Stratified local moduli of Calabi-Yau 3-folds

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Introduction

By a Calabi-Yau 3-fold X we mean, in this paper, a projective 3-fold with only terminal singularities such that $K_X \sim 0$. A Calabi-Yau 3-fold appears as a minimal model (cf. [Mo, Ka]) of a smooth projective 3-fold with Kodaira dimension 0. Let Def(X) be the Kuranishi space of X (cf. [Do, Gr]). Then by [Na 1, Theorem A] it is a smooth analytic space of $dim = Ext^1(\Omega^1_X, \mathcal{O}_X)$. Moreover, we have proved in [Na 2, Theorem(5.2)] that if X is a **Q**-factorial Calabi-Yau 3-fold, then a general point of Def(X) parametrizes a smooth Calabi-Yau 3-fold, in other words, X is smoothable by a flat deformation. In this paper we shall give a necessary and sufficient condition for a (not necessarily **Q**-factorial)Calabi-Yau 3-fold X to be smoothed and prove a structure theorem of Def(X).

Let V be the germ of a Gorenstein terminal singularity of dim 3. Then V is an isolated cDV point (i.e. its general hyperplane section is a rational double point) by Reid [Re]. Let Def(V) be the Kuranishi space of V and let $\mathcal{V}$ be a semi-universal family over Def(V). Let V_t denote its fiber over $t \in Def(V)$. We here remark that V_t is not a germ of the singularity for $t \neq 0$; it has non-zero 3-rd Betti number in general. Define $\sigma(V_t)$ to be the rank of $Weil(V_t)/Pic(V_t)$. Set $Y_i = \{t \in Def(V); \sigma(V_t) = i\}$. A small partial resolution $\pi : \hat{V} \longrightarrow V$ is, by definition, a proper birational (bimeromorphic) morphism from a normal variety $\hat{V}$ to V such that π is an isomorphism over smooth points of V and that $\pi^{-1}(0)$ is a connected curve. Since V is a rational singularity, the exceptional curve forms a tree of $\mathbf{P}^1$'s. Note that V has only finitely many small partial resolutions $\hat{V}$ and each $\hat{V}$ has only isolated cDV points. Then Def(V) has the following description:

Proposition(1.6)

(1) Let $\hat{V}$ be a small partial resolution of V and $Def(\hat{V})$ the Kuranishi space of $\hat{V}$. Then there is a natural closed immersion of $Def(\hat{V})$ into Def(V) (Wahl).

(2)$Def(V) = \coprod Y_i$, $Y_i = \bar{Y}_i - \bar{Y}_{i+1}$ and $\bar{Y}_i = \bigcup Def(\hat{V})$, where $\hat{V}$ runs through all small partial resolution such that $\rho(\hat{V}) \geq i$.

This proposition has a natural globalization to a Calabi-Yau 3-fold X with only terminal singularities. By definition, a small partial resolution $\pi : \hat{X} \longrightarrow X$ is a proper birational morphism from a normal variety $\hat{X}$ to X such that π is an isomorphism over smooth points of X and that it is a small partial resolution of every singular point of X. When π is a projective morphism, $\hat{X}$ is also a Calabi-Yau 3-fold. Let Def(X) be the Kuranishi space of X and let $\mathcal{X}$ be a semi-universal family over Def(X). We shall define $\sigma(X_t)$ and Y_i in the same way as above. Then one has:

Proposition(2.3)

(1) Let $\hat{X}$ be a small projective partial resolution of X and $Def(\hat{X})$ the Kuranishi space of $\hat{X}$. Then there is a natural closed immersion of $Def(\hat{X})$ into $Def(X)$.

(2) $Def(X) = \coprod Y_i$, $Y_i = \bar{Y}_i - \bar{Y}_{i+1}$ and $\bar{Y}_i = \bigcup Def(\hat{X})$, where $\hat{X}$ runs through all small projective partial resolution such that $\rho(\hat{X}) - \rho(X) \geq i$.

(3) Each stratum Y_i is a (Zariski) locally closed smooth subset of Def(X).

Let $\hat{X}$ be a small projective partial resolution of X. Then $\hat{X}$ is called *maximal* if for any small projective partial resolution $\tilde{X}$ of $\hat{X}$,$Def(\tilde{X})$ is a proper closed subvariety of $Def(\hat{X})$ via the natural inclusion (i.e. $Def(\tilde{X}) \longrightarrow Def(\hat{X})$ is not a surjection). We have the following criterion of the maximality:

Proposition (cf.Theorem(2.5)) Let $\{p_1, ..., p_n\} \subset Sing(\hat{X})$ be the ordinary double points on $\hat{X}$ and let $f : Z \longrightarrow \hat{X}$ be a small (not necessarily projective) partial resolution of $\hat{X}$ such that $C_i := f^{-1}(p_i) \cong \mathbf{P}^1$ and that f is an isomorphism over $\hat{X} - \{p_1, ..., p_n\}$. Then the following conditions are equivalent:

(1) There is a relation in $H_2(Z, \mathbf{C})$: $\sum \alpha_i[C_i] = 0$ with $\alpha_i \neq 0$ for all i.

(2) $\hat{X}$ is maximal.

Our main theorem now can be stated as follows.

Theorem (cf.Theorems(2.5) and (2.7)) Let $\hat{X}$ be a small projective partial resolution (possibily X itself) of X. Then we have:

(1) $\hat{X}$ is smoothable by a flat deformation if and only if $\hat{X}$ is maximal.

(2) If $\hat{X}$ is not maximal, then there is a (not necessarily unique) small projective partial resolution $\tilde{X}$ of $\hat{X}$ such that $\tilde{X}$ is maximal and $Def(\tilde{X}) \cong Def(\hat{X})$.

(3) In the situation of (2), let $\tilde{\mathcal{X}}$ (resp. $\hat{\mathcal{X}}$) denote the universal family over $Def(\tilde{X})$ (resp. $Def(\hat{X})$). Then $\hat{X}_t$ has only ordinary double points for a general point $t \in Def(\hat{X})$ and $\tilde{X}_t$ is a small resolution of it.

Let X be a $\mathbf{Q}$-factorial Calabi-Yau 3-fold. Put $\hat{X} = $ X. Then it is easily checked that

$\hat{X}$ is maximal by the criterion above. Now we can apply the Theorem to the situation and obtain:

Corollary(Na 2, Theorem(5.2)) Any **Q**-factorial Calabi-Yau 3-fold is smoothable by a flat deformation.

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§1. Isolated cDV singularity

Let V be the germ of an isolated cDV singularity. By definition, there is a holomorphic map f of V to a 1-dimensional disc Δ with a sufficiently small radius such that $f^{-1}(0) = S$ is a rational double point and other fibers are smooth. Let $\pi : \tilde{S} \longrightarrow S$ be the minimal resolution of S. We shall denote by $\mathcal{Y} \longrightarrow$ Def(V) (resp. $\mathcal{Z} \longrightarrow$ Def(S)) the semi-universal family for the deformations of V (resp. S). One can regard $\mathcal{Y}$ as a flat family of rational double points over $Def(V) \times \Delta$. Then, by the versality of Def(S), there is a holomorphic map $\varphi : Def(V) \times \Delta \longrightarrow Def(S)$ and the $\mathcal{Y}$ is obtained as the pull-back of $\mathcal{Z}$ by φ.

Let $\mathcal{V}$ be a flat deformation of V over a 1-dimensional disc Δ'. Then there is a holomorphic map $\phi : \Delta' \longrightarrow Def(V)$ and $\mathcal{V}$ is the pull-back of $\mathcal{Y}$ by ϕ. Since $\mathcal{Y}$ is a flat family of rational double points over $Def(V) \times \Delta$, $\mathcal{V}$ constitutes a flat family of rational double points over $\Delta' \times \Delta$. Let B be the discriminant divisor on Def(S) and D its inverse image in $\Delta' \times \Delta$. Let $p_1 : \Delta' \times \Delta \longrightarrow \Delta'$ be the first projection. Since V is an isolated singularity, $\{D_t\}$ is a family of Cartier divisors with $t \in \Delta'$.

Definition(1.1) A pair ($\mathcal{V}$, ϕ) is called *admissible* if $\#(D_t)$ is constant for $t \in \Delta' - 0$.

We have the following lemma.

Lemma(1.2) For $t \in Def(V)$, there is a flat deformation $g : \mathcal{V} \longrightarrow \Delta'$ of V over a 1-dimensional disc and a holomorphic map ϕ of the disc to Def(V) such that (1) $g^{-1}(0) = V$, $g^{-1}(s) = Y_t$ for some point $s \in \Delta'$ and (2) ($\mathcal{V}$, ϕ) is admissible.

Proof. Set E $= \varphi^{-1}(B)$. Take a suitable system of local coordintes $(s_1, ..., s_n)$ of Def(V) (Def(V) is smooth because V is an isolated cDV point.). Let u be the coordinate of Δ. By the Weierstrass Preparation Theorem, we may assume that E is defined as the zero locus of the function $h(u, s) = u^n + h_1(s)u^{n-1} + ... + h_n(s)$, where $h_i(0) = 0$ for all i. It can be checked that the set $W_p := \{u \in Def(V); h(u, s)$ has p different roots as a polynomial of $s\}$ forms a locally (Zariski) closed subset of Def(V) for every p and that

$\bar{W}_p \ni 0$. If we take the Def(V) sufficiently small, we can assume that $\bar{W}_p$ is connected. This implies that one can connect any point $t \in W_p$ with the origin 0 by an analytic curve $\rho : \Delta' \longrightarrow Def(V)$ in such a way that $\rho(\Delta') - 0 \subset W_p$. Q.E.D.

Let $(\mathcal{V}, \phi)$ be an admissible pair. Then there is a holomorphic map $h : \mathcal{V} \longrightarrow \Delta' \times \Delta$, and $\mathcal{V}$ can be regarded as a family of rational double points (resp. a family of isolated cDVpoints) by h (resp. $g := p_1 \circ h$).

Write $V_t = g^{-1}(t)$ for a point $t \in \Delta'$. Then one has a holomorphic map $h_t : V_t \longrightarrow \Delta$. The map h_t has exactly $\#(D_t)$ singular fibers V_{t,u_i} $(i = 1, ..., \#(D_t))$. The number $\#(D_t)$ remains constant when t varies in $\Delta' - 0$, and $\#(D_0) = 1$. We then have the following lemma.

Lemma(1.3) In the commutative diagram:

$$\begin{array}{ccc} H^2(V_t - Sing(V_t); \mathbf{Z}) & \xrightarrow{j} & H^2(V_t - \bigcup V_{t,u_i}; \mathbf{Z}) \\ \delta_1 \uparrow & & \delta_2 \uparrow \\ H^1(V_t - Sing(V_t); \mathcal{O}^*_{V_t}) & \longrightarrow & H^1(V_t - \bigcup V_{t,u_i}; \mathcal{O}^*_{V_t}) \end{array}$$

all homomorphisms are isomorphisms.

Proof. Take a suitable Galois cover $\Delta' \longrightarrow \Delta$ in such a way that it is ramified over p_i's and that the base change V_t' of V_t by the cover admits a simultaneous resolution $\pi : W \longrightarrow V_t'$. Let E be the exceptional curve of π. Since $H^3(E; \mathbf{Z}) = 0$, $H^3{}_E(W; \mathbf{Z}) = 0$ by duality. Hence the restriction map: $H^2(W; \mathbf{Z}) \longrightarrow H^2(W - E; \mathbf{Z})$ is a surjection. On the other hand, the composition $H^2(W; \mathbf{Z}) \cong H^1(W; \mathcal{O}^*) \cong H^1(V_t' - Sing(V_t'); \mathcal{O}^*) \longrightarrow H^2(V_t' - Sing(V_t'); \mathbf{Z}) \cong H^2(W - E; \mathbf{Z})$ is an injection since $H^2{}_{Sing(V_t')}(V_t'; \mathcal{O}) = 0$ by the depth argument. These implies that $H^2(W; \mathbf{Z}) \cong H^2(W - E; \mathbf{Z})$. As $H^2(W; \mathbf{Z}) \cong H^2(V_t' - \bigcup V_{t,u_i}'; \mathbf{Z})$, and $H^2(W - E; \mathbf{Z}) \cong H^2(V_t' - Sing(V_t'); \mathbf{Z})$, we have an isomorphism $H^2(V_t' - \bigcup V_{t,u_i}'; \mathbf{Z}) \cong H^2(V_t' - Sing(V_t'); \mathbf{Z})$. Take its invariant part by the Galois group. One then sees that j is an isomorphism. One also sees that the map $H^1(V_t' - Sing(V_t'); \mathcal{O}^*) \longrightarrow H^2(V_t' - Sing(V_t'); \mathbf{Z})$ is an isomorphism by the above observation. Hence we have that δ_1 is an isomorphism by taking the invariant part by the Galois group. The map δ_2 is an isomorphism because $V_t - \bigcup V_{t,u_i}$ is a Stein space and hence $H^i(V_t - \bigcup V_{t,u_i}; \mathcal{O}) = 0$ for $i > 0$. Q.E.D.

Lemma(1.4) Suppose that $\sigma(V_t) = \mathrm{rank}(Weil(V_t)/Pic(V_t)) > 0$ for some $t \in \Delta' - 0$. Then there is a projective small partial resolution $\nu : \hat{\mathcal{V}} \longrightarrow \mathcal{V}$ such that

(1) ν_s is a projective small partial resolution for every $s \in \Delta'$;

(2) $\sigma(\hat{V}_t) = 0$.

Proof. Since the number $r := \#(D_s)$ is constant for $s \in \Delta' - 0$, we have $\pi_1(\Delta' \times \Delta - D) = \bigoplus_{1 \leq i \leq r} \mathbf{Z}$, and we can take the loops γ_i in $\{t\} \times \Delta$ $(1 \leq i \leq r)$ which go around u_i in the positive direction as its basis. Hence one sees that the restriction map $H^0(\Delta' \times \Delta - D \ ; \ R^2 h_* \mathbf{Z}) \longrightarrow H^0(\{t\} \times \Delta - \{u_1, ..., u_r\} \ ; \ R^2 h_{t*} \mathbf{Z})$ is an isomorphism. Since $\sigma(V_t) > 0$, there is a $\mathbf{Q}$-factorialization $\nu_t : \hat{V}_t \longrightarrow V_t$. Take a ν_t-ample line bundle L on $\hat{V}_t$. Since $H^1(\hat{V}_t; \mathcal{O}^*) \cong H^1(V_t - Sing(V_t); \mathcal{O}^*)$, we have a non-zero element $\tau \in H^0(\{t\} \times \Delta - \{u_1, ..., u_r\} \ ; \ R^2 h_{t*} \mathbf{Z})$ corresponding to L by Lemma(1.3). The τ gives an element of $H^0(\Delta' \times \Delta - D \ ; \ R^2 h_* \mathbf{Z})$.

We now take a finite Galois cover $\alpha : T \longrightarrow \Delta' \times \Delta$ with the Galois group G in such a way that the base change $\mathcal{V}'$ of $\mathcal{V}$ by α admits a simultaneous resolution $\mu : \mathcal{W} \longrightarrow \mathcal{V}'$. Since we have $H^0(T - \alpha^{-1}(D) \ ; \ R^2 h'_* \mathbf{Z}) \cong H^0(T; R^2(\mu \circ h')_* \mathbf{Z}) \cong H^1(\mathcal{W}; \mathcal{O}^*)$, we also have an isomorphism $H^0(T - \alpha^{-1}(D) \ ; \ R^2 h'_* \mathbf{Z})^G \cong H^1(\mathcal{W}; \mathcal{O}^*)^G$.

As there is a homomorphism from $H^0(\Delta' \times \Delta - D; R^2 h_* \mathbf{Z}) \longrightarrow H^0(T - \alpha^{-1}(D) \ ; R^2 h'_* \mathbf{Z})^G$, one has a line bundle $\mathcal{L} \in H^1(\mathcal{W}; \mathcal{O}^*)^G$ corresponding to τ. We here recall that there are many choices of the simultaneous resolution $\nu : \mathcal{W} \longrightarrow \mathcal{V}'$. Two simultaneous resolutions are connected by a sequence of *flops*. Now we can specify one of them in such a way that $\mathcal{L}$ is ν-nef by [Re, §§7, 8]. Then it is easily checked that the graded $\mathcal{O}_{\mathcal{V}'}$-algebra $\bigoplus_{n \geq 0} \nu_* \mathcal{L}^{\otimes n}$ is a finitely generated $\mathcal{O}_{\mathcal{V}'}$-algebra. The line bundle $\mathcal{L}$ is G-invariant in the following sense:

The G has a meromorphic action on $\mathcal{W}$. Each element $g \in G$ induces a bimeromorphic automorphism ψ_g of $\mathcal{W}$. Note that ψ_g is an isomorphism in codimension 1 and hence there is an isomorphism ${\psi^*}_g : Pic(\mathcal{W}) \longrightarrow Pic(\mathcal{W})$. Then $\mathcal{L}$ is invariant under ${\psi^*}_g$ for every $g \in G$.

Hence $\nu_* \mathcal{L}^{\otimes n}$ is a G-sheaf for every n. We here set $\hat{\mathcal{V}} = Proj_{\mathcal{O}_{\mathcal{V}}} \bigoplus_{n \geq 0} {\nu_*}^G \mathcal{L}^{\otimes n}$. Q.E.D.

Remark(1.5) (1) In the proof of (1.4), one has a birational morphism $\varphi : \mathcal{W} \longrightarrow \hat{\mathcal{W}}$ over $\mathcal{V}'$ by using a ν-free line bundle $\mathcal{L}^{\otimes m}$ $(m >> 0)$. Then the $\hat{\mathcal{V}}$ is obtained as the quotient of $\hat{\mathcal{W}}$ by G. Let $p = \alpha^{-1}((0,0)) \in T$. Then the fiber $\hat{\mathcal{W}}_p$ of the morphism $\hat{\mathcal{W}} \longrightarrow T$ is a partial resolution S' of the rational double point S (i.e. the minimal resolution $\tilde{S}$ of S factors through S'. By the assumption, the exceptional locus of the partial resolution has exactly r irreducible components. Since G acts on $\hat{\mathcal{W}}_p$ trivially, we see that the exceptional locus of $\nu_0 : \hat{\mathcal{V}}_0 \longrightarrow V_0$ has r irreducible components.

(2) Since $R^1 \nu_{s*} \mathcal{L}^{\otimes n} = 0$ for all $s \in \Delta'$, one has the base change property: ${\nu^G}_* \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_{\mathcal{V}}} \mathcal{O}_{V_s} \cong {\nu_s}^G{}_* {\mathcal{L}_s}^{\otimes n}$. In particular, we have $\hat{V}_s = Proj \ \bigoplus_{n \geq 0} {\nu_s}^G{}_* {\mathcal{L}_s}^{\otimes n}$ for all $s \in \Delta'$.

(3) One can state the result of (1.4) in more generality as follows. With the same

assumption of (1.4), suppose that a projective small partial resolution $\nu_t : \hat{V}_t \longrightarrow V_t$ is given. Then we can extends the ν_t to a projective small partial resolution $\nu : \hat{\mathcal{V}} \longrightarrow \mathcal{V}$ with the property (1) in (1.4). In fact, we only have to replace the **Q**-factorialization with this ν_t in the proof of (1.4).

Let V be the germ of an isolated cDV point and $Def(V)$ the Kuranishi space of V. Denote by $\mathcal{Y}$ the semi-universal family over $Def(V)$. Define $\sigma(Y_t) = \mathrm{rank}(Weil(Y_t)/Pic(Y_t))$ and set $Y_i = \{t \in Def(V); \sigma(Y_t) = i\}$. Then we have the following description of $Def(V)$.

Proposition(1.6)

(1) Let $\hat{V}$ be a small partial resolution of V and $Def(\hat{V})$ the Kuranishi space of $\hat{V}$. Then there is a natural closed immersion of $Def(\hat{V})$ into $Def(V)$.

(2) $Def(V)$ has a stratification into the disjoint sums of (Zariski) lacally closed subsets: $Def(V) = \amalg Y_i, Y_i = \bar{Y}_i - \bar{Y}_{i+1}$ and $\bar{Y}_i = \bigcup Def(\hat{V})$, where $\hat{V}$ runs through all small partial resolutions such that $\rho(\hat{V}) \geq i$.

Proof (1): Since V has only rational singularity, there is a natural map $Def(\hat{V}) \longrightarrow Def(V)$ by Wahl[Wa]. So we only have to check that the homomorphism $Ext^1(\Omega^1{}_{\hat{V}}, \mathcal{O}_{\hat{V}}) \longrightarrow Ext^1(\Omega^1{}_V, \mathcal{O}_V)$ is an injection. Set $\hat{U} := \hat{V} - Sing(\hat{V})$ and $U := V - Sing(V)$. By Schlessinger[Sch], we have $Ext^1(\Omega^1{}_{\hat{V}}, \mathcal{O}_{\hat{V}}) \cong H^1(\hat{U}; \Theta_{\hat{U}})$ and $Ext^1(\Omega^1{}_V, \mathcal{O}_V) \cong H^1(U; \Theta_U)$. Denote by C the exceptional curve of the small partial resolution. Then we have an exact sequence of local cohomology:

$$H^1{}_{C\cap\hat{U}}(\hat{U}; \Theta_{\hat{U}}) \longrightarrow H^1(\hat{U}; \Theta_{\hat{U}}) \longrightarrow H^1(U; \Theta_U)$$

By the depth argument, we have $H^1{}_{C\cap\hat{U}}(\hat{U}; \Theta_{\hat{U}}) = 0$. Hence we have done.

(2): Let $t \in Y_i$. Then by Lemma(1.2) there is an admissible pair $(\mathcal{V}, \phi)$ such that $g^{-1}(0) = V$ and $g^{-1}(s) = Y_t$. We have a projective partial resolution $\nu : \hat{\mathcal{V}} \longrightarrow \mathcal{V}$ by Lemma(1.4) and Remark(1.5) such that $\rho(\hat{V}_0) = i$. This implies that $t \in Def(\hat{V}_0)$. Moreover, we have $t \in Def(\hat{V}_0) - \bigcup_{\rho(\hat{V}) \geq i+1} Def(\hat{V})$. In fact, suppose that $t \in Def(\hat{V})$for some $\hat{V}$ with $\rho(\hat{V}) > i$. We can find an analytic curve $\Gamma \subset Def(\hat{V})$ passing through t and the origin 0 in such a way that there is a flat deformation $\hat{\mathcal{V}} \longrightarrow \Gamma$ of $\hat{V}$ and a birational morphism ν from $\hat{\mathcal{V}}$ to $\mathcal{Y} \times_{Def(V)} \Gamma$. Since $\rho(\hat{V}) > i$, the exceptional locus of ν_0 has more than i irreducible components $C_1, ..., C_n$ $(n = \rho(\hat{V}))$. Each curve C_j moves sideways in the family $\hat{\mathcal{V}} \longrightarrow \Gamma$ to a curve $C_j(t)$ in $\hat{V}_t$. Since $C_j's$ are numerically independent in $\hat{V}_0$, $C_j(t)'s$ are also numerically independent in $\hat{V}_t$. This, in particular, implies that $\sigma(Y_t) > i$, which is a contradiction. Hence we have proved that $Y_i \subset \bigcup_{\rho(\hat{V}) \geq i} Def(\hat{V}) - \bigcup_{\rho(\hat{V}) \geq i+1} Def(\hat{V})$. We can also prove the converse implication by the same argument. Q.E.D.

Example(1.7) Let V be a good representative of the germ of $\{(x,y,z,w) \in \mathbf{C}^4; x^2+y^2+z^2+w^3=0\}$ at the origin. Consider the 1-parameter deformation $\mathcal{V}$ of V given by the equation $x^2+y^2+z^2+w^3+w^2t=0$. For $t \neq 0$, V_t has a singularity at $p=(0,0,0,0,t)$ and (V_t,p) is not $\mathbf{Q}$-factorial. However, V_t itself is $\mathbf{Q}$-factorial.

Let $(\mathcal{V},\phi)$ be an admissible pair such that V_t has only ordinary double points for $t \neq 0$. Assume that there is a small partial resolution $\nu : \hat{\mathcal{V}} \longrightarrow \mathcal{V}$ which satisfies

(1) ν_0 is a small partial resolution of V with n irreducible curves as the exceptional locus (or equivalently $\rho(\hat{V}_0)=n$);

(2) ν_t is a small resolution of ordinary double points of V_t for $t \neq 0$.

Note that the exceptional locus of the map ν_t is a disjoint union of $(-1,-1)$-curves for $t \neq 0$. As $(\mathcal{V},\phi)$ is an admissible pair, the number of such $(-1,-1)$-curves is independent of $t \neq 0$. We denote this number by m. In this situation, we have the following lemma.

Lemma(1.8) One has the inequality $m \geq n$, and the equality holds if and only if V is the germ of an ordinary double point and $\mathcal{V}$ is a trivial deformation of V.

Proof. As we have seen above, there is a holomorphic map $h : \mathcal{V} \longrightarrow \Delta' \times \Delta$ and $\mathcal{V}$ can be regarded as a family of rational double points. Set $S = h^{-1}((0,0))$ and $S' = (h \circ \nu)^{-1}((0,0))$. Then the minimal resolution $\pi : \tilde{S} \longrightarrow S$ factors through S' (cf. [Re]). By the versality of $Def(S)$, one has a holomorphic map of $\Delta' \times \Delta$ to $Def(S)$. In our case, this map factors through $Def(S')$. By the assumption, the partial resolution $S' \longrightarrow S$ has n irreducible curves as the exceptional divisor. Since $Ext^2(\Omega^1_{S'}, \mathcal{O}'_S) = 0$, $Def(S')$ is smooth.

Here we recall a result of Brieskorn (cf.[Br, Pi]). Let E_j $(1 \leq j \leq l)$ be the irreducible components of the exceptional locus of $\tilde{S} \longrightarrow S$. Put $\Sigma = \{D = \Sigma a_j E_j; D^2 = -2, a_j \in \mathbf{Z}\}$. The Σ forms a root system. Then $Def(\tilde{S}) \longrightarrow Def(S)$ is a finite Galois cover with Galois group $G = W(\Sigma)$, the Weyl group of Σ. Moreover, there is a one to one correspondence between the effective roots of Σ and the ramification divisors of $Def(\tilde{S})$. Since $W(\Sigma)$ acts transitively on Σ, one sees that G acts on the set of ramification divisors of $Def(\tilde{S})$ transitively. Thus, the discriminant locus B of $Def(S)$ is an irreducible divisor.

We shall prove that there are at least n irreducible component in the ramification locus $R \subset Def(S')$ of the finite cover $Def(S') \longrightarrow Def(S)$. First we factorize the partial resolution into n number of birational morphisms : $S' \longrightarrow S_{n-1} \longrightarrow, ..., S_1 \longrightarrow S$ in such a way that $\rho(S_i/S_{i-1}) = 1$ for all i. Then we have a sequence of finite covers : $Def(S') \longrightarrow Def(S_{n-1}), ..., Def(S_1) \longrightarrow Def(S)$. Renumbering the indices of E_j's, we may assume that E_i corresponds to the exceptional divisor of $S_i \longrightarrow S_{i-1}$. As we have remarked above, there is a ramification divisor $D_i \subset Def(\tilde{S})$ corresponding to E_i. Denote by $B_i \subset Def(S_i)$ its image by the map $Def(\tilde{S}) \longrightarrow Def(S_i)$, and denote by $R_i \subset Def(S')$ its image by the map $Def(\tilde{S}) \longrightarrow Def(S')$. Then it can be checked that B_i is an irreducible component of the ramification locus of $Def(S_i) \longrightarrow Def(S_{i-1})$. Since the ramification indices of ramification divisors of $Def(\tilde{S})$ all equal 1, this implies that R_i $(1 \leq i \leq n)$ are mutually *different* irreducible components of R.

Next assume that S is not of type A_1. Consider the map $f_1 : Def(S_1) \longrightarrow Def(S)$. Decompose ${f_1}^{-1}(B)$ into the two parts: the ramification locus G of f_1 and the non-ramification locus H. Both of them are Cartier divisors on $Def(S_1)$. Suppose that H is empty. Then all D_i's are mapped onto some irreducible components of G by the map $Def(\tilde{S}) \longrightarrow Def(S_1)$. But this is absurd because if so, then the ramification indices of $D_i (i \geq 2)$ are grater than one. Hence H should be non-empty and R_i's $(i \geq 2)$ are mapped onto some irreducible components of H by the map $Def(S') \longrightarrow Def(S_1)$. Here if G has more than one irreducible component, then there are at least $n+1$ irreducible components in the ramification locus $R \subset Def(S')$ of the finite cover $Def(S') \longrightarrow Def(S)$. Even if G is irreducible, we can show that there are at least $n+1$ irreducible components in R in the following way. Let $D^* \subset Def(\tilde{S})$ be the ramification divisor corresponding to the fundamental cycle of the minimal resolution $\tilde{S}$ of S. It can be checked that D^* is mapped onto G by the map $Def(\tilde{S}) \longrightarrow Def(S_1)$. Let $R^* \subset Def(S')$ be the image of D^* by the map $Def(\tilde{S}) \longrightarrow Def(S')$. We shall prove that R_1 and R^* are different divisors on $Def(S')$. Let $S' \longrightarrow S''$ be the birational morphism contracting the curve E_1 to a point. R_1 is clearly a ramification divisor of the map $Def(S') \longrightarrow Def(S'')$, but R^* is not a ramification divisor by definition. Thus, R_1 and R^* are different divosor on $Def(S')$. Now the $n+1$ divisors $R_i(1 \leq i \leq n)$ and R^* are mutually different irreducible components of R.

Assume finally that S is of type A_1. Then V is isomorphic to the germ of $\{(x,y,z,w) \in \mathbf{C}^4; x^2+y^2+z^2+w^k=0\}$ at the origin for some $k>1$. In this case, we can directly check that $m=n$ if and only if $k=2$ (cf.[Fr]). Q.E.D.

§2. Calabi-Yau 3-folds

Let X be a Calabi-Yau 3-fold with terminal singularities. As $K_X \sim 0$, X has only Gorenstein terminal singularities. Thus, X has only isolated cDV singularities by [Re]. For each singular point $p_i \in X$, we take a sufficiently small open neighborhood V_i of p_i. There is a holomorphic map f_i of V_i to a 1-dimensional disc $\triangle$ with a small radius such that $f_i^{-1}(0) = S_i$ is a rational double point and other fibers are smooth. Let $\mathcal{Y}_i \longrightarrow Def(V_i)$ be the semi-universal family for the deformations of V_i. One can regard

$\mathcal{Y}_i$ as a flat family of rational double points over $Def(V_i) \times \Delta$. By the versality of $Def(S_i)$ there is a holomorphic map $\varphi_i : Def(V) \times \Delta \longrightarrow Def(S_i)$.

Let $\mathcal{X}_{\Delta'}$ be a flat deformation of X over a 1-dimensional disc Δ'. Then there is a holomorphic map ϕ of Δ' to the Kuranishi space $Def(X)$ corresponding to this flat deformation. By composing this map with the natural map $Def(X) \longrightarrow Def(V_i)$, we obtain a holomorphic map $\phi_i : \Delta' \longrightarrow Def(V_i)$ for each singularity $p_i \in X$. We also have a holomorphic map from $\Delta' \times \Delta$ to $Def(S_i)$ by composing $\phi_i \times id$ with φ_i . By pulling back the semi-universal family $\mathcal{Z}_i$ over $Def(S_i)$ by the map, obtained is a flat family $\mathcal{V}_i$ of rational double points over $\Delta' \times \Delta$. The $\mathcal{V}_i$ can be also viwed as a flat deformation of V_i over Δ'. Note that $\mathcal{V}_i$ is an open neighborhood of $p_i \in \mathcal{X}_{\Delta'}$.

Definition(2.1) A pair $(\mathcal{X}_{\Delta'}, \phi)$ is called *admissible* if $(\mathcal{V}_i, \phi_i)$ are all admissible in the sense of (1.1).

Let $\mathcal{X}$ be the universal family over the Kuranishi space $Def(X)$ of X. By the same argument as (1.2) we have

Lemma(2.2) For $t \in Def(X)$ there is a flat deformation $g : \mathcal{X}_{\Delta'} \longrightarrow \Delta'$ of X over a 1-dimensional disc and a holomorphic map ϕ of the disc to $Def(X)$ such that(1) $g^{-1}(0) = X$, $g^{-1}(s) = X_t$ for some point $s \in \Delta'$ and (2) $(\mathcal{X}_{\Delta'}, \phi)$ is admissible.

Define $\sigma(X_t)$ to be the rank of $Weil(X_t)/Pic(X_t)$ and set $Y_i = \{t \in Def(X); \sigma(X_t) = i\}$. Then one has the following globalization of (1.6).

Proposition(2.3) (1) Let $\hat{X}$ be a small projective partial resolution of X and $Def(\hat{X})$ the Kuranishi space of $\hat{X}$. Then there is a natural closed immersion of $Def(\hat{X})$ into $Def(X)$.

(2) $Def(X) = \coprod Y_i$, $Y_i = \bar{Y}_i - \bar{Y}_{i-1}$ and $\bar{Y}_i = \bigcup Def(\hat{X})$, where $\hat{X}$ runs through all small projective resolutions such that $\rho(\hat{X}) - \rho(X) \geq i$.

(3) Each stratum Y_i is a (Zariski) locally closed smooth subset of $Def(X)$.

Proof (1): The proof is quite similar to that of (1.6)(1).

(2): Let $t \in Y_i$. We take a flat deformation $g : \mathcal{X}_{\Delta'} \longrightarrow \Delta'$ and a holomorhic map $\phi : \Delta' \longrightarrow Def(X)$ with the properties (1) and (2) of Lemma(2.2). Let $\nu_t : \hat{X}_t \longrightarrow X_t$ be a $\mathbf{Q}$-factorialization. The ν_t induces a projective small partial resolution ${\nu^i}_t : \hat{V}_{i,t} \longrightarrow V_{i,t}$. By Lemma(1.4) and Remark(1.5),(3) each ν^i_t extends to a projective small partial resolution $\nu_i : \hat{\mathcal{V}}_i \longrightarrow \mathcal{V}_i$. As a consequence, one has a small partial resolution $\nu : \hat{\mathcal{X}_{\Delta'}} \longrightarrow \mathcal{X}_{\Delta'}$. Note that $\hat{X}_{\Delta',t} = \hat{X}_t$. Since $\hat{X}_t$ is projective, there is an ample line bundle L on $\hat{X}_t$. The 2-nd Betti number (with respect to the usual cohomology) is preserved under a flat deformation of Calabi-Yau 3-folds with isolated hypersurface singularities by the vanishing cycle argument. This implies that the Picard number is also preserved because $h^1 = h^2 = 0$ in this case. Thus, the line bundle L extends to a line bundle $\mathcal{L}$ on $\hat{\mathcal{X}}_{\Delta'}$. Let $C_1, ..., C_m$ be the irreducible components of the exceptional locus of ν_0. C_j's move sideways in $\mathcal{X}_{\Delta'}$ to the curves $C_j(t)'s$ on $\hat{X}_t$. Since $(L. C_j(t)) > 0$, $(\mathcal{L}, C_j) > 0$, which means that $\hat{X}_{\Delta',0}$ is projective over X. Now the relative Picard number $\rho(\hat{X}_t/X_t) = i$ by our assumption. Hence we have $\rho(\hat{X}_{\Delta',0}/X) = i$. It follows from the observation above that $t \in \bigcup_{\rho(\hat{X}/X)\geq i} Def(\hat{X})$. Moreover, we have $t \in \bigcup_{\rho(\hat{X}/X)\geq i} Def(\hat{X}) - \bigcup_{\rho(\hat{X}/X)\geq i+1} Def(\hat{X})$. In fact, if $t \in Def(\hat{X})$ for a projective small partial resolution $\hat{X}$ with $\rho(\hat{X}/X) > i$, then we can choose an analytic curve $\Gamma \subset Def(\hat{X})$ passing throgh t and 0 in such a way that there is a flat deformation $\hat{\mathcal{X}} \longrightarrow \Gamma$ of $\hat{X}$ and a birational morphism ν from $\hat{X}$ to $\mathcal{X} \times_{Def(X)} \Gamma$. Since $\rho(\hat{X}/X) > i$, we have $\rho(\hat{X}_t/X_t) > i$, which is a contradiction.

Finally we show that if $t \in \bigcup_{\rho(\hat{X}/X)\geq i} Def(\hat{X}) - \bigcup_{\rho(\hat{X}/X)\geq i+1} Def(\hat{X})$, then $t \in Y_i$. By the assumption, $t \in Def(\hat{X})$ with a projective small resolution $\hat{X} \longrightarrow X$ for which $\rho(\hat{X}/X) = i$. Thus, $\sigma(X_t) \geq i$. On the other hand, $\sigma(X_t) \leq i$ because $t \notin \bigcup_{\rho(\hat{X}\ X)\geq i+1} Def(\hat{X})$. Hence we have done.

(3): Assume that Y_i has a singular point t. Since $Def(\hat{X})$ is a smooth subvariety of $Def(X)$ for every projective small partial resolution $\hat{X}$ of X, there are at least two different irreducible components of $\bar{Y}_i$ which contain t, say, $Def(\hat{X}_1)$ and $Def(\hat{X}_2)$, for which $\rho(\hat{X}_1/X) = \rho(\hat{X}_2/X) = i$. This means that there are two different projective small partial resolutions X_t' and X_t'' of X_t, for which $Def(X_t') \neq Def(X_t'')$ as a subvariety of $Def(X_t)$. Let W' (resp. W'') be a **Q**-factorization of X_t' (resp. X_t''). Then W' and W'' are both **Q**-factorizations of X_t, and hence they are connected by a flop. It is proved by Kollår and Mori [K-M,(11.10)] that $Def(W') \cong Def(W'')$. This, in particular, implies that $\rho(W'/X_t) > \rho(X_t'/X_t) = i$. However, it is absurd because $\sigma(X_t) = i$. Q.E.D.

Definition(2.4) Let $\hat{X}$ be a projective small partial resolution of X. Then $\hat{X}$ is called *maximal* if for any projective small partial resolution $\tilde{X}$ of $\hat{X}$, $Def(\tilde{X})$ is a proper closed subvariety of $Def(\hat{X})$ via the natural inclusion.

In view of Proposition(2.3), the stratification of $Def(X)$ is determined only by maximal projective small partial resolutions. We have the following criterion of the maximality.

Theorem(2.5) Let $\{p_1, ..., p_l\} \subset Sing(\hat{X})$ be the ordinary double points on $\hat{X}$ and let $f : Z \longrightarrow \hat{X}$ be a small (not necessarily projective) partial resolution of $\hat{X}$ such that $C_i := f^{-1}(p_i) \cong \mathbf{P}^1$ and that f is an isomorphism over $\hat{X} - \{p_1, ..., p_l\}$. Then the following three conditions are equivalent:

(1) $\hat{X}$ is maximal;

(2) $\hat{X}$ is smoothable by a flat deformation;

(3) There is a relation in $H_2(Z, \mathbf{C}) : \Sigma\alpha_i[C_i] = 0$ with $\alpha_i \neq 0$ for all i.

Proof (1) $\Rightarrow$ (2): $\hat{X}$ has a flat deformation to a Calabi-Yau 3-fold Y with only ordinary double points by [Na 2, Theorem(5.2)]. Let Y_j be the germ of a sinular point $q_j \in Y$. We may assume that $\Sigma\sigma(Y_j) = \sigma(Y)$ by [Na 2, Corollary(6.12)]. If Y has a singularity, then $\sigma(Y) > 0$, which implies that a general point of Def(X) corresponds to a non-**Q**-factorial Calabi-Yau 3-fold. Hence there is a projective small partial resolution $\tilde{X}$ of $\hat{X}$ such that $Def(\tilde{X}) \cong Def(\hat{X})$ by applying Proposition(2.3) to $Def(\hat{X})$. This contradicts the maximality of $\hat{X}$. So Y must be a smooth Calabi-Yau 3-fold.

(2) $\Rightarrow$ (1): It is obvious because smooth Calab-Yau 3-fold Y has no small partial resolutions except for Y itself.

(3) $\Rightarrow$ (2): First we shall show that all singularities of $\hat{X}$ which are not ordinary double points are smoothed under a suitable flat deformation of $\hat{X}$. Let $g : \hat{\mathcal{X}} \longrightarrow \Delta$ be a flat deformation of $\hat{X}$ over a 1-dimensional disc such that $g^{-1}(0) = \hat{X}$ and a general fiber $g^{-1}(t) := Y(t \neq 0)$ is the same as above. Suppose that when $\hat{X}$ is deformed to Y, a non-ordinary double point $p \in \hat{X}$ splits into a finite number of ordinary double points $q_1, ..., q_m$ on Y. By Proposition(2.3), there is a projective birational morphism $\nu : \tilde{\mathcal{X}} \longrightarrow \hat{\mathcal{X}}$ which satisfies (a) $\nu_0 : \tilde{X} \longrightarrow \hat{X}$ is a small partial resolution of $\hat{X}$ and (b) ν_t is a small resolution of the ordinary double points on Y for $t \neq 0$. Define n to be the number of the irreducible components of ${\nu_0}^{-1}(p)$. Then we have $m > n$ by Lemma(1.8). Hence the curves $D_i := \nu_t^{-1}(q_i)(1 \leq i \leq m)$ are not numerically independent on $\tilde{X}_t$, which contradicts the assumption $\Sigma\sigma(Y_j) = \sigma(Y)$.

We shall next prove that all ordinary double points are smoothed under a suitable flat deformation of $\hat{X}$. Let $\hat{X}_i$ be the germ of a ordinary double point $p_i \in \hat{X}$. Let $\pi : W \longrightarrow Z$ be a resolution of singularities such that $\pi^{-1}(Z - Sing(Z)) \cong Z - Sing(Z)$. Let E be the exeptional divisor of π. Then the exceptional locus of $f \circ \pi$ is a disjoint union of C_i's and E. We have the following exact commutative diagram:

(2.6)

$$\begin{array}{ccc} H^1(\hat{X} - Sing(\hat{X}); \Theta_{\hat{X}}) & \longrightarrow \oplus_i H^2{}_{C_i}(W; \Omega^2{}_W) \oplus H^2{}_E(W; \Omega^2{}_W) & -\gamma \longrightarrow H^2(W, \Omega^2{}_W) \\ \| & \beta \uparrow & \\ H^1(\hat{X} - Sing(\hat{X}); \Theta_{\hat{X}}) & -\alpha \longrightarrow \oplus H^2_{Sing(\hat{X})}(\hat{X}; \Theta_{\hat{X}}) & \end{array}$$

By the assumption of (3), there is an element $\epsilon \in Ker(\gamma)$ whose i-th component ϵ_i are all non-zero for $1 \leq i \leq l$. Then there is an element $\eta \in H^1(\hat{X} - Sing(\hat{X}); \Theta_{\hat{X}})$ such that $\alpha(\eta)_i \in H^2_{p_i}(\hat{X}; \Theta_{\hat{X}}) \cong Ext^1(\Omega^1_{\hat{X}_i}; \mathcal{O}_{\hat{X}_i})$ are all non-zero by (2.6). Since any infinitesimal deformation of $\hat{X}$ is unobstructed, 1-st order deformation of $\hat{X}$ corresponding to the η can be realized. Hence we have done.

It follows from two observations above that $\hat{X}$ is smoothable by a flat deformation because $Def(\hat{X})$ is smooth (in particular, irreducible).

(2) $\Longrightarrow$ (3): Assume that there is a positive integer $k \leq l$ and all relations in $H_2(Z;\mathbf{C})$ are of the form $\Sigma_{i\geq k+1}\alpha_i[C_i] = 0$ for some α_i's. Let $f' : Z' \longrightarrow \hat{X}$ be a small partial resolution of $\hat{X}$ obtained by contracting the curves C_i $(i \geq k+1)$ on Z to points. We shall show that $Def(Z') \cong Def(\hat{X})$. If this is proved, then we see that the ordinary double points $p_i \in \hat{X}$ $(i \leq k)$ are not smoothed by any flat deformation of $\hat{X}$ because $(-1,-1)$-curves C_i $(i \geq k+1)$ are stable under any flat deformation of Z'.

In the diagram(2.6) choose an element $\epsilon \in Ker(\gamma)$. We denote by $\epsilon_i \in H^2{}_{C_i}(W, \Omega^2{}_W)$ its i-th component and denote by $\epsilon_E \in H^2{}_E(W, \Omega^2{}_W)$ its other component. The assumption implies that ϵ_i are all zero for $1 \leq i \leq k$. Hence, for an arbitrary element $\eta \in H^1(X - Sing(X); \Theta_X)$, we see that the i-th component $\alpha(\eta)_i$ of $\alpha(\eta)$ are all zero for $1 \leq i \leq k$. Next we set $\hat{X}' = \hat{X} - (Sing(\hat{X}) - \{p_1, ..., p_k\})$ and consider the following exact commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^1(\hat{X}'; \Theta_{\hat{X}}) & \longrightarrow & H^1(Z' - Sing(Z'); \Theta_Z) & \longrightarrow & H^0(\hat{X}; R^1f_*\Theta_Z) & \\ & \| & & \downarrow & & \downarrow & \\ 0 \longrightarrow & H^1(\hat{X}'; \Theta_{\hat{X}}) & \longrightarrow & H^1(\hat{X} - Sing(\hat{X}); \Theta_{\hat{X}}) & -\ \alpha' \longrightarrow & \bigoplus_{1\leq i\leq k} H^2{}_{p_i}(\hat{X}; \Theta_{\hat{X}}) & \end{array}$$

Since $\alpha' = 0$, one has an isomorphism $H^1(Z - Sing(Z); \Theta_Z) \cong H^1(\hat{X} - Sing(\hat{X}); \Theta_{\hat{X}})$. By Schlessinger [Sch] these are isomorphic to the tangent spaces to $Def(Z)$ and $Def(\hat{X})$ at the origin respectively. As $Def(Z)$ and $Def(\hat{X})$ are both smooth, we conclude that $Def(Z) \cong Def(\hat{X})$. Q.E.D.

When a projective small partial resolution $\hat{X}$ of X is not maximal, one has the following.

Theorem(2.7) Let $\hat{X}$ be not maximal. Then there is a (not necessarily unique) small projective partial resolution $\tilde{X}$ of $\hat{X}$ such that $\tilde{X}$ is maximal and $Def(\tilde{X}) \cong Def(\hat{X})$. In this situation, let $\tilde{\mathcal{X}}$ (resp. $\hat{\mathcal{X}}$) be the universal family over $Def(\tilde{X})$ (resp. $Def(\hat{X})$). Then there is a projective birational morphism ν from $\tilde{\mathcal{X}}$ to $\hat{\mathcal{X}}$. For general $t \in Def(\hat{X})$, $\hat{X}_t$ has only ordinary double points and $\nu_t : \tilde{X}_t \longrightarrow \hat{X}_t$ is a small resolution of $\hat{X}_t$.

Proof. This is already shown in the proof of Theorem(2.5) (especially in the (1) $\Longrightarrow$ (2) part). Q.E.D.

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