 Let X denote either the n-dimensional euclidean space E^n, the n-sphere S^n or the n-dimensional hyperbolic space H^n. Embed S^n in E^{n+1}, and use for H^n the model in Lorentz space $E^{n,1}$, i.e.: If $E^{n,1}$ denotes the $(n+1)$-dimensional real vector space $\mathbf{R}^{n+1}$, together with the bilinear form

$$\langle x, y\rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \quad , \quad x\,,\, y \in \mathbf{R}^{n+1} \quad ,$$

of signature $(n, 1)$, then H^n can be interpreted as

$$H^n = \{\, x \in E^{n,1} \mid \langle x, x\rangle = -1, x_0 > 0 \,\}.$$

In the projective model, H^n is the interior of the real projective space P^n with respect to the quadric

$$Q_{n,1} := \{\, [x] \in P^n \mid \langle x, x\rangle = 0 \,\}\,.$$

The closure $\overline{H^n}$ of H^n in P^n represents the natural compactification of H^n. Points of the boundary $\partial H^n = \overline{H^n} - H^n$ are called ***points at infinity*** of H^n.

1.2 An n-***orthoscheme*** in X is a simplex in X whose vertices $P_0, \ldots, P_n$ are labelled in such a way that

$$\operatorname{span}(P_0, \ldots, P_k) \perp \operatorname{span}(P_k, \ldots, P_n) \tag{5}$$

for $1 \leq k \leq n-1$. The initial and final vertices P_0, P_n of the orthogonal edge-path $P_0P_1, \ldots, P_{n-1}P_n$ are called ***principal*** vertices and play a distinguished role. E.g. in $\overline{H^n}$, only the two principal vertices may be points at infinity in which cases the orthoscheme is called 1- or 2-***asymptotic***. Moreover, an orthoscheme has at most n non-right dihedral angles (hyperbolic orthoschemes have exactly n non-right dihedral angles $\alpha_1, \ldots, \alpha_n$ all of them being acute, i.e., $\alpha_i < \frac{\pi}{2}$).

Since orthoschemes are characterized by many orthogonality conditions, they are most conveniently described by means of weighted graphs or schemes. First, we observe that an n-orthoscheme R is a simplex bounded by hyperplanes $H_0, \ldots, H_n$ such that

$$H_i \perp H_j \qquad \text{for} \quad 2 \leq |i-j| \leq n \quad , \tag{6}$$

where H_i denotes the bounding hyperplane of R opposite to P_i. Every hyperplane H_i, $0 \leq i \leq n$, can be described by a unit normal vector e_i in the ambient space directed outwards with respect to R, say, i.e.:

$$H_i = e_i^{\perp} := \{\, x \in H^n \;\mid\; \langle x, e_i\rangle = 0 \,\} \quad \text{with} \quad \langle e_i, e_i\rangle = 1 \quad .$$

Then, the scheme $\Sigma(R)$ of R is the linear weighted graph (describing R up to congruence) whose nodes i correspond to the hyperplanes $H_i = e_i^{\perp}$ of R. The weights between adjacent nodes $i-1, i$ equal α_i, where $\cos\alpha_i = -\langle e_{i-1}, e_i\rangle_X$, while non-adjacent nodes, associated to orthogonal hyperplanes, are not joined:

$$\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{——}}\circ\text{—} \quad \cdots \quad \text{—}\circ\overset{\alpha_n}{\text{——}}\circ \quad .$$

Frequently, we shall think of orthoschemes in terms of their associated graphs.

Rank, determinant and character of definiteness of $\Sigma(R)$ are defined to be the corresponding ones of the Gram matrix $G(R) = (\langle e_i, e_j\rangle_X)_{0\le i,j\le n}$. In particular, $\Sigma(R)$ is said to be either ***elliptic***, ***parabolic***, or ***hyperbolic*** if the n-orthoscheme R is either spherical, euclidean, or hyperbolic, which is equivalent to $\Sigma(R)$ being either positive definite, positive semidefinite of rank n, or of signature $(n,1)$ (cf. [K, §14.1.2]). Every vertex P_i, $0 \le i \le n$, of $R \subset X$ is described by an $(n-1)$-dimensional vertex orthoscheme r_i formed by the vectors e_k, $0 \le k \le n$, $k \ne i$. $\Sigma(r_i)$ is obtained from $\Sigma(R)$ by discarding the node i and the edges emanating from it. If $P_i \in H^n$ is an ordinary vertex of R, then $\Sigma(r_i)$ is elliptic. If $P_i \in \partial H^n$ is a vertex at infinity of R implying that $i = 0$ or n, then $\Sigma(r_i)$ is connected and parabolic.

1.3 For the graphs of orthoschemes whose dihedral angles are commensurable with π, we use the standard notations: If two nodes are related by the weight $\frac{p\pi}{q}$, $p, q \in \mathbf{N}$ coprime with $1 \le p < q$, then they are joined by a $(q-2)$-fold line for $p = 1$ and $q = 3, 4$, and by a single line marked $\frac{q}{p}$, otherwise. From now on, let $X = \overline{H^n}$. Hyperbolic Coxeter orthoschemes ($p = 1$, i.e., all dihedral angles are submultiples of π) were classified by Coxeter (cf. [C1]). His list ends for $n = 5$ with the three examples

$$\begin{aligned} \sigma_1 \quad &: \quad \circ\text{——}\circ\text{——}\circ\text{——}\circ\text{══}\circ\text{——}\circ \\ \sigma_2 \quad &: \quad \circ\text{——}\circ\text{——}\circ\text{══}\circ\text{——}\circ\text{——}\circ \\ \sigma_3 \quad &: \quad \circ\text{——}\circ\text{══}\circ\text{——}\circ\text{——}\circ\text{══}\circ \quad . \end{aligned} \tag{7}$$

Coxeter orthoschemes are characteristic simplexes for regular honeycombs. Orthoschemes whose dihedral angles are commensurable with π are related to characteristic simplexes for regular star-honeycombs (cells and vertex figures are regular star-polytopes); in case of finite density (covering the space a finite number of times), they were completely enumerated by Coxeter (cf. [C1, p.161 ff]) and exist only up to $n = 4$. If one allows infinite density, then one finds among the regular star-honeycombs whose characteristic simplexes are 2-asymptotic exactly five examples, and their characteristic simplexes are given by the schemes

$$
\begin{array}{lcl}
\mu_1 & : & \circ\text{——}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\text{——}\circ\overset{5}{\text{——}}\circ \\
\mu_2 & : & \circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\text{——}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ \\
\mu_3 & : & \circ\overset{\frac{5}{2}}{\text{——}}\circ\text{——}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\text{——}\circ \\
\nu_1 & : & \circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ \\
\nu_2 & : & \circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ\overset{5}{\text{——}}\circ\overset{\frac{5}{2}}{\text{——}}\circ \quad .
\end{array}
\tag{8}
$$

That these schemes are the only 2-asymptotic ones, is easily seen using the list 14.14 in Coxeter's classification of regular star-honeycombs of finite densities (see [C2, §14]).

1.4 Let $\mathcal{P}(X)$ denote the n-th scissors congruence group of polytopes in X (see [Sa, §1]). Then, for $n \geq 2$, $\mathcal{P}(H^n)$ is isomorphic to $\mathcal{P}(\overline{H^n})$ (see [DS, Theorem 2.1, p.162]), and, for $d \geq 3$ odd, $\mathcal{P}(\overline{H^d})$ is generated by the classes of 2-asymptotic orthoschemes (see [Sa, Remark 3.10 and p.199]). This latter property was reproved by Debrunner [D, p.125] using a certain dissection of a d-orthoscheme into $d+1$ orthoschemes ($d \geq 2$ arbitrary). This dissection process will be helpful later (cf. **1.5**, **3.2**).

1.5 Consider a five-dimensional 2-asymptotic orthoscheme $R = P_0 \cdots P_5$ with vertices $P_0, \ldots, P_5$ and with graph

$$
\Sigma(R) \quad : \quad \circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ \quad .
$$

It is characterized by three independent dihedral angles $\alpha_2, \alpha_3, \alpha_4$, say, while α_1, α_5 are given by the relations

$$
\det(\circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ) = \det(\circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ) = 0 \quad . \tag{9}
$$

An angle α_i ($1 \leq i \leq 5$) is formed by the facet orthoschemes $R_{i-1} = H_{i-1} \cap R = P_0 \cdots \widehat{P_{i-1}} \cdots P_5$ and $R_i = H_i \cap R = P_0 \cdots \widehat{P_i} \cdots P_5$; it is attached to the apex orthoscheme $F_i = R_{i-1} \cap R_i = P_0 \cdots \widehat{P_{i-1} P_i} \cdots P_5$, and, by the orthogonality conditions (5), can be seen as planar or spatial angle (cf. Figure 1).

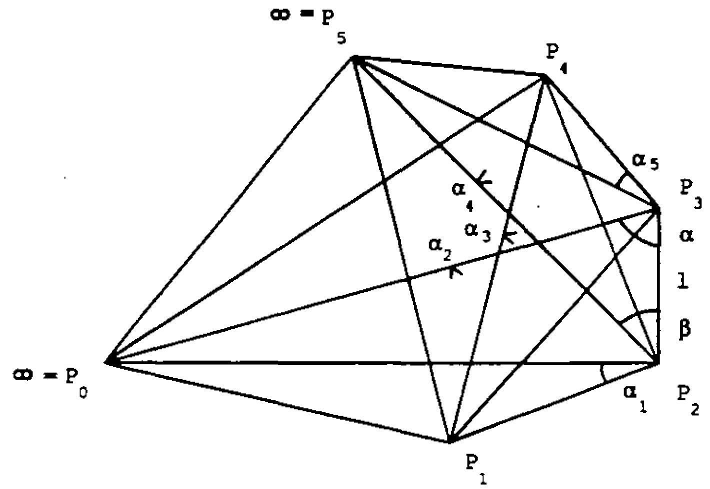

Fig. 1

For the subsequent volume investigations, we are interested in the graphs $\Sigma(F_i)$ of F_i $(1 \le i \le 5)$. First, we observe that F_1, F_5 resp. F_2, F_3, F_4 are 1- resp. 2-asymptotic. Moreover, it is easy to see (cf. Figure 1) that

$$\begin{array}{ll} \Sigma(F_1)\ :\ \circ\overset{\frac{\pi}{2}-\alpha_4}{\text{———}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ & \Sigma(F_4)\ :\ \circ\overset{\alpha_1}{\text{——}}\circ\overset{\frac{\pi}{2}-\alpha_1}{\text{———}}\circ\overset{\alpha_1}{\text{——}}\circ \\ \Sigma(F_2)\ :\ \circ\overset{\alpha_5}{\text{——}}\circ\overset{\frac{\pi}{2}-\alpha_5}{\text{———}}\circ\overset{\alpha_5}{\text{——}}\circ & \Sigma(F_5)\ :\ \circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ\overset{\frac{\pi}{2}-\alpha_2}{\text{———}}\circ \end{array} \quad . \tag{10}$$

To determine the scheme $\Sigma(F_3)$, we define first the following auxiliary angle:

Definition. *Let $\alpha_6 \in (0, \frac{\pi}{2})$ be such that the graph*

$$\Sigma\ :\ \circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ\overset{\alpha_6}{\text{——}}\circ$$

is the graph $\Sigma(Q)$ of a 2-asymptotic orthoscheme $Q \subset \overline{H^5}$, i.e.,

$$\det(\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ\overset{\alpha_6}{\text{——}}\circ) = 0 \quad . \tag{11}$$

Moreover, we need the following

LEMMA.

Let $\Sigma(R)\ :\ \circ\overset{\alpha_1}{\text{——}}\circ\overset{\alpha_2}{\text{——}}\circ\overset{\alpha_3}{\text{——}}\circ\overset{\alpha_4}{\text{——}}\circ\overset{\alpha_5}{\text{——}}\circ$ denote the graph of a 2-asymptotic hyperbolic 5-orthoscheme R. Then,

$$\tan\alpha_1 \tan\alpha_2 = \tan\alpha_4 \tan\alpha_5 \quad . \tag{12}$$

Proof: Denote by $P_0, \ldots, P_5$ the vertices of R satisfying (5). Consider the 1-asymptotic face orthoscheme $P_0P_1P_2P_3$ of dimension three and its spherical vertex orthoscheme at P_3 with angles α_1, α_2 (cf. Figure 1) whose hypotenuse of length α satisfies

$$\cos\alpha = \cot\alpha_1 \cot\alpha_2 \quad .$$

But α is also the parallel angle in the orthoscheme $P_0P_2P_3$, i.e.,

$$\cos\alpha = \tanh l \quad ,$$

where l denotes the length of the edge P_2P_3. On the other hand, this edge belongs to the 1-asymptotic 3-orthoscheme $P_2P_3P_4P_5$ whose spherical vertex orthoscheme at P_3 has angles α_4, α_5. If β denotes the parallel angle in $P_2P_3P_5$, we deduce that

$$\cos\beta = \cot\alpha_4 \cot\alpha_5 = \tanh l \quad .$$

Hence, $\tanh l = \cot\alpha_1 \cot\alpha_2 = \cot\alpha_4 \cot\alpha_5$.

Q.E.D.

Now, the apex orthoscheme F_3 associated to α_3 is given by

$$\Sigma(F_3) \quad : \quad \circ\overset{\frac{\pi}{2}-\alpha_6}{\text{———}}\circ\overset{\alpha_6}{\text{——}}\circ\overset{\frac{\pi}{2}-\alpha_6}{\text{———}}\circ \quad . \tag{13}$$

This follows from (11) written in the form

$$\cot^2\alpha_6 = \frac{\sin^2\alpha_3 \sin^2\alpha_5 - \cos^2\alpha_4}{\cos^2\alpha_3 \cos^2\alpha_5} \cot^2\alpha_3 \quad ,$$

which satisfies Böhm's general formula (4.4) relating apex angles to angles of R (see [B, p.303-304]). It can also be seen by the following dissection comparing R with Q (see Definition above): Denote by $Q_0, \ldots, Q_5$ the vertices of Q and assume that Q_5 coincides with P_0 and that Q_0 is the point at infinity on the ray from $P_0 = Q_5$ through P_1. If H denotes the four-dimensional plane through Q_0 orthogonal to the segment P_0P_5, we have (cf. Figure 2)

$$Q_{i-1} = H \cap P_0P_i \quad , \quad i = 1, \ldots, 5 \quad .$$

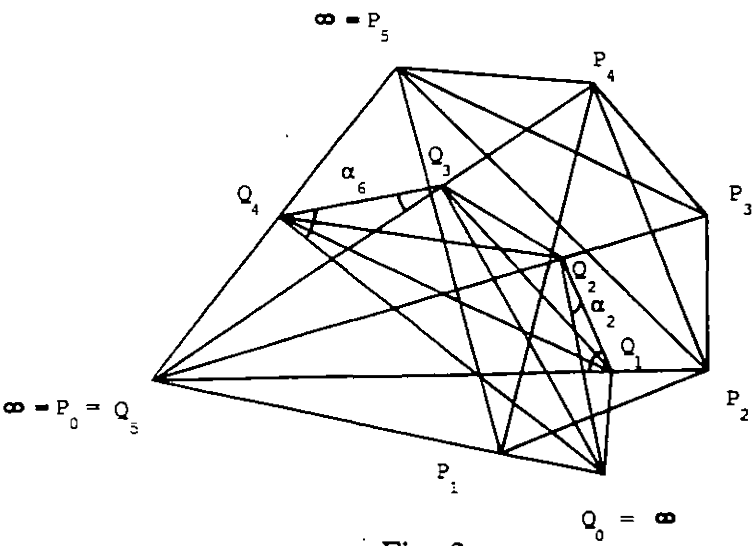

Fig. 2

Note that R, Q have identical (euclidean) vertex orthoschemes $r_0 = q_5$ at $P_0 = Q_5$. If we form the simplexes

$$R_k := Q_0 \cdots Q_{k-1} P_k \cdots P_5 \quad , \quad k = 1, \ldots, 5 \quad , \tag{14}$$

in $\overline{H^5}$, then, by a result of Debrunner (see [D, Theorem (2.6) (i)]), we see that

(a) R_k is a 2-asymptotic orthoscheme;

(b) On the scissors congruence level, there is the relation $[R] + [R_1] = [Q] + \sum_{i=2}^{5} [R_i]$.

Moreover, by the above Lemma (see also Figure 2), one can deduce that

$$\Sigma(R) = \Sigma(R_1) \quad , \quad \Sigma(Q) = \Sigma(R_5) \quad , \tag{15}$$

i.e., R, R_1 and Q, R_5 are congruent, and that

$$\begin{array}{lll} \Sigma(R_2) & : & \circ \overset{2\alpha_1}{\text{———}} \circ \overset{x_1}{\text{———}} \circ \overset{y_1}{\text{———}} \circ \overset{\alpha_4}{\text{———}} \circ \overset{\alpha_5}{\text{———}} \circ \\ \Sigma(R_3) & : & \circ \overset{\alpha_2}{\text{———}} \circ \overset{x_2}{\text{———}} \circ \overset{y_2}{\text{———}} \circ \overset{z_1}{\text{———}} \circ \overset{\alpha_5}{\text{———}} \circ \\ \Sigma(R_4) & : & \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \overset{y_3}{\text{———}} \circ \overset{z_2}{\text{———}} \circ \overset{\pi - 2\alpha_6}{\text{—————}} \circ \quad . \end{array} \tag{16}$$

Here $x_i, y_i, z_i \in (0, \frac{\pi}{2})$ satisfy

$$\begin{aligned} \tan x_1 &= \cot(2\alpha_1) \tan\alpha_4 \tan\alpha_5 \quad , \quad x_1 + x_2 = \alpha_2 \quad , \\ \tan z_1 &= \cot\alpha_5 \tan\alpha_2 \tan x_2 \quad , \\ \tan z_2 &= \cot(\pi - 2\alpha_6) \tan\alpha_2 \tan\alpha_3 \quad , \end{aligned} \tag{17}$$

and y_i are such that the parabolicity conditions (cf. (9)) are satisfied. Hence,

$$2[R] = 2[Q] + [R_2] + [R_3] + [R_4] \quad . \tag{18}$$

Finally, one reads off Figure 2 that the scheme $\Sigma(F_3)$ is given by (13), since the angle at Q_4 between Q_0, Q_3 equals α_6, and because the plane through these three points is orthogonal to P_0P_5 in F_3.

1.6 Among the set of 2-asymptotic orthoschemes in $\overline{H^5}$, there is a particular family of orthoschemes R given by graphs

$$\Sigma(R) \quad : \quad \circ \overset{\alpha}{\text{——}} \circ \overset{\beta}{\text{——}} \circ \overset{\gamma}{\text{——}} \circ \overset{\alpha}{\text{——}} \circ \overset{\beta}{\text{——}} \circ \quad \text{with} \tag{19}$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad . \tag{20}$$

Condition (20) guarantees that R is 2-asymptotic, and it implies that the auxiliary angle α_6 (see Definition **1.5**) satisfies $\alpha_6 = \gamma$.
By a result of Gordan (cf. [C2, p.109]), the only solutions (α, β, γ) of (20) with ingredients commensurable with π are – up to permutations – $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{4})$ and $(\frac{\pi}{3}, \frac{\pi}{5}, \frac{2\pi}{5})$ yielding five different orthoscheme realizations in $\overline{H^5}$, namely, σ_2, σ_3 and μ_1, μ_2, μ_3 (see (7) and (8)).
The connected subschemes of (19) of order four were studied by Schläfli and Coxeter (cf. [S, p.281 ff] and [C2, §6.7]); they occur as characteristic simplexes of the three-dimensional spherical regular honeycombs and regular star-honeycombs of finite density.

2. Polylogarithms and higher Lobachevsky functions

2.1 Let $z \in \mathbf{C}$, $|z| \leq 1$. Then,

$$\mathrm{Li}_n(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^n} \quad , \quad n = 1, 2, \ldots , \tag{21}$$

denotes the Polylogarithm function with the properties (cf. [L, §7.1 and 7.3]), for $n \geq 2$,

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(t)}{t}\, dt \quad , \tag{22}$$

$\mathrm{Li}_n(1) = \zeta(n)$, Riemann's zeta function, and

$$\frac{1}{k^{n-1}}\mathrm{Li}_n(z^k) = \mathrm{Li}_n(z) + \mathrm{Li}_n(\omega z) + \cdots + \mathrm{Li}_n(\omega^{k-1} z) \quad \text{for} \quad \omega = e^{2\pi i/k},\ k \geq 1 \quad . \tag{23}$$

2.2 The Dilogarithm $\mathrm{Li}_2(z)$ at arguments $z = e^{2i\theta}$, θ real, leads to the Lobachevsky function

$$Л(\theta) = \frac{1}{2}\,\mathrm{Im}(\mathrm{Li}_2(e^{2i\theta})) = -\int_0^\theta \log|2\sin t|\, dt \quad , \tag{24}$$

which is known to represent volumes of polyhedra in hyperbolic 3-space: If R denotes a hyperbolic 3-orthoscheme with graph

$$\Sigma(R) \quad : \quad \circ \overset{\alpha_1}{\text{———}} \circ \overset{\alpha_2}{\text{———}} \circ \overset{\alpha_3}{\text{———}} \circ \quad ,$$

then, Lobachevsky showed that (cf. [K, Introduction and Theorem 14.5])

$$\begin{aligned}\mathrm{vol}_3(R) = \frac{1}{4}\{ & Л(\alpha_1 + \theta) - Л(\alpha_1 - \theta) + Л(\frac{\pi}{2} + \alpha_2 - \theta) + Л(\frac{\pi}{2} - \alpha_2 - \theta) + \\ & + Л(\alpha_3 + \theta) - Л(\alpha_3 - \theta) + 2Л(\frac{\pi}{2} - \theta)\} \quad , \quad \text{where}\end{aligned} \tag{25}$$

$$0 \le \theta := \arctan\left(\frac{\cos^2\alpha_2 - \sin^2\alpha_1\sin^2\alpha_3}{\cos^2\alpha_1\cos^2\alpha_3}\right)^{\frac{1}{2}} \le \frac{\pi}{2} \quad .$$

The Lobachevsky function is closely related to the Clausen function (see [L, §4])

$$\mathrm{Cl}_2(\theta) := \sum_{r=1}^{\infty} \frac{\sin(r\theta)}{r^2} = -\int_0^\theta \log|\sin\frac{t}{2}|\,dt$$

according to

$$\text{Л}(\theta) = \frac{1}{2}\,\mathrm{Cl}_2(2\theta) \quad , \quad \forall\theta \in \mathbf{R} \quad .$$

Analogous to the case of higher Clausen functions $\mathrm{Cl}_n(\theta)$ (see [L, §7.1.4]), we define higher Lobachevsky functions as follows:

Definition. *For* $m \ge 1$, $\theta \in \mathbf{R}$, *the higher Lobachevsky functions are defined by*

$$\begin{aligned} \text{Л}_{2m}(\theta) &= \frac{1}{2^{2m-1}}\,\mathrm{Im}(\mathrm{Li}_{2m}(e^{2i\theta})) = \frac{1}{2^{2m-1}}\sum_{r=1}^{\infty}\frac{\sin(2r\theta)}{r^{2m}} \quad , \\ \text{Л}_{2m+1}(\theta) &= \frac{1}{2^{2m}}\,\mathrm{Re}(\mathrm{Li}_{2m+1}(e^{2i\theta})) = \frac{1}{2^{2m}}\sum_{r=1}^{\infty}\frac{\cos(2r\theta)}{r^{2m+1}} \quad . \end{aligned} \tag{26}$$

It follows that

$$\text{Л}_{2m}(\theta) = \int_0^\theta \text{Л}_{2m-1}(t)\,dt \quad , \quad \text{Л}_{2m+1}(\theta) = \frac{1}{2^{2m}}\zeta(2m+1) - \int_0^\theta \text{Л}_{2m}(t)\,dt \quad . \tag{27}$$

Moreover, $\text{Л}_m(\theta)$ is π-periodic, even for m odd, and odd for m even, respectively. By means of (23) (see also [L, (7.46)]), one deduces the following distribution law

$$\frac{1}{k^{m-1}}\text{Л}_m(k\theta) = \sum_{r=0}^{k-1}\text{Л}_m(\theta + \frac{r\pi}{k}) \tag{28}$$

and, as a particular case, the duplication formula

$$\frac{1}{2^{m-1}}\text{Л}_m(2\theta) = \text{Л}_m(\theta) + \text{Л}_m(\theta + \frac{\pi}{2}) \quad . \tag{29}$$

2.3 In connection with volumes of five-dimensional hyperbolic polytopes, we are mainly interested in the Lobachevsky function of order three. By the above, we obtain the following results for $\text{Л}_3(\theta)$:

$$\text{Л}_3(0) = \frac{1}{4}\zeta(3) \quad , \quad \text{Л}_3(\frac{\pi}{4}) = -\frac{3}{128}\zeta(3) \quad , \quad \text{Л}_3(\frac{\pi}{2}) = -\frac{3}{16}\zeta(3) \quad ; \tag{30}$$

$$Л_3(\frac{\pi}{6}) = \frac{1}{12}\zeta(3) \quad , \quad Л_3(\frac{\pi}{3}) = -\frac{1}{9}\zeta(3) \quad ; \tag{31}$$

$$Л_3(\frac{\pi}{5}) + Л_3(\frac{2\pi}{5}) = -\frac{3}{25}\zeta(3) \quad . \tag{32}$$

3. The volume formula and applications

3.1 In order to derive volume formulae for orthoschemes in terms of their angles, we make use of the hyperbolic analog of Schläfli's volume differential representation: For a family of orthoschemes R in H^n $(n \geq 2)$ with dihedral angles α_i attached to the apices F_i $(1 \leq i \leq n)$, the volume differential $d\mathrm{vol}_n(R)$ can be represented by

$$d\mathrm{vol}_n(R) = \frac{1}{1-n} \sum_{r=1}^{n} \mathrm{vol}_{n-2}(F_r)\, d\alpha_r \quad , \quad \mathrm{vol}_0(F_r) := 1 \quad . \tag{33}$$

Schläfli proved the spherical version of this formula for arbitrary simplexes. For a proof of both, the spherical and hyperbolic case, we refer to Kneser [Kn]. Plainly, formula (33) is still valid for a family of orthoschemes in $\overline{H^n}$, $n \neq 3$, with one or two of the principal vertices at infinity. With these preliminaries, we are ready to prove the following

THEOREM.

Let R denote the 2-asymptotic 5-orthoscheme given by

$$\Sigma(R) \quad : \quad \circ \overset{\alpha}{\text{—}} \circ \overset{\beta}{\text{—}} \circ \overset{\gamma}{\text{—}} \circ \overset{\alpha}{\text{—}} \circ \overset{\beta}{\text{—}} \circ \quad \textit{with} \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \quad .$$

Then,

$$\begin{aligned} \mathrm{vol}_5(R(\alpha,\beta,\gamma)) =& \frac{1}{4}\{Л_3(\alpha) + Л_3(\beta) - \frac{1}{2}Л_3(\frac{\pi}{2} - \gamma)\} - \\ & -\frac{1}{16}\{Л_3(\frac{\pi}{2} + \alpha + \beta) + Л_3(\frac{\pi}{2} - \alpha + \beta)\} + \frac{3}{64}\zeta(3) \quad , \end{aligned} \tag{34}$$

where $Л_3(\omega)$, $\omega \in$ R, *denotes the Lobachevsky function of order three.*

Proof: We use Schläfli's volume differential (33) for a family of 2-asymptotic orthoschemes R given by graphs

$$\Sigma(R) \ : \ \circ \overset{\alpha_1}{\text{——}} \circ \overset{\alpha_2}{\text{——}} \circ \overset{\alpha_3}{\text{——}} \circ \overset{\alpha_4}{\text{——}} \circ \overset{\alpha_5}{\text{——}} \circ \quad \text{with} \quad \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3 = 1\,. \tag{35}$$

Then, by the asymptoticity conditions, $\alpha_1 = \alpha_4 =: \alpha$, $\alpha_2 = \alpha_5 =: \beta$. Moreover, we see that $\alpha_3 = \alpha_6 =: \gamma$. Now, assume that β is constant and that α is the independent variable, i.e., $\gamma = \gamma(\alpha)$. In order to determine the coefficients of $d\alpha_1 = d\alpha_4 = d\alpha$, $d\alpha_3 = d\gamma$ in (33), we observe that the corresponding apex orthoschemes F_1, F_3, F_4 are characterized by the graphs (see (10),(13))

$$
\begin{aligned}
\Sigma(F_1) \quad &: \quad \circ \overset{\frac{\pi}{2}-\alpha}{\rule[0.5ex]{3em}{0.4pt}} \circ \overset{\alpha}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \overset{\beta}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \\
\Sigma(F_3) \quad &: \quad \circ \overset{\frac{\pi}{2}-\gamma}{\rule[0.5ex]{3em}{0.4pt}} \circ \overset{\gamma}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \overset{\frac{\pi}{2}-\gamma}{\rule[0.5ex]{3em}{0.4pt}} \circ \qquad \circ \\
\Sigma(F_4) \quad &: \quad \circ \overset{\alpha}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \overset{\frac{\pi}{2}-\alpha}{\rule[0.5ex]{3em}{0.4pt}} \circ \overset{\alpha}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \quad .
\end{aligned}
$$

Therefore, by Lobachevsky's formula (see **2.2**, (25)), their volumes are given by

$$
\begin{aligned}
\mathrm{vol}_3(F_1) &= \frac{1}{2}\text{Л}(\alpha) + \frac{1}{4}\{\text{Л}(\frac{\pi}{2} - \alpha + \beta) - \text{Л}(\frac{\pi}{2} + \alpha + \beta)\} \quad , \\
\mathrm{vol}_3(F_3) &= \frac{1}{2}\text{Л}(\frac{\pi}{2} - \gamma) \quad , \\
\mathrm{vol}_3(F_4) &= \frac{1}{2}\text{Л}(\alpha) \quad .
\end{aligned}
$$

Hence, Schläfli's formula (33) yields

$$
\begin{aligned}
(-4)\, d\mathrm{vol}_5(R) = \text{Л}(\alpha)\, d\alpha &+ \frac{1}{4}\{\text{Л}(\frac{\pi}{2} - \alpha + \beta) - \text{Л}(\frac{\pi}{2} + \alpha + \beta)\}\, d\alpha + \\
&+ \frac{1}{2}\,\text{Л}(\frac{\pi}{2} - \gamma)\, d\gamma \quad ,
\end{aligned}
$$

where $\gamma = \gamma(\alpha)$. Since $\text{Л}_3(\omega) = \frac{1}{4}\zeta(3) - \int\limits_0^\omega \text{Л}(t)\, dt$ is an even function, and since a volume formula for $\Sigma(R)$ has to be symmetric in α, β, we obtain the following expression

$$
\begin{aligned}
4\mathrm{vol}_5(R) = \text{Л}_3(\alpha) + \text{Л}_3(\beta) &- \frac{1}{2}\text{Л}_3(\frac{\pi}{2} - \gamma) - \\
&- \frac{1}{4}\{\text{Л}_3(\frac{\pi}{2} + \alpha + \beta) + \text{Л}_3(\frac{\pi}{2} - \alpha + \beta)\} + c \quad .
\end{aligned} \tag{36}
$$

Here, c denotes the constant of integration which can be computed by evaluating (36) in the degenerate case of an orthoscheme R_{deg} in $\overline{H^5}$ satisfying (35) such that $\mathrm{vol}_5(R_{deg}) = 0$. For this, we consider the following class of orthoschemes $R_{\varepsilon,\varepsilon'} \subset \overline{H^5}$ given by

$$
\Sigma_{\varepsilon,\varepsilon'} \quad : \quad \circ \overset{\varepsilon'}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \overset{\frac{\pi}{2}-\varepsilon}{\rule[0.5ex]{3em}{0.4pt}} \circ \overset{\frac{\pi}{2}-\varepsilon}{\rule[0.5ex]{3em}{0.4pt}} \circ \overset{\varepsilon'}{\rule[0.5ex]{1.5em}{0.4pt}} \circ \overset{\frac{\pi}{2}-\varepsilon}{\rule[0.5ex]{3em}{0.4pt}} \circ
$$

with $0 < \varepsilon < \frac{\pi}{6}$, $\varepsilon < \varepsilon' < \frac{\pi}{2}$ and $\sin^2 \varepsilon' = 2\sin^2 \varepsilon$. Then, property (35) is satisfied, and $R_{\varepsilon,\varepsilon'}$ is 2-asymptotic. Since, for $\varepsilon \longrightarrow 0$, $\varepsilon'(\varepsilon) \longrightarrow 0$ and $\det(\Sigma_{\varepsilon,\varepsilon'}) = -\sin^2 \varepsilon \longrightarrow$

0, $R_{e,e'}$ converges to an orthoscheme R_{deg} with $\mathrm{vol}_5(R_{deg}) = 0$. This implies that $c = -\frac{3}{16}\,\zeta(3)$ which finishes the proof.

Q.E.D.

3.2 The above Theorem, combined with certain dissection properties of orthoschemes (cf. **1.4**, **1.5**), enables us to compute explicitly the volumes of the three Coxeter orthoschemes (7) as well as the volumes of the characteristic simplexes (8) of all regular star-honeycombs (being necessarily of infinite density) in $\overline{H^5}$.

The two Coxeter orthoschemes of (7),

$$\sigma_2 \quad : \quad \circ - \circ - \circ = \circ - \circ - \circ \quad ,$$

$$\sigma_3 \quad : \quad \circ - \circ = \circ - \circ - \circ = \circ \quad ,$$

satisfy the conditions of the Theorem. Using **2.3**, we get for their volumes $\mathrm{vol}_5(\sigma_i)$, $i = 2, 3$:

$$\mathrm{vol}_5(\sigma_2) = \frac{7}{9216}\,\zeta(3) \simeq 0.000913 \quad , \quad \mathrm{vol}_5(\sigma_3) = \frac{7}{4608}\,\zeta(3) \simeq 0.001826 \quad . \tag{37}$$

Before we compute the volume $\mathrm{vol}_5(\sigma_1)$ of the remaining Coxeter orthoscheme

$$\sigma_1 \quad : \quad \circ - \circ - \circ - \circ = \circ - \circ \quad ,$$

which is 1-asymptotic, we make the following remark.

Remark. Let $\alpha_n := \arccos \frac{1}{\sqrt{n}} \in [0, \frac{\pi}{2}]$, $n \geq 3$, and consider the schemes

$$\rho_i^n(\alpha) \;:\; \circ - \circ - \cdots - \circ - \circ = \circ \overset{\alpha}{-} \underset{i}{\circ} \overset{\alpha}{-} \circ = \circ - \circ - \cdots - \circ - \circ \,,$$

of order $n+1$, $i \in [0, n]$, which describe either spherical, euclidean or compact hyperbolic n-orthoschemes if either $\alpha_n < \alpha < \pi - \alpha_n$, $\alpha = \alpha_n$, or $\alpha_{n-1} < \alpha < \alpha_n$ (see [D, (7.9)]). In the spherical case, Schläfli (cf. [S, p.270]) derived the following volume relations

$$\mathrm{vol}_n(\rho_i^n(\alpha)) = \binom{n}{i} \mathrm{vol}_n(\rho_0^n(\alpha)) \quad , \quad i \in [0, n] \quad , \tag{38}$$

which were generalized by Debrunner (cf. [D, Theorem (7.8)]) to all three cases using a dissection argument: The orthoschemes $\rho_i^n(\alpha)$ tile the regular cross-polytope with dihedral angle 2α.

But by continuity, we see that (38) holds even in the hyperbolic (asymptotic) limiting case $\alpha = \alpha_{n-1}$; in particular, for $n = 5$ (i.e., $\alpha = \frac{\pi}{3}$), where $\rho_0^5(\frac{\pi}{3}) = \sigma_1$, $\rho_1^5(\frac{\pi}{3}) = \sigma_2$ and $\rho_2^5(\frac{\pi}{3}) = \sigma_3$, we obtain the relations (see (37))

$$\mathrm{vol}_5(\sigma_1) = \frac{1}{5}\mathrm{vol}_5(\sigma_2) = \frac{1}{10}\mathrm{vol}_5(\sigma_3) = \frac{7}{46080}\zeta(3) \simeq 0.000183 \quad . \tag{39}$$

Hence, the volumes of the three Coxeter orthoschemes in $\overline{H^5}$ are rational multiples of $\zeta(3)$ and therefore commensurable. Considering the associated reflection groups and passing over to torsionfree subgroups, which, by a result of Borel (see [Bo, Theorem B (ii), p.345]), always exist, we obtain 1- and 2-cusped hyperbolic manifolds of dimension five whose volumes are rational multiples of $\zeta(3)$.

Finally, consider the orthoschemes presented in (8),

$$\begin{array}{lcl}
\mu_1 & : & \circ\text{———}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ \\
\mu_2 & : & \circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ \\
\mu_3 & : & \circ\overset{\frac{5}{2}}{\text{———}}\circ\text{———}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\text{———}\circ \\
\nu_1 & : & \circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\overset{5}{\text{———}}\circ \\
\nu_2 & : & \circ\overset{\frac{5}{2}}{\text{———}}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ\overset{5}{\text{———}}\circ\overset{\frac{5}{2}}{\text{———}}\circ \quad .
\end{array}$$

Since $\cos^2\frac{\pi}{3} + \cos^2\frac{\pi}{5} + \cos^2\frac{2\pi}{5} = 1$, we can use our Theorem to calculate the volumes of the first three schemes making use of **2.3**:

$$\begin{aligned}
\mathrm{vol}_5(\mu_1) &= \frac{\zeta(3)}{1200} \simeq 0.001002 \quad ; \\
\mathrm{vol}_5(\mu_2) &= \frac{1}{144}\left\{ Л_3(\frac{2\pi}{5}) + \frac{\zeta(3)}{5}\right\} = \frac{1}{144}\left\{ - Л_3(\frac{\pi}{5}) + \frac{2\zeta(3)}{25}\right\} \simeq 0.000339 \quad ; \\
\mathrm{vol}_5(\mu_3) &= \frac{1}{144}\left\{ Л_3(\frac{\pi}{5}) + \frac{\zeta(3)}{5}\right\} \simeq 0.001998 \quad .
\end{aligned} \tag{40}$$

For the computation of the values $\mathrm{vol}_5(\nu_1), \mathrm{vol}_5(\nu_2)$, we use the orthoscheme dissection presented in **1.5** and the above results. Let R denote the orthoscheme with graph $\Sigma(R) = \nu_1$, which we take as starting simplex with respect to the dissection $2[R] = 2[Q] + [R_2] + [R_3] + [R_4]$ (see Definition, (15) and (18) of **1.5**). Then, Q is the 2-asymptotic orthoscheme given by the graph $\Sigma(Q) = \nu_2$. Using the Lemma, (16) and (17) of **1.5**, we obtain the following identities between the schemes of R_2, R_3, R_4

$$\Sigma(R_2) = \Sigma(R_3) = \Sigma(R_4) = \mu_2 \quad ,$$

which, by (18), imply that

$$\mathrm{vol}_5(\nu_1) = \mathrm{vol}_5(\nu_2) + \frac{3}{2}\mathrm{vol}_5(\mu_2) \quad . \tag{41}$$

Repeating this process by starting with the orthoscheme R given by $\Sigma(R) = \mu_2$, we deduce that $\Sigma(Q) = \mu_3$, and that

$$\Sigma(R_2) = \Sigma(R_3) = \nu_2 \quad , \quad \Sigma(R_4) = \mu_3 \quad .$$

Therefore, we have

$$\mathrm{vol}_5(\mu_2) = \mathrm{vol}_5(\nu_2) + \frac{3}{2}\mathrm{vol}_5(\mu_3) \quad , \tag{42}$$

which together with (40) and (41) yields

$$\begin{aligned} \mathrm{vol}_5(\nu_1) &= \frac{1}{96}\, Л_3(\frac{\pi}{5}) + \frac{\zeta(3)}{800} \simeq 0.001996 \quad ; \\ \mathrm{vol}_5(\nu_2) &= \frac{1}{96}\, Л_3(\frac{\pi}{5}) \simeq 0.000493 \quad . \end{aligned} \tag{43}$$

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