CORNER VIEW ON THE CROWN DOMAIN

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1. **Introduction**

Our concern is with the crown domain, henceforth denoted by Ξ. We recall that Ξ is an equivariant complexification of a Riemannian symmetric space $X = G/K$ of the non-compact type. Most naturally one defines Ξ by the theory of unitary K-spherical representations of the symmetry group G (see the introduction of [3]). Geometrically, one can define Ξ as the maximal G-invariant domain in the affine complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$ which can be equivariantly embedded into the tangent bundle TX.

As a complex manifold, Ξ has the property that bounded holomorphic functions separate points. Therefore we may define its distinguished (i.e. Shilov) boundary $\partial_d\Xi$ as the smallest closed subset of the topological boundary $\partial\Xi \subset X_{\mathbb{C}}$ on which bounded plurisubharmonic functions on $\mathrm{cl}(\Xi)$ attend their maximum. We know by [1] and [3] that $\partial_d\Xi$ is a finite (and explicite) union of G-orbits, say

$$\partial_d\Xi = \mathcal{O}_1 \amalg \ldots \amalg \mathcal{O}_s \,.$$

From now on we shall identify each $\mathcal{O}_j$ with a homogeneous space: G/H_j. The main result of [1] was:

If G/H_j is a symmetric space, then it is a non-compactly causal symmetric space. Moreover, every non-compactly causal symmetric space $Y = G/H$ appears in the distinguished boundary of the corresponding crown domain for $X = G/K$.

The aim of this paper is to understand this result better. To be more concise: what is the reason that precisely non-compactly causal (NCC) symmetric spaces appear in the boundary?

NCC-spaces are very special among all semisimple symmetric spaces. We recall their definition. We assume the Lie algebra of G to be simple and write $\mathfrak{q}$ for the tangent space of Y at the standard base point $y_o = H \in Y$. We note that $\mathfrak{q}$ is a linear H-module. Now, non-compactly causal means that $\mathfrak{q}$ admits an non-empty open H-invariant convex cone, say C, which is hyperbolic and does not contain any affine lines.

The theme of this paper is to view Ξ from the corner point $y_o \in Y$ and not as a thickening of X as customary. Now a slight precision is of need. In general $\partial_d\Xi$ has several connected components. If this happens to be the case, then we shrink Ξ to a G-domain Ξ_H whose distinguished boundary is precisely Y, see [2].

For C being the minimal cone we form in the tangent bundle $TY = G \times_H \mathfrak{q}$ the cone-subbundle

$$\mathcal{C} = G \times_H C$$

and with that its boundary cone-bundle

$$\partial \mathcal{C} = G \times_H \partial C\,.$$

In this context we ask the following

Question: Is there a G-equivariant, generically injective, proper continuous surjection $p : \partial\mathcal{C} \to \partial \Xi_H$?

In other words, we ask if there exists an equivariant "resolution" of the boundary in terms of the geometrically simple boundary cone bundle $\partial\mathcal{C}$.

In this paper we give an affirmative answer to this question if X is a Hermitian tube domain. In this simplified situation the crown domain is $\Xi = X \times \overline{X}$ with $\overline{X}$ denoting X but endowed with the opposite complex structure (i.e., if X is already complex, then the crown is the complex double). On top of that $\partial_d \Xi = Y$ is connected, i.e. $\Xi = \Xi_H$.

I wish to point out that the presented method of proof will not generalize. In order to advance one has to understand more about the structure of the minimal cone C; one might speculate that some sort of "$H \cap K$-invariant theory" for C could be useful.

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2. **Main part**

Let $X = G/K$ be a Hermitian symmetric space of tube type. This means that there is an Euclidean (or formally real) Jordan algebra V with positive cone $W \subset V$ such that

$$X = V + iW \subset V_{\mathbb{C}}\,.$$

The action of G is by fractional linear transformation and our choice of K is such it fixes the base point $x_0 = ie$ with $e \in V$ the identity element of the Jordan algebra.

It is no loss of generality if we henceforth restrict ourselves to the basic case of $G = \mathrm{Sp}(n, \mathbb{R})$ – the more general case is obtained by using standard dictionary which can be found in text books.

For our specific choice, the Jordan algebra is $V = \mathrm{Sym}(n, \mathbb{R})$ and $W \subset V$ is the cone of positive definite symmetric matrices. The identity element e is I_n, the $n \times n$ identity matrix. The group G acts on X by standard fractional linear transformations: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with appropriate $a, \ldots, d \in M(n, \mathbb{R})$ acts as

$$g \cdot z = (az + b)(cz + d)^{-1} \qquad (z \in X).$$

The maximal compact subgroup K identifies with $U(n)$ under the standard embedding

$$U(n) \to G, \quad u + iv \mapsto \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \qquad (u, v \in M(n, \mathbb{R})).$$

It is then clear that $K = U(n)$ is the stabilizer of $x_0 = iI_n$. In the sequel we consider $V_\mathbb{C}$ as the affine piece of the projective variety $\mathcal{L}$ of Lagrangians in $\mathbb{C}^{2n}$; the embedding is given by

$$V_\mathbb{C} \mapsto \mathcal{L}, \quad T \mapsto L_T := \{(T(v), v) \mid v \in \mathbb{C}^n\}.$$

It is then clear that $G_\mathbb{C} = \mathrm{Sp}(n, \mathbb{C})$ acts on $\mathcal{L}$; in symbols: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\mathbb{C}$ with appropriate $a, \ldots, d \in M(n, \mathbb{C})$ acts as

$$g \cdot L = \{(av + bw, cv + dw) \mid (v, w) \in \mathbb{C}^n \oplus \mathbb{C}^n = \mathbb{C}^{2n}\} \qquad (L \in \mathcal{L}).$$

The space $\mathcal{L}$ is homogeneous under $G_\mathbb{C}$. If we choose the base point

$$x_0 \leftrightarrow L_0 = \{(iv, v) \mid v \in \mathbb{C}^n\},$$

then the stabilizer of x_0 in $G_\mathbb{C}$ is the Siegel parabolic

$$S^+ = K_\mathbb{C} \ltimes P^+ \quad \text{and} \quad P^+ = \left\{ \mathbf{1} + \begin{pmatrix} u & -iu \\ -iu & -u \end{pmatrix} \mid u \in V_\mathbb{C} \right\}.$$

Thus we have

$$\mathcal{L} = G_\mathbb{C} \cdot L_0 \simeq G_\mathbb{C}/S^+.$$

Sometimes it is useful to take the conjugate base point $\overline{x_0} = -iI_n$. Then the stabilizer of $\overline{L_0}$ in $\mathcal{L}$ is the oppposite Siegel parabolic

$$S^- = K_\mathbb{C} \ltimes P^- \quad \text{and} \quad P^- = \left\{ \mathbf{1} + \begin{pmatrix} u & iu \\ iu & -u \end{pmatrix} \mid u \in V_\mathbb{C} \right\}$$

and

$$\mathcal{L} = G_{\mathbb{C}} \cdot \overline{L_0} \simeq G_{\mathbb{C}}/S^- \,.$$

Next we come to the realization of the affine complexification of $X_{\mathbb{C}} = G_{\mathbb{C}}/K_{\mathbb{C}}$. We consider the $G_{\mathbb{C}}$-equivariant embedding

$$X_{\mathbb{C}} \to \mathcal{L} \times \mathcal{L}, \quad gK_{\mathbb{C}} \mapsto (g \cdot L_0, g \cdot \overline{L_0}) \,.$$

It is not hard to see that

$$X_{\mathbb{C}} = \{(L, L') \in \mathcal{L} \times \mathcal{L} \mid L + L' = \mathbb{C}^{2n}\},$$

i.e., $X_{\mathbb{C}}$ is the affine variety of pairs of transversal Lagrangians.

Set $\overline{X} = V - iW$ and note that the map $z \mapsto \overline{z}$ identifies X with $\overline{X}$ in a G-equivarinat, but antiholomorphic manner.

Next we come to the subject matter, the crown domain of X:

$$\Xi = X \times \overline{X} \subset X_{\mathbb{C}} \,.$$

Let us denote by $\partial\Xi$ the topological boundary of Ξ in $X_{\mathbb{C}}$. The goal is to resolve $\partial\Xi$ by a cone bundle over the affine symmetric space $Y = G/H$ where $H = \mathrm{Gl}(n, \mathbb{R})$ is the structure group of the Euclidean Jordan algebra V.

We define an involution τ on G by

$$\tau(g) = I_{n,n} g I_{n,n} \quad \text{where} \quad I_{n,n} = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix} .$$

The fixed point set of τ is

$$H = \left\{ \begin{pmatrix} a & \\ & a^{-t} \end{pmatrix} \mid a \in \mathrm{Gl}(n, \mathbb{R}) \right\} = \mathrm{Gl}(n, \mathbb{R}) \,.$$

We write $\mathfrak{h}$ for the Lie algebra of H and denote by τ as well the derived involution on $\mathfrak{g}$. The τ-eigenspace decomposition on $\mathfrak{g}$ shall be denoted by

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q} \quad \text{where} \quad \mathfrak{q} = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix} .$$

Write $\mathfrak{q}^+ = \begin{pmatrix} 0 & V \\ 0 & 0 \end{pmatrix}$ and $\mathfrak{q}^- = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}$ and note that

$$\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$$

is the splitting of $\mathfrak{q}$ into two inequivalent irreducible H-modules.

The affine space $Y = G/H$ admits (up to sign) a unique H-invariant convex open cone $C \subset \mathfrak{q}$, containing no affine lines and consisting of hyperbolic elements. Explicitely:

$$C = \begin{pmatrix} 0 & W \\ W & 0 \end{pmatrix} = W \oplus W \subset \mathbf{q}^+ \oplus \mathbf{q}^- \,.$$

We form the cone bundle

$$\mathcal{C} = G \times_H W$$

and note that there is a natural G-equivariant map

$$P : G \times_H C \to \Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, (iy_2)^{-1})\,.$$

Let us verify that this map is in fact defined. For that one needs to check that for $h \in H$ and $y_1, y_2 \in W$, the elements (h, y_1, y_2) and $(\mathbf{1}, hy_1h^t, h^{-t}y_2h^{-1})$ have the same image. Indeed,

$$h \cdot (iy_1, (iy_2)^{-1}) = (ihy_1h^t, h(iy_2)^{-1}h^t) = (ihy_1h^t, (ih^{-t}y_2h^{-1})^{-1})$$

which was asserted.

Lemma 2.1. *The map $P : \mathcal{C} \to \Xi$ is onto.*

Proof. Write A for the group of diagonal matrices in G with positive entries. Note that the Lie algebra $\mathfrak{a}$ of A is a maximal flat in $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}(2n, \mathbb{R})$. In general, we know that $\mathfrak{p} = \mathrm{Ad}(K)\mathfrak{a}$. Furthermore, if W_d denotes the diagonal part of W , then $iW_d = A \cdot x_0$. From $G = KAK$ it mow follows that for any two points $(z, w) \in X$ there exist a $g \in G$ such that $g \cdot (z, w) = (x_0, w')$ with $w' \in iW_d$. As a consequence we obtain that

$$\Xi = G \cdot (iW_d, -iI_n)\,.$$

Clearly the right hand side is contained in the image of P and this finishes the proof. □

Remark 2.2. *(a) The map P is not injective. We shall give two different arguments for this assertion, beginning with an abstract one. If P were injective, then P establishes an homeoporphism between Ξ and $\mathcal{C} = G \times_H C$. In particular Ξ is homotopy equivalent to $Y = G/H$. But we know that Ξ is contractible; a contradiction.*

More concretely for $k \in K, k \neq \mathbf{1}$, the elements $[k, (iI_n, -iI_n)] \neq [\mathbf{1}, (iI_n, -iI_n)]$ have the same image in Ξ. It should be remarked however, that the map is generically injective.

(b) As H acts properly on C, it follows that G acts properly on the cone-bundle $G \times_H C$. Further it is not hard to see that the map P is proper.

We need a more invariant formulation of the map P. For that, note that the rational map

$$V_{\mathbb{C}} \to V_{\mathbb{C}}, \quad z \mapsto -z^{-1}$$

belongs to K. Its extension to $\mathcal{L}$, shall be denoted by s_0 and is given by

$$s_0(L) = \{(-w, v) \in \mathbb{C}^{2n} \mid (v, w) \in L\}.$$

Also, the anti-symplectic map $V_{\mathbb{C}} \to V_{\mathbb{C}}, z \mapsto -z$ has a natural extension to $\mathcal{L}$ given by

$$L \mapsto -L := \{(-v, w) \in \mathbb{C}^{2n} \mid (v, w) \in L\}.$$

In this way, we can rewrite P as

$$P : G \times_H C \to \Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2))$$

and we see that P extends to a continuous map

$$\tilde{P} : G \times_H \mathfrak{q} \to \mathcal{L} \times \mathcal{L}, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2)).$$

We restrict $\tilde{P}$ to $G \times_H \partial C$ and call this restriction p. It is clear that $\operatorname{im} p$ is contained in the boundary of Ξ in $\mathcal{L} \times \mathcal{L}$. But even more is true: the following proposition constitutes a G-equivariant "resolution" of $\partial\Xi$.

Proposition 2.3. $\operatorname{im} p \subset \partial\Xi$ *and the G-equivariant map*

$$p : G \times_H \partial C \to \partial\Xi, \quad [g, (y_1, y_2)] \mapsto g \cdot (iy_1, -s_0(iy_2))$$

is onto and proper.

Proof. We first show that $imp \subset \partial\Xi$. This means that $\operatorname{im} p \subset X_{\mathbb{C}}$. So we have to verify that for $y_1, y_2 \in cl(W)$ the Lagrangians

$$L_1 = \{(iy_1 v, v) \mid v \in \mathbb{C}^n\} \quad \text{and} \quad L_2 = \{(w, iy_2 w) \mid w \in \mathbb{C}^n\}$$

are transversal. We use the structure group H to bring y_1 in normal form

$$y_1 = \operatorname{diag}(\underbrace{1, \ldots, 1}_{p-\text{times}}, 0, \ldots, 0).$$

Thus $(iy_1 v, v) = (w, iy_2 w)$ for some $v, w \in \mathbb{C}^n$ means explicitely that

$$(iv_1, iv_2, \ldots, iv_p, 0, \ldots, 0; v_1, \ldots, v_n) = (w_1, \ldots, w_n; iy_2(w)).$$

We conclude that $w_{p+1} = \ldots = w_n = 0$. If $p = 0$, then we are finished. So let us assume that $p > 0$. But then

$$y_2 = \begin{pmatrix} -I_p & * \\ * & * \end{pmatrix}$$

and this contradicts the fact that y_2 is positive semi-definite.

We turn our attention to the onto-ness of p. First we note that

$$\partial C = W \times \partial W \amalg \partial W \times \partial W \amalg \partial W \times W\,. \tag{2.1}$$

Next we note that the closure $cl(X)$ in $\mathcal{L}$ equals the geodesic compactification. As a result $\partial X = K \cdot (i\partial W_d) = K \cdot (i\partial W)$. Likewise $\partial\overline{X} = K \cdot (-i\partial W)$. Observe that

$$\partial\Xi = \left[X \times \partial\overline{X} \amalg \partial X \times \partial\overline{X} \amalg \partial X \times \overline{X}\right] \cap X_{\mathbb{C}}\,. \tag{2.2}$$

We first show that $X \times \partial\overline{X} \subset \operatorname{im} p$, even more precisely $p(G \times_H (W \times \partial W)) = X \times \partial\overline{X}$. In fact,

$$X \times \partial\overline{X} = G \cdot (iI_n, K \cdot i\partial W) = G \cdot (iI_n, i\partial W)$$

and the claim is implied by (2.1). In the manner one verifies that $\partial X \times \overline{X} \subset \operatorname{im} p$.

In order to conclude the proof it is now enough to show that p is proper. This is because proper maps are closed and we have already seen that $\operatorname{im} p$ contains the dense piece $X \times \partial\overline{X} \amalg \partial X \times \overline{X} \subset \partial\Xi$. Now to see that p is proper, it is enough to show that inverse images of compact subsets in $[\partial X \times \partial\overline{X}] \cap X_{\mathbb{C}}$ are compact. For the other pieces in $\partial\Xi$ this is more or less automatic: Use that G acts properly on X, resp. $\overline{X}$ which implies that G acts properly on $X \times \partial\overline{X}$ resp. $\partial X \times \overline{X}$; likewise G acts properly on $G \times_H (W \times \partial W)$ and $G \times_H (\partial W \times W)$. Thus we are about to show that preimages of compacta in $[\partial X \times \partial\overline{X}] \cap X_{\mathbb{C}}$ are again compact. But this is more or less immediate from transversality; I allow myself to skip the details. □

Remark 2.4. *For $n = 1$ the map p is in fact a homeomorphism which we showed in* [3]. *If $n > 1$, the map p fails to be injective by the same computational reason shown in the preceeding remark. However, we emphasize that the map is generically injective and that $p|_{\partial C}$ is injective.*

References

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