MAASS-JACOBI FORMS

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Abstract. In this paper, we introduce the notion of new automorphic forms on $GL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)}$ generalizing that of automorphic forms on the general linear group $GL(n,\mathbb{R})$.

1. Introduction

Let n and m be two fixed positive integers. Let

$$Sp(n,\mathbb{R}) = \left\{ M\in\mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \right\}$$

be the real symplectic group of degree n, where

$$J_n := \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Here we denote by ${}^tA$ the transpose of any matrix A and by $F^{(k,l)}$ the set of all $k\times l$ matrices with entries in a commutative ring F. Let

$$H_n := \left\{ Z\in\mathbb{C}^{(n,n)} \mid Z = {}^tZ,\ \ \mathrm{Im}\, Z>0 \right\}$$

be the Siegel upper half plane of degree n. Then $Sp(n,\mathbb{R})$ acts on H_n transitively by

$$M<Z>:=(AZ+B)(CZ+D)^{-1}, \tag{1.1}$$

wher $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in Sp(n,\mathbb{R})$ and $Z\in H_n$. We let

$$\mathcal{P}_n := \left\{ Y\in\mathbb{R}^{(n,n)} \mid Y = {}^tY>0 \right\}$$

be the open convex cone of positive definite matrices of degree n in the Euclidean space $\mathbb{R}^{n(n+1)/2}$. Then the action (1.1) gives a natural action of the general linear group $GL(n,\mathbb{R})$ on $\mathcal{P}_n$ defined by

$$(g,Y)\longmapsto g<Y>:=gY\,{}^tg,\qquad g\in GL(n,\mathbb{R}),\ Y\in\mathcal{P}_n. \tag{1.2}$$

This work was supported financially by TGRC-KOSEF and Max-Planck Institut für Mathematik.

1991 Mathematics Subject Classification. Primary 11F55, 32C55, 32N10, 43A85, 43A90.

In his famous paper [S1], Selberg proved that the algebra $\mathbb{D}(\mathcal{P}_n)$ of all differential operators on $\mathcal{P}_n$ which are invariant under the action (1.2) of $GL(n,\mathbb{R})$ has a commutative ring structure and is generated by n algebraically independent ones (see Remark 4.13). Thus $\mathbb{D}(\mathcal{P}_n)$ may be identified with the commutative ring $\mathbb{C}[Z_1,\cdots,Z_n]$ of polynomials in n indeterminates. Using the action (1.2) and the $GL(n,\mathbb{R})$-invariant differential operators on $\mathcal{P}_n$, the concept of automorphic forms for $GL(n,\mathbb{Z})$ is defined together with a certain suitable growth condition. These automorphic forms on $GL(n,\mathbb{R})$ have been investigated intensively by Langlands school (cf. [B-J], [G], [J-L]) adelically and by some people (cf. [Gr], [I-T]) analytically.

The aim of this paper is to introduce a new notion of automorphic forms on the new group $GL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)}$ generalizing the concept of automorphic forms on $GL(n,\mathbb{R})$ and also the notion of the so-called *Maass-Jacobi forms*. Now I will explain the procedure in some detail. We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)}:=\left\{\,(\lambda,\mu;\kappa)\mid \lambda,\mu\in\mathbb{R}^{(m,n)},\ \kappa\in\mathbb{R}^{(m,m)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric }\right\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa)\circ(\lambda',\mu';\kappa'):=(\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G_{n,m}^J(\mathbb{R}):=Sp(n,\mathbb{R})\ltimes H_{\mathbb{R}}^{(n,m)} \tag{1.3}$$

endowed with the following multiplication law

$$\begin{aligned}&(M,(\lambda,\mu;\kappa))\cdot(M',(\lambda',\mu';\kappa'))\\ &:=(MM',(\tilde\lambda+\lambda',\tilde\mu+\mu',;\kappa+\kappa'+\tilde\lambda\,{}^t\mu'-\tilde\mu\,{}^t\lambda')),\end{aligned}$$

where $M,\,M'\in Sp(n,\mathbb{R})$ and $(\tilde\lambda,\tilde\mu):=(\lambda,\mu)M'$. The group $G_{n,m}^J(\mathbb{R})$ is called the *Jacobi group* of degree (n,m). It is easy to see that the Jacobi group $G_{n,m}^J(\mathbb{R})$ acts on the homogeneous space $H_n\times\mathbb{C}^{(m,n)}$ transitively by

$$(M,(\lambda,\mu;\kappa))\cdot(Z,W):=(M<Z>,(W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{1.4}$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W)\in H_n\times\mathbb{C}^{(m,n)}$. We let

$$G_{n,m}(\mathbb{R}):=GL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)} \tag{1.5}$$

be the semidirect product of $GL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,a)*(h,b):=(gh,a\,{}^th^{-1}+b), \tag{1.6}$$

where $g,h\in GL(n,\mathbb{R})$ and $a,b\in\mathbb{R}^{(m,n)}$. Then the action (1.4) of $G_{n,m}^J(\mathbb{R})$ on $H_n\times\mathbb{C}^{(m,n)}$ gives a *canonical* action of $G_{n,m}(\mathbb{R})$ on the *reductive* homogeneous space $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ given by

$$(g,a)\cdot(Y,V):=(gY\,{}^tg,\ (V+a)\,{}^tg), \tag{1.7}$$

where $g\in GL(n,\mathbb{R}),\ a\in\mathbb{R}^{(m,n)},\ Y\in\mathcal{P}_n$ and $V\in\mathbb{R}^{(m,n)}$. At this stage we naturally consider the discrete subgroup

$$G_{n,m}(\mathbb{Z}):=GL(n,\mathbb{Z})\ltimes\mathbb{Z}^{(m,n)} \tag{1.8}$$

of $G_{n,m}(\mathbb{R})$ and investigate the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ of all differential operators on $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ which is invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$.

Now it is natural for us to introduce the *new concept* of automorphic forms for $GL(n,\mathbb{Z})\ltimes\mathbb{Z}^{(m,n)}$ generalizing that of well-known automorphic forms for $GL(n,\mathbb{Z})$ and the new notion of Maass-Jacobi forms generalizing that of Maass wave forms on $SL(2,\mathbb{R})$. With some modifications, we may define the notion of Eisenstein series, L-functions and other related materials on $G_{n,m}(\mathbb{R})$ analytically or adelically.

The main result of this paper is the following theorem.

Main Theorem. The algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ is generated by the following invariant differential operators

$$D_j := \sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\right),\quad 1\le j\le n$$

and

$$\Psi_{kp} := \left\{\frac{\partial}{\partial V}Y\,{}^t\left(\frac{\partial}{\partial V}\right)\right\}_{kp},\quad 1\le k\le p\le m.$$

These differential operators satisfies the following relations:

$$D_iD_j = D_jD_i,$$

$$\Psi_{kp}\Psi_{lq}-\Psi_{lq}\Psi_{kp}=0,$$

$$D_j\Psi_{kp}-\Psi_{kp}D_j\neq 0$$

for all $1\le i,j\le n$, $1\le k\le p\le m$ and $1\le l\le q\le m$. The operators D_j's are algebraically independent and the operators Ψ_{kp}'s are also algebraically independent.

Here $\sigma(A)$ denotes the trace of a square matrix A. For the precise definitions of D_j and Ψ_{kp}, we refer to (4.1), (4.17) and (4.18).

This paper is organized as follows. In Section 2, we obtain some basic properties of the new group $G_{n,m}(\mathbb{R})$ which are to be used in the subsequent sections. In Section 3, we give some geometric properties of $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$. In Section 4, we investigate the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ of all $G_{n,m}(\mathbb{R})$-invariant differential operators on $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$. According to Helgason [He2], Theorem 4.9, p. 287, the problem of finding the complete description of the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ is reduced to that of finding all the invariants of the action of the orthogonal group $O(n)$ on the symmetric algebra $S(\mathfrak{p})$ of $\mathfrak{p}$ induced from the adjoint action Ad of $O(n)$ on $\mathfrak{p}$ given by

$$\mathrm{Ad}(k)((X,Z)) := (kX\,{}^tk,\ Z\,{}^tk),\quad k\in O(n),\quad (X,Z)\in\mathfrak{p}. \tag{1.9}$$

Here $\mathfrak{p}$ denotes a certain subspace of the Lie algebra of $G_{n,m}(\mathbb{R})$ which can be identified with the tangent space of $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ at the origin $(E_n,0)$. For more detail, we refer the reader to (4.5) and (4.6) in Section 4. We give a complete description of algebraically independent generators of the subalgebra $S(\mathfrak{p})^K$ of K-invariants and generators of the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$. We also provide many invariant differential operators on $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$. And we show that for any positive integers n and m, the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ is *not* commutative and hence $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ is not a *weakly* *symmetric* Riemannian manifold in the sense of Selberg [S1], p. 51. It is well known that if G is a noncompact semisimple real Lie group and K is a maximal compact subgroup of G, then the algebra $\mathbb{D}(G/K)$ of all G-invariant differential operators on G/K is a commutative ring and that actually it is a polynomial ring with rank (G) algebraically independent generators. But in our case $G_{n,m}(\mathbb{R})$ is not semisimple and the algebra $\mathbb{D}(G_{n,m}(\mathbb{R})/O(n)) = \mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$ is *not* commutative. So we need a new approach to the study of subjects related to $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$. In Section 5, we define the spherical functions on $G_{n,m}(\mathbb{R})$ and characterize these functions following the general theory of spherical functions on a Lie group. We describe the spherical functions on $G_{n,m}(\mathbb{R})$ explicitly. In Section 6, using the results of Section 2 and 4, we define the concept of new automorphic forms on the new group $G_{n,m}(\mathbb{R})$. In Section 7, we define the notion of *Maass-Jacobi forms* on the group $SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$. This notion generalizes that of Maass wave forms on $SL(2,\mathbb{R})$. The reason why I name "Maass-Jacobi" to such new automorphic forms is that the new automorphic forms are implicitly related to the concepts of *Maass wave forms* (cf. [M1]) and *Jacobi forms* (cf. [E-Z], [K], [Y1-6], [Z]). We deal with the simple case $n = 2$ and $m = 1$. In this case, we characterize Maass-Jacobi forms as smooth eigenfunctions on $\mathbb{H} \times \mathbb{C}$ satisfying a certain invariance property, where $\mathbb{H}$ denotes the Poincaré upper half plane. Also we discuss the actions of the group $SL_{2,1}(\mathbb{R}) := SL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}$ on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ and $\mathbb{H} \times \mathbb{C}$ (cf. see (7.4) and (7.5)). We provides the algebra of all differential operators on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ or $\mathbb{H} \times \mathbb{C}$ invariant under the actions (7.4) and (7.5) of $SL_{2,1}(\mathbb{R})$ without proof. And we present $SL_{2,1}(\mathbb{R})$-invariant Riemannian metrics on $\mathbb{H} \times \mathbb{C}$ and $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ together with their Laplace-Beltrami operators. The detail can be found in [Y7-8]. In the final section, we briefly remark a geometrical aspect of the quotient space $G_{n,m}(\mathbb{Z})\backslash\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ and provide some problems related to our subjects.

Acknowledgement. I would like to thank Professor R. Bott, W. Schmid, R. Howe, S. Helgason and B. Kostant for some useful discussions and their kind advices about the invariant theory. This work was mostly done during my stay in Cambridge. I am very grateful to Department of Mathematics, Harvard University for its wonderful hospitality during my stay in Cambridge. The final writing was completed in a way during my stay at Max-Planck-Institut für Mathematik in Bonn. I would like to give my hearty thanks to the institute for its financial support and its wonderful hospitality during my stay in Bonn.

Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ denotes the set of all positive integers. The symbol ":=" means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix A, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of M. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = {}^tABA$. E_n denotes the identity matrix of degree n.

2. The Group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$

In this section, we obtain some basic properties of the group

$$G_{n,m}(\mathbb{R}) := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

endowed with the multiplication law (1.6). Clearly the stabilizer K of $G_{n,m}(\mathbb{R})$ at $(E_n,0)$ is given by

$$K := \{\, (k,0) \in G_{n,m}(\mathbb{R}) \mid k \in O(n)\,\}$$

which is isomorphic to the orthogonal group $O(n)$. Since the action (1.7) of $G_{n,m}(\mathbb{R})$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ is transitive, $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ is diffeomorphic to the homogeneous space $G_{n,m}(\mathbb{R})/O(n)$. We note that $G_{n,m}(\mathbb{R})$ acts on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ effectively. It is easy to see that the Lie algebra $\mathfrak{g}$ of $G_{n,m}(\mathbb{R})$ is given by

$$\mathfrak{g} = \Big\{ \,(X,Z) \mid X \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)}\, \Big\} \tag{2.1}$$

equipped with the following Lie bracket

$$[(X_1,Z_1),\,(X_2,Z_2)] := ([X_1,X_2]_0,\, Z_2\,{}^tX_1 - Z_1\,{}^tX_2), \tag{2.2}$$

where $[X_1,X_2]_0 := X_1X_2 - X_2X_1$ denotes the usual matrix bracket and $(X_1,Z_1),\,(X_2,Z_2) \in \mathfrak{g}$. The adjoint representation Ad of $G_{n,m}(\mathbb{R})$ is given by

$$\mathrm{Ad}\,((g,a))(X,Z) = (gXg^{-1},\,(Z - a\,{}^tX)\,{}^tg), \tag{2.3}$$

where $(g,a) \in G_{n,m}(\mathbb{R})$ and $(X,Z) \in \mathfrak{g}$. And the adjoint representation ad of $\mathfrak{g}$ on $\mathrm{End}\,(\mathfrak{g})$ is given by

$$\mathrm{ad}\,((X,Z))((X_1,Z_1)) = [(X,Z),(X_1,Z_1)]. \tag{2.4}$$

We easily see that the Killing form B of $\mathfrak{g}$ is given by

$$B((X_1,Z_1),(X_2,Z_2)) = 4(n+m+1)\,\sigma(X_1X_2). \tag{2.5}$$

The Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \big\{\, (X,0) \in \mathfrak{g} \mid X + {}^tX = 0\,\big\}. \tag{2.6}$$

We let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{p} := \Big\{ \,(X,Z) \in \mathfrak{g} \mid X = {}^tX \in \mathbb{R}^{(n,n)},\ Z \in \mathbb{R}^{(m,n)}\, \Big\}. \tag{2.7}$$

Then we have the following relation

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \qquad \text{and} \qquad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}. \tag{2.8}$$

In addition, we have

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \qquad (\,\text{direct sum}\,). \tag{2.9}$$

Thus the coset space $G_{n,m}(\mathbb{R})/O(n) \cong \mathcal{P}_n \times \mathbb{R}^{(m,n)}$ is a *reductive* homogeneous space in the sense of [He2], p. 284. We note that the restriction of the Killing form B to $\mathfrak{k}$ is negative definite and the restriction of B to the abelian subalgebra $\mathfrak{r} := \{\,(0,Y) \in \mathfrak{g}\,\}$ is identically zero. Since $\mathfrak{r}$ is the radical of B, B is degenerate. The subspace $\mathfrak{p}$ of $\mathfrak{g}$ may be identified with the tangent space of the reductive homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)} = G_{n,m}(\mathbb{R})/O(n)$ at the origin $(E_n,0)$.

An Iwasawa decomposition of the group $G_{n,m}(\mathbb{R})$ is given by

$$G_{n,m}(\mathbb{R}) = NAK, \quad N = N_0 \ltimes \mathbb{R}^{(m,n)}, \tag{2.10}$$

where A is an abelian subgroup of $G_{n,m}(\mathbb{R})$ consisting of all $n \times n$ real diagonal matrices with positive real diagonals and N_0 is the subgroup of $GL(n,\mathbb{R})$ consisting of all $n \times n$ upper triangular real matrices whose diagonal entries are all equal to one. The Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of $G_{n,m}(\mathbb{R})$ is given by

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}, \quad \mathfrak{n} = \mathfrak{n}_0 \oplus \mathbb{R}^{(m,n)}, \tag{2.11}$$

where $\mathfrak{a}$ is the subalgebra of $\mathfrak{g}$ consisting of all $n \times n$ real diagonal matrices and $\mathfrak{n}_0$ denotes the subalgebra of $\mathfrak{g}$ consisting of all $n \times n$ upper triangular real matrices whose diagonal entries are all equal to zero. In fact, $\mathfrak{a}$ is the Lie algebra of A and $\mathfrak{n}$ is the Lie algebra of N.

Now we will determine the roots of $\mathfrak{g}$. Let $E_{ij}\,(\,1 \leq i, j \leq n\,)$ be the $n \times n$ matrix with entry 1 where the i-throw and the j-th column meet, all other entries being 0. We let $F_{kl}\,(\,1 \leq k \leq m,\ 1 \leq l \leq n\,)$ be the $m \times n$ matrix with 1 in the (k,l)-entry and zero elsewhere. We put

$$\begin{aligned} H_i &:= (E_{ii}, 0), \quad 1 \leq i \leq n, \\ E_{ij}^0 &:= (E_{ij}, 0), \quad 1 \leq i, j \leq n, \ \ i \neq j, \\ F_{kl}^0 &:= (0, F_{kl}), \quad 1 \leq k \leq m, \ \ 1 \leq l \leq n. \end{aligned}$$

Then the set $\{\, H_i,\, E_{ij}^0,\, F_{kl}^0 \,\}$ forms a basis for $\mathfrak{g}$. Clearly $\mathfrak{a} = \sum_{i=1}^n \mathbb{R}\, H_i$ and $\mathfrak{n} = \sum_{i<j} \mathbb{C}\, E_{ij}^0 + \sum_{k,l} \mathbb{R}\, F_{kl}^0$. Using the rules

$$E_{ij}E_{kl} = \delta_{jk}E_{il}, \quad [E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kl},$$

we obtain the following

$$\begin{aligned} [H, E_{ij}^0] &= (\, e_i(H) - e_j(H))\, E_{ij}^0, \quad i \neq j, \\ [H, F_{kl}^0] &= \ e_l(H) F_{kl}^0, \quad 1 \leq k \leq m, \ 1 \leq l \leq n \end{aligned}$$

for all $H \in \mathfrak{a}$, where $e_i\,(\,1 \leq i \leq n\,)$ is the linear functional $e_i : \mathfrak{a} \longrightarrow \mathbb{C}$ on $\mathfrak{a}$ defined by

$$e_i(H_j) = \delta_{ij}, \quad 1 \leq j \leq n.$$

Thus the roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ are given by

$$\pm e_l,\ \pm(e_i - e_j), \quad 1 \leq l \leq n,\ 1 \leq i < j \leq n. \tag{2.12}$$

We denote by $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{a}$. The set Σ^+ of positive roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ is given by

$$\Sigma^+ = \{\, e_l,\ e_i - e_j \mid 1 \leq l \leq n,\ 1 \leq i < j \leq n \,\}. \tag{2.13}$$

The roots

$$e_1 - e_2,\, e_2 - e_3, \cdots, e_{n-1} - e_n,\, e_n$$

form the set of simple roots. We observe that the root subspaces of $\mathfrak{g}$ are given by

$$\mathfrak{g}_{-e_l} = \mathfrak{g}_{e_l} = \sum_{k=1}^{m} \mathbb{R}\, F^0_{kl}, \quad 1 \le l \le n,$$
$$\mathfrak{g}_{e_i - e_j} = \mathbb{R}\, E^0_{ij}, \quad 1 \le i,j \le n,\ i \neq j.$$

So we obtain

$$\dim \mathfrak{g}_{e_l} = \dim \mathfrak{g}_{-e_l} = m, \quad \dim \mathfrak{g}_{e_i - e_j} = 1, \quad 1 \le i,j,l \le n,\ i \neq j. \tag{2.14}$$

We define

$$\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\, \dim \mathfrak{g}_\alpha\,) \cdot \alpha. \tag{2.15}$$

By an easy computation, we have

$$\rho = \frac{1}{2} \sum_{h=1}^{n} (n+m+1-2h)\, e_h. \tag{2.16}$$

3. Some geometry on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$

For a variable $(Y,V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}$ with $Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$, we put

$$Y = (y_{\mu\nu}), \quad V = (v_{kl}), \quad dY = (dy_{\mu\nu}), \quad dV = (dv_{kl})$$

and

$$[dY] := \prod_{\mu \le \nu} dy_{\mu\nu}, \qquad [dV] := \prod_{k,l} dv_{kl},$$

where $1 \le \mu, \nu,\ l \le n$ and $1 \le k \le m$.

Lemma 3.1. The metric ds^2 on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ defined by

$$ds^2 := \sigma(Y^{-1} dY\, Y^{-1} dY) + \sigma(Y^{-1}\, {}^t(dV)\, dV) \tag{3.1}$$

is invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. So $G_{n,m}(\mathbb{R})$ is a subgroup of the isometry group of the Riemannian manifold $(\mathcal{P}_n \times \mathbb{R}^{(m,n)},\, ds^2)$. The Laplace-Beltrami operator of $(\mathcal{P}_n \times \mathbb{R}^{(m,n)},\, ds^2)$ is a differential operator of order 2 which is invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$.

Proof. For a fixed element $(g,a) \in G_{n,m}(\mathbb{R})$, we set

$$(Y^*, V^*) := (g,a) \cdot (Y,V).$$

The first statement follows immediately from the fact that

$$dY^* = g dY g^{-1} \quad \text{and} \quad dV^* = dV\, {}^t g.$$

The last statement follows from the fact that a diffeomorphism Φ on a Riemannian manifold leaves the Laplace-Beltrami operator invariant if and only if it is an isometry (see [He1], proposition 2, p. 245). $\square$

Lemma 3.2. The following volume element dv on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ defined by

$$dv := (\det Y)^{-\frac{n+m+1}{2}}[dY][dV] \tag{3.2}$$

is invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$.

Proof. For a fixed element $(g,a) \in G_{n,m}(\mathbb{R})$, we set

$$(Y^*, V^*) := (g,a)\cdot(Y,V) = (gY\,{}^tg, (V+a)\,{}^tg).$$

Let $\frac{\partial(Y^*,V^*)}{\partial(Y,V)}$ be the Jacobian determinant of the action (1.7) of $G_{n,m}(\mathbb{R})$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. It is known that the Jacobian determinant of the action $Y \longmapsto Y^*$ is given by

$$\frac{\partial(Y^*)}{\partial(Y)} = (\det g)^{n+1}.$$

Take the diagonal matrix $g = (d_1, \cdots, d_n)$ with distinct real numbers d_i. Obviously if $a = (a_{kl})$, $V = (v_{kl})$ and $V^* = (v^*_{kl})$, then $v^*_{kl} = (v_{kl} + a_{kl})d_l$ for all k,l. Thus we have

$$\frac{\partial(V^*)}{\partial(V)} = (d_1 \cdots d_n)^m = (\det g)^m. \tag{3.3}$$

Since the set of all $n\times n$ real matices whose eigenvalues are all distinct is everywhere dense in $GL(n,\mathbb{R})$, and $\frac{\partial(V^*)}{\partial(V)}$ is a rational function, the relation (3.3) holds for any $g \in GL(n,\mathbb{R})$. It is easy to see that

$$\frac{\partial(Y^*,V^*)}{\partial(Y,V)} = \frac{\partial(Y^*)}{\partial(Y)}\cdot\frac{\partial(V^*)}{\partial(V)}.$$

Thus we obtain

$$[dY^*][dV^*] = (\det g)^{m+n+1}[dY][dV].$$

Since $\det Y^* = (\det g)^2 \det Y$, we have

$$(\det Y^*)^{-\frac{m+n+1}{2}}[dY^*][dV^*] = (\det Y)^{-\frac{m+n+1}{2}}[dY][dV].$$

Hence the volume element (3.2) is invariant under the action (1.7). □

Theorem 3.3. Any geodesic through the origin $(E_n, 0)$ is of the form

$$\gamma(t) = \left(\lambda(2t)[k],\, Z\left(\int_0^t \lambda(t-s)ds\right)[k]\right),$$

where k is a fixed element of $O(n)$, t is a real variable, $\lambda_1, \lambda_2, \cdots, \lambda_n$ are fixed real numbers but not all zero and

$$\lambda(t) := \operatorname{diag}(e^{\lambda_1 t}, \cdots, e^{\lambda_n t}).$$

Furthermore, the tangent vector $\gamma'(0)$ of the geodesic $\gamma(t)$ at $(E_n, 0)$ is $(D[k], Z)$, where $D = \operatorname{diag}(2\lambda_1, \cdots, 2\lambda_n)$.

Proof. Let $W := (X, Z)$ be an element of $\mathfrak{p}$ with $X \neq 0$. Then the curve

$$\alpha(t) := \exp tW = \left(e^{tX},\, Z\left(\left(\int_0^t e^{-sX} ds\right)\right)\right), \quad t \in \mathbb{R}$$

is a geodesic in $G_{n,m}(\mathbb{R})$ with $\alpha'(0) = W$ passing through the identity of $G_{n,m}(\mathbb{R})$. Thus the curve

$$\gamma(t) := \alpha(t) \cdot (E_n, 0) = \left(e^{2tX},\, Z\left(\int_0^t e^{-sX} ds\right) e^{tX} \right)$$

is a geodesic in $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ passing through the origin $(E_n, 0)$. Since X is a symmetric real matrix, there is a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ such that

$$X = {}^t k \Lambda k \quad \text{for some } k \in O(n),$$

where $\lambda_1, \cdots, \lambda_n$ are real numbers and not all zero. Thus we may write

$$\gamma(t) = \left((\delta_{kl} e^{2\lambda_k t})[k],\, Z\left(\int_0^t e^{(t-s)\Lambda} ds\right)[k] \right).$$

Hence this completes the proof. □

4. Proof of Main Theorem

In this section, we investigate the differential operators on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ which are invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$.

We set

$$\frac{\partial}{\partial Y} := \left(\frac{1+\delta_{\mu\nu}}{2} \frac{\partial}{\partial y_{\mu\nu}} \right), \quad \frac{\partial}{\partial V} := \left(\frac{\partial}{\partial v_{kl}} \right), \tag{4.1}$$

where $1 \leq \mu, \nu, l \leq n,\ 1 \leq k \leq m$ and $\delta_{\mu\nu}$ denotes the Kronecker delta symbol. For a fixed element $(g, a) \in G_{n,m}(\mathbb{R})$, we set

$$(Y^*, V^*) := (g, a) \cdot (Y, V) = (gY\,{}^t g,\, (V + a)\,{}^t g),$$

where $(Y, V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}$. Then we have

$$Y^* = gY\,{}^t g, \quad V^* = (V + a)\,{}^t g \tag{4.2}$$

and

$$\frac{\partial}{\partial Y^*} = {}^t g^{-1} \frac{\partial}{\partial Y} g^{-1}, \quad \frac{\partial}{\partial V^*} = \frac{\partial}{\partial V} \cdot g^{-1}. \tag{4.3}$$

In [Y4], I introduced the following differential operator $M_{n,m,\mathcal{M}}$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ defined by

$$M_{n,m,\mathcal{M}} := \det(Y) \cdot \det\left(\frac{\partial}{\partial Y} + \frac{1}{8\pi}\, {}^t\left(\frac{\partial}{\partial V}\right) \mathcal{M}^{-1} \left(\frac{\partial}{\partial V}\right) \right), \tag{4.4}$$

where $\mathcal{M}$ is a positive definite, symmetric half-integral matrix of degree m. This differential operator characterized *singular Jacobi forms*. For more detail, we refer the reader to [Y4].

Lemma 4.1. The differential operator $M_{n,m,\mathcal{M}}$ is invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$.

Proof. It follows immediately from (4.2) and (4.3). □

We recall that the adjoint action of $K := O(n)$ on $\mathfrak{p}$ is given by

$$\mathrm{Ad}\,(k)((X,Z)) = (kX\,{}^tk,\ Z\,{}^tk), \tag{4.5}$$

where $k \in K$ and $(X,Z) \in \mathfrak{p}$ with $X = {}^tX \in \mathbb{R}^{(n,n)}$. The action (4.5) induces uniquely the action ρ of K on the symmetric algebra $S(\mathfrak{p})$

$$\rho\,:\,K \longrightarrow \mathrm{Aut}\,(S(\mathfrak{p})). \tag{4.6}$$

Now we have the following natural questions:

Problem A. Find the algebra $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$ of all differential operators on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ which are invariant under the action (1.7) of $G_{n,m}(\mathbb{R})$ on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. In other words, this problem is to find a set of algebraically independent differential operators which generate the algebra $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

Problem B. Compute all the joint eigenfunctions of $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

According to [He2], Problem A is reduced to the following problem:

Problem A'. Find all the invariants of the action (4.6) of K on $S(\mathfrak{p})$.

We define the nondegerate symmetric bilinear form $\tilde{B}$ on $\mathfrak{p}$ by

$$\tilde{B}((X_1,Z_1),(X_2,Z_2)) := \sigma(X_1X_2) + \sigma(Z_1\,{}^tZ_2),\quad (X_i,Z_i)\in\mathfrak{p},\quad i=1,2. \tag{4.7}$$

We identify the dual space $\mathfrak{p}^*$ of $\mathfrak{p}$ with $\mathfrak{p}$ via $\tilde{B}$ by letting $v \mapsto v^*\,(\,v \in \mathfrak{p}\,)$, where $v^*(w) = \tilde{B}(w,v)$ for all $w \in \mathfrak{p}$. Let Ad^* be the contragredient of the adjoint representation Ad of K defined by

$$\mathrm{Ad}^*(k)v^*(w) := v^*(\mathrm{Ad}\,(k^{-1})w) = \tilde{B}(\mathrm{Ad}\,(k^{-1})w,\,v),\quad k\in K,\quad v,w\in\mathfrak{p}. \tag{4.8}$$

We also identify $\mathfrak{p}^{**}$ with $\mathfrak{p}$ via $\tilde{B}$ by letting $v \mapsto v^{**}\,(\,v\in\mathfrak{p}\,)$, where $v^{**}(w^*) = w^*(v) = \tilde{B}(v,w),\ v,w\in\mathfrak{p}$. Let $\mathrm{Pol}\,(\mathfrak{p}^*) = S(\mathfrak{p}^{**})$ be the polynomial algebra on $\mathfrak{p}^*$, i.e., the algebra of all polynomial functions on $\mathfrak{p}^*$. Then K acts on $\mathrm{Pol}\,(\mathfrak{p}^*)$ by

$$\mathrm{Ad}^0(k)P(v) := P(\mathrm{Ad}(k^{-1})v),\quad k\in K,\quad P\in\mathrm{Pol}\,(\mathfrak{p}^*),\quad v\in\mathfrak{p}. \tag{4.9}$$

Then it is easy to see that $S(\mathfrak{p})$ is isomorphic to $\mathrm{Pol}\,(\mathfrak{p}^*)$ and $S(\mathfrak{p})^K$ is isomorphic to $\mathrm{Pol}\,(\mathfrak{p}^*)^K$. Here $S(\mathfrak{p})^K$ denotes the subalgebra of all K-invariants under the action (4.6) of K and $\mathrm{Pol}\,(\mathfrak{p}^*)^K$ denotes the subalgebra of all K-invariants under the action (4.9) of K.

By the above argument, the problem of finding the K-invariants of the action (4.6) of K is equivalent to that of finding the polynomials on $\mathfrak{p}$ which are invariant under the action (4.5) of K.

Before we give a complete set of algebraically independent generators of the algebra $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$, we give some examples of K-invariants under the action (4.6) and invariant differential operators on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. Also we describe the relation between the two invariants.

For variables (X,Z) in $\mathfrak{p}$, we define the polynomials P_j and Q_k on $\mathfrak{p}$ by

$$P_j(X,Z) := \sigma(X^j), \quad 1 \leq j \leq n, \tag{4.10}$$

$$\xi_{kp}(X,Z) := (Z\,{}^tZ)_{kp}, \qquad 1 \leq k \leq p \leq m \tag{4.11}$$

and

$$Q_k(X,Z) := \sigma(({}^tZZ)^k)), \quad 1 \leq k \leq m, \tag{4.12}$$

where $(Z\,{}^tZ)_{kp}$ denotes the (k,p)-entry of the matrix function $Z\,{}^tZ$.

And we define the polynomials R_{jk} on $\mathfrak{p}$ by

$$R_{jk}(X,Z) := \sigma(X^j({}^tZZ)^k), \quad 1 \leq j \leq n, \;\; 1 \leq k \leq m. \tag{4.13}$$

For any real matrix S of degree m, we define the polynomials $M_{j,S}$, $Q_{k,S}$ and $R_{j,k,l,S}$ on $\mathfrak{p}$ by

$$M_{j,S}(X,Z) := \sigma\left((X + {}^tZSZ)^j\right), \quad 1 \leq j \leq n, \tag{4.14}$$

$$Q_{k,S}(X,Z) := \sigma\left(({}^tZSZ)^k\right), \qquad 1 \leq k \leq m \tag{4.15}$$

and

$$R_{j,k,l,S}(X,Z) := \sigma(X^j({}^tZSZ)^k(X + {}^tZSZ)^l), \;\; 1 \leq l \leq n. \tag{4.16}$$

Proposition 4.2. The set $\{\, P_j,\, \xi_{kp} \,|\, 1 \leq j \leq n,\; 1 \leq k \leq p \leq m \,\}$ is a set of algebraically independent K-invariants. Furthermore the polynomials Q_k, R_{jk}, $M_{j,S}$, $Q_{k,S}$ and $R_{j,k,l,S}$ are also K-invariants, where S is any real matrix of degree m.

Proof. The proof follows immediately from (4.5) and (4.6). □

We define the differential operators D_j and L_k on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ by

$$D_j := \sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\right), \quad 1 \leq j \leq n, \tag{4.17}$$

$$\Psi_{kp} := \left\{ \frac{\partial}{\partial V}\, Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\right\}_{kp}, \quad 1 \leq k \leq p \leq m \tag{4.18}$$

and

$$L_k := \sigma\left(\left\{ Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^k\right), \quad 1 \leq k \leq m. \tag{4.19}$$

Also we define the differential operators S_{jk} by

$$S_{jk} := \sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^j \left\{ Y\, {}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right\}^k\right), \tag{4.20}$$

where $1 \leq j \leq n$ and $1 \leq k \leq m$.

For any real matrix S of degree m, we define the differential operators $\Phi_{j,S}$ and $L_{k,S}$ by

$$\Phi_{j,S} := \sigma\left(\left\{ Y\left(2\frac{\partial}{\partial Y} + {}^t\left(\frac{\partial}{\partial V}\right) S \left(\frac{\partial}{\partial V}\right)\right)\right\}^j\right), \quad 1 \leq j \leq n, \tag{4.21}$$

$$L_{k,S} := \sigma\left(\left\{ Y\, {}^t\left(\frac{\partial}{\partial V}\right) S \left(\frac{\partial}{\partial V}\right)\right\}^k\right), \qquad 1 \leq k \leq m \tag{4.22}$$

and

$$\Phi_{j,k,l,S}(X,Z) := \tag{4.23}$$
$$\sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^j \left(Y\, {}^t\left(\frac{\partial}{\partial V}\right) S \left(\frac{\partial}{\partial V}\right)\right)^k \left\{ Y\left(2\frac{\partial}{\partial Y} + {}^t\left(\frac{\partial}{\partial V}\right) S \left(\frac{\partial}{\partial V}\right)\right)\right\}^l\right).$$

Proposition 4.3. The set $\{\, D_j,\ \Psi_{kp} \mid 1 \leq j \leq n,\ 1 \leq k \leq p \leq m \,\}$ is a set of algebraically independent $G_{n,m}(\mathbb{R})$-invariant differential operators. Furthermore, $L_k,\ S_{jk},\ \Phi_{j,S},\ L_{k,S}$ and $\Phi_{j,k,l,S}\,(\, 1 \leq j \leq n,\ 1 \leq k \leq m\,)$ are invariant differential operators under the action (1.7) of $G_{n,m}(\mathbb{R})$.

Proof. The proof follows immediately from (4.2) and (4.3). □

We recall that $S(\mathfrak{p})^K$ is the algebra of all invariants of the action (4.6) of K. According to [He2], Theorem 4.9, p. 287, there exists a canonical linear bijection λ of $S(\mathfrak{p})^K$ onto $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$. We recall that if (ξ_α) is any basis of $\mathfrak{p}$ and $P \in S(\mathfrak{p})^K$, then

$$\left(D_{\lambda(P)} f\right)(gK) = \left[P\left(\frac{\partial}{\partial t_\alpha}\right) f\left(g \exp\left(\sum_{\alpha=1}^N t_\alpha \xi_\alpha\right) K\right)\right]_{(t_\alpha)=0}, \tag{4.24}$$

where $f \in C^\infty(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$ and $N = n(n+1)/2 + mn$ is the dimension of $\mathfrak{p}$.

Proposition 4.4. $D_j,\ \Psi_{kp},\ L_k,\ S_{jk},\ \Phi_{j,S},\ L_{k,S}$ and $\Phi_{j,k,l,S}$ are the $G_{n,m}(\mathbb{R})$-invariant differential operators on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ corresponding to the K-invariants $P_j,\ \xi_{kp},\ Q_k,\ R_{jk}$, $M_{j,S},\ Q_{k,S}$ and $R_{j,k,l,S}$ respectively under a canonical linear bijection $\lambda : S(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

Proof. Using the bijection λ in (4.24), it follows from a direct computation. □

Now we prove the main theorem mentioned in the introduction. The proof will be divided into several steps.

First we let G be a reductive real group and H a closed subgroup of G with their Lie algebras $\mathfrak{g}_0$ and $\mathfrak{h}$ respectively. We assume that $\mathfrak{g}_0$ has a decomposition $\mathfrak{g}_0 = \mathfrak{h} + \mathfrak{m}$ (the direct sum) satisfying the following conditions

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}.$$

The adjoint representation Ad_G of H on $\mathfrak{m}$ induces the action Ad_G of H on the polynomial algebra $\mathrm{Pol}(\mathfrak{m})$ on $\mathfrak{m}$. Let $\mathrm{Pol}(\mathfrak{m})^H$ be the subalgebra of all $\mathrm{Ad}_G(H)$-invariants in $\mathrm{Pol}(\mathfrak{m})$ and $\mathbb{D}(G/H)$ the algebra of all G-invariant differential operators on the homogeneous space G/H. According to [He2], there is a canonical linear bijection $\mathrm{Pol}(\mathfrak{m})^H \longrightarrow \mathbb{D}(G/H)$

$$P \longmapsto D_P, \qquad P \in \mathrm{Pol}(\mathfrak{m})^H \tag{*}$$

defined in a similiar way as (4.24).

Lemma 4.5. Assume that $\mathbb{D}(G/H)$ is commutative. Let $R_1, R_2, \cdots, R_q$ be algebraically independent generators of $\mathrm{Pol}(\mathfrak{m})^H$ and $\hat{R}_i := D_{R_i}\,(\,1 \leq i \leq q\,)$ the elements of $\mathbb{D}(G/H)$ given by (*). Then $\hat{R}_1, \cdots, \hat{R}_q$ are algebraically independent generators of $\mathbb{D}(G/H)$.

Proof. We can prove the above lemma following the line of the argument in [G-H], Lemma 4.2 and [He1], pp.269-270. For the convenience of the reader, we will give a proof of the above lemma. First we show that the operators $\hat{R}_i := D_{R_i}\,(\,1 \leq i \leq q\,)$ are algebraically independent. Suppose that they are not algebraically independent. Then there exists a nonzero polynomial

$$P = \sum a_{n_1,\cdots,n_q} x_1^{n_1} \cdots x_q^{n_q}$$

such that $P(\hat{R}_1, \cdots, \hat{R}_q) = 0$. Let $r_i\,(\,1 \leq i \leq q\,)$ be the degree of the polynomial R_i. We put $N := \max(\sum r_i n_i)$, where the maximum is taken over the set of all q-tuples $(n_1, \cdots, n_q)$ with $a_{n_1,\cdots,n_q} \neq 0$. We consider the following two polynomials S and Q defined by

$$S := \sum a_{n_1,\cdots,n_q} R_1^{n_1} \cdots R_q^{n_q}$$

and

$$Q := \sum_{\sum d_i n_i = N} a_{n_1,\cdots,n_q} R_1^{n_1} \cdots R_q^{n_q}.$$

Obviously $Q \neq 0$ by assumption and the polynomial $R := S - Q$ has degree less than N. It is easy to see that the operator

$$U := \sum a_{n_1,\cdots,n_q} \hat{R}_1^{n_1} \cdots \hat{R}_q^{n_q} - D_S$$

has order less than N. And we note that since $P(\hat{R}_1, \cdots, \hat{R}_q) = 0$, we have

$$U = 0 - D_S = -D_Q - D_R.$$

Therefore the operator U has order N by the linear bijection (*). This contradicts our assumption. It remains to prove that the operators $\hat{R}_i\,(\,1 \leq i \leq q\,)$ are generators. Suppose D is an element of $\mathbb{D}(G/H)$. Then there exists a polynomial $F \in \mathrm{Pol}(\mathfrak{m})^H$ such that $D = D_F$ via the linear bijection (*). We write the polynomial F as

$$F = \sum \beta_{m_1,\cdots,m_q} R_1^{m_1} \cdots R_q^{m_q}, \quad \beta_{m_1,\cdots,m_q} \in \mathbb{C}.$$

We put $M := \max(\sum r_i m_i)$, where the maximum is taken over the set of all q-tuple $(m_1, \cdots, m_q)$ with $\beta_{m_1,\cdots,m_q} \neq 0$. Then it is easy to see that the differential operator

$$D - \sum_{\sum r_i m_i = M} \beta \hat{R}_1^{N_1} \cdots \hat{R}_q^{N_q}$$

is of lower order than D. Hence it follows from the induction on the order that the $\hat{R}_i\,(\,1 \leq i \leq q\,)$ are generators of $\mathbb{D}(G/H)$. $\square$

Secondly, we review the well-known classical results. Let $V_1 := \mathrm{Symm}\,(\mathbb{R}^n)$ be the vector space of all $n \times n$ symmetric real matrices. The polynomial algebra $\mathrm{Pol}\,(V_1)$ on V_1 has a standard grading:

$$\mathrm{Pol}\,(V_1) \cong \sum_{d=0}^{\infty} \mathrm{Pol}^d(V_1).$$

The elements of $\mathrm{Pol}^d\,(V_1)\,(\,d \geq 0\,)$ is called *homogeneous* of *degree d*. We consider the action of K on V_1 given by

$$(k, X) \longmapsto kX\,{}^tk, \quad k \in K,\ X \in V_1. \tag{4.25}$$

Then this action extends uniquely to the action of K on $\mathrm{Pol}\,(V_1)$

$$\rho_1\,:\,K \longrightarrow \mathrm{Aut}\,(\mathrm{Pol}\,(V_1)) \tag{4.26}$$

by algebra automorphisms preserving the grading. For a typical element $X = (x_{ij}) \in V_1$ with $x_{ij} = x_{ji}\,(\,1 \leq i, j \leq n\,)$, we define the polynomials P_j on V_1 by

$$P_j(X) := \sigma(X^j), \quad 1 \leq j \leq n. \tag{4.27}$$

Theorem 4.6. Let $\mathrm{Pol}\,(V_1)^K$ be the subalgebra of $\mathrm{Pol}\,(V_1)$ consisting of K-invariants under the action (4.26) of K. Then the polynomials $P_j(X)\,(\,1 \leq j \leq n\,)$ are algebraically independent generators of $\mathrm{Pol}\,(V_1)^K$.

Proof. It is obvious that each polynomial $P_j(X)$ is an element of $\mathrm{Pol}\,(V_1)^K$. The $GL(n, \mathbb{R})$-invariant differential operators D_j^0 for the action (1.2) of $GL(n, \mathbb{R})$ on $\mathcal{P}_n$ corresponding to the K-invariant polynomials $P_j(X)$ under a canonical linear bijection

$$\lambda_1\,:\,\mathrm{Pol}\,(V_1)^K \longrightarrow \mathbb{D}(\mathcal{P}_n), \quad \mathcal{P}_n = GL(n, \mathbb{R})/K$$

are given by

$$D_j^0 \;=\; \sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^j\right), \quad 1 \leq j \leq n.$$

According to Maass [M2], pp. 64-66, $D_j^0\,(\,1 \leq j \leq n\,)$ are algebraically independent generators of the algebra $\mathbb{D}(\mathcal{P}_n)$ consisting of all $GL(n, \mathbb{R})$-invariant differential operators on $\mathcal{P}_n$ for the action (1.2). According to Lemma 4.5 and the commutativity of the algebra $\mathbb{D}(\mathcal{P}_n)\,($ cf. [M2] and [S1] $)$, the polynomials $P_j(X)\,(\,1 \leq j \leq n\,)$ must be algebraically independent generators of $\mathrm{Pol}\,(V_1)$. □

Another proof of Theorem 4.6. We first observe that $K := O(n)$ is a maximal compact subgroup of $GL(n, \mathbb{R})$. Then the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ of $GL(n, \mathbb{R})$ has the Cartan decomposition

$$\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{k} + V_1, \quad V_1 := \mathrm{Symm}(\mathbb{R}^n),$$

where $\mathfrak{k}$ is the subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ consisting of all skew-symmetric real matrices of degree n. Let $\mathfrak{a}$ be the maximal abelian subspace of V_1 consisting of all $n \times n$ diagonal real matrices. It is well known that the Weyl group W of $GL(n, \mathbb{R})$ is isomorphic to the symmetric group S_n of degree n. A classical result on invariant theory tells us that the following *symmetric* polynomials

$$C_1(\lambda) := \lambda_1 + \cdots + \lambda_n, \quad C_2(\lambda) := \lambda_1^2 + \cdots \lambda_n^2, \cdots, C_n(\lambda) := \lambda_1^n + \cdots + \lambda_n^n$$

are algebraically independent generators of the algebra $\mathrm{Pol}(\mathfrak{a})^W$ of all W-invariant polynomials on $\mathfrak{a}$, where $\lambda_1, \cdots, \lambda_n$ are n variables on $\mathfrak{a}$. It is also known that $\mathrm{Pol}(V_1)^K$ is isomorphic to $\mathrm{Pol}(\mathfrak{a})^W$ via the restriction to $\mathfrak{a}$ (cf. [He2], corollary 5.12, p. 299). It is easily seen that the polynomials on V_1 corresponding to W-invariant polynomials $C_j(\lambda),\ 1 \le j \le n$ are given by $P_j(X) := \sigma(X^j),\ 1 \le j \le n$ respectively. Hence the polynomials $P_1, \cdots, P_n$ are algebraically independent generators of $\mathrm{Pol}(V_1)^K$. □

Thirdly we let $V_2 := \mathbb{R}^{(m,n)}$ be the vector space of all $m \times n$ real matrices. The polynomial algebra $\mathrm{Pol}\,(V_2)$ on V_2 has a natural grading

$$\mathrm{Pol}\,(V_2) \cong \sum_{d=0}^{\infty} \mathrm{Pol}^d(V_2).$$

We consider the action of K on V_2 given by

$$(k, Z) \longmapsto Z\,{}^t k, \quad k \in K,\ Z \in V_2. \tag{4.28}$$

Then the above action extends uniquely to the action of K on $\mathrm{Pol}\,(V_2)$

$$\rho_2\ :\ K \longrightarrow \mathrm{Aut}(\mathrm{Pol}\,(V_2)) \tag{4.29}$$

by algebra automorphisms preserving the grading. For a typical element $Z = (z_{kl}) \in V_2$ with $1 \le k \le m,\ 1 \le l \le n$, we define the polynomials ξ_{kp} on V_2 by

$$\xi_{kp}(Z) :=\ (Z\,{}^t Z)_{kp}, \quad 1 \le k, p \le m, \tag{4.30}$$

where $(Z\,{}^tZ)_{kp}$ denotes the (k,p)-entry of the matrix $Z\,{}^tZ$. Obviously $\xi_{kp} = \xi_{pk}$ for all $1 \le k, p \le m$.

Theorem 4.7. Let $\mathrm{Pol}\,(V_2)^K$ be the subalgebra of $\mathrm{Pol}\,(V_2)$ consisting of K-invariants under the action (4.29) of K. Then the polynomials $\xi_{kp}\ (\,1 \le k \le p \le m\,)$ are algebraically independent generators of $\mathrm{Pol}\,(V_2)^K$.

Proof. The proof of the above theorem can be found in [Ho], Theorem 3.4.4., pp. 34-35. □

Fourthly we prove the following theorem.

Theorem 4.8. The following polynomials

$$P_j(X,Z) :=\ \sigma(X^j), \quad 1 \le j \le n \quad (\,\text{see (4.10)}\,)$$

and

$$\xi_{kp}(X,Z) :=\ (Z\,{}^tZ)_{kp}, \quad 1 \le k \le p \le m \quad (\,\text{see (4.11)}\,)$$

are algebraically independent generators of $\mathrm{Pol}\,(\mathfrak{p})^K$.

Proof. Since $\mathfrak{p} = V_1 \oplus V_2$, we have the following natural isomorphism

$$\mathrm{Pol}\,(\mathfrak{p}) \,\cong\, \mathrm{Pol}\,(V_1) \otimes \mathrm{Pol}\,(V_2). \tag{4.31}$$

Using the gradings of $\mathrm{Pol}\,(V_1)^K$ and $\mathrm{Pol}\,(V_2)^K$, we can show that

$$\mathrm{Pol}\,(\mathfrak{p})^K = \mathrm{Pol}\,(V_1)^K \otimes \mathrm{Pol}\,(V_2)^K. \tag{4.32}$$

According to Theorem 4.6, Theorem 4.7 and (4.32), we see that the elements $P_j(X) \otimes 1\,(\,1 \leq j \leq n\,)$ and $1 \otimes \xi_{kp}(Z)\,(\,1 \leq k \leq p \leq m\,)$ generate $\mathrm{Pol}\,(\mathfrak{p})^K$. It remains that the elements $P_j(X,Z) = P_j(X) \otimes 1\,(\,1 \leq j \leq n\,)$ and $\xi_{kp}(X,Z) := 1 \otimes \xi_{kp}(Z)\,(\,1 \leq k \leq p \leq m\,)$ are algebraically independent. Assume that there exists a nonconstant polynomial Q such that

$$Q(P_1(X,Z), \cdots, \xi_{11}(X,Z), \cdots, \xi_{mm}(X,Z)) = 0. \tag{4.33}$$

Taking a bigrading of the left hand side of (4.33), we have the following grading

$$Q(P_1, P_2, \cdots, \xi_{11}, \cdots, \xi_{mm}) = \sum_{a,b} Q_{a,b},$$

where

$$Q_{a,b} = \sum_{\alpha} P_{a,\alpha} \otimes \xi_{b,\alpha}, \quad P_{a,\alpha} = P_{a,\alpha}(X),\ \xi_{b,\alpha} = \xi_{b,\alpha}(Z)$$

is a homogeneous component of bidegree (a,b). Since $Q(P_1, P_2, \cdots, \xi_{mm}) = 0$, the following conditions have to be satisfied

$$Q_{a,b} = 0 \quad \text{for all the bidegree } (a,b),$$

that is,

$$P_{a,\alpha} = 0 \quad \text{or} \quad \xi_{b,\alpha} = 0 \quad \text{for all } a, b, \alpha.$$

Therefore Q is identically zero. This contradicts our assumption. Our theorem is completely proved. □

Finally we prove the main theorem.

Proof of Main Theorem. By Proposition 4.4, D_j and $\Psi_{kp}\,(\,1 \leq j \leq n,\ 1 \leq k \leq p \leq m\,)$ are invariant differential operators on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ corresponding to the elements P_j and ξ_{kp} of $\mathrm{Pol}(\mathfrak{p})^K$ via (4.24). By the argument similiar to that of the latter part in the proof of Lemma 4.5, we can prove that D_j and $\Psi_{kp}\,(\,1 \leq j \leq n,\ 1 \leq k \leq p \leq m\,)$ generate the algebra $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$. It follows from [M2] and [S1] that the operators D_j's commute with each other and that they are algebraically independent. It is easy to see that the operators $\Psi_{kp}\,(\,1 \leq k \leq p \leq m\,)$ commute with each other and also that $D_j \Psi_{kp} \neq \Psi_{kp} D_j$ for all j, k, p. We also see easily that the Ψ_{kp}'s are algebraically independent. □

Example 4.9. We consider the case such that $n = 1$ and m is an arbitrary positive integer. We let $G^0 := \mathbb{R}^+ \ltimes \mathbb{R}^{(m,1)}$ be the connected Lie group equipped with the following multiplication

$$(g_1, a_1) * (g_2, a_2) := (g_1 g_2,\, g_2^{-1} a_1 + a_2), \quad g_1, g_2 \in \mathbb{R}^+, \quad a_1, a_2 \in \mathbb{R}^{(m,1)},$$

where $\mathbb{R}^+$ denotes the multiplicative group of all positive real numbers. G^0 acts on $\mathcal{P}_1 \times \mathbb{R}^{(m,1)}$ transitively by

$$(g, a) \cdot (y, v) := (g^2 y,\, g(v + a)), \quad y \in \mathbb{R}^+, \quad v = (v_k) \in \mathbb{R}^{(m,1)}.$$

The stabilizer K^0 of G^0 at $(1, 0)$ is just the identity element $\{1\}$. Thus $\mathfrak{k}^0 = (0)$ and $\mathfrak{p}^0 = \mathfrak{g}^0$ (cf. (2.6) and (2.7)). Here $\mathfrak{k}^0$ and $\mathfrak{g}^0$ denotes the Lie algebra of K^0 and G^0 respectively. Since the adjoint representation of K^0 on $\mathfrak{p}^0$ is the trivial one, the symmetric algebra $S(\mathfrak{g}^0)$ is none other than the algebra of all invariants of the action (4.6) of $K^0 = \{1\}$ on $S(\mathfrak{g}^0)$. We denote by e_k ($1 \leq k \leq m$) the vector ${}^t(0, \cdots, 1, \cdots, 0) \in \mathbb{R}^{(m,1)}$ with 1 in the k-th position. We let

$$Y_0 := (1, 0), \quad X_1 := (0, e_1), \cdots, X_m := (0, e_m) \tag{4.34}$$

be the standard basis of $\mathfrak{g}^0$. We denote by $\mathbb{D}^0(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$ the algebra of all G^0-invariant differential operators on $\mathcal{P}_1 \times \mathbb{R}^{(m,1)}$. By a simple computation, the G^0-invariant differential operators $L^0_{Y_0}, L^0_{X_1}, \cdots, L^0_{X_m}$ on $\mathcal{P}_1 \times \mathbb{R}^{(m,1)}$ corresponding to $Y_0, X_1, \cdots, X_m$ under a canonical linear bijection $\lambda : S(\mathfrak{g}^0) \longrightarrow \mathbb{D}^0(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$ are given by

$$L^0_{Y_0} = 2y \frac{\partial}{\partial y}, \; L^0_{X_1} = y^{1/2} \frac{\partial}{\partial v_1}, \cdots, L^0_{X_m} = y^{1/2} \frac{\partial}{\partial v_m} \tag{4.35}$$

respectively. Obviously the above $(n + 1)$ G^0-invariant differential operators (4.35) generate $\mathbb{D}^0(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$. Since

$$L^0_{Y_0} L^0_{X_j} - L^0_{X_j} L^0_{Y_0} = L^0_{X_j}, \quad 1 \leq j \leq m,$$

the algebra $\mathbb{D}^0(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$ is *not* commutative.

On the other hand, for the group $G_{1,m}(\mathbb{R}) = GL(1, \mathbb{R}) \ltimes \mathbb{R}^{(m,1)}$ with $GL(1, \mathbb{R}) = \mathbb{R}^* - \{0\}$, the stabilizer $K := O(1)$ of $G_{1,m}(\mathbb{R})$ at $(1, 0)$ is $\{\pm 1\}$. Thus $\mathfrak{k} = 0$ and $\mathfrak{p} = \mathfrak{g}$ (cf. (2.6) and (2.7)). We let $Y_0, X_1, \cdots, X_m$ as (4.34). Then $Y_0, X_1, \cdots, X_m$ is the standard basis of $\mathfrak{p}$. The algebra $S(\mathfrak{p})^K$ of K-invariants under the action (4.6) is generated by algebraically independent elements $Y_0,\ X_k X_l$ ($1 \leq k \leq l \leq m$). By a simple computation, the $G_{1,m}(\mathbb{R})$-invariant differential operators $L_{Y_0},\ L_{X_k X_l}$ $(1 \leq k \leq l \leq m)$ corresponding to the the K-invariants $Y_0,\ X_k X_l$ under a canonical linear bijection $\lambda : S(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$ are given by

$$L_{Y_0} = 2y \frac{\partial}{\partial y}, \quad L_{X_k X_l} = y \frac{\partial^2}{\partial v_k \partial v_l}, \quad 1 \leq k \leq l \leq m. \tag{4.36}$$

It is easy to see that $L_{Y_0},\ L_{X_k X_l}$ ($1 \leq k \leq l \leq m$) generate $\mathbb{D}(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$. Although $L_{X_k X_l}$ ($1 \leq k \leq l \leq m$) commute with each other, L_{Y_0} does not commute with any $L_{X_k X_l}$. Indeed, we have the noncommutation relation

$$L_{Y_0} L_{X_k X_l} - L_{X_k X_l} L_{Y_0} = 2 L_{X_k X_l}.$$

Hence the algebra $\mathbb{D}(\mathcal{P}_1 \times \mathbb{R}^{(m,1)})$ is *not* commutative.

Example 4.10. We consider the case $n = 2$ and $m = 1$. In this case,

$$G_{2,1}(\mathbb{R}) := GL(2,\mathbb{R}) \ltimes \mathbb{R}^{(1,2)}, \quad K := O(2) \ \text{ and } \ G_{2,1}(\mathbb{R})/K = \mathcal{P}_2 \times \mathbb{R}^{(1,2)}.$$

According to (2.7),

$$\mathfrak{p} = \left\{ (X, Z) \mid X = {}^tX \in \mathbb{R}^{(2,2)}, \quad Z \in \mathbb{R}^{(1,2)} \right\}.$$

We put

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$f_1 := (1, 0), \qquad f_2 := (0, 1).$$

Then $\{ e_1, e_{12}, e_2, f_1, f_2 \}$ forms a basis for $\mathfrak{p}$. We write for variables $(X, Z) \in \mathfrak{p}$ by

$$X = \begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix} \quad \text{and} \quad Z = (z_1, z_2).$$

The following polynomials

$$P_1(X, Z) := \sigma(X) = x_1 + x_2, \qquad P_2(X, Z) := \sigma(X^2) = x_1^2 + x_2^2 + 2x_3^2$$

and

$$\xi_{11}(X, Z) := z_1^2 + z_2^2$$

are invariant under the action (4.6) of K.

Now we will compute the $G_{2,1}(\mathbb{R})$-invariant differential operators D_1, D_2, Ψ_{11} on $\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ corresponding to the K-invariants P_1, P_2, ξ_{11} respectively under a canonical linear bijection

$$\lambda : S(\mathfrak{p})^K \longrightarrow \mathbb{D}(\mathcal{P}_2 \times \mathbb{R}^{(1,2)}).$$

For real variables $t = (t_1, t_2, t_3)$ and $s = (s_1, s_2)$, we have

$$\begin{aligned} &\exp\left(t_1 e_1 + t_2 e_2 + t_3 e_3 + s_1 f_1 + s_2 f_2\right) \\ &= \left(\begin{pmatrix} a_1(t,s) & a_3(t,s) \\ a_3(t,s) & a_2(t,s) \end{pmatrix}, \left(b_1(t,s),\, b_2(t,s) \right) \right), \end{aligned}$$

where

$$\begin{aligned} a_1(t,s) &= 1 + t_1 + \frac{1}{2!}(t_1^2 + t_3^2) + \frac{1}{3!}(t_1^3 + 2t_1t_3^2 + t_2t_3^2) + \cdots, \\ a_2(t,s) &= 1 + t_2 + \frac{1}{2!}(t_2^2 + t_3^2) + \frac{1}{3!}(t_1t_3^2 + 2t_2t_3^2 + t_2^3) + \cdots, \\ a_3(t,s) &= t_3 + \frac{1}{2!}(t_1t_3 + t_2t_3) + \frac{1}{3!}(t_1t_2t_3 + t_1^2t_3 + t_2^2t_3 + t_3^3) + \cdots, \\ b_1(t,s) &= s_1 - \frac{1}{2!}(s_1t_1 + s_2t_3) + \frac{1}{3!}\left\{ s_1(t_1^2 + t_3^2) + s_2(t_1t_3 + t_2t_3) \right\} - \cdots, \\ b_2(t,s) &= s_2 - \frac{1}{2!}(s_1t_3 + s_2t_2) + \frac{1}{3!}\left\{ s_1(t_1t_3 + t_2t_3) + s_2(t_2^2 + t_3^2) \right\} - \cdots. \end{aligned}$$

For brevity, we write a_i, b_k for $a_i(t,s)$, $b_k(t,s)$ ($i=1,2,3,\ k=1,2$) respectively. We now fix an element $(g,c)\in G_{2,1}(\mathbb{R})$ and write

$$g=\begin{pmatrix} g_1 & g_{12} \\ g_{21} & g_2 \end{pmatrix} \quad \text{and} \quad c=(c_1,c_2).$$

We put

$$(Y(t,s),\,V(t,s)):=\left((g,c)\cdot\exp\left(\sum_{i=1}^{3}t_ie_i+\sum_{k=1}^{2}s_kf_k\right)\right)\cdot(E_2,0)$$

with

$$Y(t,s):=\begin{pmatrix} y_1(t,s) & y_3(t,s) \\ y_3(t,s) & y_2(t,s) \end{pmatrix} \quad \text{and} \quad V(t,s):=(v_1(t,s),\,v_2(t,s)).$$

By an easy computation, we obtain

$$\begin{aligned}
y_1 &= (g_1a_1+g_{12}a_3)^2+(g_1a_3+g_{12}a_2)^2,\\
y_2 &= (g_{21}a_1+g_2a_3)^2+(g_{21}a_3+g_2a_2)^2,\\
y_3 &= (g_1a_1+g_{12}a_3)(g_{21}a_1+g_2a_3)+(g_1a_3+g_{12}a_2)(g_{21}a_3+g_2a_2),\\
v_1 &= (c_1+b_1a_1+b_2a_3)g_1+(c_2+b_1a_3+b_2a_2)g_{12},\\
v_2 &= (c_1+b_1a_1+b_2a_3)g_{21}+(c_2+b_1a_3+b_2a_2)g_2.
\end{aligned}$$

Using the chain rule, we can easily compute the $G_{2,1}(\mathbb{R})$-invariant differential operators $D_1:=D_{\lambda(P_1)},\ D_2:=D_{\lambda(P_2)}$ and $\Psi_{11}:=D_{\lambda(\xi_{11})}$. They are given by

$$D_1\;=\;2\,\sigma\left(Y\frac{\partial}{\partial Y}\right)=2\left(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}+y_3\frac{\partial}{\partial y_3}\right),$$

$$\begin{aligned}
D_2 \;&=\; \sigma\left(\left(2Y\frac{\partial}{\partial Y}\right)^2\right)\\
&=\;3\,D_1\;+\;8\left(y_3^2\,\frac{\partial^2}{\partial y_1\partial y_2}\;+\;y_1y_3\,\frac{\partial^2}{\partial y_1\partial y_3}\;+\;y_2y_3\,\frac{\partial^2}{\partial y_2\partial y_3}\right)\\
&\quad+\;4\left\{y_1^2\,\frac{\partial^2}{\partial y_1^2}\;+\;y_2^2\,\frac{\partial^2}{\partial y_2^2}\;+\;\frac{1}{2}(y_1y_2+y_3^2)\,\frac{\partial^2}{\partial y_3^2}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{11}\;&=\;\sigma\left(Y^{\,t}\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial V}\right)\right)\\
&=\;y_1\,\frac{\partial^2}{\partial v_1^2}\;+\;2y_3\,\frac{\partial^2}{\partial v_1\partial v_2}\;+\;y_2\,\frac{\partial^2}{\partial v_2^2}.
\end{aligned}$$

Clearly D_1 commutes with D_2 but Ψ_{11} does not commute with D_1 and D_2. Indeed, we have the following noncommutation relations

$$\begin{aligned}
[D_1,\,\Psi_{11}]:&=D_1\Psi_{11}-\Psi_{11}D_1\\
&=\;2\,\Psi_{11}
\end{aligned}$$

and

$$\begin{aligned}[D_2,\,\Psi_{11}] :=\ & D_2\Psi_{11}-\Psi_{11}D_2\\ =\ & 2\,(\,2\,D_1\,-1\,)\,\Psi_{11}\\ &-\,8\det(Y)\cdot\det\left(\frac{\partial}{\partial Y}+{}^t\!\left(\frac{\partial}{\partial V}\right)\frac{\partial}{\partial V}\right)\\ &+\,8\det(Y)\cdot\det\left(\frac{\partial}{\partial Y}\right)\,-\,4\,(y_1y_2+y_3^2)\,\frac{\partial^3}{\partial y_3\partial v_1\partial v_2}.\end{aligned}$$

Hence the algebra $\mathbb{D}(\mathcal{P}_2\times\mathbb{R}^{(1,2)})$ is *not* commutative. According to Theorem 4.8, D_1, D_2 and Ψ_{11} generate the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$.

Remark 4.11. It is easily seen that for all $1\le k\le p\le m$,

$$[D_1,\,\Psi_{kp}]:=\,D_1\Psi_{kp}\,-\,\Psi_{kp}D_1\,=\,2\,\Psi_{kp}.$$

However in general for $j\ge 2$, it is not easy to express the following operators

$$[D_j,\,\Psi_{kp}]:=\,D_j\Psi_{kp}\,-\,\Psi_{kp}D_j$$

in terms of the generators D_j's and Ψ_{kp}'s.

Definition 4.12. Let G be a locally compact group of isometries of a Riemannian manifold M which acts transitively on M. We assume that there is a fixed isometry ϕ of M (possibly not in G). We say that M is a *weakly symmetric* Riemannian manifold with respect to G and ϕ if the following two conditions (WS1) and (WS2) are satisfied :

(WS1) ϕ commutes with G and ϕ^2 belongs to G.

(WS2) For any pair of points x and y of M, there exists an element $\lambda\in G$ such that $\lambda(x)=\phi(y)$ and $\lambda(y)=\phi(x)$.

Remark 4.13. Selberg [S1] proved that the algebra $\mathbb{D}(M)$ of all G-invariant differential operators on a weakly symmetric Riemannian manifold M is *commutative.* It is well known [S1] that the algebra $\mathbb{D}(\mathcal{P}_n)$ of all differential operators on $\mathcal{P}_n$ which are invariant under the action (1.2) of $GL(n,\mathbb{R})$ are generated by the following n algebraically independent differential operators

$$\sigma\left(\left(Y\frac{\partial}{\partial Y}\right)^j\right),\qquad 1\le j\le n. \tag{4.37}$$

Furthermore, $\mathbb{D}(\mathcal{P}_n)$ has a *commutative* ring structure which is isomorphic to the ring $\mathbb{C}[Z_1,\cdots,Z_n]$ of polynomials in n indeterminates $Z_1,\cdots,Z_n$. In fact, $\mathcal{P}_n$ is a weakly symmetric Riemannian manifold. For more detail, we refer to [M2] and [S1].

Proposition 4.14. For any positive integers n and m, the algebra $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ is *not* commutative. Consequently the homogeneous space $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ is not a weakly symmetric Riemannian manifold in the sense of the above definition.

Proof. We already know that the operators D_j and L_k (cf. (4.17) and (4.19)) do not commute. Since the ring $\mathbb{D}(\mathcal{P}_n\times\mathbb{R}^{(m,n)})$ is not commutative, according to Selberg's theorem, $\mathcal{P}_n\times\mathbb{R}^{(m,n)}$ is not weakly symmetric. We will leave the detail to the reader. $\square$

5. Spherical Functions

In this section, we define the spherical functions on the group $G_{n,m}(\mathbb{R})$ and characterize these functions.

Let M be a smooth manifold and let $\phi : M \longrightarrow M$ be a diffeomorphism of M onto itself. We put

$$f^{\phi} = f \circ \phi^{-1}, \quad f \in C^{\infty}(M), \tag{5.1}$$

where $C^{\infty}(M)$ denotes the space of all smooth functions on M. If D is a differential operator on M, we define D^{ϕ} by

$$D^{\phi}(f) := (D(f \circ \phi)) \circ \phi^{-1}, \quad f \in C^{\infty}(M). \tag{5.2}$$

We note that if D is invariant under ϕ, then $D^{\phi} = D$, i.e., $D(f \circ \phi) = (Df) \circ \phi$ for all $f \in C^{\infty}(M)$.

Now we return to our stage. For an element g of $G_{n,m}(\mathbb{R})$, we denote by L_g and R_g the left translation and the right translation of $G_{n,m}(\mathbb{R})$ by g. Let $\mathbb{D}(G_{n,m}(\mathbb{R}))$ be the algebra of all differential operators on $G_{n,m}(\mathbb{R})$ invariant under all the left translations L_g of $G_{n,m}(\mathbb{R})$. We note that $\mathbb{D}(G_{n,m}(\mathbb{R}))$ is generated by all the left invariant vector fields on $G_{n,m}(\mathbb{R})$. We define

$$\mathbb{D}_K(G_{n,m}(\mathbb{R})) := \left\{ D \in \mathbb{D}(G_{n,m}(\mathbb{R})) \mid D^{R_k} = D \ \text{ for all } k \in K \right\}. \tag{5.3}$$

Let $\pi : G_{n,m}(\mathbb{R}) \longrightarrow G_{n,m}(\mathbb{R})/K$ be the natural projection. We put for any function f on $G_{n,m}(\mathbb{R})/K$

$$\tilde{f} := f \circ \pi. \tag{5.4}$$

So $\tilde{f}$ is a function on $G_{n,m}(\mathbb{R})$. Since $G_{n,m}(\mathbb{R})/K$ is reductive, we have a natural homomorphism $\mu : u \mapsto D_u$ of $\mathbb{D}_K(G_{n,m}(\mathbb{R}))$ onto $\mathbb{D}(G_{n,m}(\mathbb{R})/K)$ defined by

$$(\mu(u)f)^{\tilde{}} = (D_u f)^{\tilde{}} = u\tilde{f}, \quad f \in C^{\infty}(G_{n,m}(\mathbb{R})/K). \tag{5.5}$$

The kernel of μ is given by

$$\ker \mu = \mathbb{D}_K(G_{n,m}(\mathbb{R})) \cap \mathbb{D}(G_{n,m}(\mathbb{R}))\mathfrak{k}. \tag{5.6}$$

So we have the isomorphism

$$\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)}) \cong \mathbb{D}(G_{n,m}(\mathbb{R})/K) \cong \mathbb{D}_K(G_{n,m}(\mathbb{R}))/\ker \mu. \tag{5.7}$$

For the proof of (5.5), (5.6) and (5.7), we refer the reader to [He2], pp. 285-286.

Now we are ready to define the concept of spherical functions on $G_{n,m}(\mathbb{R})$.

Definition 5.1. A function

$$\phi : G_{n,m}(\mathbb{R}) \longrightarrow \mathbb{C}$$

is called a *spherical* function if ϕ satisfies the following properties (S1)-(S3):

(S1) $\phi(k_1 g k_2) = \phi(g)$ for all $k_1, k_2 \in K$ and $g \in G_{n,m}(\mathbb{R})$, i.e., ϕ is bi-invariant under K.

(S2) ϕ is an eigenfunction for all the differential operators in $\mathbb{D}_K(G_{n,m}(\mathbb{R}))$.

(S3) $\phi(E_n, 0) = 1$.

We can say that a spherical function is a rotation-invariant eigenfunction for the algebra $\mathbb{D}_K(G_{n,m}(\mathbb{R}))$ which is normalized to have the value 1 at the identity $(E_n, 0)$.

Let us descend a spherical function ϕ on $G_{n,m}(\mathbb{R})$ to the coset space $\mathcal{P}_n \times \mathbb{R}^{(m,n)} \cong G_{n,m}(\mathbb{R})/K$. We define the function $f_\phi : \mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{C}$ by

$$f_\phi(Y,V) := \phi(g,\, V\,{}^t g^{-1}), \quad (Y,V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}, \tag{5.8}$$

where $Y = g\,{}^t g$ with $g \in GL(g,\mathbb{R})$. We note that g is unique up to K. Then f_ϕ satisfies the following properties:

(S1)^0 $f_\phi(kY\,{}^t k,\, V\,{}^t k) = f_\phi(Y,V)$ for all $(Y,V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}$ and $k \in K$.

(S2)^0 f_ϕ is an eigenfunction for all the differential operators in $\mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

(S3)^0 $f_\phi(E_n, 0) = 1$.

We also call a function on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ satisfying the above three properties *spherical*.

First we recall the Iwasawa decompositions (2.10) and (2.11) of $G_{n,m}(\mathbb{R})$ and $\mathfrak{g}$ respectively. For $g \in G_{n,m}(\mathbb{R})$, we let $A(g)$ denote the unique element of $\mathfrak{a}$ such that

$$g \in N \exp A(g)\, K.$$

Since the algebra $\mathbb{D}(A)$ of all invariant differential operators on A is a commutative ring, each linear function $\nu : \mathfrak{a} \longrightarrow \mathbb{C}$ extends uniquely to a homomorphism of $\mathbb{D}(A)$ into $\mathbb{C}$, also denoted by $\nu : \mathbb{D}(A) \longrightarrow \mathbb{C}$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and let $\mathfrak{a}_c^*$ be the set of all linear functionals of $\mathfrak{a}$ into $\mathbb{C}$.

Lemma 5.2. For each $\nu \in \mathfrak{a}_c^*$, the function p_ν on $G_{n,m}(\mathbb{R})$ defined by

$$p_\nu(g) := e^{\nu(A(g))}, \qquad g \in G_{n,m}(\mathbb{R}) \tag{5.9}$$

is an eigenfunction of each $D \in \mathbb{D}_K(G_{n,m}(\mathbb{R}))$ wirh eigenvalue $\nu(D_{\mathfrak{a}})$, where $D_{\mathfrak{a}}$ denotes the radial part of D. The function p_ν is called the *power function* on $G_{n,m}(\mathbb{R})$ associated with $\nu \in \mathfrak{a}_c^*$. Moreover, the function φ_ν on $G_{n,m}(\mathbb{R})$ defined by

$$\varphi_\nu(g) := \int_K e^{\nu(A(kg))}\, dk, \qquad g \in G_{n,m}(\mathbb{R}) \tag{5.10}$$

is an eigenfunction of each $D \in \mathbb{D}_K(G_{n,m}(\mathbb{R}))$ with eigenvalue $\nu(D_{\mathfrak{a}})$, where dk is the Haar measure on K normalized to be $\int_K dk = 1$.

Proof. The proof of the above lemma can be found in [He2], p. 303. □

Theorem 5.3. For each $\nu \in \mathfrak{a}_c^*$, we define the function ϕ_ν on $G_{n,m}(\mathbb{R})$ by

$$\phi_\nu(g) := \int_K e^{(i\nu+\rho)(A(kg))} dk, \quad g \in G_{n,m}(\mathbb{R}), \tag{5.11}$$

where ρ is defined in (2.15). Then

$$\phi_{s\nu} = \phi_\nu \qquad \text{for all } s \in W,$$

where W denotes the Weyl group and $(s\nu)(H) := \nu(s^{-1}H),\ H \in \mathfrak{a}$. The functions ϕ_ν ($\nu \in \mathfrak{a}_c^*$) define the class of spherical functions on $G_{n,m}(\mathbb{R})$. Moreover, two such functions ϕ_ν and ϕ_λ are identical if and only if $\nu = s\lambda$ for some $s \in W$.

Proof. We can prove the above theorem following the line of the argument in [He2], pp. 303-304 and p. 418. □

We describe the functions mentioned above in more detail. Let $\nu \in \mathfrak{a}_c^*$ be the linear functional $\nu : \mathfrak{a} \longrightarrow \mathbb{C}$ given by

$$\nu(D(a_1, \cdots, a_n)) := \nu_1 a_1 + \cdots + \nu_n a_n, \quad \nu_1, \cdots, \nu_n \text{ are fixed complex numbers,}$$

where $D(a_1, \cdots, a_n)$ denotes the $n \times n$ diagonal real matrix with entries $a_1, \cdots, a_n$. Now we define the function $\tilde{p}_\nu : \mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{C}$ by

$$\tilde{p}_\nu(Y, V) := p_\nu(g_0,\, V\,{}^t g_0), \quad (Y, V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}, \tag{5.12}$$

where $Y = g_0\,{}^t g_0$ with $g_0 \in GL(n, \mathbb{R})$. We observe that $\tilde{p}_\nu(Y, V)$ is independent of the choice of $g_0 \in GL(n, \mathbb{R})$ with $Y = g_0\,{}^t g_0$ and hence $\tilde{p}_\nu$ is well defined. We may choose an upper triangular matrix g_0 with diagonal entries $t_1, \cdots, t_n > 0$ such that $Y = g_0\,{}^t g_0$. Then we have

$$\begin{aligned} p_\nu(g_0,\, V\,{}^t g_0) &= e^{\nu(A(g_0,\, V\,{}^t g_0^{-1}))} = e^{\nu(D(\log t_1, \cdots, \log t_n))} \\ &= e^{\nu_1 \log t_1 + \cdots + \nu_n \log t_n} \\ &= t_1^{\nu_1} \cdots t_n^{\nu_n}. \end{aligned}$$

We define the n-tuple $s = (s_1, \cdots, s_n) \in \mathbb{C}^n$ of complex numbers by

$$\nu_h := 2(s_{n-h+1} + \cdots + s_n), \quad h = 1, 2, \cdots, n. \tag{5.13}$$

Then we have

$$s_h := \frac{1}{2}(\nu_{n-h+1} - \nu_{n-h}), \;\; h = 1, \cdots, n-1 \;\text{ and }\; s_n = \frac{1}{2}\nu_1. \tag{5.14}$$

For each $h \in \mathbb{Z}^+$ with $1 \leq h \leq n$, we denote by Y_h the lower right corner $h \times h$ matrix of Y so that

$$Y = \begin{pmatrix} * & * \\ * & Y_h \end{pmatrix}.$$

From the fact that

$$\det Y_h \;=\; (t_{n-h+1} t_{n-h+2} \cdots t_n)^2,$$

it follows immediately that

$$\tilde{p}_\nu(Y, V) = p_\nu(g_0,\, V\,{}^t g_0) = \prod_{h=1}^{n} (\,\det Y_h\,)^{s_h}\,. \tag{5.15}$$

For $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{C}^n$ we define the function $\Phi_\nu : \mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{C}$ by

$$\Phi_\nu(Y, V) := \int_K e^{(i\tilde{\nu} + \rho)(A(kg))} dk, \quad (Y, V) \in \mathcal{P}_n \times \mathbb{R}^{(m,n)}, \tag{5.16}$$

where $g = (g_0,\, V\,{}^t g_0^{-1})$ with $Y = g_0\,{}^t g_0,\; g_0 \in GL(n, \mathbb{R})$ and $\tilde{\nu} : \mathfrak{a} \longrightarrow \mathbb{C}$ is the linear function on $\mathfrak{a}$ defined by

$$\tilde{\nu}(D(a_1, \cdots, a_n)) = -i(\nu_1 a_1 + \cdots + \nu_n a_n), \quad D(a_1, \cdots, a_n) \in \mathfrak{a}.$$

Then it is easily seen that Φ_ν is a joint eigenfunction for all $D \in \mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$. If $Y = g_0\,{}^t g_0$ such that g_0 is a upper triangular $n \times n$ matrix with diagonals $t_1, \cdots, t_n > 0$, then according to (2.16) we have

$$\Phi_\nu(Y, V) = \prod_{h=1}^{n} t_h^{\frac{1}{2}(n+m+1-2h) + \nu_h}. \tag{5.17}$$

Summarizing the above, we have

Theorem 5.4. For any $\nu \in \mathfrak{a}_c^*$, the function

$$\Phi_\nu(Y,V) = (\det Y)^{\frac{1}{4}(n+m+1)} \prod_{h=1}^{n} (\det Y_h)^{s_h-\frac{1}{2}} \tag{5.18}$$

is a joint eigenfunction for all $D \in \mathbb{D}(\mathcal{P}_n \times \mathbb{R}^{(m,n)})$, where the n-tuple $s=(s_1,\cdots,s_n)$ of complex numbers is given by (5.14).

Proof. The proof follows from (5.17) and the fact that

$$\det Y_h = (t_{n-h+1}t_{n-h+2}\cdots t_n)^2, \quad h=1,\cdots,n.$$

□

6. Automorphic Forms on $G_{n,m}(\mathbb{R})$

In this section, we define the concept of automorphic forms on $G_{n,m}(\mathbb{R})$.

First of all, following the article of Borel and Jacquet [B-J] we define

Definition 6.1. A norm $||\ ||$ on $G_{n,m}(\mathbb{R})$ is a function of the form

$$||x|| = \{ \operatorname{tr}(\rho(x)^*\rho(x)) \}^{\frac{1}{2}}, \qquad x \in G_{n,m}(\mathbb{R}),$$

where $\rho : G_{n,m}(\mathbb{R}) \longrightarrow GL(V_\rho)$ is a finite dimensional complex representation with finite kernel whose image is closed in $\operatorname{End}(V_\rho)$ and $\rho(x)^*$ denotes the adjoint of $\rho(x)$ with respect to a Hilbert space structure on V_ρ invariant under K. We say that a function ϕ on $G_{n,m}(\mathbb{R})$ is *slowly increasing* if there exists a norm $||\ \ ||$ on $G_{n,m}(\mathbb{R})$, a positive constant C and a positive integer ℓ such that

$$|\phi(x)| \leq C \cdot ||x||^\ell \quad \text{for all } x \in G_{n,m}(\mathbb{R}). \tag{6.2}$$

Borel and Jacquet [B-J] introduced the notion of automorphic forms on a real reductive Lie group. We will define the notion of automorphic forms on the group $G_{n,m}(\mathbb{R})$ slightly different from theirs.

Definition 6.2. A smooth function

$$f : G_{n,m}(\mathbb{R}) \longrightarrow \mathbb{C}$$

is called an *automorphic form* with respect to $G_{n,m}(\mathbb{Z})$ if f satisfies the following conditions:

(a) $f(\gamma x) = f(x)$ for all $\gamma \in G_{n,m}(\mathbb{Z})$ and $x \in G_{n,m}(\mathbb{R})$.

(b) f is K-finite on the right.

(c) f is an eigenfunction for all the differential operators in $\mathbb{D}_K(G_{n,m}(\mathbb{R}))$.

(d) f is slowly increasing.

For a system $\lambda = (\lambda_\alpha)$ of eigenvalues for $\mathbb{D}_K(G_{n,m}(\mathbb{R}))$, we denote by $\mathrm{A}(G_{n,m}(\mathbb{Z}),\lambda)$ the vector space of all automorphic forms f with respect to $G_{n,m}(\mathbb{Z})$ such $L_\alpha f = \lambda_\alpha f$ for all $L_\alpha \in \mathbb{D}_K(G_{n,m}(\mathbb{R}))$. And an automorphic form ϕ is said to be *cuspidal* if

$$\int_{N_{\mathbb{Z}}\backslash N(\mathbb{R})} f(nx)\,dn \,=\, 0, \qquad N_{\mathbb{Z}} := N(\mathbb{R}) \cap G_{n,m}(\mathbb{Z})$$

for all $x \in G_{n,m}(\mathbb{R})$, where N is the unipotent radical of any proper $\mathbb{Q}$-subgroup of $G_{n,m}(\mathbb{R})$. In fact, it suffices to require this for any proper maximal parabolic $\mathbb{Q}$-subgroup (cf. [HC], Lemma 3, p. 8). We denote by $\mathrm{A}(G_{n,m}(\mathbb{Z}),\lambda)_0$ the subspace of all cuspidal forms in $\mathrm{A}(G_{n,m}(\mathbb{Z}),\lambda)$.

Remark 6.3. The condition (b) means that by definition the right translates of f by elements of K span a finite dimensional vector space of functions. The condition (c) in the above definition is slightly different from the condition 1.3 (c) in [B-J].

Theorem 6.4. The space $A(G_{n,m}(\mathbb{Z}),\lambda)$ is finite dimensional.

Proof. The proof follows from [HC], Theorem 1, p. 8 and [B-J], p. 191. □

7. Maass-Jacobi Forms

We let

$$SL_{n,m}(\mathbb{R}) := SL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)} \tag{7.1}$$

be the semidirect product of the special linear group $SL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ with the multiplication law (1.6). Obviously $SL_{n,m}(\mathbb{R})$ is a subgroup of $G_{n,m}(\mathbb{R})$. We let

$$SL_{n,m}(\mathbb{Z}) := SL(n,\mathbb{Z}) \ltimes \mathbb{Z}^{(m,n)} \tag{7.2}$$

be the discrete subgroup of $SL_{n,m}(\mathbb{R})$ and $K_0 := SO(n,\mathbb{R})$.

Now we introduce the notion of Maass-Jacobi forms.

Definition 7.1. A smooth bounded function

$$\phi \,:\, SL_{n,m}(\mathbb{R}) \longrightarrow \mathbb{C}$$

is called a *Maass-Jacobi form* if ϕ satisfies

(MJ1) $\phi(\gamma x k) = \phi(x)$ for all $\gamma \in SL_{n,m}(\mathbb{Z}),\ x \in SL_{n,m}(\mathbb{R})$ and $k \in K_0$.

(MJ2) ϕ is an eigenfunction for all the differential operators in $\mathbb{D}_{K_0}(SL_{n,m}(\mathbb{R}))$.

(MJ3) ϕ is slowly increasing.

For a fixed system $\lambda = (\lambda_\alpha)$ of eigenvalues for $\mathbb{D}_{K_0}(SL_{n,m}(\mathbb{R}))$, we denote by $\mathrm{MJ}(SL_{n,m}(\mathbb{Z}),\lambda)$ the vector space of all Maass-Jacobi forms ϕ such that $L_\alpha\phi = \lambda_\alpha\phi$ for all Ł$_\alpha \in \mathbb{D}_{K_0}(SL_{n,m}(\mathbb{R}))$.

We set
$$S\mathcal{P}_n := \{\, Y \in \mathcal{P}_n \mid \det Y = 1 \,\}\,.$$

Let us descend a Maass-Jacobi form ϕ to the coset space $S\mathcal{P}_n \times \mathbb{R}^{(m,n)} \cong SL_{n,m}(\mathbb{R})/K_0$. We define the function $f_\phi : S\mathcal{P}_n \times \mathbb{R}^{(m,n)} \longrightarrow \mathbb{C}$ by

$$f_\phi(Y,V) := \phi(g,\, V\,{}^t g^{-1}), \quad (Y,V) \in S\mathcal{P}_n \times \mathbb{R}^{(m,n)}, \tag{7.3}$$

where $Y = g\,{}^t g$ for some $g \in SL(n,\mathbb{R})$. (7.3) is well-defined. Then f_ϕ satisfies the following conditions:

$(MJ1)^0$ $f_\phi(\gamma Y\,{}^t\gamma,\, (V+\alpha)\,{}^t\gamma) = f_\phi(Y,V)$ for all $(\gamma,\alpha) \in SL_{n,m}(\mathbb{Z})$.

$(MJ2)^0$ f_ϕ is an eigenfunction for all the differential operators in $\mathbb{D}(S\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

$(MJ3)^0$ f_ϕ has a suitable polynomial growth condition.

For a system $s = (s_\alpha)$ of eigenvalues for $\mathbb{D}(S\mathcal{P}_n \times \mathbb{R}^{(m,n)})$, we denote by $\mathrm{MJ}_s(SL_{n,m}(\mathbb{Z}))$ the vector space of all Maass-Jacobi forms f on $S\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ such that $L_\alpha f = s_\alpha f$ for all $L_\alpha \in \mathbb{D}(S\mathcal{P}_n \times \mathbb{R}^{(m,n)})$.

Theorem 7.2. For any fixed system $\lambda = (\lambda_\alpha)$, the space $\mathrm{MJ}(SL_{n,m}(\mathbb{Z}),\lambda)$ is finite dimensional.

The proof of the above theorem is due to Harish-Chandra [HC], Theorem 1, p. 8 and [B-J], p 191.

Now we consider the simple case $n = 2$ and $m = 1$. We recall that $SL_{2,1}(\mathbb{R})$ acts on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ transitively by

$$(g,\alpha)\cdot(Y,V) := (\, gY\,{}^t g,\, (V+\alpha)\,{}^t g\,), \tag{7.4}$$

where $g \in SL(2,\mathbb{R}),\ \alpha \in \mathbb{R}^{(1,2)},\ Y \in S\mathcal{P}_2$ and $V \in \mathbb{R}^{(1,2)}$. We recall that $SL(2,\mathbb{R})$ acts on $\mathbb{H}$ transitively by

$$g < \tau > := (a\tau+b)(c\tau+d)^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}), \quad \tau \in \mathbb{H},$$

where $\mathbb{H}$ denotes the Poincaré upper half plane.

Now we define the action of $SL_{2,1}(\mathbb{R})$ on $\mathbb{H}\times\mathbb{C}$ by

$$(g,\alpha)\circ(\tau,z) := (\,{}^t g^{-1} < \tau >,\, (z+\alpha_1\tau+\alpha_2)(-b\tau+a)^{-1}\,), \tag{7.5}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^{(1,2)}$, and $(\tau,z) \in \mathbb{H}\times\mathbb{C}$. Since the action (7.5) is transitive and the special orthogonal real group $SO(2)$ is the stabilizer of this action at the origin $(i,0)$, $\mathbb{H}\times\mathbb{C}$ can be identified with the homogeneous space $SL_{2,1}(\mathbb{R})/SO(2)$ as follows:

$$SL_{2,1}(\mathbb{R})/SO(2) \ni (g,\alpha)\,SO(2) \longmapsto (g,\alpha)\circ(i,0). \tag{7.6}$$

We observe that we can express an element Y of $S\mathcal{P}_2$ uniquely as

$$Y = \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}\left[\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} y^{-1} & -xy^{-1} \\ -xy^{-1} & x^2y^{-1}+y \end{pmatrix} \tag{7.7}$$

with $x,y \in \mathbb{R}$ and $y > 0$.

Proposition 7.3. We define the mapping $T : S\mathcal{P}_2 \times \mathbb{R}^{(1,2)} \longrightarrow \mathbb{H} \times \mathbb{C}$ by

$$T(Y,V) := (x+iy,\, v_1(x+iy)+v_2\,), \tag{7.8}$$

where Y is of the form (7.7) and $V=(v_1,v_2)\in\mathbb{R}^{(1,2)}$. Then the mapping T is a bijection which is compatible with the above two actions (7.4) and (7.5).

For any $Y\in S\mathcal{P}_2$ of the form (7.7), we put

$$g_Y := \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}\begin{pmatrix} y^{-1/2} & 0 \\ 0 & y^{1/2} \end{pmatrix} = \begin{pmatrix} y^{-1/2} & 0 \\ -xy^{-1/2} & y^{1/2} \end{pmatrix}. \tag{7.9}$$

and

$$\alpha_{Y,V} := V\,{}^t g_Y^{-1}. \tag{7.10}$$

Then we have

$$T(Y,V) = (\,g_Y,\,\alpha_{Y,V}\,)\circ(i,0). \tag{7.11}$$

Proof. It is obvious that T is a bijection. Let $T_0 : S\mathcal{P}_2 \longrightarrow \mathbb{H}$ be the mapping defined by

$$T_0(Y) := x + i\,y,$$

where Y is of the form (7.7). We observe that

$$Y = g_Y\,{}^t g_Y \quad \text{and} \quad T_0(Y) = {}^t g_Y^{-1} < i >,$$

where g_Y is an element of $SL(2,\mathbb{R})$ defined by (7.9).

For $g\in SL(2,\mathbb{R})$ and $Y\in S\mathcal{P}_2$,

$$Y_* := gY\,{}^t g = g_*\,{}^t g_* \quad \text{for some } g_*\in SL(2,\mathbb{R}).$$

Since $Y_* = g_{Y_*}\,{}^t g_{Y_*}$, there is an element $k\in K$ such that $g_{Y_*} = g_* k$. So we have

$$\begin{aligned} T_0(Y_*) &= {}^t g_{Y_*}^{-1} < i > = {}^t (g_* k)^{-1} < i > \\ &= {}^t g_*^{-1}\,{}^t k^{-1} < i > = {}^t g_*^{-1} < i > . \end{aligned}$$

So far we showed that if $Y_0\in S\mathcal{P}_2$ such that $Y_0 = g_0\,{}^t g_0$ with $g_0\in SL(2,\mathbb{R})$, then $T_0(Y_0) = g_0 < i >$. If Y is of the form (7.7), we take

$$g_* := g\cdot g_Y.$$

Then $g_*\,{}^t g_* = gY\,{}^t g$. Therefore we have

$$T_0(gY\,{}^t g) = {}^t g_*^{-1} < i > = {}^t g^{-1}\cdot{}^t\, g_Y^{-1} < i > = {}^t g^{-1} < \tau >,$$

where we put $\tau := x + i\,y$.

Let $(g,\alpha)\in SL_{2,1}(\mathbb{R})$ with $g:=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),\ \alpha:=(\alpha_1,\,\alpha_2)\in\mathbb{R}^{(1,2)}$ and $(Y,V)\in \mathcal{SP}_2\times\mathbb{R}^{(1,2)}$ such that $Y\in\mathcal{SP}_2$ is of the form (7.7) and $V:=(v_1,\,v_2)\in\mathbb{R}^{(1,2)}$ with $v_1,\,v_2\in\mathbb{R}$. Now we have, if $\tau = x+i\,y\in\mathbb{H}$,

$$\begin{aligned}
&(g,\,\alpha)\circ T(Y,V)\\
=\ &(g,\,\alpha)\circ(\tau,\,v_1\tau+v_2)\\
=\ &(\,{}^tg^{-1}<\tau>,\,\{\,(\,v_1+\alpha_1\,)\,\tau+(\,v_2+\alpha_2\,)\,\}\,(\,-\,b\,\tau+a\,)^{-1}\,).
\end{aligned}$$

On the other hand.

$$\begin{aligned}
T\,(\,(g,\alpha)\cdot(Y,V)\,)\ &=\ T\,(\,gY\,{}^tg,\,(\,V+\alpha\,)\,{}^tg\,)\\
&=\ (\,T_0(gY\,{}^tg),\ \tilde v_1\cdot T_0(gY\,{}^tg)+\tilde v_2\,)\\
&=\ (\,{}^tg^{-1}<\tau>,\,\tilde v_1\cdot{}^t g^{-1}<\tau>+\tilde v_2\,),
\end{aligned}$$

where $(\,\tilde v_1,\,\tilde v_2\,):=\ (V+\alpha)\,{}^tg.$

Since

$$\begin{aligned}
\tilde v_1\,:=\ &(v_1+\alpha_1)\,a+(v_2+\alpha_2)\,b,\\
\tilde v_2\,:=\ &(v_1+\alpha_1)\,c+(v_2+\alpha_2)\,d,
\end{aligned}$$

by a simple calculation, we get

$$\tilde v_1\,{}^tg^{-1}<\tau>+\tilde v_2\ =\ \{\,(v_1+\alpha_1)\,\tau+(v_2+\alpha_2)\,\}\ (-b\tau+a\,)^{-1}.$$

Hence we obtain

$$T((g,\,\alpha)\cdot(Y,V))\ =\ (g,\alpha)\circ T(Y,V).$$

Consequently T is compatible with the actions (7.4) and (7.5). It is easily checked that (7.11) holds. □

Theorem 7.4. Let $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ be the algebra of all differential operators on $\mathbb{H}\times\mathbb{C}$ invariant under the action (7.5) of $SL_{2,1}(\mathbb{R})$. Then the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is generated by the following differential operators

$$\begin{aligned}
D :=\,&y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+v^2\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)\\
&+2\,y\,v\left(\frac{\partial^2}{\partial x\partial u}+\frac{\partial^2}{\partial y\partial v}\right)
\end{aligned}\tag{7.12}$$

and

$$\Psi := y\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right),\tag{7.13}$$

where $\tau = x+iy$ and $z=u+iv$ with real variables x,y,u,v. Moreover, we have

$$\begin{aligned}
[D,\,\Psi] := D\Psi-\Psi D\ =\,&2\,y^2\,\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)-4\,y^2\,\frac{\partial^3}{\partial x\partial u\partial v}\\
&-2\left(v\,\frac{\partial}{\partial v}\Psi+\Psi\right).
\end{aligned}$$

In particular, the algebra $\mathbb{D}(\mathbb{H}\times\mathbb{C})$ is not commutative.

Proof. The proof can be found in [Y7]. □

For any right K-invariant function $\phi : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$ on $SL_{2,1}(\mathbb{R})$, we define the function $f_\phi : \mathbb{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ by

$$f_\phi(\tau,z) := \phi(g,\alpha), \quad \tau\in\mathbb{H}, \quad z\in\mathbb{C}, \tag{7.14}$$

where (g,α) is an element of $SL_{2,1}(\mathbb{R})$ with $(g,\alpha)\circ(i,0)=(\tau,z)$. Clearly it is well defined because (7.14) is independent of the choice of $(g,\alpha)\in SL_{2,1}(\mathbb{R})$ such that $(g,\alpha)\circ(i,0)=(\tau,z)$.

Theorem 7.5. Let $\phi : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$ be a nonzero Maass-Jacobi form. Then the function f_ϕ defined by (7.14) satisfies the following conditions:

$(\mathrm{MJ}1)^0$ $f_\phi(\tilde\gamma\circ(\tau,z)) = f_\phi(\tau,z)$ for all $\tilde\gamma\in SL_{2,1}(\mathbb{Z}) := SL(2,\mathbb{Z})\ltimes\mathbb{Z}^{(1,2)}$.

$(\mathrm{MJ}2)^0$ f_ϕ is a joint eigenfunction for two invariant differential operators D and Ψ.

$(\mathrm{MJ}3)^0$ f_ϕ has a polynomial growth.

Conversely, if a smooth function $f : \mathbb{H}\times\mathbb{C}\longrightarrow\mathbb{C}$ satisfies the above conditions $(\mathrm{MJ}1)^0 - (\mathrm{MJ}3)^0$, then the function $\phi_f : SL_{2,1}(\mathbb{R})\longrightarrow\mathbb{C}$ defined by

$$\phi_f(g,\alpha) := f((g,\alpha)\circ(i,0)), \quad (g,\alpha)\in SL_{2,1}(\mathbb{R}) \tag{7.15}$$

is a Maass-Jacobi form.

Proof. Take $\tilde g\in SL_{2,1}(\mathbb{R})$ such that $\tilde g\circ(i,0)=(\tau,z)$. Then for any $\tilde\gamma\in SL_{2,1}(\mathbb{Z})$, we have

$$\begin{aligned} f_\phi(\tilde\gamma\circ(\tau,z)) &= f_\phi((\tilde\gamma*\tilde g)\circ(i,0)) \\ &= \phi(\tilde\gamma*\tilde g) = \phi(\tilde g) = f_\phi(\tau,z). \end{aligned}$$

So $(\mathrm{MJ}1)^0$ is proved.

Let $\pi : SL_{2,1}(\mathbb{R}) \longrightarrow SL_{2,1}(\mathbb{R})/SO(2)$ be the natural projection. For any function f on $SL_{2,1}(\mathbb{R})/SO(2)$, we put

$$\tilde f := f\circ\pi.$$

Thus $\tilde f$ is a function on $SL_{2,1}(\mathbb{R})$. Since $SL_{2,1}(\mathbb{R})/SO(2)$ is a reductive homogeneous space, we have a natural homomorphism $\mu : u\longmapsto D_u$ of $\mathbb{D}_{SO(2)}(SL_{2,1}(\mathbb{R}))$ onto $\mathbb{D}(SL_{2,1}(\mathbb{R})/SO(2))$ defined by

$$(\mu(u)f)^{\tilde{}} = (D_uf)^{\tilde{}} := u\tilde f, \quad f\in C^\infty(SL_{2,1}(\mathbb{R})/SO(2)). \tag{7.16}$$

The kernel of μ is given by

$$\ker\mu = \mathbb{D}_{SO(2)}(SL_{2,1}(\mathbb{R}))\cap\mathbb{D}(SL_{2,1}(\mathbb{R}))\,\mathfrak{so}(2), \tag{7.17}$$

where $\mathfrak{so}(2)$ denotes the Lie algebra of $SO(2)$. Therefore we have the isomorphism

$$\mathbb{D}(SL_{2,1}(\mathbb{R})/SO(2)) = \mathbb{D}(\mathbb{H}\times\mathbb{C}) \cong \mathbb{D}_{SO(2)}(SL_{2,1}(\mathbb{R}))/\ker\mu. \tag{7.18}$$

For the proof of the above facts (7.16), (7.17) and (7.18), we refer to [He2], pp. 285-286. It follows from (7.18) that the function f_ϕ is a joint eigenfunction for $\mathbb{D}(SL_{2,1}(\mathbb{R})/SO(2))$ $=\mathbb{D}(\mathbb{H}\times\mathbb{C})$.

It is easy to check $(\mathrm{MJ}3)^0$. We leave the detail to the reader. If we note $\phi = f_\phi\circ\pi$, the proof of the converse can be done in a similiar way. $\square$

For a right K-invariant function $\phi : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$, we define another function $h_\phi : S\mathcal{P}_2 \longrightarrow \mathbb{C}$ by

$$h_\phi(Y,V) := \phi(g,\, V\,{}^t g^{-1}\,), \quad Y \in S\mathcal{P}_2, \quad V \in \mathbb{R}^{(1,2)}, \tag{7.19}$$

where $Y = g\,{}^t g$ with some $g \in SL(2,\mathbb{R})$. The definition (7.19) is well defined because it is independent of the choice of g with $Y = g\,{}^t g$. Indeed, such g is unique up to $SO(2)$.

Theorem 7.6. Let $\phi : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$ be a nonzero Maass-Jacobi form. Then the function h_ϕ defined by (7.19) satisfies the following conditions:

(MJ1)* $h_\phi(\gamma Y\,{}^t\gamma,\, (V+\delta)\,{}^t\gamma\,) = h_\phi(Y,V)$ for all $(\gamma,\delta) \in SL_{2,1}(\mathbb{Z})$ with $\gamma \in SL(2,\mathbb{Z})$ and $\delta \in \mathbb{Z}^{(1,2)}$.

(MJ2)* h_ϕ is a joint eigenfunction for all the differential operators in $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$.

(MJ3)* h_ϕ has a suitable polynomial growth.

Conversely, if a smooth function $h : S\mathcal{P}_2 \times \mathbb{R}^{(1,2)} \longrightarrow \mathbb{C}$ satisfies the conditions (MJ1)* – (MJ3)*, the function $\phi_h : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$ defined by

$$\phi_h(g,\alpha) := h(g\,{}^t g,\, \alpha\,{}^t g), \quad g \in SL(2,\mathbb{R}), \quad \alpha \in \mathbb{R}^{(1,2)} \tag{7.20}$$

is a Maass-Jacobi form.

Proof. Let $(\gamma,\delta) \in SL_{2,1}(\mathbb{Z})$ and $(Y,V) \in S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$. We take $g \in SL(2,\mathbb{R})$ such that $g\,{}^t g = Y$. Then we have

$$\begin{aligned} h_\phi(\gamma Y\,{}^t\gamma,\, (V+\delta)\,{}^t\gamma) &= h_\phi(\gamma g\,{}^t(\gamma g),\, (V+\delta)\,{}^t\gamma\,) \\ &= \phi(\gamma g,\, (V+\delta)\,{}^t g^{-1}) \qquad (\text{ by } (7.19)\,) \\ &= \phi((\gamma,\,\delta) * (g,\, V\,{}^t g^{-1})) \\ &= \phi(g,\, V\,{}^t g^{-1}) \;=\; h_\phi(Y,V). \end{aligned}$$

So (MJ1)* is proved. Following the line of the proof of (MJ2)^0 in Theorem 7.5, we can prove (MJ2)*. It is easy to check (MJ3)*. The converse may be proved in a similiar way. We leave the detail of the rest to the reader. □

Definition 7.7. A smooth function on $\mathbb{H} \times \mathbb{C}$ is also called a Maass-Jacobi form if it satisfies the conditions (MJ1)^0 – (MJ3)^0. For a system $\lambda = (\lambda_\alpha)$ of eigenvalues for $\mathbb{D}(\mathbb{H} \times \mathbb{C})$, we also denote by $MJ(SL_{2,1}(\mathbb{Z}),\, \lambda)$ the vector space of all Maass-Jacobi forms f such that $L_\alpha f = \lambda_\alpha f$ for all $L_\alpha \in \mathbb{D}(\mathbb{H} \times \mathbb{C})$.

Remark 7.8. We note that Maass wave forms are special ones of Maass-Jacobi forms. Thus the number of λ's with $MJ(SL_{2,1}(\mathbb{Z}),\lambda) \neq 0$ is infinite.

Theorem 7.9. The Riemannian metric ds^2 on $\mathbb{H} \times \mathbb{C}$ defined by

$$\begin{aligned} ds^2 &= \frac{y\,+\,v^2}{y^3}\,(\,dx^2\,+\,dy^2\,)\,+\,\frac{1}{y}\,(\,du^2\,+\,dv^2\,) \\ &\quad -\,\frac{2v}{y^2}\,(\,dx\,du\,+\,dy\,dv\,) \end{aligned}$$

is invariant under the action (7.5) of $SL_{2,1}(\mathbb{R})$. The Laplace-Beltrami operator Δ of the Riemannian space $(\mathbb{H}\times\mathbb{C},\, ds^2)$ is given by

$$\begin{aligned}\Delta =&\, y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (\, y + v^2\,) \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\ &+ 2\, y\, v \left(\frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right).\end{aligned}$$

That is, $\Delta = D + \Psi$. In particular, $SL_{2,1}(\mathbb{R})$ is a subgroup of the isometry group of $(\mathbb{H}\times\mathbb{C},\, ds^2)$.

Proof. The proof can be found in [Y7]. □

Now we introduce the notion of *weak Maass-Jacobi forms.*

Definition 7.10. A smooth bounded function $\phi : SL_{2,1}(\mathbb{R}) \longrightarrow \mathbb{C}$ is called a *weak Maass-Jacobi form* if it satisfies the conditions (MJ1), (MJ3) and the following condition (WMJ2) :

(WMJ2) ϕ is an eigenfunction for the Laplace-Beltrami operator in $\mathbb{D}_{SO(2)}(SL_{2,1}(\mathbb{R}))$.

In other words, the function $f_\phi : \mathbb{H}\times\mathbb{C} \longrightarrow \mathbb{C}$ defined by (7.14) from a weak Maass-Jacobi form ϕ satisfies the conditions $(\mathrm{MJ1})^0$, $(\mathrm{MJ3})^0$ and the following condition $(\mathrm{WMJ2})^0$:

$(\mathrm{WMJ2})^0$ f_ϕ is an eigenfunction for the Laplace-Beltrami operator $\Delta := D + \Psi$.

Obviously Maass-Jacobi forms are weak Maass-Jacobi forms but in general the converse does not hold. We observe that Maass wave forms are the special ones of weak Maass-Jacobi forms.

Theorem 7.11. Let $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ be the algebra of all differential operators on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ invariant under the action (7.4) of $SL_{2,1}(\mathbb{R})$. Then the algebra $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ is generated by the following differential operators

$$\tilde{D} := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \tag{7.21}$$

and

$$\tilde{\Psi} := y^{-1} \left\{ \frac{\partial^2}{\partial v_1^2} - 2\, x\, \frac{\partial^2}{\partial v_1 \partial v_2} + (\, x^2 + y^2\,)\, \frac{\partial^2}{\partial v_2^2} \right\}, \tag{7.22}$$

where Y is of the form (7.7) and $V = (\, v_1,\, v_2\,)$ with real variables x, y, v_1, v_2 and $y > 0$. Moreover, we have

$$\begin{aligned}[\tilde{D},\, \tilde{\Psi}] :=&\; \tilde{D}\,\tilde{\Psi} - \tilde{\Psi}\,\tilde{D} \\ =&\; -4\, y\, \frac{\partial^3}{\partial x \partial v_1 \partial v_2} + 4\, x\, y\, \frac{\partial^3}{\partial x \partial v_2^2} - 2\, \frac{\partial^3}{\partial y \partial v_1^2} + 4\, x\, \frac{\partial^3}{\partial y \partial v_1 \partial v_2} \\ &+ 2\,(\, y^2 - x^2\,)\, \frac{\partial^3}{\partial y \partial v_2^2} + \frac{2\, x}{y}\, \frac{\partial^2}{\partial v_1 \partial v_2} + 2\, \Psi.\end{aligned}$$

In particular, the algebra $\mathbb{D}(S\mathcal{P}_2 \times \mathbb{R}^{(1,2)})$ is not commutative.

Proof. The proof can be found in [Y8]. □

Now we present a $SL_{2,1}(\mathbb{R})$-invariant metric on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$.

Theorem 7.12. The following Riemannian metric ds_0^2 on $S\mathcal{P}_2 \times \mathbb{R}^{(1,2)}$ defined by

$$ds_0^2 := \frac{1}{y^2}\,(\,dx^2 + dy^2\,) + \frac{1}{y}\,\{\,(\,x^2 + y^2\,)\,dv_1^2 + 2\,x\,dv_1\,dv_2 + dv_2^2\,\}$$

is invariant under the action (7.4) of $SL_{2,1}(\mathbb{R})$, where we used the coordinates $(\,Y,\,V\,)$ such that Y is of the form (7.7) and $V = (\,v_1,\,v_2\,)$ with $v_1,\,v_2$ real variables. The Laplace-Beltrami operator Δ_0 of the Riemannian space $(\,S\mathcal{P}_2 \times \mathbb{R}^{(1,2)},\,ds_0^2\,)$ is given by

$$\Delta_0 = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{1}{y}\left\{\frac{\partial^2}{\partial v_1^2} - 2\,x\,\frac{\partial^2}{\partial v_1 \partial v_2} + (\,x^2 + y^2\,)\,\frac{\partial^2}{\partial v_2^2}\right\}.$$

That is, $\Delta_0 = \tilde{D} + \tilde{\Psi}$. In particular, $SL_{2,1}(\mathbb{R})$ is a subgroup of the isometry group of $(\,S\mathcal{P}_2 \times \mathbb{R}^{(1,2)},\,ds_0^2\,)$.

Proof. The proof can be found in [Y8]. □

8. Final Remarks

In this final section, we make some comments on a fundamental domain for $G_{n,m}(\mathbb{Z})\backslash\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ and provide some problems related to the previous subjects.

We define the domain $\mathfrak{R}_{n,m}$ in $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ by

$$\mathfrak{R}_{n,m} = \Big\{\,(Y,\,\lambda) \mid Y \in \mathfrak{R}_n,\ \lambda = (\lambda_{kl}) \in \mathbb{R}^{(m,n)},\ 0 \le \lambda_{kl} \le 1 \text{ for all } k, l\,\Big\},$$

where $\mathfrak{R}_n$ denotes the Minkowski fundamental domain for $GL(n,\mathbb{Z})\backslash\mathcal{P}_n$. For more detail on $\mathfrak{R}_n$, we refer the reader to [M2] or [Si1]. It is easy to see that $\mathfrak{R}_{n,m}$ is a fundamental domain for $G_{n,m}(\mathbb{Z})\backslash\mathcal{P}_n \times \mathbb{R}^{(m,n)}$. We see that $\mathfrak{R}_{n,m}$ may be regarded as the fibre bundle over the Minkowski fundamental domain $\mathfrak{R}_n$ with fibres as tori of dimension mn. Thus we call the fundamental domain $\mathfrak{R}_{n,m}$ the *universal torus* or the *universal family of tori.* The concept of the universal torus is analogous to that of the universal abelian variety (cf. [F-C] and [Y6]).

We propose the following problem:

Problem 8.1. Describe the boundary of the universal torus in terms of automorphic forms on $G_{n,m}(\mathbb{R})$.

We review Maass wave forms. Let $s \in \mathbb{C}$ and $\Gamma := \Gamma_0(N)$ for some positive integer $N > 0$. We denote by $W_s(\Gamma)$ the vector space of all smooth bounded functions $\phi : SL(2,\mathbb{R}) \longrightarrow \mathbb{C}$ such that

(a) $\phi(\gamma g k) = \phi(g)$ for all $\gamma \in \Gamma,\ g \in SL(2,\mathbb{R})$ and $k \in SO(2)$.

(b) $\Delta\phi = \frac{1-s^2}{4}\,\phi$, where

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\partial\theta}$$

is the Laplace-Beltrami operator on $SL(2,\mathbb{R})$ whose coordinates $x, y, \theta\,(\,x \in \mathbb{R},\ y > 0,\ 0 \le \theta < 2\pi\,)$ are given by

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad g \in SL(2,\mathbb{R})$$

by means of the Iwasawa decomposition of $SL(2,\mathbb{R})$. The functions in $W_s(\Gamma)$ are called *Maass wave forms*. It is well known that $W_s(\Gamma)$ is nontrivial for infinitely many values of s.

Let π_s^+ and π_s^c be the irreducible unitary representations of $SL(2,\mathbb{R})$ induced by the characters

$$\begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix} \longmapsto |a|^s, \quad \mathrm{Re}\,(s) = 0\ ;\ \text{ a unitary character of } B$$

and

$$\begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix} \longmapsto \mathrm{sgn}\,(a)\,|a|^s, \quad \mathrm{Re}\,(s) = 0\ ;\ \text{ a non-unitary character of } B$$

respectively, where $B := \left\{ \begin{pmatrix} a & u \\ 0 & a^{-1} \end{pmatrix} \mid a > 0,\ u \in \mathbb{R} \right\}$ is a Borel subgroup of $SL(2,\mathbb{R})$. Let $L_0^2\,(\Gamma\backslash SL(2,\mathbb{R}))$ be the subspace of all cuspidal forms in $L^2\,(\Gamma\backslash SL(2,\mathbb{R}))$. It is well known that the multiplicity of π_s^+ or π_s^c in the right regular representation of $SL(2,\mathbb{R})$ on $L_0^2\,(\Gamma\backslash SL(2,\mathbb{R}))$ is equal to the dimension of $W_s(\Gamma)$. For more detail, we refer to [G] and [I].

Problem 8.2. Decompose the Hilbert space $L^2\,(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$ into irreducibles. Let $L_0^2\,(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$ be the subspace of all cuspidal automorphic forms in $L^2\,(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$. Investigate the role which Maass-Jacobi forms play in the decomposition of the right regular representation of $SL_{n,m}(\mathbb{R})$ on $L_0^2(SL_{n,m}(\mathbb{Z})\backslash SL_{n,m}(\mathbb{R}))$.

Let $\mu : \mathbb{D}\,(G_{n,m}(\mathbb{R})/K) \longrightarrow \mathbb{C}$ be a homomorphism. Let

$$E_\mu := \{\, f \in C^\infty\,(G_{n,m}(\mathbb{R})/K) \mid Df = \mu(D) \ \text{ for all } D \in \mathbb{D}\,(G_{n,m}(\mathbb{R})/K)\,\}$$

be the joint eigenspace associated with μ. Then we have the natural representation of $G_{n,m}(\mathbb{R})$ on E_μ defined by

$$(\,T_\mu(g)f\,)\,(xK) := f(g^{-1}xK), \quad g, x \in G_{n,m}(\mathbb{R}),\ f \in E_\mu.$$

We call T_μ the eigenspace representation of $G_{n,m}(\mathbb{R})$ on E_μ.

Problem 8.3. For which μ is T_μ irreducible ? Decompose T_μ into irreducibles.

Problem 8.4. Write down $M_{n,m,\mathcal{M}}$ $(\,cf.\,4.1\,)$ in terms of D_j's and Ψ_{kp}'s.

Problem 8.5. Construct automorphic forms $G_{n,m}(\mathbb{R})$ and Maass-Jacobi forms. Develope the Fourier expansion of automorphic forms on $G_{n,m}(\mathbb{R})$ and Maass-Jacobi forms. And develope the theory of these automorphic forms adelically.

Selberg [S2] discussed the invariant differential operators and the integral operators on bounded symmetric domains in the sense of Elie Cartan (cf. [Si2]). We propose the following problem.

Problem 8.6. Discuss the $G^J_{n,m}(\mathbb{R})$-invariant differential operators on $H_n \times \mathbb{C}^{(m,n)}$ under the action of (1.4) of $G^J_{n,m}(\mathbb{R})$ and the integral operators on $H_n \times \mathbb{C}^{(m,n)}$. We observe that $H_n \times \mathbb{C}^{(m,n)}$ is not a symmetric space and not a bounded domain in $\mathbb{C}^d,\ d = n(n+1)/2 + mn$.

Finally we review the results of G. Shimura [Sh]. We let G/K be a hermitian symmetric space of classical and noncompact type. Here G is classical, usually a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup of G. Let ρ be a continuous representation of K on a finite dimensional complex vector space V_ρ. We denote by $C^\infty(\rho)$ the set of all V_ρ-valued smooth functions $f : G \longrightarrow V_\rho$ such that

$$f(gk^{-1}) = \rho(k)f(g) \quad \text{for all } k \in K \text{ and } g \in G.$$

We let $\mathbb{D}(\rho)$ the ring of all left-invariant differential operators on G which map $C^\infty(\rho)$ into itself.

G. Shimura proved that if ρ is one-dimensional, then $\mathbb{D}(\rho)$ is commutative and there exists a canonically defined set of generators $L_1, L_2, \cdots, L_r$ for $\mathbb{D}(\rho)$ which are algebraically independent. Thus we have $\mathbb{D}(\rho) = \mathbb{C}[L_1, \cdots, L_r]$. Here r denotes the rank of the hermitian symmetric space G/K.

Returning to our case, since $G^J_{n,m}(\mathbb{R})$ acts on $H_n \times \mathbb{C}^{(m,n)}$ transitively, $H_n \times \mathbb{C}^{(m,n)}$ may be regarded as a homogeneous space $G^J_{n,m}(\mathbb{R})/K^J_{n,m}$ via the identification

$$G^J_{n,m}(\mathbb{R})/K^J_{n,m} \longrightarrow H_n \times \mathbb{C}^{(m,n)}, \quad g \cdot K^J_{n,m} \longmapsto g \cdot (iE_n, 0), \quad g \in G^J_{n,m}(\mathbb{R}),$$

where $K^J_{n,m} := U(n) \times \mathrm{Symm}(\mathbb{R}^m)$ is the stabilizer of $G^J_{n,m}(\mathbb{R})$ at $(iE_n, 0)$. Let τ be a continuous representation of $K^J_{n,m}$ on a finite dimensional complex vector space V_τ. We denote by $C^\infty(\tau)$ the set of all V_τ-valued smooth functions on $G^J_{n,m}(\mathbb{R})$ such that

$$f(gk^{-1}) = \tau(k)f(g), \quad g \in G^J_{n,m}(\mathbb{R}), \quad k \in K^J_{n,m}.$$

We let $\mathbb{D}(\tau)$ the ring of all left-invariant differential operators on $G^J_{n,m}(\mathbb{R})$ which map $C^\infty(\tau)$ into itself.

Problem 8.7. Discuss the 1st and 2nd main theorems for $\mathbb{D}(\tau)$ in the sense of invariant theory. Also discuss the existence of a canonically defined set of algebraically independent generators for $\mathbb{D}(\tau)$.

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