# On the Intersection Ring of the Grassmannian of Complex Structures

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## Contents



# 1 Introduction

The Grassmannian of complex structures parametrizes orthogonal automorphisms of the Euclidean space $\mathbf{R}^{2n}$ whose square is the minus identity. Equivalently, it parametrizes minimal geodesics from the identity to the minus identity in the orthogonal group $SO(2n, \mathbf{R})$ [Mi]. This space is usually denoted by $CS_n$. It played a significant role in several important achievements in topology:

- in the investigation of orthonormal vector fields on spheres by Hurewicz and Adams;
- in the study of the existence of complex structures on even dimensional spheres by Borel and Serre;
- in the Bott's discovery of the eight-periodicity of homotopy groups of the stable real orthogonal groups.

Also, $CS_n$ serves as the classifying space of all complex bundles whose real reduction is trivial, by a result of the first author [Du1].

The goal of the present paper is to develop in a systematic way a Schubert calculus for $CS_n$. We hope that it will be useful also for topologists.

The space $CS_n$ has two connected components, each isomorphic to the homogeneous space

$$SO(2n, \mathbf{C})/U(n) \ \text{ or } \ SO(2n, \mathbf{C})/P\,,$$

where $P$ is the maximal parabolic corresponding to omitting the "right end root". This space is a connected component of the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. to the orthogonal form induced by the scalar product, and as such, it is also known as the orthogonal Grassmannian.

With the help of the group-theoretic description, we can use the characteristic map of Borel [Bo], and - via the theory of Bernstein-Gelfand-Gelfand [BGG] and Demazure [De] - divided differences of type $D$ to study the intersection theory on the space in question. In order to make the work with the characteristic map efficient, one needs a proper family of "invariant" polynomials that are well suited to divided differences, and also to geometry/topology at the same time.

A result of the second author [P1] identified Schubert classes in the homogeneous space $SO(2n, \mathbf{C})/U(n)$ with suitable Schur $P$-polynomials. In loc. cit. this identification used a geometric argument, namely an isomorphism $SO(2n, \mathbf{C})/U(n) \simeq SO(2n-1, \mathbf{C})/U(n-1)$, and an identification of the Schubert classes for the latter Grassmannian with Schur $P$-functions. (This last identification was based on comparison of the Pieri-type formulas from [HB] and [Mo].)

In the present paper we revisit the identification for the Schubert classes for $SO(2n, \mathbf{C})/U(n)$ with Schur $P$-functions via a direct *group-theoretic* argument based on the calculus of divided differences of type $D$. More precisely, we give a group-theoretic proof of a Pieri-type formula that is based on some vanishing results for operators composed of divided differences of type $D$ and simple reflections from the Weyl group of type $D$ (i.e. products of elements in the nil Hecke ring of type $D$). These last results form the most technical part of the present work. (We recommend to omit them by the first reading for the reader not having a sufficient combinatorial background.) Our proof of the Pieri-type formula follows a strategy for deriving similar formulas for various homogeneous spaces worked out by Ratajski and the second autor in a series of papers summarized in [P2]. This particular proof was promised in [PR2] – a paper that is now under revision. The proof uses esentially an iteration of the Leibniz-type formula for a simple divided difference applied to the product of two functions.

Combining the Pieri formula with a combinatorial lemma of Schur for the projective characters of the symmetric groups, we get a formula for the degree of Schubert varieties in $SO(2n,\mathbf{C})/U(n)$. (Occasionally, we also sketch another proof of the lemma of Schur.)

We remark that there exists now a refinement of Schur $P$-functions that seems to be even better adapted for some aspects of geometry. These are the so called $\widetilde{P}$-functions of [PR1], which are modeled on Schur $P$-functions. In [LP], A. Lascoux and the second author worked out a connection of orthogonal divided differences to $\widetilde{P}$-functions using vertex operators. This has led to *orthogonal Schubert polynomials* that are useful in various cohomological computations (cf. [KT] and [T2]).

After presenting the Schubert varieties in a group-theoretic way, we also describe them via Schubert-type conditions relative to some flag of linear subspaces, and finally we study them in terms of complex structures. To this end, we are guided by the Mahowald-Vassiljev-type formula ([DV],[V]):

$$H_i(CS_n) = \oplus_{k=0}^{n} H_{i-k(k-1)}(G_k(\mathbf{C}^n))\,,$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex $k$-planes through zero in $\mathbf{C}^n$. This is done by some vector bundle argument.

We end the paper by illustrating how the Schubert calculus developed here can be used to solve problems about enumeration of complex structures which satisfy some natural conditions of "partial overlaping" with a certain number of complex structures in general position in $\mathbf{R}^{2n}$. One of the applications leads to an interesting algebraic conjecture about homomorphisms between the cohomology ring of $CS_n^+$ and that of the Grassmannian $G_k(\mathbf{C}^n)$.

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# 2 Preliminaries, notation, and conventions.

We start with some algebraic preliminaries on even orthogonal groups. We fix a positive integer $n$. Suppose that $H = SO(2n,\mathbf{C})$ is the orthogonal group (of type $D_n$) over the field of complex numbers. Let us use the following notation:

$B$ - a fixed Borel subgroup of $H$,

$T \subset B$ - a fixed maximal torus.
$\mathcal{R}$ - the root system of $H$ associated with $T$,
$\Sigma$ - a set of simple roots of $\mathcal{R}$ associated with $B$,
$W$ - the Weyl group of $(H, T)$,
In a standard Bourbaki [Bu] realization, we have:

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\} \subset \mathbf{R}^n = \oplus_{i=1}^n \mathbf{R} e_i,$$

$$\Sigma = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\},$$
$$W = S_n \ltimes \mathbf{Z}_2^{n-1}.$$

A typical element of $W$ can be written as a pair $(\tau, \epsilon)$, where $\tau \in S_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ is a sequence of elements of $\mathbf{Z}_2 = \{-1, 1\}$ such that $\#\{i : \epsilon_i = -1\}$ is even. Multiplication in $W$ is given by

$$(\tau, \epsilon) \cdot (\tau', \epsilon') = (\tau \circ \tau', \delta),$$

where "$\circ$" denotes the composition of permutations and $\delta_i = \epsilon_{\tau'(i)} \cdot \epsilon'_i$. The following lemma can be easily verified (and is pretty well-known). For $w \in W$, let $l(w)$ denote the length of w takem w.r.t. to the above $\Sigma$.

**Lemma 2.1** *For any $w \in W$, $l(w)$ is*

$$\sum_{i=1}^{n} \#\{j \mid j > i \,\&\, w(j) < w(i)\} + \sum_{\epsilon_p = -1} 2\#\{q \mid q > p \,\&\, w(q) > w(p)\}.$$

We will use the "barred-permutation" notation, indicating by a bar a place in the permutation where $\epsilon_i = -1$.

The poset $W^*$ of the minimal length left coset representatives of $S_n$ in $W$ consists o f elements of the form

$$[y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1]$$

where $k$ runs over the set of even integers between 1 and $n$, $y_1 < \cdots < y_{n-k}$, and $z_k > \cdots > z_1$. The following lemma, that is easy to prove, gives us the lengths of some barred permutations important for this paper.

**Lemma 2.2** *Let, as above, $y_1 < \cdots < y_{n-k}$ and $z_k > \cdots > z_1$ be sequences of integers that are complementary in $\{1, \ldots, n\}$. Assume that $k$ is even. Then in $W$ we have*

$$l([y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1]) = \sum_{j=1}^{k} (n - z_j).$$

We set $\rho(k) := (k, k-1, \ldots, 1)$, a "triangular partition" of length $k$.

Given a strict partition $\alpha \subset \rho(n-1)$, we set

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_l + 1)$$

if $l = l(\alpha)$ is even, and

$$\alpha^+ := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_l + 1, 1)$$

if $l = l(\alpha)$ is odd. Note that $\alpha^+$ is of even length.

Given a strict partition $\mu \subset \rho(n)$ of even length $k$, we associate with it the following element $w_\mu$ of $W^*$. We set

$$w_\mu := [y_1, y_2, \ldots, y_{n-k}, \overline{n+1-\mu_1}, \overline{n+1-\mu_2}, \ldots, \overline{n+1-\mu_k}]\,.$$

Note that $l(w_\mu) = |\mu| - k$.

Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $l(w_\lambda) = |\alpha|$.

Finally, we adopt a convention that all homology or cohomology groups in the present paper are taken with integer coefficients.

# 3 Vanishing in nil Hecke ring of type $D$.

We define simple divided differences of type $D_n$ which are operators $\partial_i : \mathbf{Z}[X] \to \mathbf{Z}[X]$, $i = 1, \ldots, n$, of degree $-1$ acting on the ring of polynomials $\mathbf{Z}[X]$ where $X$ is a fixed set of indeterminates $X = \{x_1, x_2, \ldots, x_n\}$. To this end, we denote by $s_i$, $1 \le i \le n-1$, the transposition

$$[1, \ldots, i-1, i+1, i, i+2, \ldots, n] \in S_n \subset W\,,$$

acting on $X$ by interchanging $x_i$ and $x_{i+1}$. Moreover, let

$$s_n = [1, \ldots, n-2, \overline{n}, \overline{n-1}]$$

be the reflection which transposes $x_{n-1}$ with $x_n$ and changes the signs of both the variables. The remaining variables are invariant under the action of these transpositions. This action is extended multiplicatively to the ring $\mathbf{Z}[X]$. Note that $s_n$ commutes with $s_i$, $i \neq n-2$, and

$$s_{n-2} \cdot s_n \cdot s_{n-2} = s_n \cdot s_{n-2} \cdot s_n\,.$$

Simple divided differences of type $D$ are defined as follows:

$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1}), \qquad i = 1, \ldots, n-1;$$

$$\partial_n(f) = (f - s_n f)/(x_{n-1} + x_n).$$

The algebra generated by the divided diferences $\partial_i$, $1 \le i \le n-1$ that goes back to Newton, has been recently extensively studied by Lascoux and Schutzenberger [LSc]. Here we investigate the effect of adding one extra divided difference $\partial_n$ (which changes the picture in a drastic way).

For every $f, g \in \mathbf{Z}[X]$ and any $i$, we have:

$$\partial_i(f \cdot g) = f \cdot (\partial_i g) + (\partial_i f) \cdot (s_i g) \tag{1}$$

(a Leibniz-type formula).

For a given $\mathbf{a} = (a_n, a_{n-1}, \ldots, a_2, a_1) \in \{-1, 0, 1\}^n$, we define the generating function:

$$E_{\mathbf{a}} = \prod_{i=1}^{n} (1 + a_i x_i). \tag{2}$$

In particular, for $\mathbf{a} = (1, \ldots, 1)$, the resulting generating function, denoted by $E$, is the generating function for the elementary symmetric polynomials $e_i(X) = e_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$.

**Lemma 3.1** a) *We have* $s_i(E_{\mathbf{a}}) = E_{\mathbf{a}'}$, *where*

$$\mathbf{a}' = \begin{cases} (a_n, \ldots, a_{i+2}, a_i, a_{i+1}, a_{i-1}, \ldots, a_1) & i < n, \\ (-a_{n-1}, -a_n, a_{n-2}, \ldots, a_1) & i = n. \end{cases}$$

b) *For* $i = 1, 2, \ldots, n-1$,

$$\partial_i(E_{\mathbf{a}}) = d \cdot E_{\mathbf{a}'} \qquad \text{if } \ a_i = a_{i+1} + d \qquad (d = -2, -1, 0, 1, 2),$$

*where* $\mathbf{a}' = (a_n, \ldots, 0, 0, \ldots, a_1)$ *is the sequence* $\mathbf{a}$ *with* $a_{i+1}$, $a_i$ *replaced by zeros.*

c) $\partial_n(E_{\mathbf{a}}) = (a_n + a_{n-1}) \cdot E_{(0,0,a_{n-2},\ldots,a_1)}$.

*In particular if* $\Delta$ *is a composition of some* $s$*- and* $\partial$*-operations, then for every* $\mathbf{a}$, $\Delta(E_{\mathbf{a}}) = (\mathit{scalar}) \cdot E_{\mathbf{a}'}$, *where* $\mathbf{a}'$ *is uniquely determined if this scalar is not zero.*

**Proof.** We prove e.g. c). We have, with $\mathbf{a}' = (0, 0, a_{n-2}, \ldots, a_1)$,

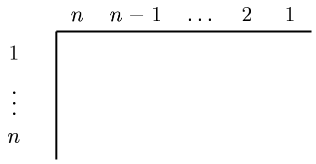
$$\partial_n(E_{\mathbf{a}}) = \frac{(1 + a_{n-1}x_{n-1})(1 + a_n x_n) - (1 - a_{n-1}x_n)(1 - a_n x_{n-1})}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'}$$

$$= \frac{(a_{n-1} + a_n)(x_{n-1} + x_n)}{x_{n-1} + x_n} \cdot E_{\mathbf{a}'} = (a_n + a_{n-1}) \cdot E_{\mathbf{a}'} ,$$

as desired. $\square$

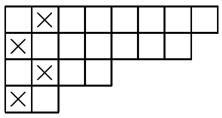
We now recall the following fact from [BGG] and [De]. For any $w \in W$ and any reduced decomposition $w = s_{i_1} \cdots s_{i_l}$ one can define $\partial_w = \partial_{i_1} \circ \cdots \circ \partial_{i_l}$ – an operator on $\mathbf{Z}[X]$ of degree $-l(w)$. In fact, since divided differences satisfy braid relations, $\partial_w$ does not depend on the chosen reduced decomposition of $w$.

Suppose a strict partition $\mu$ with even $l(\mu)$ is given. Let us use the following coordinates for boxes in the Ferrers' diagram $D_\mu$ of $\mu$:




We associate with $\mu$ a certain distinguished reduced decomposition of $w_\mu \in W$. To this end, let us modify the diagram $D_\mu$ in the following way. Remove one box from each row of $D_\mu$ : from rows with even numbers remove the box in the $n$-th column, and from rows with odd numbers remove the box in the $(n-1)$-th column.

We display the removed boxes in the picture using the symbol $\times$ and denote the so obtained diagram by $\mathring{D}_\mu$. For example, $\mathring{D}_{(8,7,4,2)}$ is:

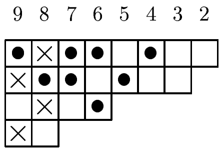


Assume now, that a subset $D \subset \mathring{D}_\mu$ is given. A box belonging to $D$ will be called *a $D$-box* and a box from the difference $D_\mu \setminus D$ will be called *a $\sim$$D$-box.*

**Definition 3.2** *Read $\mathring{D}_\mu$ row by row from left to right and from top to bottom. Every $D$-box (resp. $\sim$ $D$-box) in the $i$-th column gives us $s_i$ (resp. $\partial_i$). Then $\partial_\mu^D$ is the composition of the resulting $s_i$'s and $\partial_i$'s (the composition written from right to left).*

**Definition 3.3** *Read $\mathring{D}_\mu$. Every $D$-box in the $i$-th column gives us $s_i$. $\sim$$D$-boxes give no contribution. Then, $r_D$ is the word obtained by writing the resulting $s_i$'s from right to left. ( In other words, one obtains $r_D$ by erasing all the $\partial_i$'s from $\partial_\mu^D$.)*

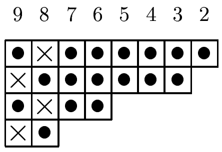
For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,



$$\partial_\mu^D = \partial_8 \circ s_6 \circ \partial_7 \circ \partial_9 \circ \partial_3 \circ \partial_4 \circ s_5 \circ \partial_6 \circ s_7 \circ s_8 \circ \partial_2 \circ \partial_3 \circ s_4 \circ \partial_5 \circ s_6 \circ s_7 \circ s_9$$

$$r_D = s_6 \cdot s_5 \cdot s_7 \cdot s_8 \cdot s_4 \cdot s_6 \cdot s_7 \cdot s_9 \,.$$

One can easily prove that for $D = \mathring{D}_\mu$, we have $r_D \in R(w_\mu)$ - the set of reduced decompositions of $w_\mu$. This is our *distinguished reduced decomposition* of $w_\mu$. For example, for $n = 9$ and $\mu = (8, 7, 4, 2)$,

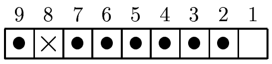


$$w_\mu = s_8 \cdot s_6 \cdot s_7 \cdot s_9 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_8 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdot s_7 \cdot s_9 \,.$$

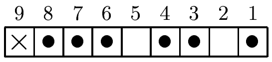
Let now $\mu$ be a strict partition with even length. We will examine subsets $D \subset \mathring{D}_\mu$ for which $\partial_\mu^D(E) = 0$ (we say: " $D$ causes the vanishing").

In many computations in this section, we will apply compositions of the operators of boxes of $D_\mu$ to the generating functions $E_{\mathbf{a}}$. With the following example we illustrate how such operators act.

**Example 3.4** *Let $n = 9$. We apply the operators of boxes from left to right to $E_{\mathbf{a}}$, where $\mathbf{a} = (a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1)$, and obtain $E_{\mathbf{a}'}$; "$\bullet$" denotes a D-box and empty boxes are $\sim$D-boxes. We give 2 examples of the action of the operators associated with a row in $D_\mu$:*



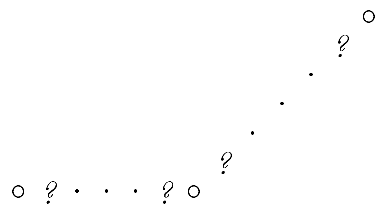
$$\mathbf{a}' = (-a_8, a_7, a_6, a_5, a_4, a_3, a_2, 0, 0)$$



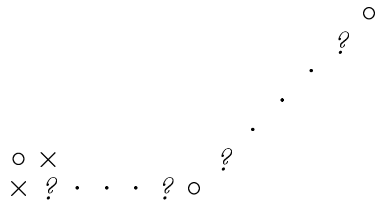
$$\mathbf{a}' = (a_8, a_7, a_6, 0, a_4, a_3, 0, a_1, 0)$$

**Lemma 3.5** *The following configurations of $\sim$D-boxes (noted by "$\circ$") in $D_\mu$ give the vanishing:*

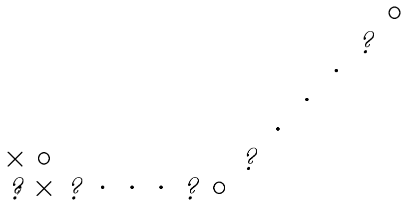
*1. (A three ∼D-box configuration)*



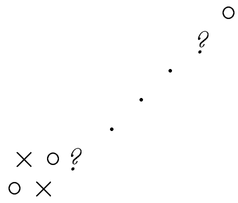
*2. (A three ∼D-box configuration)*



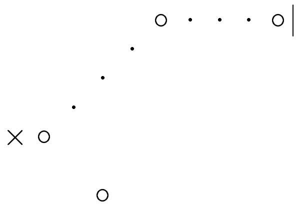
*3. (A three ∼D-box configuration)*



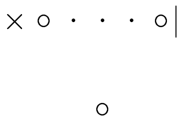
also:



*4.*



in particular:



5.

$$
\begin{array}{ccccccc}
 & & & & \circ & \cdots & \circ| \\
 & & & \cdot & & & \\
 & & \cdot & & & & \\
 & \cdot & & & & & \\
 & \circ & & & & & \\
\circ\ \times & & & & & & \\
 & \circ & & & & &
\end{array}
\qquad \text{in particular:} \qquad
\begin{array}{c}
\circ \times \circ \cdots \circ| \\
\circ
\end{array}
$$

(Above, "?" can be $\circ$, $\bullet$, or $\times$; in 4. and 5., the symbol "|" denotes the end of a row; the skew directions are all parallel to the antidiagonal.)

**Proof.** Direct calculation using Lemma 3.1. $\square$

We state this lemma in a stronger form than is needed in the present paper. This stronger form will be used in the revised version of [PR2].

Let us fix an element $w = [y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1] \in W^*$. Recall that $k$ is even. We treat a given reduced decomposition $w = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_l}$ as a sequence of simple transposition operations, which produces the element in question from the identity permutation:

$$[y_1, y_2, \ldots, y_{n-k}, \overline{z}_k, \overline{z}_{k-1}, \ldots, \overline{z}_1] = (\cdots([1, 2, \ldots, n] \cdot s_{i_1}) \cdots) \cdot s_{i_l}.$$

In the following, the simple transpositions involved will be called "the $s_{i_h}$-operations" ($h = 1, \ldots, l$).

**Proposition 3.6** *We have the following two possibilities for the action of $s_{i_h}$-operations on the $z$'s:*

1. *If $i_h = n$ then this operation is:*

$$[\ldots, z, z'] \ \rightarrow \ [\ldots, \overline{z}', \overline{z}]$$

2. *If $i_h < n$, then this operation is:*

$$[\ldots, z, x, \ldots] \rightarrow \ [\ldots, x, z, \ldots]$$

*where $x \neq z_j$ for $j = 1, \ldots, k$.*

**Proof.** The most economical way (i.e. the one that requires the minimal number of $s_{i_h}$-operations) to supply all the $z$'s with bars, is the following one. We move the pair $[\ldots, z_{k-1}, \ldots z_k, \ldots]$ towards the end of the permutation (i.e.

towards the $n-1$-th and $n$-th places), and then the $s_n$-operation transforms $[\dots, z_{k-1}, z_k]$ to $[\dots, \overline{z}_k, \overline{z}_{k-1}]$. This requires at least

$$(n - z_k) + (n - 1 - z_{k-1}) + 1 = (n - z_k) + (n - z_{k-1})$$

$s_{i_h}$-operations. Then, we do the same with the pairs:

$$[\dots, z_{k-3}, \dots, z_{k-2} \dots] \ , \ [\dots, z_{k-5}, \dots, z_{k-4}, \dots] \ , \ \dots$$

up to the last pair $[\dots, z_1, \dots, z_2, \dots]$. We see that $l(w)$ (which is the number of $s_{i_h}$-operations) is greater than or equal to

$$\sum_{j=1}^{k} (n - z_j) \, .$$

But by Lemma 2.2, this last number is actually equal to $l(w)$. This means that no $s_{i_h}$-operation, for $i_h < n$, interchanges two (bar-free) $z$'s (on their way towards the end of the permutation), and that exactly $k/2$ $s_{i_h}$-operations with $i_h = n$ appear. This implies immediately both assertions of the proposition. □

**Corollary 3.7** *1. No $s_{i_h}$-operation for $i_h < n$ interchanges two (bar-free) $z$'s. (Also, it does not interchange two barred $z$'s.)*

*2. If $i_h = n$ then the $s_{i_h}$-operation interchanges two (bar-free) $z$'s on the $n$-th and $n-1$-th places (more precisely, these are $z$'s od the form: $z_{2i-1}$ and $z_{2i}$), and supplies them with bars.*

Now we assume that $\lambda$ is another strict partition, that $D \subset \mathring{D}_\mu$, and that $r_D \in R(w_\lambda)$. Suppose that a $D$-box appears in the $i$-th column where $i < n$. We define the *mark* of this box to be $p$, if the corresponding $s_{i_h}$-operation acts on the $i$-th and $i+1$-th places as follows:

$$[\dots, z_p, x, \dots] \to [\dots, x, z_p, \dots] \, ,$$

where $x \neq z_j$ , $j = 1, \dots, k$. A $D - box$ in the $n$-th column has mark $p$ if the corresponding $(s_{i_h} = s_n)$-operation acts via

$$[\dots, z_{p-1}, z_p] \to [\dots, \overline{z}_p, \overline{z}_{p-1}] \, .$$

In particular, boxes in the $n$-th column have only odd marks. In the following lemma, we collect some simple properties of marks.

**Lemma 3.8** *1. (Conectedness) The D-boxes with a fixed mark in one row form a connected set. (The boxes marked by $\times$ are treated as empty boxes).*

2. *(Separation) In a fixed row, the two sets of D-boxes equipped with different marks are disconnected (i.e. there is at least one $\sim$ D-box between them; a box marked by $\times$ is not treated as as $\sim$D-box).*

3. *The sequence of boxes with odd mark $p$ is of the form:*

$$(t_n, n), (t_{n-2}, n-2), \ldots, (t_{z_p}, z_p)\,,$$

*where $p \leq t_n \leq t_{n-2} \leq \cdots \leq t_{z_p}$.*

4. *The sequence of boxes with even mark $p$ is of the form:*

$$(t_{n-1}, n-1), (t_{n-2}, n-2), \ldots, (t_{z_p}, z_p)\,,$$

*where $p \leq t_{n-1} \leq t_{n-2} \leq \cdots \leq t_{z_p}$.*

5. *The marks of boxes in a fixed column (strictly) increase from top to bottom.*

6. *The marks of boxes in a fixed row (weakly) decrease from left to right.*

**Definition 3.9** *The set of D-boxes with mark $p$ is called the* ribbon of mark *$p$.*

We have two basic operations of deforming ribbons.

- ("Push down") Let $i$ be odd and suppose that the boxes

$$(i, n), (i, n-2), \ldots, (i, j)$$

form an entire ribbon. The operation transforms them to

$$(i-2, n), (i-2, n-2), \ldots, (i-2, j)\,.$$

Let $i$ be even and suppose that the boxes

$$(i, n-1), (i, n-2), \ldots, (i, j)$$

form an entire ribbon. The operation transforms them to

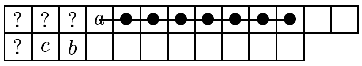
$$(i-2, n-1), (i-2, n-2), \ldots, (i-2, j)\,.$$

For example, the ribbons

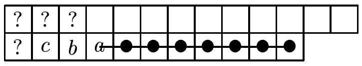
can be pushed down to

- ("Breaking a ribbon") Let $j \leq n-2$. The operation transforms a final segment
$$(i,j),(i,j-1),\ldots,(i,h)$$
of a ribbon to
$$(i-1,j),(i-1,j-1),(i-1,h)\,,$$
provided $(i-1,j+1)$ is a $\sim D$-box, or it is $\times$ and $(i-1,j+2)$ is a $\sim D$-box. The box $(i,j)$ (before the operation) is called the *breaking box.*

  For example, for a breaking box $a$, $b$ a $\sim D$-box, or $b = \times$ and $c$ a $\sim D$-box:




  can be broken at $a$ and transformed to




Using the braid relations in $W$ one easily shows that if $r_D \in R(w_\lambda)$, then, after breaking a ribbon in $D$, we get $D'$ such that $r_{D'} \in R(w_\lambda)$. In the case of the push down operation, it is clear that we get $D'$ with $r_D = r_{D'}$. Note that any configuration of boxes $D \subset D_\mu$ such that $r_D \in R(w_\lambda)$ can be obtained from $\mathring{D}_\lambda \subset \mathring{D}_\mu$ by a sequence of operations of the above described type.

**Definition 3.10** *(" Maximal deformation" of $\mathring{D}_\lambda \subset \mathring{D}_\mu$)*

- *Pick the lowest ribbon. Push it down as many times as possible. Then choose the leftmost breaking box on this ribbon (if it exists) and break the ribbon.*

- *Pick a ribbon and suppose that lower ribbons in $\mathring{D}_\lambda$ have been already deformed. Push down this ribbon as many times as possible. Let $a$ be the leftmost breaking box on the ribbon. Break the ribon at $a$ as many times as possible. Then choose the next leftmost breaking box $b$ and break the ribbon at $b$ as many times as possible etc.*

For some examples of maximal deformations of diagrams, see Example 4.7.

**Proposition 3.11** *Let $D \subset \mathring{D}_\mu$ be such that $r_D \in R(w_\lambda)$. If $\partial_\mu^D(E) \neq 0$ then $D$ is the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$.*

**Proof.** The proof is by descending induction on the mark of a ribbon. Pick the ribbon with mark $p$. Assume that the ribbons with marks $p+1, \ldots, l(\lambda)$ have been already maximally deformed. Suppose that we have either a possibility of pushing down of a ribbon or breaking a ribbon. In the case of the former operation we will refer to boxes of the three involved rows; in the case of the latter operation, we will refer to the two involved rows. We note that

- any box directly to the right of the end of the (deformed) ribbon is a $\sim D$-box;
- any box directly below a box of the (deformed) ribbon is a $\sim D$-box;
- $\sim D$-boxes in the $n$-th or $n-1$-th column cannot be supplied by marks smaller that $p$.

This implies that if we not perform the operations in a maximal way, then we will either obtain a configuration

$$\begin{array}{cccccccc} & & & & & & & \circ \\ \circ & ? & \cdot & \cdot & \cdot & ? & \circ & \end{array}$$

or, we will get one of the following two possibilities:

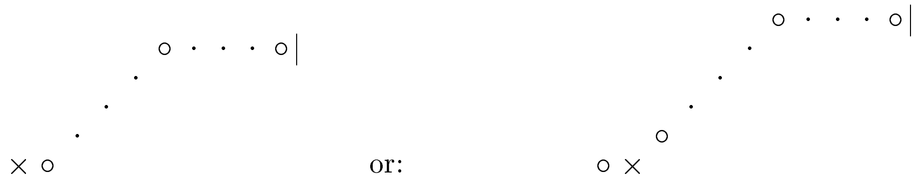
$$\begin{array}{cccccccc} \bullet & \times & & & & & & \\ \times & b & c & & & & & \\ a & \times & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \end{array} \qquad\qquad \begin{array}{cccccccc} \times & & & & & & & \\ a & \times & b & & & & & \\ \times & c & \bullet & \cdot & \cdot & \cdot & \cdot & \bullet \end{array}$$

where $a, b$ and $c$ are $\sim D$-boxes. By Lemma 3.5 all these three configurations of $\sim D$-boxes give the vanishing. The obtained contradiction means that the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$ is necessary to avoid the vanishing. $\Box$

As a corollary of this proposition, we get the following two results bounding the size of $\mu$ w.r.t. $\lambda$, if one wants to avoid the vanishing. We will need some additional notation. Given a strict partition $\mu$ of even length $l$, we denote by $\mu^-$ the strict partition $(\mu_1 - 1, \ldots, \mu_l - 1)$. Note that $(\mu^-)^+ = \mu$. We also set $\mathring{l}(\mu) := l(\mu^-)$. Setting for a strict partition $\alpha \subset \rho(n-1)$, $\lambda := \alpha^+$, we have $\mathring{l}(\lambda) = l(\alpha) = \# D_\alpha = \# \mathring{D}_\lambda$.

**Proposition 3.12** *If, for the maximal deformation $D$ of $\mathring{D}_\lambda \subset \mathring{D}_\mu$, one has $\partial^D_\mu(E) \neq 0$, then $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$.*

**Proof.** Pick the highest row from which some ribbon has been pushed down in the maximal deformation. After breaking some higher ribbons we get the following configuration of $\sim D$-boxes:



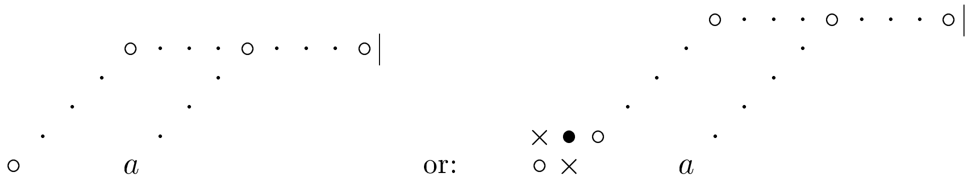
If we suppose $\mathring{l}(\mu) \geq \mathring{l}(\lambda) + 2$, then there exists a $\sim D$ box in some lower row than the rows of the displayed boxes. By Lemma 3.5, we get the vanishing – contradiction. $\square$

**Proposition 3.13** *Assume that $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$. If for the maximal deformation $D$ of $\mathring{D}_\lambda \subset \mathring{D}_\mu$ one has $\partial^D_\mu(E) \neq 0$, then $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip.*

**Proof.** Suppose that $\lambda_i < \mu_{i+1}$ for some $i$. We can assume that for some $j < i$,

$$\mu_i = \lambda_{i-1}, \mu_{i-1} = \lambda_{i-2}, \ldots, \mu_{j+2} = \lambda_{j+1} \quad \text{but} \quad \lambda_j > \mu_{j+1}\,.$$

After the maximal deformation, we get in the consecutive rows with numbers $i+1, i, \ldots, j+1$ the following configuration of $\sim D$-boxes:



where $a$ denotes a $\sim D$-box. We get the vanishing by Lemma 3.5 – contradiction. $\square$

The maximal deformation is obtained by breaking each row such that $\lambda_i = \mu_{i+1}$, at one breaking point.

Among the connected components of $\mathring{D}_\mu \setminus \mathring{D}_\lambda$ we have those which do not meet the $n$-th column: they are ordinary horizontal strips [Ma]. Those which meet the $n$-th component are of the form:

$$\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&&&&&&&&\circ&\cdot&\cdot&\cdot&\circ\\
&&&&&&&&&&&&&&&&&.\\
&&&&&&&&&&&&&&&&.\\
&&&&&&&&&&&&&&&.\\
&&&&&&&&&&\circ&\cdot&\cdot&\cdot&\circ\\
&&&&&&&&&.\\
&&&&&&&&.\\
&&&&&&&.\\
\circ&\times&\circ&\cdot&\cdot&\cdot&\circ\\
\times
\end{array} \tag{3}$$

or of the form:

$$\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&&&&&&&&\circ&\cdot&\cdot&\cdot&\circ\\
&&&&&&&&&&&&&&&&&.\\
&&&&&&&&&&&&&&&&.\\
&&&&&&&&&&&&&&&.\\
&&&&&&&&&&\circ&\cdot&\cdot&\cdot&\circ\\
&&&&&&&&&.\\
&&&&&&&&.\\
&&&&&&&.\\
\times&\bullet&\circ&\cdot&\cdot&\cdot&\circ\\
\circ&\times
\end{array} \tag{4}$$

Note the following particular case of (4):

$$\begin{array}{cccccccc}
\times&\bullet&\bullet&\cdot&\cdot&\cdot&\bullet&\bullet\\
\circ&\times&\circ&\cdot&\cdot&\cdot&\circ|\\
\times
\end{array} \tag{5}$$

After the maximal deformation an ordinary horizontal strip becomes

$$
\begin{array}{ccccccccc}
 & & & & \circ & \cdot & \cdot & \cdot & \circ| \\
 & & & \cdot & & & & & \\
 & & \cdot & & & & & & \\
 & \cdot & & & & & & & \\
\circ & & & & & & & &
\end{array}
$$

In turn, configuration (4) which is not of the form (5), and configuration (3) are deformed to:

$$
\begin{array}{ccccccccccc}
 & & & & & & \circ & \cdot & \cdot & \cdot & \circ| \\
 & & & & & \cdot & & & & & \\
 & & & & \cdot & & & & & & \\
 & & & \cdot & & & & & & & \\
\times & \bullet & \circ & & & & & & & & \\
\circ & \times & & & & & & & & &
\end{array}
$$

Of course, configuration (5) does not change under the maximal deformation.

**Proposition 3.14** *Suppose that $\lambda \subset \mu \subset \rho(n)$ are strict partitions such that $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular, $\mathring{l}(\mu) \leq \mathring{l}(\lambda) + 1$). Let $D$ be the maximal deformation of $\mathring{D}_\mu \setminus \mathring{D}_\lambda$. Then $\partial_\mu^D(E) = 2^m$, where $m$ is the number of connected components of $\mathring{D}_\mu \setminus D$.*

**Proof.** Different connected components of $\mathring{D}_\mu \setminus D$ lie in separate rows and separate columns. Let us number these components from top to bottom. Pick a connected component of $\mathring{D}_\mu \setminus D$. The part of $\partial_\mu^D$ associated with the boxes in the rows preceding the rows of the component, transform $E$ into $2^{m'} E_{\mathbf{a}}$, where $m'$ is the number of components preceding the given one. If the first row of its appearance has odd (resp. even) number, then $\mathbf{a} = (1, 1, \ldots, 1, *, \ldots, *)$ (resp. $\mathbf{a} = (-1, 1, \ldots, 1, *, \ldots, *)$ ) and the cardinality of displayed $\pm 1$'s is the length of the first row of $\mu$ supporting the component, the count including the box $\times$. In turn, the operators of rows supporting the component transform $E_{\mathbf{a}}$ to $2E_{\mathbf{a}'}$ for some $\mathbf{a}'$. The multiplicity 2 comes from the highest leftmost box of the component; the operators of all remaining boxes give the multiplicity 1. If such highest leftmost box lies in the $h$-th column where $h < n$, then one gets the multiplicity 2 by applying $\partial_h$ to $E_{\mathbf{b}}$ where $\mathbf{b} = (\ldots, -1, 1, \ldots)$, the displayed entries being in $(h+1)$-th and $h$-th places. If the component is of the form (5), then we get the multiplicity 2 by applying $\partial_n$ to $E_{\mathbf{c}}$ where $\mathbf{c} = (1, 1, \ldots)$. This proves the proposition. $\square$

We summarize the results of this section in the following theorem (observe that the connected components of $\mathring{D}_\mu \setminus D$ are in one-to-one correspondence with the connected components of $D_{\mu^-} \setminus D_{\lambda^-}$):

**Theorem 3.15** *Let $\lambda \subset \mu \subset \rho(n)$ be strict partitions. Then for $D \subset \mathring{D}_\mu$, one has $\partial_\mu^D(E) \neq 0$ iff $D_{\mu^-} \setminus D_{\lambda^-}$ is a horizontal strip (in particular $\mathring{l}(\mu) < \mathring{l}(\lambda) + 1$), and $D$ is obtained by the maximal deformation of $\mathring{D}_\lambda \subset \mathring{D}_\mu$. In this case, $\partial_\mu^D(E) = 2^m$, where $m$ is the number of connected components of $D_{\mu^-} \setminus D_{\lambda^-}$.*

# 4 A group-theoretic approach to Schubert calculus for $CS_n$.

We first introduce some notation. Recall that $H = SO(2n, \mathbf{C})$ and $B \subset H$ is a Borel subgroup of $H$. We denote by $P$ - the maximal parabolic subgroup of $H$ containing $B$ and corresponding to the subset $\Sigma$ of simple roots minus the right end root, by $F$ – an "isotropic" orthogonal flag manifold $H/B$, and by $G$ – the orthogonal Grassmannian $H/P$.

Moreover, the Schubert variety $X_w$, $w \in W$, is defined as the closure of the Schubert cell $B^- wB/B$ in $H/B$ ($B^-$ is the opposite Borel subgroup to $B$). We record the following well-known result:

**Lemma 4.1** *$X_w$ is a (closed) subvariety of $H/B$ of (complex) codimension $l(w)$.*

Let $X = \{x_1, \ldots, x_n\}$ be a sequence of variables. For brevity, we denote also by the symbol $X_w$ the class of the variety $X_w$ in $H^{2l(w)}(F)$. Let $\alpha \subset \rho(n-1)$ be a strict partition and put $\lambda := \alpha^+$; one has $X_{w_\lambda} \in H^{2|\alpha|}(F)$. In fact, $X_{w_\lambda}$ belongs already to $H^{2|\alpha|}(G) \subset H^{2|\alpha|}(F)$. Let us denote this element in $H^{2|\alpha|}(G)$ (as well as the representing it Schubert variety in $G$), by $\sigma_\alpha$.

There exists a surjective ring homomorphism $c : \mathbf{Z}[1/2][X] \to H^*(F)$ (called the *Borel characteristic map*) such that for a homogeneous $f \in \mathbf{Z}[X]$ one has

$$c(f) = \sum_{l(w)=\deg f} \partial_w(f)\ X_w\,. \tag{6}$$

(The original Borel's definition [Bo] of the characteristic map was different; this description comes from the work of Bernstein-Gelfand-Gelfand [BGG] and Demazure [De].)

Note (cf. e.g. [Bo]) that the ring $H^*(G)$ can be identified algebraically as

$$H^*(G) = \mathbf{Z}[X]^{S_n}/(e_i(X^2), i = 1, \ldots, n-1; x_1x_2\cdots x_n)\,,$$

where $X^2 = (x_1^2, \ldots, x_n^2)$.

We have also another identification of $H^*(G)$ stemming from [Du1] and [P1]:

**Lemma 4.2** *Let $S$ be the tautological rank $n$ subbundle on $G$. The Chern classes $c_i(S)$ are all divisible by 2, and one has the identification $\sigma_i = \frac{1}{2}c_i(S^*)$ for $i = 1, \ldots, n-1$. Moreover,*

$$H^*(G) = \mathbf{Z}[\sigma_1, \ldots, \sigma_{n-1}]/(R_i, 1 \le i \le n-1)\,,$$

*where, with the convention $\sigma_i = 0$ for $k > n-1$, the relations $R_i$ are given by*

$$R_i = \sigma_i^2 - 2\sigma_{i-1}\sigma_{i+1} + 2\sigma_{i-2}\sigma_{i+2} - \cdots + 2(-1)^{i-1}\sigma_1\sigma_{2i-1} + (-1)^i\sigma_{2i}\,.$$

It turns out that after restriction to $\mathbf{Z}[X]^{S_n} \otimes \mathbf{Z}[1/2]$, the map $c$ goes onto $H^*(G)$, and we have the following fact. Let, from now on, $e_r = e_r(X)$ denote the $r$-th elementary symmetric function in $X = \{x_1, \ldots, x_n\}$.

**Lemma 4.3** *For every $r = 1, \ldots, n-1$, one has $c(e_r) = 2\sigma_r$.*

**Proof.** We have

$$\partial_{n-r}\cdots\partial_{n-2}\partial_n(e_r) = 2\,.$$

Any other divided difference operator of degree $r$ applied to $e_r$ gives 0. This implies the assertion. □

For a strict partition $\alpha \subset \rho(n-1)$, we choose a homogeneous $f_\alpha \in \mathbf{Z}[1/2][X]$ such that $c(f_\alpha) = \sigma_\alpha$. Then, for $w \in W^*$ with $l(w) = l(\alpha)$, one has $\partial_w(f_\alpha) \neq 0$ iff $w = w_\lambda$ and $\partial_{w_\lambda}(f_\alpha) = 1$ for $\lambda = \alpha^+$. We want to find the coefficients $d_\beta$ in the expansion

$$c(f_\alpha \cdot e_r) = \sum d_\beta\sigma_\beta\,. \tag{7}$$

**Proposition 4.4** *In the above notation, setting $\mu := \beta^+$, one has*

$$d_\beta = \sum \partial_\mu^D(e_r),$$

*where the sum is over all $D \subset \mathring{D}_\mu$ such that $r_D \in R(w_\lambda)$ (here, $\lambda = \alpha^+$).*

**Proof.** We have $d_\beta = \partial_{w_\mu}(f_\alpha \cdot e_r)$, and $\partial_{w_\mu} = \partial_\mu^\emptyset$. The integer $d_\beta = \partial_\mu^\emptyset(f_\alpha \cdot e_r)$ is computed by a consecutive application of the Leibniz-type formula (1): we apply only the $\partial_i$'s (and the identity operators) to $f_\alpha$, and both the $s_i$'s and $\partial_i$'s to the factor $e_r$. We get

$$d_\beta = \sum \partial_{r_D}(f_\alpha) \cdot \partial_\mu^D(e_r)\,,$$

the sum over all $D \subset \mathring{D}_\mu$. The summand corresponding to a subset $D \subset \mathring{D}_\mu$ is not zero only if $\#D = \deg f_\alpha$ and $\#(D_\mu \setminus D) = r$. By the choice of $f_\alpha$, $\partial_{r_D}(f_\alpha) = 0$ if $r_D \notin R(w_\lambda)$, and equals 1 if $r_D \in R(w_\lambda)$, and thus we get the desired equality. $\square$

**Remark 4.5** *This use of an iterated Leibniz-type formula to compute the multiplicities $d_\beta$ stems from a series of papers of Ratajski and the second author (cf. [P2]). It was also known to Kostant and Kumar – see [KK].*

Combining this proposition with Theorem 3.15, and taking into account Lemma 4.3, we get a group-theoretic proof of the following result (that is refered to as a "Pieri-type formula"):

**Theorem 4.6** *Let $\alpha \subset \rho(n-1)$ be a strict partition. Then for any $1 \le r \le n-1$,*

$$\sigma_\alpha \cdot \sigma_r = \sum_\beta 2^{m_\beta}\ \sigma_\beta\,,$$

*where the sum is over all strict partitions $\beta \subset \rho(n-1)$ such that $D_\beta \setminus D_\alpha$ is a horizontal strip of length $r$ and $m_\beta$ is the number of connected components of $D_\beta \setminus D_\alpha$ minus 1.*

(Cf. also [P1, Theorem 6.17'].)

**Example 4.7** *Let $n = 8$. We examine the product $\sigma_{5,3} \cdot \sigma_4$:*

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \end{matrix} \qquad \text{times} \qquad \begin{matrix} \circ & \circ & \circ & \circ \end{matrix}$$

*On the LHS we depict the $\beta$'s; on the RHS we display the unique $D \subset \mathring{D}_\mu$ such that $\partial_\mu^D(E) \neq 0$:*

$$\begin{matrix} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & & \end{matrix} \qquad \text{and} \qquad \begin{matrix} \bullet & \times & \bullet & \bullet & \bullet & \circ & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & \bullet & & \end{matrix}$$

$$\begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & \circ & & & \\ \circ & & & & & & \end{array} \qquad \text{and} \qquad \begin{array}{llllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & & & \\ \circ & \times & & & & & & \\ \times & & & & & & & \end{array}$$

$$\begin{array}{lllllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \bullet & \bullet & \bullet & & & & \\ \circ & \circ & & & & & \end{array} \qquad \text{and} \qquad \begin{array}{llllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & & & & \\ \circ & \times & \circ & & & & & \\ \times & & & & & & & \end{array}$$

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \circ & \circ & \\ \circ & & & & & \end{array} \qquad \text{and} \qquad \begin{array}{lllllll} \bullet & \times & \bullet & \bullet & \bullet & \circ & \circ \\ \times & \bullet & \bullet & \bullet & \circ & \bullet & \\ \circ & \times & & & & & \\ \times & & & & & & \end{array}$$

$$\begin{array}{llllll} \bullet & \bullet & \bullet & \bullet & \bullet & \circ \\ \bullet & \bullet & \bullet & \circ & & \\ \circ & \circ & & & & \end{array} \qquad \text{and} \qquad \begin{array}{lllllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet & \circ \\ \times & \bullet & \bullet & \bullet & \circ & & \\ \circ & \times & \circ & & & & \\ \times & & & & & & \end{array}$$

$$\begin{array}{lllll} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \circ & \\ \circ & \circ & \circ & & \end{array} \qquad \text{and} \qquad \begin{array}{llllll} \bullet & \times & \bullet & \bullet & \bullet & \bullet \\ \times & \bullet & \circ & \circ & \circ & \\ \circ & \times & \bullet & \bullet & & \\ \times & & & & & \end{array}$$

*Thus we get:*

$$\sigma_{5,3} \cdot \sigma_4 = \sigma_{7,5} + 4\sigma_{7,4,1} + 2\sigma_{7,3,2} + 2\sigma_{6,5,1} + 4\sigma_{6,4,2} + \sigma_{5,4,3}\,.$$

A fundamental invariant of a projective variety $X \subset \mathbf{P}^N_{\mathbf{C}}$ is its *degree*, defined by

$$\deg X = \int_X {\omega_X}^n\,,$$

where $n = \dim_{\mathbf{C}} X$ and $\omega_X$ is the restriction of the standard Kähler form on $\mathbf{P}^N$ to $X$. The importance of this invariant is seen from its various interpretations [GH, p.171]:

(i) The number $\deg X$ equals to the number of intersection points of $X$ with a general linear subspace in $\mathbf{P}^N$ of complementary dimension.

(ii) The number $n!\deg X$ agrees with the volume of $X$ .

It is well known that $\sigma_1$ is the generator of $\mathrm{Pic}(G)$ and also $\sigma_1$ is the Kähler class of $G$.

So by (i),

$$\deg(\sigma_\alpha) = \sigma_\alpha \cdot {\sigma_1}^{n(n-1)/2-|\alpha|}\,.$$

We invoke Schur $P$-functions whose definition reads:

1) For a nonnegative integer $i$, $P_i := \sum s_\lambda$, where the sum is over all hook partitions $\lambda$ of $i$, and $s_\lambda$ denotes the corresponding Schur $S$-function (cf., e.g., [Ma]).

2) For integers $i > j > 0$,

$$P_{(i,j)} := P_i P_j + 2 \sum_{1 \leq q \leq i-1} (-1)^q P_{j+q} P_{i-q} + (-1)^{i+j} P_{i+j} .$$

3) For a strict partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ written with an even $k$ (by putting $\lambda_k = 0$ if necessary),

$$P_{(\lambda_1,\ldots,\lambda_k)} := Pf[P_{(\lambda_p,\lambda_q)}]_{1 \leq p < q \leq k} .$$

(See [S], [Ma], [HH], and [P1] for more on Schur $P$-functions.)

Comparing the Pieri-type formula for $P$-functions, extracted in [P1] from [Mo], with Theorem 4.6, we get that $\deg \sigma_\alpha$ is the coefficient of $P_{\rho(n-1)}$ in

$$P_\alpha \cdot P_1^{n(n-1)/2-|\alpha|} ,$$

or the coefficient of $P_{\bar\alpha}$ in

$$P_1^{n(n-1)/2-|\alpha|} = P_1^{|\bar\alpha|} .$$

Here $\bar\alpha$ is the partition whose part complement the parts of $\alpha$ in $\{1, \ldots, n-1\}$.

We define for a partition $\gamma = (\gamma_1, \gamma_2, \ldots)$,

$$g^\gamma = \frac{|\gamma|!}{\gamma!} \prod_{i<j} \frac{\gamma_i - \gamma_j}{\gamma_i + \gamma_j} , \qquad (8)$$

where $\gamma! = \gamma_1! \gamma_2! \cdots$.

**Proposition 4.8** *One has $deg(\sigma_\alpha) = g^\gamma$ for $\gamma = \bar\alpha$.*

**Remark 4.9** *Certain special cases of this formula were obtained by Hiller [Hi]. Some related computations were performed by Tamvakis [Ta2] in the context of heights of homogeneous spaces in arithmetic intersection theory.*

The proposition follows from the following lemma due essentially to Schur.

**Lemma 4.10** *One has*

$$P_1^k = \sum g^\gamma P_\gamma ,$$

*the sum over strict partitions $\gamma$ of $k$.*

**Proof.** Here is a quick proof of the lemma invoking the following results from [Ma]: combine [Ma, Ex.III.8.1 p.259; Formula III.7.1 p.246; and Ex.III.8.12 p. 266]. $\square$

**Remark 4.11** *As a matter of fact the key in the original Schur's calculation [S] (see also [Ma, p.267] and in more detail [HH]) is the proof of the following equality:*

$$g^{\gamma} = \sum g^{\gamma^{(i)}}\,, \tag{9}$$

*where $\gamma^{(i)}$ runs over strict partitions obtained from $\gamma$ by subtracting 1 from a part of the partition $\gamma$. (The fact that this last equation implies the assertion of the lemma follows directly from the Pieri-type formula for $P$-functions.) The original argument rests on the expansion into partial fractions of the function*

$$(2t-1)\prod_i \frac{(t+\gamma_i)(t-\gamma_i-1)}{(t+\gamma_i-1)(t-\gamma_i)}\,.$$

*For those who prefer Lagrange interpolation to the expansion into partial fractions, we sketch another proof of the lemma of Schur.*

*We use the following specialization. In the notation of [Ma], we put $p_1 = 1/2$ and $p_i = 0$ for $i \geq 2$ (in the notation of [L] this reads: $\psi_1 = 1/2$ and $\psi_i = 0$ for $i \geq 2$). To work with this specialization it is handy to define Schur $Q$-functions by $Q_\lambda = 2^{l(\lambda)}P_\lambda$, where $\lambda$ is a strict partition. Under this specialization, $Q_i = 1/i!$. Denote the resulting under this specialization $Q_\lambda$ by the same symbol $Q_\lambda$ for brevity. The following equality was proved in [DP]:*

$$Q_\lambda = \frac{1}{\lambda!}\prod_{i<j}\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\,. \tag{10}$$

*Now, we proceed as follows. By simple manipulations with Schur $Q$-functions, Equation (9) can be rewritten in the following form:*

$$|\gamma| Q_\gamma = \sum_{i=1}^{l} (-1)^i Q_{\gamma_i - 1} Q_{\gamma_1,\ldots,\gamma_{i-1},\gamma_{i+1},\ldots,\gamma_l}\,, \tag{11}$$

*where $\gamma = (\gamma_1 > \cdots > \gamma_l > 0)$. This is a formal identity in the $\gamma$'s, so we can assume that the $\gamma$'s are indeterminates and the last equation (to be proven!) holds in the field of rational functions $\mathbf{Z}(\gamma_1,\ldots,\gamma_l)$. Equation (11) can be rewritten, with the help of (10), as the following equation:*

$$|\gamma| \prod_{i<j}(\gamma_i-\gamma_j) = \sum_p (-1)^p \gamma_p \prod_{\{i,j\neq p;\ i<j\}} (\gamma_i-\gamma_j) \prod_{i\neq p}(\gamma_p+\gamma_i)\,. \tag{12}$$

*But this is exactly (cf. [L]) the content of the Lagrange interpolation for $|\gamma|$. (We note that in this particular case, the formula can be obtained rather easily, by remarking that the RHS of (12) is an alternating function in the $\gamma$'s.)*

Up to now, we have considered Schubert varieties as purely group-theoretic objects. We end this section by recalling their interpretation in terms of Schubert-type conditions. This description is a recollection from [LSe] and [P1], and we will need it in the next section.

Let $U$ be a $2n$-dimensional vector space endowed with a nondegenerate orthogonal form $\xi : U \times U \to \mathbf{C}$. Consider

$$Z = \{L \subset U : L \text{ is maximal isotropic subspace in } U\}\,.$$

This subvariety is canonically embedded in the Grassmannian $G_n(U)$. This last variety has Schubert (sub)varieties which are defined w.r.t. to flag

$$U_1 \subset U_2 \subset \cdots \subset U_{2n} = U$$

(where $\dim(U_i) = i$), in the following way: Given a sequence $1 \le i_1 < \cdots < i_n \le 2n$, we set

$$\Omega(i_1, \ldots, i_n) = \{L \in G_n(U) : \dim(L \cap U_{i_p}) \ge p \ \forall p = 1, \ldots, n\}\,.$$

One has

$$\dim \Omega(i_1, \ldots, i_n) = i_1 + \cdots + i_n - n(n+1)/2\,.$$

It is known that $Z$ has two connected components which are isomorphic to $G = H/P$. Let $v_1, \ldots, v_n, w_1, \ldots, w_n$ be a basis of $U$ such that $\xi(v_i, v_j) = \xi(w_i, w_j) = 0, \xi(v_i, w_j) = \xi(w_j, v_i) = \delta_{i,j}$. Let $V_i$ be the vector space spanned by the first $i$ vectors of the above basis. Then the Schubert varieties in $G_n(U)$ (determined by the flag $V_1 \subset \cdots \subset V_{2n} = V$) which *give rise* to the Schubert varieties in $G$ (in the sense of [LSe] and [P1]) are indexed by the sequences $(i_1, \ldots, i_n)$ where $i_p \ne 2n + 1 - i_q$ for $p, q = 1, \ldots, n$, and if $k$ denotes the largest number such that $i_k \le n$, then $n - k$ is even. Let us denote by $\Omega[i_1, \ldots, i_k]$ the Schubert variety in $G$ determined (via restriction to $G$) by this Schubert variety in $G_n(U)$, that is:

$$\Omega[i_1, \ldots, i_k] := \{L \in G : \dim(L \cap V_{i_p}) \ge p \ \forall \ p = 1, \ldots, k\}\,.$$

(Instead refering to the flag $V_1 \subset \cdots \subset V_n$, we will also say that this Schubert variety is defined w.r.t. the ordered basis $\{v_1, \ldots, v_n\}$.) One has

$$\dim \Omega[i_1, \ldots, i_k] = i_1 + \cdots + i_k + n(n-k) - n(n+1)/2\,.$$

The Schubert classes in $H^*(G)$ determined by these Schubert varieties are related in following way to the Schubert classes $\sigma_\alpha$ considered earlier in this section. For a strict partition $\alpha = (\alpha_1, \ldots, \alpha_k) \subset \rho(n-1)$, one has $\sigma_\alpha = \Omega[n-\alpha_1, \ldots, n-\alpha_k]$ if $n-k$ is even, and $\sigma_\alpha = \Omega[n-\alpha_1, \ldots, n-\alpha_k, n]$ if $n-k$ is odd.

The corresponding Schubert variety $\sigma_\alpha$ in $G$ can be defined in the following way w.r.t. the above flag $V_1 \subset \cdots \subset V_n$; it is:

$$\{L \in G : \dim(L \cap V_{n-\alpha_p}) \geq i \ \forall\ p = 1, \ldots, k \ \&\ \mathrm{codim}_{V_n}(L \cap V_n) \text{ is even}\}.$$

# 5 Schubert cycles of complex structures on $\mathbf{R}^{2n}$.

We will adopt the following convention. Let $\mathbf{R}^{2n}$ be the real Euclidean $2n$-space with the standard orthonormal basis $\{e_1, \ldots, e_{2n}\}$.

The $2n$-dimensional complex Euclidean space $\mathbf{C}^{2n}$ will be considered as the complexification of $\mathbf{R}^{2n}$; $\mathbf{C}^{2n} = \mathbf{R}^{2n} \otimes_{\mathbf{R}} \mathbf{C}$. Note the following simple facts:

(a) The set $\{e_1 \otimes 1, \ldots, e_{2n} \otimes 1\}$ is an orthonormal basis for $\mathbf{C}^{2n}$.

(b) If $L, K \subset \mathbf{R}^{2n}$ are two linear subspaces satisfying $\dim_{\mathbf{R}}(L \cap K) \geq i$, then their complexifications $L^{\mathbf{C}}, K^{\mathbf{C}} \subset \mathbf{C}^{2n}$ satisfy $\dim_{\mathbf{C}}(L^{\mathbf{C}} \cap K^{\mathbf{C}}) \geq i$.

(c) Corresponding to an orthogonal decomposition $L = L_1 \oplus L_2$ of a subspace $L \subset \mathbf{R}^{2n}$, one has the orthogonal decomposition $L^{\mathbf{C}} = L_1^{\mathbf{C}} \oplus L_2^{\mathbf{C}}$ of $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

(d) An $\mathbf{R}$-linear endomorphism of a subspace $L \subset \mathbf{R}^{2n}$ induces a $\mathbf{C}$-linear endomorphism of the subspace $L^{\mathbf{C}} \subset \mathbf{C}^{2n}$.

Let $V$ be an oriented even dimensional real Euclidean space and $Iso(V)$, the group of orientation preserving isometries of $V$. Consider

$$CS(V) = \{A \in Iso(V) \ : \ A^2 = -Id_V\}.$$

It is the space of complex structures on $V$.

If $V = \mathbf{R}^{2n}$, one has the identification $Iso(R^{2n}) = SO(2n, \mathbf{R})$, the special orthogonal group of order $2n$, and

$$CS(\mathbf{R}^{2n}) = \{A \in SO(2n, \mathbf{R}) \mid A^2 = -I_{2n}\}.$$

Note that if $A \in CS(\mathbf{R}^n)$ then $A$ is a skew-symmetric matrix. Let us abbreviate $CS(\mathbf{R}^{2n})$ by $CS_n$ as is common. The space $CS_n$ has two connected components which are distinguished by the Pfaffian function

$$Pf : CS_n \to \{\pm 1\}.$$

We write $CS_n = CS_n^+ \sqcup CS_n^-$ with $Pf(CS_n^{\pm}) = \pm 1$.( The symbol $\sqcup$ denotes the disjoint union.) Both manifolds $CS_n^{\pm}$ are isometric to $SO(2n, \mathbf{C})/U(n)$.

Our goal in this section is to interpret Schubert varieties in $SO(2n, \mathbf{C})/U(n)$ in terms of complex structures. We will give an interpretation, in terms of Schubert varieties, of the following Mahowald-Vassiljev-type formula

$$H_p(CS_n) = \oplus_{k=0}^{n} H_{p-k(k-1)}(G_k(\mathbf{C}^n)) \,,$$

where $G_k(\mathbf{C}^n)$ is the Grassmannian of all complex $k$-planes through zero in $\mathbf{C}^n$ (see [DV] and also [V]).

We will now define Schubert varieties of complex structures. Let us fix a complex structure $J_0 \in CS_n$. By convention, we will denote by $CS_n^+$ the connected component of $CS_n$ that contains $J_0$. We will work here with the component $CS_n^+$, leaving to the reader details concerning the other component $CS_n^-$. The results about the component $CS_n^-$ will be summarized in Proposition 5.6.

Let

$$\mathbf{R}^{2n} = L_1 \oplus L_2 \oplus \cdots \oplus L_n \,,$$

where $\dim_{\mathbf{R}} L_i = 2$, be an invariant subspace decomposition of the orthogonal operator $J_0 : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$. This yields a flag in $\mathbf{R}^{2n}$

$$F_1 \subset F_2 \subset \cdots \subset F_n = \mathbf{R}^{2n} \,, \tag{13}$$

where $F_i = L_1 \oplus L_2 \oplus \cdots \oplus L_i$. Furnishing $\mathbf{R}^{2n}$ with the complex structure $J_0$, we get an $n$-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$. Since each $F_i$ is an invariant subspace w.r.t. $J_0$, the flag (13) gives rise to a complex flag

$$W_1 \subset W_2 \subset \cdots \subset W_n = \mathbf{C}^n \,,$$

where $\dim_{\mathbf{C}} W_i = i$.

Consider the Grassmannian $G_l(\mathbf{C}^n)$ of all complex $l$-planes through zero in $\mathbf{C}^n$. For a sequence $1 \le j_1 < \cdots < j_l \le n$, one defines a Schubert variety

$$\Omega(j_1, \ldots, j_l) = \{L \in G_l(\mathbf{C}^n) : \dim(L \cap W_{i_p}) \ge p \ \forall p = 1, \ldots, l\} \,.$$

One has

$$\dim_{\mathbf{C}} \Omega(j_1, \ldots, j_l) = j_1 + \cdots + j_l - l(l+1)/2 \,.$$

For an even $l$, we define $CS_n^+(j_1, \ldots, j_l)$ as

$$\{A \in CS_n^+ : \exists L \in \Omega(j_1, \ldots, j_l) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}}^{\perp} = J_0|L_{\mathbf{R}}^{\perp}\} \,,$$

where $L_{\mathbf{R}}^{\perp}$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of $L$. This is a closed subvariety in $CS_n^+$. One has

$$\dim_{\mathbf{C}} CS_n^+(j_1, \ldots, j_l) = j_1 + \cdots + j_l - l\,.$$

One verifies easily that the class of the variety $CS_n^+(j_1, \ldots, j_l)$ is independent of the choice of $J_0$. Indeed, a path in $CS_n^+$ joining $J_0$ to another $J \in CS_n^+$ yields a one-parameter family of varieties from $CS_n^+(j_1, \ldots, j_l)$ attached to $J_0$, to that attached to $J$.

Define the complex structure $J_n$ in the initial basis $\{e_1, \ldots, e_{2n}\}$ by

$$J_n = J_1 \oplus J_1 \oplus \cdots \oplus J_1 \quad n \text{ times}\,,$$

where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

**Example 5.1** *Suppose that $J_0 = J_n$. For a sequence $1 \leq i_1 < \cdots < i_k \leq n$, we put*

$$J(i_1, \ldots, i_k) = \epsilon_1 J_1 \oplus \epsilon_2 J_1 \oplus \cdots \oplus \epsilon_n J_1$$

*where $\epsilon_h = -1$ if $h = i_p$ for $1 \leq p \leq k$ Then $J(i_1, \ldots, i_k) \in CS_n^+$ iff $k$ is even.*

We now want to identify the variety $CS_n^+(j_1, \ldots, j_l)$ with a suitable Schubert variety $\Omega[i_1, \ldots, i_k]$ in $G$. To this end, we first describe an imbedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$. Every $A \in CS_n$ has two eigenvalues $\pm i$ (here $i$ is the pure imaginary complex number) with equal multiplicities $n$. Thus, as an endomorphism of $\mathbf{C}^{2n}$, (cf. (d)), $A$ has the eigensubspace decomposition

$$\mathbf{C}^{2n} = L(A, +) \oplus L(A, -) \quad \text{where} \quad \dim L(A, +) = \dim L(A, -) = n\,,$$

with

$$A(v) = iv \text{ for all } v \in L(A, +) \quad \text{and} \quad A(v) = -iv \quad \text{for all } v \in L(A, -)\,.$$

The embedding $\iota : CS_n \to G_n(\mathbf{C}^{2n})$ defined by $A \to L(A, +)$ has as its image the Grassmannian of all isotropic subspaces of $\mathbf{C}^{2n}$ w.r.t. the orthogonal form induced by the scalar product.

For an Euclidean complex $n$-vector bundle $p : E \to X$ over a connected space $X$, let $p_{\mathbf{R}} : E_{\mathbf{R}} \to X$ be the real reduction of $p$, i.e. $E_x = p_{\mathbf{R}}^{-1}(x)$, $x \in X$, is the real reduction of the complex $n$-space $p^{-1}(x)$. We regard a complex structure on $p$ as a bundle isometry $J : E_{\mathbf{R}} \to E_{\mathbf{R}}$ with $J^2 = -id$.

Consider the space $CS(E) = \{J_x \in CS(E_x) : x \in X\}$. The map $\pi : CS(E) \to X$ defined by $\pi(J_x) = x$, is a fiber bundle projection whose fiber over a point $x \in X$ is $CS(E_x)$. The set of all sections of $\pi$ is in one-to-one correspondence with the set of all complex structures on the real $2n$-vector bundle $p_{\mathbf{R}} : E_{\mathbf{R}} \to X$. In particular, a complex structure $K : E_{\mathbf{R}} \to E_{\mathbf{R}}$ gives a section $\sigma : X \to CS(E)$.

Let $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$, so we can form $G_l(\mathbf{C}^n)$ – the Grassmannian of all complex $l$-planes through zero in $\mathbf{C}^n$. Let $R$ be the tautological complex $l$-bundle on $G_l(\mathbf{C}^n)$. Of course, $J_0 : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ induces a complex structure on $R$, denoted by the same symbol. Let $CS^+(R)$ denote the connected component of $CS(R)$ containing the section $\sigma$ corresponding to this complex structure $J_0 : R \to R$. So we have the fibrations

$$\pi : CS(R) \to G_l(\mathbf{C}^n) \quad \text{or} \quad \pi^+ : CS^+(R) \to G_l(\mathbf{C}^n)\,.$$

A point in $CS(R)$ consists of a pair $(L, J)$ with $L \in G_l(\mathbf{C}^n)$ and $J : L_{\mathbf{R}} \to L_{\mathbf{R}}$ – a complex structure on the real reduction $L_{\mathbf{R}}$ of $L$.

Finally, by furnishing $\mathbf{R}^{2n}$ with the complex structure $J_0 = J_n$, we have an identification $\mathbf{C}^n = (\mathbf{R}^{2n}, J_n)$. Let $c_h = e_{2h-1}$ for $h = 1, \ldots, n$, and consider the complex flag

$$W_1 \subset W_2 \subset \cdots \subset W_n = \mathbf{C}^n\,,$$

where $W_h = \operatorname{span}\{c_1, \ldots, c_h\}$. Let $\Omega(j_1, \ldots, j_l) \subset G_l(\mathbf{C}^n)$ be the Schubert variety defined w.r.t. to this flag; we will also say that this Schubert variety is defined w.r.t. the ordered basis $\{c_1, \ldots, c_n\}$ of $\mathbf{C}^n$.

Assume that $l$ is even. Let $CS^+(R; j_1, \ldots, j_l) \subset CS^+(R)$ be the inverse image of $\Omega(j_1, \ldots, j_l)$ under $\pi^+$. An easy dimension count shows that

$$\dim CS^+(R; j_1, \ldots, j_l) = j_1 + \cdots + j_l - l\,.$$

Consider now the map $\varphi : CS^+(R) \to CS_n^+$ defined on a point $(L, J)$ by

$$\varphi((L, J)) = J \oplus (J_n | L_{\mathbf{R}}^{\perp}) : \mathbf{R}^{2n} = L_{\mathbf{R}} \oplus L_{\mathbf{R}}^{\perp} \to \mathbf{R}^{2n} = L_{\mathbf{R}} \oplus L_{\mathbf{R}}^{\perp}\,.$$

(Note that for each $L \in G_l(\mathbf{C}^n)$, the real reduction $L_{\mathbf{R}} \subset \mathbf{R}^{2n}$, and hence its orthogonal complement $L_{\mathbf{R}}^{\perp} \subset \mathbf{R}^{2n}$ is an invariant subspace of $J_n$.)

The restriction of $\varphi$ to the subvariety $CS^+(R; j_1, \ldots, j_l)$ will be denoted by $\varphi_{\mathbf{j}}$. By construction, $\varphi_{\mathbf{j}}$ maps $CS_n^+(R; j_1, \ldots, j_l)$ onto $CS_n^+(j_1, \ldots, j_l)$.

On the other hand, consider now the complexification $\mathbf{C}^{2n} = \mathbf{R}^{2n} \otimes_{\mathbf{R}} \mathbf{C}$. Complete the basis $\{c_1, \ldots, c_n\}$ of $\mathbf{C}^n$ to the basis $\{c_1, \ldots, c_n, d_1, \ldots, d_n\}$ of $\mathbf{C}^{2n}$ by setting $d_h = e_{2h} \otimes 1$ for $h = 1, \ldots, n$.

Pick $(L, J) \in CS_n^+(R; j_1, \ldots, j_n)$. By using a simple linear algebra, one shows that

$$\iota \circ \varphi_{\mathbf{j}}((L, J)) \in \Omega(n + 1 - t_{n-l}, \ldots, n + 1 - t_1, n + j_1, \ldots, n + j_l)\,,$$

where the Schubert variety is defined w.r.t. ordered basis

$$\{c_n, \ldots, c_1, d_1, \ldots, d_n\},$$

and $t_1 < \cdots < t_{n-l}$ is the complement of $j_1 < \cdots < j_l$ in $\{1, \ldots, n\}$.

We point out the following fact which is used in this argument (see, e.g., [HP, p.321]).

**Lemma 5.2** *The map $G_l(\mathbf{C}^n) \to G_{n-l}(\mathbf{C}^n)$ defined by $L \to L^\perp$, sends the Schubert variety $\Omega(j_1, \ldots, j_l) \subset G_l(\mathbf{C}^n)$ defined w.r.t. the ordered basis $\{c_1, \ldots, c_n\}$ to the Schubert variety $\Omega(n+1-t_{n-l}, \ldots, n+1-t_1) \subset G_{n-l}(\mathbf{C}^n)$ defined w.r.t. the ordered basis $\{c_n, \ldots, c_1\}$.*

Since one has

$$\dim \mathrm{Im}(\varphi_{\mathbf{j}}) = \dim CS_n^+(j_1, \ldots, j_n) = \dim \Omega[n+1-t_{n-l}, \ldots, n+1-t_1],$$

the image of $\varphi_{\mathbf{j}}$ must be actually equal to $\Omega[n+1-t_{n-l}, \ldots, n+1-t_1]$, defined w.r.t. the ordered basis $\{c_n, \ldots, c_1\}$.

Summing up, we have proved the following identification:

**Proposition 5.3** *Let $l$ be even. Then one has*

$$CS_n^+(j_1, \ldots, j_l) = \Omega[n+1-t_{n-l}, \ldots, n+1-t_1],$$

*where $t_1 < \cdots < t_{n-l}$ is the complement of $j_1 < \cdots < j_l$ in $\{1, \ldots, n\}$.*

**Remark 5.4** *The equality in this proposition is a refinement of a result announced by Dynnikov and Veselov in [DV], where some variant of Morse theory was used.*

As a consequence of Proposition 4.8, we get:

**Corollary 5.5** *The degree of $CS_n^+(j_1, \ldots, j_l)$ is equal to the number $g^\gamma$, where $\gamma = (j_l - 1, \ldots, j_1 - 1)$.*

Since, for even $l$, $\Omega[n+1-t_{n-l}, \ldots, n+1-t_1]$ is equal to

$$\{A : \exists K \in \Omega(j_1, \ldots, j_l) \text{ s.t. } A(K_{\mathbf{R}}) = K_{\mathbf{R}} \ \&\ A|K_{\mathbf{R}}^\perp = J_0|K_{\mathbf{R}}^\perp\},$$

then, rewriting it for even $n-k$, $\Omega[i_1, \ldots, i_k]$ is equal to

$$\{A : \exists K \in \Omega(n+1-s_{n-k}, \ldots, n+1-s_1) \text{ s.t. } A(K_{\mathbf{R}}) = K_{\mathbf{R}} \ \&\ A|K_{\mathbf{R}}^\perp = J_0|K_{\mathbf{R}}^\perp\}.$$

By taking $L = K^{\perp}$, we can present this $\Omega[i_1, \dots, i_k]$ as

$$\{A : \exists L \in \Omega(i_1, \dots, i_k) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,. \tag{14}$$

This last identification (14) seems to be the most handy for applications.

In the following proposition, keeping the notation from this section, we collect properties of Schubert varieties in $CS_n^-$. For an odd $l$, we define $CS_n^-(j_1, \dots, j_l)$ as

$$\{A \in CS_n^- : \exists L \in \Omega(j_1, \dots, j_l) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}}^{\perp} = J_0|L_{\mathbf{R}}^{\perp}\}\,,$$

where $L_{\mathbf{R}}^{\perp}$ is the orthogonal complement of the real reduction $L_{\mathbf{R}}$ of $L$.

**Proposition 5.6** *(i) $CS_n^-(j_1, \dots, j_l)$ is a closed subvariety in $CS_n^-$ of dimension $j_1 + \cdots + j_l - l$.*
*(ii) $CS_n^-(j_1, \dots, j_l)$ can be identified with the restriction to $CS_n^-$, properly embedded in $G_n(\mathbf{C}^{2n})$, of the Schubert variety*

$$\Omega(n+1-t_{n-l}, \dots, n+1-t_1, n+j_1, \dots, n+j_l)$$

*in this last Grassmannian.*
*(iii) $CS_n^-(j_1, \dots, j_l)$ is also identified with*

$$\{A \in CS_n^- : \exists L \in \Omega(i_1, \dots, i_k) \text{ s.t. } A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

*(iv) The degree of $CS_n^-(j_1, \dots, j_l)$ is equal to $g^{\gamma}$, where $\gamma = (j_l - 1, \dots, j_1 - 1)$.*

**Example 5.7** *We describe the Schubert varieties in $CS_n^+$ which are divisors. We have different description according to the parity of $n$. If $n$ is odd, then the divisor $\sigma_1 = \Omega[n-1]$ is*

$$\{A : \exists L \subset W_{n-1} \ s.t. \ \dim_{\mathbf{C}} L = 1, \ A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

*If $n$ is even, then the divisor $\sigma_1 = \Omega[n-1, n]$ is*

$$\{A : \exists L \subset W_n \ s.t. \ \dim_{\mathbf{C}} L = 2, \ A(L_{\mathbf{R}}) = L_{\mathbf{R}} \ \& \ A|L_{\mathbf{R}} = J_0|L_{\mathbf{R}}\}\,.$$

We will end this paper with some applications. The identification made in (14) allows us to solve enumerative problems about the number of general complex structures satisfying some constrains. To this end, we need the following definition.

**Definition 5.8** *Let $A$ and $B$ be two orthogonal operators on $\mathbf{R}^{2n}$. A linear subspace $L \subset \mathbf{R}^{2n}$ is said to be a common $k$-space of $A$ and $B$ iff*

$$A(L) = B(L) = L \ , \ A|L = B|L \ \& \ \dim_{\mathbf{R}} L = k \, .$$

We will work in $CS_n^+$. Let $n$ and $2 \leq k \leq n$ be even integers. Let $1 \leq i_1 < \cdots < i_k \leq n$ be a sequence of integers. Set $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \leq i \leq d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. $A$ has a common $2k$-space from $\Omega(i_1, \ldots, i_k)$ with $B_0$, and $A$ has a common 4-space with any other $B_i$ from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

As a particular case, we have:

**Proposition 5.9** *Let $n$ and $2 \leq k \leq n$ be even integers. Suppose that a list $\{B_i\}$ of general complex structures on $\mathbf{R}^{2n}$ is given, where*

$$0 \leq i \leq 1 + (n-k)(n-k-1)/2 \, .$$

*Then the number of complex structures $A \in CS_n^+$ having a common fixed $2k$-space $(W_k)_{\mathbf{R}}$ with $B_0$ and a common 4-space with any other $B_i$ is given by*

$$g^{(n-k-1, n-k-2, \ldots, 2, 1)} = [(n-k)(n-k-1)/2]! \prod_{i=1}^{n-k-1} \frac{(i-1)!}{(2i-1)!} \, .$$

Indeed, this is a restatement of the formula about the degree of

$$\Omega[1, 2, \ldots, k-1, k] = CS_n^+(1, 2, \ldots, n-k) \, ,$$

which is simplified in this case of a triangular partition, cf. [DP].

**Example 5.10** *For $n = 8$ and $k = 4$, a list of 7 complex structures $\{B_0, B_1, B_2, B_3, B_4, B_5, B_6\}$ is given. There exist exactly 2 complex structures $A \in CS_n^+$ s.t. $A$ has a common fixed 8-space with $B_0$ and at least a common 4-space with every $B_i$, where $1 \leq i \leq 6$.*

*If $n = 10$ and $k = 4$, a list of 16 structures $\{B_0, B_1, \ldots, B_{15}\}$ is given. There exist exactly 286 complex structures $A \in CS_n^+$ s.t. $A$ has a common fixed 8-space with $B_0$ and at least a common 4-space with every $B_i$, where $1 \leq i \leq 15$.*

Let now $n$ and $2 < k \leq n$ be odd integers. Let $1 \leq i_1 < \cdots < i_k \leq n$ be a sequence of integers. Put $d = \dim \Omega[i_1, \ldots, i_k]$. Suppose that a list $\{B_i\}$, $0 \leq i \leq d$, of general complex structures on $\mathbf{R}^{2n}$ is given. Then the number of complex structures $A \in CS_n^+$ s.t. $A$ has a common $2k$-space from

$\Omega(i_1, \ldots, i_k)$ with $B_0$, and $A$ has a common 2-space from $(W_{n-1})_{\mathbf{R}}$ with any other $B_i$ from the list, is equal to $\deg \Omega[i_1, \ldots, i_k]$.

We leave it to the reader to deduce from it a result analogous to the one in the last proposition.

We will give now another example of enumerating complex structures satisfying some constraints and state some conjecture.

For a complex structure $B_0 \in CS_n$, we have an $n$-dimensional complex space $\mathbf{C}^n = (\mathbf{R}^{2n}, B_0)$. Note that for all $L \in G_k(\mathbf{C}^n)$, both $L_{\mathbf{R}}$ and $L_{\mathbf{R}}^{\perp}$ are invariant subspaces of $B_0$. We have then an embedding $\alpha : G_k(\mathbf{C}^n) \to CS_n$ defined by

$$\alpha(L) = (B_0 | L_{\mathbf{R}}) \oplus (-B_0 | L_{\mathbf{R}}^{\perp}) .$$

Without loss of generality, we can assume that the image of $\alpha$ lies in $CS_n^+$.

Let $R$ be the canonical complex $k$-bundle over $G_k(\mathbf{C}^n)$ and $R^{\perp}$ its orthogonal complement in the trivial complex $n$-bundle. Let $S$ be the canonical complex $n$-bundle over $CS_n^+$. From the definition of $\alpha$ we have

$$\alpha^* S = R \oplus \overline{R^{\perp}} ,$$

where $\overline{R^{\perp}}$ denotes the complex conjugation of $R^{\perp}$. We infer that the pullback of the total Chern class of $S$, $\alpha^*(1 + c_1(S) + \cdots + c_n(S))$, is equal to

$$(1 + c_1(R) + \cdots + c_k(R))(1 - c_1(R) + c_2(R) - \cdots + (-1)^k c_k(R))^{-1} .$$

From this we get that the induced homomorphism

$$\alpha^* : H^*(CS_n) \to H^*(G_k(\mathbf{C}^n))$$

satisfies $\alpha^*(\frac{1}{2} c_1(S)) = -\alpha^*(\sigma_1) = c_1(R)$. That is, the embedding $\alpha$ preserves the classes of hyperplane sections, or, the Kähler classes of both varieties.

As a consequence, we get results summarized in the following proposition.

**Proposition 5.11** *(i) Let $n$ be an even integer. Suppose that $2 \leq k \leq n$ is another integer. Let $\{B_i\}$, $0 \leq i \leq k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. $A$ and $B_0$ have a common $2k$-space, $A$ and $(-B_0)$ have a common $2(n-k)$-space, and $A$ and each $B_i$, $i \geq 1$ have at least a common 4-space, is equal to the degree of $G_k(\mathbf{C}^n)$.*
*(ii) Let now $n$ be an odd integer. Suppose that $2 \leq k \leq n$ is another integer. Let $\{B_i\}$, $0 \leq i \leq k(n-k)$, be a list of general complex structures on $\mathbf{R}^{2n}$. Then the number of complex structures $A \in CS_n^+$ s.t. $A$ and $B_0$ have a common $2k$-space, $A$ and $(-B_0)$ have a common $2(n-k)$-space, and $A$ and each $B_i$, $i \geq 1$ have at least a common 2-space in $(W_{n-1})_{\mathbf{R}}$, is equal to the degree of $G_k(\mathbf{C}^n)$.*

(Recall that

$$\deg G_k(\mathbf{C}^n) = \frac{1!2!\cdots(k-1)![k(n-k)]!}{(n-k)!(n-k+1)!\cdots(n-1)!},$$

a result which goes back to Schubert (1886).)

It is well known that the Grassmannian $G_k(\mathbf{C}^n)$ is an approximation space for the classifying space $BU(k)$ of all complex $k$-bundles. On the other hand, the space $CS_n$ serves as the classifying space for all complex $n$-bundles with a trivial real reduction [Du1]. Thus a homotopy classification of continuous maps $G_k(\mathbf{C}^n) \to CS_n$ may suggest possible interesting operators between these two vector bundle theories.

Let $\beta : G_k(\mathbf{C}^n) \to CS_n$ be a continuous map. Since $\sigma_1 = -\frac{1}{2}c_1(S) \in H^*(CS_n^{\pm}) = \mathbf{Z}$ and $c_1(R) \in H^*(G_k(\mathbf{C}^n)) = \mathbf{Z}$ are the only generators in dimension 2, then the induced map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ satisfies

$$\beta^*(\frac{1}{2}c_1(S)) = -\beta^*(\sigma_1) = m \cdot c_1(R)$$

for some $m \in \mathbf{Z}$.

We finish this paper by stating the following conjecture.

**Conjecture 5.12** *If $m \neq 0$, then the map $\beta^* : H^*(CS_n^{\pm}) \to H^*(G_k(\mathbf{C}^n))$ is given by*

$$\beta^*(x) = m^p \alpha^*(x),$$

*for $x \in H^p(CS_n^{\pm})$.*

We refer the reader to [Du2] and [Ho] for some background related to this conjecture. Note that this conjecture is purely algebraic.

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