ON THE NONEXCELLENCE OF THE FUNCTION FIELDS OF SEVERI-BRAUER VARIETIES

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§0. Introduction

Let F be a field of characteristic different from 2 and ϕ be a non-degenerate quadratic form over F. It is an important problem to study the behavior of the anisotropic part of forms over F under a field extension L/F. A field extension L/F is called *excellent* if for any quadratic form ϕ over F the anisotropic part $(\phi_L)_{\mathrm{an}}$ of ϕ over L is defined over F (i.e., there is a form ξ over F such that $(\phi_L)_{\mathrm{an}} \cong \xi_L$).

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Any quadratic extension is excellent. Since any anisotropic quadratic form ψ over F is still anisotropic over the field of rational functions $F(t)$, every purely transcendental field extension is excellent.

Let $F(X)$ be the field of rational functions on a geometrically integral variety X. One of the important problems is to find conditions on X so that the field extension $F(X)/F$ is excellent. We say that $F(X)/F$ is *universally excellent* if for any extension K/F the extension $K(X)/K$ is excellent. The following varieties are most important in the algebraic theory of quadratic forms: quadric hypersurfaces, Severi–Brauer varieties, varieties of totally isotropic flags, and products of such varieties.

If X is rational (or unirational) then $F(X)/F$ is purely transcendental (respectively, unirational), and it follows from Springer's theorem that $F(X)/F$ is excellent and moreover that it is universally excellent.

In the case of a hyper-surface $X = X_q$ defined by the equation $q = 0$ where q is a non-degenerate quadratic form, the following results are known: 1) if q is isotropic, then $F(X_q)/F$ is universally excellent (for in this case X_q is rational); 2) if the field extension $F(X_q)/F$ is excellent and q is anisotropic, then q is a Pfister neighbor [Kn2]; 3) if $\dim q \leqslant 3$ (or $\dim q = 4$ and $\det q = 1$), then X_q is universally excellent (see [ELW, Appendix II] or [Ro2], [LVG]); 4) if q is anisotropic, then $F(X_q)/F$ is universally excellent if and only if q is a Pfister neighbor of dimension $\leqslant 4$ (see [Izh1] or [H2]).

Thus the problem whether the field extension $F(X)/F$ is universally excellent is completely solved in the case where X is a quadric surface X_q.

In this paper we study the case where X is a Severi–Brauer variety. In the simpliest case where X is the Severi– Brauer variety of a quaternion algebra (a, b), the field extension $F(X)/F$ is excellent. Indeed, in this case the variety X coincides with the quadric hypersurface X_ϕ, where $\phi = \langle 1, -a, -b\rangle$.

The next interesting case is the case of a biquaternion division algebra A. We study this case in Sections 3 and 5. In Section 3 we prove that the field extension $F(SB(A))/F$ is not universally excellent for any biquaternion division F-algebra A. Moreover we construct a unirational field extension E/F such that $E(SB(A))/E$ is not excellent (see Theorem 3.3). Applying this result, we find a condition on a central simple algebra A under which $F(SB(A))/F$ is universally excellent. Theorem 3.10 asserts that the field extension $F(SB(A))/F$ is universally excellent only in the following two cases: 1) the index of A is odd; 2) the algebra A has the form $Q \otimes_F D$, where Q is a quaternion algebra and D is of odd index. In addition, we show that the field extension $F(SB(A))/F$ is not excellent for an arbirtary algebra A of index 8 and exponent 2 (see Theorem 3.11).

In our proof of the main result of Section 3 we apply some deep results of E. Peyre and N. Karpenko concerning the groups $\ker\left(H^3(F, \mathbb{Z}/2\mathbb{Z}) \to H^3(F(X), \mathbb{Z}/2\mathbb{Z})\right)$ and $\operatorname{Tor}_2 CH^2(X)$, where X is a product of Severi–Brauer varieties of algebras of exponent 2 (see [Pe], [Kar1], [Kar2]). In Section 2 and Appendix A we prove some results concerning Chow grops and Galois cohomology. In particular, in Appendix A we prove the following

Theorem. *Let A and B be central simple algebras of exponent 2 over F. Let*

$X = SB(A) \times SB(B)$. *Then the homomorphism*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A] \cup H^1(F) + [B] \cup H^1(F)} \xrightarrow{\bar{\varepsilon}_2} \operatorname{Tor}_2 CH^2(X).$$

is an isomorphism. Here $H^*(F) = H^*(F, \mathbb{Z}/2\mathbb{Z})$ *and the homomorphism* $\bar{\varepsilon}_2$ *is induced by the homomorphism* $\varepsilon\colon H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \to CH^2(X)$ *defined in* [Su].

This theorem plays an important part in the proof of the non universal excellence of the function fields of the Severi–Brauer varieties of biquaternion division algebras.

In Section 4 we prove the following statement: For any central simple F-algebra A the field extension $F(SB(A))/F$ is 5-excellent (this means that if $\dim\phi \leqslant 5$ then $(\phi_{F(SB(A))})_{\text{an}}$ is defined over F). We prove that if $u(F) \leqslant 6$ then the field extension $F(SB(A))/F$ is excellent. In §5 we construct explicit examples of a biquaternion division algebra A such that the field extension $F(SB(A))/F$ is not excellent[1]. In particular, we prove that the biquaternion algebra $A = (a,b) \otimes (c,d)$ over the field of rational functions in 4 variables $F(a,b,c,d)$ yields such an example (see Corollary 5.11). In Appendix B we study the excellence property for generic splitting fields. In particular, we find a criterion of universal excellence for the function fields of integer varieties of totally isotropic subspaces (see Theorem B.21).

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§1. Main notation and facts

1.1. Quadratic forms and central simple algebras. By $\phi \perp \psi$, $\phi \cong \psi$, and $[\phi]$ we denote respectively orthogonal sum of forms, isometry of forms, and the class of ϕ in the Witt ring $W(F)$ of the field F. The maximal ideal of $W(F)$ generated by the classes of even dimensional forms is denoted by $I(F)$. We write $\phi \sim \psi$ if ϕ is similar to ψ, i.e., $k\phi = \psi$ for some $k \in F^*$. The anisotropic part of ϕ is denoted by ϕ_{an} and $i_W(\phi)$ denotes the Witt index of ϕ. We denote by $\langle\langle a_1, \dots, a_n\rangle\rangle$ the n-fold Pfister form

$$\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$$

and by $P_n(F)$ the set of all n-fold Pfister forms. The set of all forms similar to n-fold Pfister forms we denote by $GP_n(F)$. The fundamental Arason–Pfister Hauptsatz (APH for short) states that if $\phi \in I^n(F)$ and $\dim\phi < 2^n$ then $[\phi] = 0$; if $\phi \in I^n(F)$ and $\dim\phi = 2^n$ then $\phi \in GP_n(F)$. An easy corollary from Arason–Pfister Hauptsatz (APH' for short in what follows) states that if $\phi, \psi \in GP_n(F)$ satisfy the condition $\phi \equiv \psi \pmod{I^{n+1}(F)}$ and the intersection $D_F(\phi) \cap D_F(\psi)$ is not empty then $\phi = \psi$. For any field extension L/F we put $\phi_L = \phi \otimes L$, $W(L/F) = \ker(W(F) \to W(L))$, and $I^n(L/F) = \ker(I^n(F) \to I^n(L))$.

[1] Another example (a little more complicated than ours) was independently constructed by A. Sivatskii.

Let ϕ be a quadratic form such that $\dim\phi \geqslant 2$ and $\phi \not\cong \mathbb{H}$. The function field $F(\phi)$ of the form ϕ over F is the function field of the projective variety X_ϕ given by equation $\phi = 0$. In the case where $\dim\phi \leqslant 1$ or $\phi \cong \mathbb{H}$, we set $F(\phi) \overset{\text{def}}{=} F$.

Let A be a central simple algebra (CS algebra for short) over F. By $\deg(A)$, $\operatorname{ind}(A)$, and $[A]$ we denote respectively the degree of A, the Schur index of A, and the class of A in the Brauer group $\operatorname{Br}(F)$. By $SB(A)$ we denote the Severi–Brauer variety of an algebra A.

We recall that two field extensions E/F and K/F are stably isomorphic if and only if there exist indeterminates $x_1, \dots, x_s, y_1, \dots, y_r$ and an isomorphism $E(x_1, \dots, x_r) \cong K(y_1, \dots, y_s)$ over F. We will write $E/F \overset{\text{st}}{\sim} K/F$ if E/F is stably isomorphic to K/F.

If $[A] = [A']$ in $\operatorname{Br}(F)$ then the field extensions $F(SB(A))/F$ and $F(SB(A'))/F$ are stably isomorphic. Moreover we have the following

Lemma 1.2. *Let $A_1, \dots, A_k$ and $A'_1, \dots, A'_l$ be SC algebras over F. Suppose that the subgroup $\langle [A_1], \dots, [A_k] \rangle$ of the Brauer group $\operatorname{Br}(F)$ generated by the classes of algebras $A_1, \dots, A_k$ coincides with the subgroup $\langle [A'_1], \dots, [A'_l] \rangle$ generated by the classes of algebras $A'_1, \dots, A'_l$. Then the field extensions*

$$F(SB(A_1) \times \cdots \times SB(A_k))/F \qquad \text{and} \qquad F(SB(A'_1) \times \cdots \times SB(A'_l))/F$$

are stably isomorphic.

Let ϕ be a quadratic form. We denote by $C(\phi)$ the Clifford algebra of ϕ. If $\phi \in I^2(F)$ then $C(\phi)$ is a CS algebra. Hence we get a well defined element $[C(\phi)]$ of $\operatorname{Br}_2(F)$ which we will denote by $c(\phi)$.

Good references for the basic theory of quadratic forms and central simple algebras are books of T. Y. Lam [Lam], W. Scharlau [Sch], P. K. Draxl [Dr], and R. S. Pierce [Pi].

1.3. Cohomology groups. Let F be a field of characteristic $\neq 2$. By $H^n(F)$ we denote the cohomology group $H^n(F, \mathbb{Z}/2\mathbb{Z})$. The groups $H^n(F)$ $(n \geqslant 0)$ form a graded ring, with the multiplication given by the cup product.

Obviously $H^0(F) \cong \mathbb{Z}/2\mathbb{Z}$. By Hilbert theorem 90 we have $H^1(F) \cong F^*/F^{*2}$. Thus any element $a \in F^*$ gives rise to an element of $H^1(F)$ which we will denote by (a). The cup product $(a_1) \cup \cdots \cup (a_n)$ we will denote by $(a_1, \dots, a_n)$.

The group $H^2(F)$ is isomorphic to $\operatorname{Br}_2(F)$. This isomorphism maps the element $(a, b) = (a) \cup (b)$ of the group $H^2(F)$ to the class of the quaternion algebra (a, b) in the Brauer group $\operatorname{Br}_2(F)$. We will identify the groups $\operatorname{Br}_2(F)$ with the group $H^2(F)$. Thus for any CS algebra A of exponent 2 we get an element $[A]$ of the group $H^2(F)$.

If the field extensions E/F and E/K are stably isomorphic then $\ker(H^i(F) \to H^i(E)) = \ker(H^i(F) \to H^i(K))$.

For $n = 0, 1, 2, 3, 4$ there is a homomorphism

$$e^n : I^n(F)/I^{n+1}(F) \to H^n(F)$$

which is uniquely determined by the condition $e^n(\langle\langle a_1, \dots, a_n \rangle\rangle) = (a_1, \dots, a_n)$. This homomorphism was constructed by Arason [Ar2] for $n \leqslant 3$, and by Jacob, Rost [JR] and Szyjewsky [Sz] for $n = 4$. The homomorphism e^n is a isomorphism for $n = 0, 1, 2, 3$ (see [Me], [MS], and [Rol])[2]. The homomorphism e^2 maps a quadratic form $\phi \in I^2(F)$ to $c(\phi) \in \mathrm{Br}_2(F)$.

1.4. The group $\widetilde{H}^n(F)$. Let $A_1, \dots, A_k$ be CS algebras of exponent 2 over F. We have $[A_1], \dots, [A_k] \in \mathrm{Br}_2(F) = H^2(F)$. Let $X_1 = SB(A_1), \dots, X_k = SB(A_k)$. Let us denote by $\widetilde{H}^n(F)$ the group

$$\widetilde{H}^n(F) \stackrel{\text{def}}{=} \ker\left(H^n(F) \to H^n(F(X_1 \times \cdots \times X_k))\right).$$

Clearly $\widetilde{H}^*(F)$ is a ideal in $H^*(F)$, i.e., for any m, n we have $\widetilde{H}^n(F)H^m(F) \subset \widetilde{H}^{n+m}(F)$.

Obviously $\widetilde{H}^0(F) = \widetilde{H}^1(F) = 0$. The group $\widetilde{H}^2(F)$ coincides with the subgroup $\langle [A_1], \dots, [A_k] \rangle$ of $H^2(F)$ generated by the classes of the algebras $A_1, \dots, A_k$. The first nontrivial group is $\widetilde{H}^3(F)$. This group contains the group

$$\widetilde{H}^2(F)H^1(F) = [A_1]H^1(F) + \cdots + [A_k]H^1(F).$$

It is a natural question whether the group $\widetilde{H}^3(F)$ coincides with $\widetilde{H}^2(F)H^1(F)$. This question gives rise to the study of the following factor group

$$\frac{\widetilde{H}^3(F)}{\widetilde{H}^2(F)H^1(F)} = \frac{\ker\left(H^3(F) \to H^3(F(SB(A_1) \times \cdots \times SB(A_k)))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)}.$$

We denote this factor group by $\Gamma(F; A_1, \dots, A_k)$.

It follows from Lemma 1.2 that the group $\Gamma(F; A_1, \dots, A_k)$ depends only on the subgroup $\langle [A_1], \dots, [A_k] \rangle$ of $\mathrm{Br}_2(F)$ generated by $[A_1], \dots, [A_k]$. More precisely, if CS algebras $A'_1, \dots, A'_l$ satisfy $\langle [A_1], \dots, [A_k] \rangle = \langle [A'_1], \dots, [A'_l] \rangle$, then

$$\Gamma(F; A_1, \dots, A_k) = \Gamma(F; A'_1, \dots, A'_l)$$

In particular, for any algebras A_1, A_2, and B with $[A_1] + [A_2] + [B] = 0$, we have

$$\Gamma(F; A_1, A_2, B) = \Gamma(F; A_1, A_2) = \Gamma(F; A_1, B) = \Gamma(F; A_2, B).$$

In the case $k = 1$ the following result is known

Theorem 1.5. (see [Ar1, Pe]). *If* $\mathrm{ind}(A) \leqslant 4$ *and* $\exp(A) = 2$, *then* $[A]H^1(F) = \ker\left(H^3(F) \to H^3(F(SB(A)))\right)$.

Applying this theorem and the injectivity of the homomorphism e^3, we get the following

Corollary 1.6. *Let A be a biquaternion algebra and q be a corresponding Albert form. Then* $I^3(F(SB(A))/F) \subset [q]I(F) + I^4(F)$. □

[2]Bijectivity of e^4 was announced by M. Rost. Recently V. Voevodsky proved that there is a well defined bijective homomorphism e^n for all $n \geqslant 0$. We do not use these resuts in our paper.

1.7. Chow groups. For any smooth projective variety X, the homomorphism from the group $\varepsilon_X:\ \ker\left(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X),\mathbb{Q}/\mathbb{Z}(2))\right)$ to the group $CH^2(X)$ was constructed in [Su, Sec. 23]

We need the following

Theorem 1.8. (see [Pe, Th. 4.1]). *Let $A_1,\dots,A_k$ be CS algebras over F. Let $X = SB(A_1)\times\cdots\times SB(A_k)$.*

1) *The homomorphism ε induces an isomorphism*

$$\frac{\ker\left(H^3(F,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(X,\mathbb{Q}/\mathbb{Z}(2))\right)}{[A_1]H^1(F,\mathbb{Q}/\mathbb{Z}) + \cdots + [A_k]H^1(F,\mathbb{Q}/\mathbb{Z})} \xrightarrow{\sim} \operatorname{Tor}(CH^2(X)).$$

which we will denote by $\bar\varepsilon_X$ or $\bar\varepsilon$.

2) *If all the algebras $A_1,\dots,A_k$ have exponent 2 then the homomorphism ε induces a monomorphism*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)} \to \operatorname{Tor}_2 CH^2(X),$$

which we will denote by $\bar\varepsilon_{X,2}$ or $\bar\varepsilon_2$.

Thus $\bar\varepsilon_2\colon \Gamma(F;A_1,\dots,A_k) \to \operatorname{Tor}_2 CH^2(SB(A_1)\times\cdots\times SB(A_k))$ is a monomorphism.

It is not difficult to show that for any CS algebras $A_1,\dots,A_k$ the torsion subgroup of $CH^2(SB(A_1)\times\cdots\times SB(A_k))$, depends only on the subgroup $\langle [A_1],\dots,[A_k]\rangle$ of $\operatorname{Br}(F)_2$ generated by $[A_1],\dots,[A_k]$. More precisely, if CS algebras $A'_1,\dots,A'_l$ satisfy $\langle [A_1],\dots,[A_k]\rangle = \langle [A'_1],\dots,[A'_l]\rangle$, then

$$\operatorname{Tor} CH^2(SB(A_1)\times\cdots\times SB(A_k)) \cong \operatorname{Tor} CH^2(SB(A'_1)\times\cdots\times SB(A'_l)).$$

In particular, for any algebras A_1, A_2, and B with $[A_1]+[A_2]+[B]=0$ we have

$$\operatorname{Tor}_2 CH^2(SB(A_1)\times SB(A_2)\times SB(B)) \cong \operatorname{Tor}_2 CH^2(SB(A_1)\times SB(B)).$$

The group $\operatorname{Tor} CH^2(SB(A))$ was studied by Karpenko. One of his reults asserts that for any algebra A of exponent 2 the group $\operatorname{Tor} CH^2(SB(A))$ (and hence the group $\Gamma(F;A)$) is either zero or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see [Kar1, Proposition 4.1]). It is an interesting question to give an explicit description for an element of $H^3(F)$ which determines a generator of the group

$$\Gamma(F;A) = \ker\left(H^3(F) \to H^3(F(SB(A)))\right)/[A]H^1(F).$$

In the case $k>1$ the groups $\operatorname{Tor} CH^2(SB(A_1)\times\cdots\times SB(A_k))$ were also investigated by N. Karpenko. In our paper we need the following particular case of the main theorem from [Kar2].

Theorem 1.9. *Let A and B be algebras of exponent 2 such that* $\operatorname{ind}(A) \leqslant 4$ *and* $\operatorname{ind}(B) \leqslant 2$. *Let* $X = SB(A) \times SB(B)$. *Then*

1) *The group* $\operatorname{Tor} CH^2(X)$ *is trivial or isomorphic to* $\mathbb{Z}/2\mathbb{Z}$.
2) *If the group* $\operatorname{Tor} CH^2(X)$ *is not trivial then* $\operatorname{ind}(A) = 4$, $\operatorname{ind}(B) = 2$ *and* $\operatorname{ind}(A \otimes_F B) = 4$. *In particular, if at least one of the algebras A and B is not a division algebra then* $\operatorname{Tor} CH^2(X) = 0$.
3) *If* $\operatorname{ind}(A \otimes_F B) = 8$ *then there is a field extension E/F such that* $\operatorname{ind}(A \otimes_F B)_E = 4$ *and* $\operatorname{Tor} CH^2(X_E) = \mathbb{Z}/2\mathbb{Z}$. *Moreover we can take for E the function field $F(Y)$ of the generalized Severi–Brauer variety* $Y = SB(A \otimes_F B, 4)$.

Corollary 1.10. *Let A_1 and A_2 be biquaternion algebras and B be a quaternion algebra such that* $[A_1] + [A_2] + [B] = 0$. *Let* $X = SB(A_1) \times SB(A_2) \times SB(B)$. *Then the group* $\operatorname{Tor} CH^2(SB(X))$ *is trivial or equals to* $\mathbb{Z}/2\mathbb{Z}$. *Moreover if at least one of the algebras A_1, A_2, and B is not a division algebra then the group* $\operatorname{Tor} CH^2(X)$ *is trivial.*

1.11. The group $\Gamma(F; q_1, \dots, q_1)$. Let $q_1, \dots, q_k \in I^2(F)$. The Clifford algebras $C(q_1), \dots, C(q_k)$ are CS algebras of exponent 2 over F. Let us define the group $\Gamma(F; q_1, \dots, q_k)$ by the formula

$$\Gamma(F; q_1, \dots, q_k) = \Gamma(F; C(q_1), \dots, C(q_k)).$$

Note that for another collection $q'_1, \dots, q'_l \in I^2(F)$ with

$$[q_1]W(F) + \dots + [q_k]W(F) + I^3(F) = [q'_1]W(F) + \dots + [q'_l]W(F) + I^3(F),$$

we have $\Gamma(F; q_1, \dots, q_k) = \Gamma(F; q'_1, \dots, q'_l)$. In particular, for any $q_1, q_2, q_3 \in I^2(F)$ satisfying $q_1 \perp q_2 \perp q_3 \in I^3(F)$, we have

$$\Gamma(F; q_1, q_2, q_3) = \Gamma(F; q_1, q_2) = \Gamma(F; q_1, q_3) = \Gamma(F; q_2, q_3).$$

Let $X = SB(C(q_1)) \times \dots \times SB(C(q_k))$. By the Peyre's Theorem 1.8 we have the embedding $\bar{\varepsilon}_2 : \Gamma(F; q_1, \dots, q_k) \hookrightarrow \operatorname{Tor}_2 CH^2(X)$. Therefore we have a well-defined homomorphism,

$$I^3(F(X)/F) \xrightarrow{e^3} \ker\big(H^3(F) \to H^3(F(X))\big) \twoheadrightarrow \Gamma(F; q_1, \dots, q_k) \overset{\bar{\varepsilon}_2}{\hookrightarrow} \operatorname{Tor}_2 CH^2(X).$$

Thus for any $\phi \in I^3(F(X)/F)$ we get the elements $e^3(\phi) \in \Gamma(F; q_1, \dots, q_k)$ and $\bar{\varepsilon}_2 \circ e^3(\phi) \in \operatorname{Tor}_2 CH^2(X)$.

Lemma 1.12. *Let* $X = SB(C(q_1) \times \dots \times SB(C(q_k)))$ *and* $\phi \in I^3(F(X)/F)$. *The following assertions are equivalent:*

1) $e^3(\phi) = 0$ *in* $\Gamma(F; q_1, \dots, q_k)$.
2) $\bar{\varepsilon}_2 \circ e^3(\phi) = 0$ *in* $\operatorname{Tor}_2 CH^2(X)$.
3) $\phi \in [q_1]I(F) + \dots + [q_k]I(F) + I^4(F)$.

Proof. 1)$\Longleftrightarrow$2) since $\bar{\varepsilon}_2$ is injective. To prove 1)$\Longleftrightarrow$3) it suffices to show that the isomorphism $e^3 : I^3(F)/I^4(F) \to H^3(F)$ induces an isomorphism

$$\frac{I^3(F)}{[q_1]I(F) + \dots + [q_k]I(F) + I^4(F)} \to \frac{H^3(F)}{[C(q_1)]H^1(F) + \dots + [C(q_k)]H^1(F)}. \qquad \square$$

1.13. The case $\dim(q_1),\dots,\dim(q_k)\leqslant 6$ **and** $q_1\perp\dots\perp q_k\in I^3(F)$**.** Let $X=SB(C(q_1))\times\dots\times SB(C(q_k))$. Obviously $(q_1)_{F(X)},\dots,(q_k)_{F(X)}\in I^3(F(X))$ The assumption $\dim(q_i)\leqslant 6$ $(i=1,\dots,k)$ and APH imply that $[(q_1)_{F(X)}]=\dots=[(q_k)_{F(X)}]=0$. Thus $q_1,\dots,q_k\in W(F(X)/F)$. Hence $q_1\perp\dots\perp q_k\in W(F(X)/F)$. Since $q_1\perp\dots\perp q_k\in I^3(F)$, we have $q_1\perp\dots\perp q_k\in I^3(F(X)/F)$. Thus we get the elements $e^3(q_1\perp\dots\perp q_k)\in\Gamma(F;q_1,\dots,q_k)$ and $\bar{\varepsilon}_2\circ e^3(q_1\perp\dots\perp q_k)\in\operatorname{Tor}_2 CH^2(X_{q_1,\dots,q_k})$.

§2. Special triples

Definition 2.1. Let F be a field of characteristic $\neq 2$.

1) We say that a triple (q_1,q_2,π) of quadratic forms over F is *special* if the following conditions hold:
   a) q_1 and q_2 are Albert forms and π is a 2-fold Pfister form.
   b) $q_1\perp q_2\perp\pi\in I^3(F)$
2) We say that a triple (A_1,A_2,B) of F-algebras is *special* if the following conditions hold:
   a) A_1 and A_2 are biquaternion F-algebras and B is a quaternion algebra.
   b) $[A_1]+[A_1]+[B]=0\in\operatorname{Br}_2(F)$.
3) We say that a triple (q_1,q_2,π) is *anisotropic* if all the forms q_1, q_2, and π are anisotropic. We say that a special triple of forms (q_1,q_2,π) *corresponds* to a special triple of algebras (A_1,A_1,B) if $c(q_1)=[A_1]$, $c(q_2)=[A_2]$ and $c(\pi)=[B]$.

It is clear that for any special triple of forms (q_1,q_2,π) there exists a unique special triple of algebras (A_1,A_2,B) which corresponds to (q_1,q_2,π). Converserly, for any special triple of algebras (A_1,A_2,B) there exists a special triple of forms (q_1,q_2,π), which corresponds to the triple (A_1,A_2,B). In the latter case, the quadratic forms q_1, q_2, and π are uniquely defined up to similarity.

In view of 1.13 we have a well defined element $e^3(q_1\perp q_2\perp\pi)\in\Gamma(F;q_1,q_2,\pi)$.

Proposition 2.2. *Let (q_1,q_2,π) be a special triple. Then:*

1) $\Gamma(F;q_1,q_2,\pi)=\Gamma(F;q_1,q_2)=\Gamma(F;q_1,\pi)=\Gamma(F;q_2,\pi)$.
2) *The group $\Gamma(F;q_1,q_2,\pi)$ is either 0 or $\mathbb{Z}/2\mathbb{Z}$.*
3) *The element $e^3(q_1\perp q_2\perp\pi)$ generates the group $\Gamma(F;q_1,q_2,\pi)$.*
4) *The homomorphism*

$$\bar{\varepsilon}_2\colon\Gamma(F;q_1,q_2,\pi)\to\operatorname{Tor}_2 CH^2(SB(C(q_1))\times SB(C(q_2))\times SB(C(\pi)))$$

   is an isomorphism.

Before we adduce the proof, we want to note that the proof of the assertion 3) in Proposition 2.2 presented below is a slight modification of Laghribi's proof of the following result:

Proposition 2.3. (*see* [Lag]). *Let A be a biquaternion algebra and B be a quaternion algebra over F such that* $\operatorname{ind}(A\otimes B)=8$. *Let $X=SB(A)\times SB(B)$. Then*

$$\ker\left(H^3(F)\to H^3(F(X))\right)=[A]H^1(F)+[B]H^1(F).\quad\square$$

In our paper we need the following

Lemma 2.4. *Let A be a biquaternion algebra and B be a quaternion algebra over F such that* $\operatorname{ind}(A \otimes B) = 4$. *Then*

$$\ker\left(H^3(F) \to H^3(F(SB(A) \times SB(B)))\right) = [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F),$$

where the quadratic form ϕ is defined as follows: $\phi = q \perp q' \perp \pi$, where q and q' are Albert forms corresponding to the algebras A and $A \otimes_F B$, and π is a 2-fold Pfister form, corresponding to B.

In other words, the element $e^3(\phi)$ generates the group $\Gamma(F; A, B)$.

Proof. We actually have rewritten the first part of the proof from the paper of Laghribi cited above. Let $X = SB(A)$, $Y = SB(B)$, and $L = F(Y) = F(SB(B))$. Since $\operatorname{ind}(A), \operatorname{ind}(B) \leqslant 4$, Theorem 1.5 implies that

$$\ker\left(H^3(L) \to H^3(L(X))\right) = [A_L]H^1(L),$$
$$\ker\left(H^3(F) \to H^3(F(Y))\right) = [B]H^1(F).$$

Let $u \in \ker\left(H^3(F) \to H^3(F(X \times Y))\right)$. We need to prove that $u \in [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F)$.

We have $u_L \in \ker\left(H^3(L) \to H^3(L(X))\right) = [A_L]H^1(L)$. Hence there is $f \in L^*$ such that $u_L = [A_L] \cup (f) = e^3(q_L\langle\langle f\rangle\rangle)$, where q is an Albert form corresponding to A. Since the homomorphism e^3 is surjective, there exists $\phi \in I^3(F)$ such that $u^3(\phi) = u$. We have

$$e^3(\phi_L) = u_L = [A_L] \cup (f) = e^3(q_L\langle\langle f\rangle\rangle) = e^3(q_L \perp -f \cdot q_L).$$

Hence $\phi_L - q_L + f \cdot q_L \in \ker(I^3(L) \xrightarrow{e^3} H^3(L)) = I^4(L)$. Let $\tau = f \cdot q_{F(Y)}$. Since $L = F(Y)$, we have $\tau = f \cdot q_{F(Y)} \equiv (q \perp -\phi)_{F(Y)} \pmod{I^4(F(Y))}$. Hence for any 0-dimensional point $y \in Y$ we have $\partial_y^2(\tau) \equiv 0 \pmod{I^3(F(y))}$. Since $\dim \tau = 6 < 8$, it follows from APH that $\partial_y^2(\tau) = 0$. Since $\partial_y^2(\tau) = 0$ for each 0-dimensional point y on the projective conic Y, it follows from [CTS, Lemma 3.1] that the form τ is defined over the field F (see also [Ge]). This means that there exists a 6-dimensional form $\widetilde{q}$ over F such that $\widetilde{q}_L = \tau = f \cdot q_L$. Therefore $c(\widetilde{q})_L = c(q)_L = [A_L]$. Hence $c(\widetilde{q}) - [A] \in \operatorname{Br}_2(L/F)$. Since $L = F(SB(B))$, we have $\operatorname{Br}_2(L/F) = \{0, [B]\}$. Therefore $c(\widetilde{q}) \in \{[A], [A \otimes B]\}$.

Consider the case $c(\widetilde{q}) = [A]$. Since $[A] = c(q)$, we have $c(\widetilde{q}) = c(q)$. Thus $\widetilde{q} \sim q$. Let $k \in F^*$ be such that $\widetilde{q} = kq$. Then $f \cdot q_L = \widetilde{q}_L = kq_L$. We have

$$u_L = e^3(q_L \perp -f \cdot q_L) = e^3(q_L \perp -kq_L) = (e^3(q\langle\langle k\rangle\rangle))_L = ([A] \cup (k))_L.$$

Hence $u - [A] \cup (k) \in \ker\left(H^3(F) \to H^3(F(Y))\right) = [B]H^1(F)$. Therefore $u \in [A]H^1(F) + [B]H^1(F)$.

Suppose now that $c(\widetilde{q}) = [A \otimes_F B]$. By the assumption of the lemma, $c(q') = [A \otimes_F B]$. We have $c(\widetilde{q}) = c(q')$. Hence $\widetilde{q} \sim q'$. Choose $k \in F^*$ such that $\widetilde{q} = kq'$. Then $fq_L = \widetilde{q}_L = kq'_L$. Since $[\pi_L] = 0$, we have

$$\begin{aligned} u_L &= e^3(q_L \perp -fq_L) = e^3(q_L \perp -kq'_L) = e^3((q + q' + \pi) - q'\langle\langle k\rangle\rangle)_L \\ &= (e^3(\phi) - [c(q')] \cup (k))_L = (e^3(\phi) - [A] \cup (k) - [B] \cup (k))_L. \end{aligned}$$

Thus $u + [A] \cup (k) + [B] \cup (k) - e^3(\phi) \in \ker\left(H^3(F) \to H^3(F(Y))\right) = [B]H^1(F)$. Therefore $u \in [A]H^1(F) + [B]H^1(F) + e^3(\phi)H^0(F)$. □

Proof of Proposition 2.2. The assertion 1) was proved in 1.11. The assertion 3) follows immediately from Lemma 2.4 since $\Gamma(F; q_1, q_2, \pi) = \Gamma(F; q_1, \pi)$. Obviously 3) implies 2). The assertion 4) is proved in Appendix A (see Corollary A.11). □

Remark **2.5.** Both Proposition 2.3 and assertion 2) in Proposition 2.2 are obvious consequences of the results of E. Peyre and N. Karpenko (see Theorem 1.8 and Corollary 1.10).

Lemma 2.6. *Let (q_1, q_2, π) be a special anisotropic triple over F and let (A_1, A_2, B) be the corresponding triple of algebras. Let $E = F(SB(A_1))$. Then*

1) *$(q_2)_E$ is isotropic, and $\dim((q_2)_E)_{\text{an}} = 4$.*
2) *For any $s \in D_E(((q_2)_E)_{\text{an}})$ we have $((q_2)_E)_{\text{an}} = s \cdot \pi_E$.*
3) *If $((q_2)_E)_{\text{an}}$ is defined over F, then there exists $s \in F^*$ such that $((q_2)_E)_{\text{an}} = s \cdot \pi_E$.*

Proof. 1),2). Since $[A_1] + [A_2] = [B] \in \text{Br}_2(F)$ and $[(A_1)_E] = 0 \in \text{Br}_2(E)$, we have $[(A_2)_E] = [B_E]$. Therefore the $(A_2)_E$ is not a division algebra. Hence its Albert form $(q_2)_E$ is isotropic and $\dim((q_2)_E)_{\text{an}} \leqslant 4$.

We claim that $\dim((q_2)_E)_{\text{an}} = 4$ (and hence $((q_2)_E)_{\text{an}} \in GP_2(E)$). Otherwise we would have $[(q_2)_E] = 0$, and hence $[(A_2)_E] = 0$. Then $[A_2] \in \text{Br}_2(E/F) = \text{Br}_2(F(SB(A_1))/F) = \{0, [A_1]\}$. Therefore either $[A_2] = 0$, or $[B] = [A_1]+[A_2] = 0$, which is a contradiction.

Let $s \in D_E(((q_2)_E)_{\text{an}})$. Since $c(q_2)_E = [(A_2)_E] = [B_E] = c(\pi)_E = c(s\pi_E)$, it follows that $((q_2)_E)_{\text{an}} \equiv s\pi_E \pmod{I^3(E)}$. By APH' we have $((q_2)_E)_{\text{an}} = s \cdot \pi_E$.

3). If $((q_2)_E)_{\text{an}}$ is defined over F, we can choose s in $D_E(((q_2)_E)_{\text{an}}) \cap F^*$. □

Proposition 2.7. *Let (q_1, q_2, π) be a special anisotropic triple over F and let (A_1, A_2, B) be the corresponding triple of algebras. The following conditions are equivalent:*

1) *$((q_2)_{F(SB(A_1))})_{\text{an}}$ is defined over F,*
2) *$((q_1)_{F(SB(A_2))})_{\text{an}}$ is defined over F,*
3) *$q_1 \perp q_2 \perp \pi \in [q_1]I(F) + [q_2]I(F) + [\pi]I(F) + I^4(F)$.*
4) *There exist $k_1, k_2 \in F^*$ such that*

$$k_1 q_1 \perp k_2 q_2 \perp \pi \in I^4(F).$$

5) *The group $\Gamma(F; q_1, q_2, \pi)$ is trivial.*
6) *The group $\text{Tor}_2 CH^2(SB(A_1) \times SB(A_2) \times SB(B))$ is trivial.*

Proof. It suffices to prove that 1) ⇒ 3) ⇒ 4) ⇒ 1) and 3) ⟺ 5) ⟺ 6).

1) ⇒ 3). Let $E = SB(A_1)$. It follows from Lemma 2.6 that there exists $s \in F^*$ such that $[(q_2)_E] = [s\pi_E]$. Hence $(q_2 \perp -s\pi) \in W(E/F)$. Since $q_1 \in W(E/F)$, we have $(q_1 \perp q_2 \perp -s\pi) \in W(E/F)$. Therefore $(q_1 \perp q_2 \perp \pi) \in W(E/F) + [\pi]I(F)$. Since $\phi = q_1 \perp q_2 \perp \pi \in I^3(F)$, we have $\phi \in I^3(E/F) + [\pi]I(F)$. It follows from Corollary 1.6 that $I^3(E/F) \subset [q_1]I(F) + I^4(F)$. Hence

$$\phi \in [q_1]I(F) + [\pi]I(F) + I^4(F) \subset [q_1]I(F) + [q_2]I(F) + [\pi]I(F) + I^4(F).$$

3) $\Rightarrow$ 4). Since $\phi \in [q_1]I(F)+[q_2]I(F)+[\pi]I(F)+I^4(F)$, there exist $\mu_1,\mu_2,\mu_3 \in I(F)$ such that $[\phi]-[q_1\mu_1]-[q_2\mu_2]-[\pi\mu_3] \in I^4(F)$. Let $r_i = \det_\pm \mu_i$ $(i=1,2,3)$. Since $\mu_i \equiv \langle\langle r_i\rangle\rangle \pmod{I^2(F)}$, we have $[\phi]-[q_1\langle\langle r_1\rangle\rangle]-[q_2\langle\langle r_2\rangle\rangle]-[\pi\langle\langle r_3\rangle\rangle] \in I^4(F)$. Since $[\phi]=[q_1]+[q_2]+[\pi]$, we have $[r_1q_1]+[r_2q_2]+[r_3\pi] \in I^4(F)$. Setting $k_1 = r_1/r_3$ and $k_2 = r_2/r_3$, we have $[k_1q_1]+[k_2q_2]+[\pi] \in I^4(F)$.

4) $\Rightarrow$ 1). Let $E = SB(A_1)$. We have $(k_1q_1 \perp k_2q_2 \perp \pi)_E \in I^4(E)$ and $[(q_1)_E]=0$. Using APH, we have $[(k_1q_1)_E]+[\pi_E]=0$. Hence $((q_1)_E)_{\text{an}} = -k_1\pi_E$ is defined over F.

3) $\Longleftrightarrow$ 5). Obvious in view of Lemma 1.12 and Proposition 2.2.

5) $\Longleftrightarrow$ 6). See Proposition 2.2. □

§3. A criterion of universal excellence for the function fields of Severi–Brauer varieties.

In this section for any biquaternion division algebra A over F we construct a field extension E/F such that the field extension $E(SB(A))/E$ is not excellent. The construction is based on the following obvious consequence of Propositions 2.2 and 2.7:

Lemma 3.1. *Let (q_1,q_2,π) be an anisotropic special triple over E and (A_1,A_2,B) be the corresponding triple of E-algebras. The following conditions are equivalent:*

1) *For any $k_1,k_2 \in F^*$ we have $k_1q_1 \perp k_2q_2 \perp \pi \notin I^4(E)$,*
2) *The group $\Gamma(E;q_1,q_2,\pi)=\Gamma(E;A_1,A_2,B)$ is not trivial.*
3) *$\Gamma(E;q_1,q_2,\pi)=\Gamma(E;A_1,A_2,B)\cong \mathbb{Z}/2\mathbb{Z}$.*
4) *The group $\operatorname{Tor}_2 CH^2(SB(A_1)\times SB(A_2)\times SB(B))$ is not trivial.*

If these conditions hold then the field extension $E(SB(A_1))/E$ is not excellent. □

Proposition 3.2. *Let A be a biquaternion division algebra. Then there exists a unirational field extension E/F, a biquaternion algebra A' over E, and a quaternion algebra B over E such that $[A_E]+[A']+[B]=0\in \operatorname{Br}_2(E)$ and $\operatorname{Tor}_2 CH^2(SB(A_E)\times SB(A')\times SB(B))\cong \mathbb{Z}/2\mathbb{Z}$.* □

Proof. Let $K=F(u,v)$ be the field of rational functions in 2 variables. Let B_0 be the quaternion algebra (u,v) over K. Clearly, $\operatorname{ind}(A_K\otimes_K B_0)=8$. Let E be the function field $F(Y)$ of the generalized Severi–Brauer variety $Y=SB(A_K\otimes B_0,4)$. Let $B=(B_0)_E=(u,v)_E$. By Theorem 1.9, we have $\operatorname{Tor}_2 CH^2(SB(A_E)\times_E SB(B))\cong \mathbb{Z}/2\mathbb{Z}$.

It follows from the properties of the generalized Severi–Brauer varieties [Bla] that the algebra $A_E\otimes_E B$ has the form $M_2(A')$ where A' is a biquaternion E-algebra. Obviously $[A_E]+[A']+[B]=0\in \operatorname{Br}_2(E)$. Hence the triple (A_E,A',B) is special and $\operatorname{Tor}_2 CH^2(SB(A_E)\times SB(A')\times SB(B))\cong \operatorname{Tor}_2 CH^2(SB(A_E)\times SB(B))\cong \mathbb{Z}/2\mathbb{Z}$.

Now we need to verify that the field extension E/F is unirational. Let $\widetilde{K}=K(\sqrt{u})$. Since $[(B_0)_{\widetilde{K}}]=0$, we see that $\operatorname{ind}((A_K\otimes_K B_0)_{\widetilde{K}})=\operatorname{ind}(A_{\widetilde{K}})\leqslant 4$. Hence the variety $Y_{\widetilde{K}}=SB((A_K\otimes_K B_0)_{\widetilde{K}},4)$ is rational. Therefore the field extension $\widetilde{K}E/\widetilde{K}=\widetilde{K}(Y)/\widetilde{K}$ is purely transcendental. Obviously $\widetilde{K}/F$ is purely transcendental. Hence $\widetilde{K}E/F$ is purely transcendental too, and hence the field extension E/F is unirational. □

Theorem 3.3. *Let A be a biquaternion division algebra. Then there exists a unirational field extension E/F such that the field extension $E(SB(A))/E$ is not excellent.*

Proof. Take E/F, A' and B as in Proposition 3.2. Let $A_1 = A_L$ and $A_2 = A'$. Obviously the triple (A_1, A_2, B) is special over E, and $\operatorname{Tor}_2 CH^2(SB(A_1) \times SB(A_2) \times SB(B)) = \mathbb{Z}/2\mathbb{Z}$. It follows from Lemma 3.1 that the field extension $E(SB(A))/E$ is not excellent. □

Definition 3.4. We say that the field extensions E_1/F and E_2/F are q-equivalent (and write $E_1/F \stackrel{q}{\sim} E_2/F$) if the following conditions hold:

1) For any quadratic form ϕ over F, the form ϕ_{E_1} is isotropic if and only if ϕ_{E_2} is isotropic.
2) $W(E_1/F) = W(E_2/F)$.

We have the following examples of q-eqivalent field extensions.

Lemma 3.5. *Field extensions E_1/F and E_2/F are always q-equivalent in the following cases:*

(1) *$E_1 \subset E_2$ and E_2/E_1 is a finite odd extension.*
(2) *$E_1 \subset E_2$ and E_2/E_1 is a purely transcendental field extension.*
(3) *If E_1/F and E_2/F are stable isomorphic.*

Proof. (1) Obvious in view of Springer's theorem [Lam, Ch. VII, Th. 2.3]; (2) follows from [Lam, Ch. IX, Lemma 1.1]. (3) Since E_1/F and E_2/F are stable isomorphic, there is a field K such that K/E_1 and K/E_2 are purely transcendental. By (2), we have $E_1/F \stackrel{q}{\sim} K/F \stackrel{q}{\sim} E_2/F$. □

Lemma 3.6. (see [ELW, Lemma 2.6]) *Let E_1/F and E_2/F are field extensions such that $E_1/F \stackrel{q}{\sim} E_2/F$. Then E_1/F is excellent if and only if E_2/F is excellent.*

Lemma 3.7. *Let A_1 and A_2 be CS algebras such that* $\operatorname{ind}(A_1 \otimes_F A_2^{op})$ *is odd. Then*

1) *The field extensions $F(SB(A_1))/F$ and $F(SB(A_2))/F$ are q-equivalent.*
2) *The field extension $F(SB(A_1))/F$ is excellent if and only if $F(SB(A_2))/F$ is excellent.*

Proof. 1) Let $X_1 = F(SB(A_1))$ and $X_2 = F(SB(A_2))$. Since $\operatorname{ind}(A_1 \otimes_F A_2^{op})$ is odd, there is an odd field extension K/F such that $[(A_1 \otimes_F A_2^{op})_K] = 0$. Then $[(A_1)_K] = [(A_2)_K]$. By Lemma 1.2, the field extensions $K(X_1)/K$ and $K(X_2)/K$ are stably isomorphic. Therefore $K(X_1)/F$ and $K(X_2)/F$ are stably isomorphic too. By Lemma 3.5, we have $K(X_1)/F \stackrel{q}{\sim} K(X_2)/F$. Since $[K(X_1) : F(X_1)] = [K(X_2) : F(X_2)] = [K : F]$ is odd, it follows from Lemma 3.5 that $F(X_1)/F \stackrel{q}{\sim} K(X_1)/F \stackrel{q}{\sim} K(X_2)/F \stackrel{q}{\sim} F(X_2)/F$.

2) Obvious in view of Lemma 3.6. □

Corollary 3.8. *Let A and B be CS algebras over F such that $[A] = [B]$ in* $\operatorname{Br}(F)$*. Then the field extension $F(SB(A))/F$ is excellent if and only if $F(SB(B))/F$ is excellent.* □

Corollary 3.9. *Let A be a CS algebra over F and let $A\{2\}$ denote the 2-prime component of A. Then the following conditions are equivalent:*

1) *The field extension $F(SB(A))/F$ is excellent,*
2) *The field extension $F(SB(A\{2\}))/F$ is excellent.* □

Theorem 3.10. *Let A be a CS algebra over F. Let $X = SB(A)$. The following conditions are equivalent:*

1) *$F(X)/F$ is universally excellent,*
2) $\operatorname{ind}(A)$ *is not divisible by* 4.

In other words, the field extension $F(SB(A))/F$ is universally excellent only in the following two cases: 1) index of A is odd; 2) algebra A has the form $Q \otimes_F D$, where Q is a quaternion algebra and the index of D is odd.

Proof. 1) $\Rightarrow$ 2). Suppose that $\deg(A)$ has the form $\deg(A) = 4k$. Let $Y = SB(A, k) \times SB(A^{\otimes 2})$ and $K = F(Y)$. Obviously $\operatorname{ind}(A_K) \leqslant 4$ and $2[A_K] = 0$. By the Blanchet's index reduction formula (see [Bla] or [MPW]), we have $\operatorname{ind}(A_K) = 4$. Hence there is a biquaternion algebra $\widetilde{A}$ over K such that $[A_K] = [\widetilde{A}]$. It follows from Theorem 3.3, that there is a field extension E/K such that $E(SB(\widetilde{A}))/E$ is not excellent. By Corollary 3.8 the field extension $E(SB(A))/E$ is not excellent too.

2) $\Rightarrow$ 1). In view of Corollary 3.9, we can suppose that A as a division algebra and $\deg A = 2^n$. Since $\operatorname{ind}(A)$ is not divisible by 4, we see that A is a quaternion algebra or $A = F$. Hence $F(SB(A))/F$ is universally excellent. □

For algebras of index 8 we have the following

Theorem 3.11. *Let A be a CS algebra of index 8 and exponent 2. Then the field extension $F(SB(A))/F$ is not excellent.*

Since any algebra of index 8 and exponent 2 is Brauer equivalent to a 4-quaternion algebra, it suffices to prove the following lemma.[3]

Lemma 3.12. *Let $A = (a_1, b_1) \otimes_F (a_2, b_2) \otimes_F (a_3, b_3) \otimes_F (a_4, b_4)$ be a 4-quaternion algebra over F such that* $\operatorname{ind} A \geqslant 8$. *Then the field extension $F(SB(A))/F$ is not excellent.*

In the proof of this lemma we will use the following deep theorem.

Theorem 3.13. (see [EKLV, Corollary 9.3]) *Let ϕ be a quadratic form over F such that* $\operatorname{ind} C(\phi) \geqslant 8$. *Let $K = F(SB(C(\phi)))$. Then $\phi_K \notin I^4(K)$ (and hence $[\phi_{F(SB(C(\phi)))}] \neq 0$).*

Proof of Lemma 3.12. Let $E = F(SB(A))$ and $q \in I^2(F)$ be an arbitrary 10-dimensional quadratic form such that $c(q) = [A]$. Since $q_E \in I^3(E)$ and $\dim q_E = 10$, the form q_E is anisotropic (see [Pf]). Hence there is $\gamma \in GP_3(E)$ such that $[q_E] = [\gamma] \in W(E)$. Suppose at the moment that the field extension E/F is excellent. Then γ is defined over F. It follows from Lemma 3.14 bellow that there is $\alpha \in GP_3(F)$ such that $\gamma = \alpha_E$. We have $[q_E] = [\gamma] = [\alpha_E]$. Let $\phi = q \perp -\alpha$. Then

[3] We adduce here the proof suggested by D. Hoffmann which is essentially shorter than the original author's proof.

$[\phi_E] = 0$. Since $\alpha \in I^3(F)$, it follows that $c(\phi) = c(q) = [A]$. Therefore the field extension $F(SB(C(\phi)))/F$ is equivalent to E/F. Hence it follows from $[\phi_E] = 0$ that $[\phi_{F(SB(C(\phi)))}] = 0$, which provides a contradiction to Theorem 3.13. □

Lemma 3.14. *Let E/F be an excellent field extension and $\gamma \in GP_n(E)$ be a form defined over F. Then there is $\alpha \in GP_n(F)$ such that $\gamma = \alpha_E$.*

Proof. Since γ is defined over F, there is $c \in D_E(\gamma) \cap F^*$. Then the form $\phi = c\gamma$ is an n-fold E-Pfister form which is defined over F. By [ELW, Proposition 2.10] there is an n-fold F-Pfister form β such that $\phi = \beta_E$. Setting $\alpha = c\beta$, we have $\gamma = \alpha_E$, $\alpha \in GP_n(E)$. □

§4. Five-excellence of $F(SB(A))/F$

Let n be a positive integer. We say that a field extension L/F is n-excellent if for any quadratic form ϕ over F of dimension $\leqslant n$ the quadratic form $(\phi_L)_{\text{an}}$ is defined over F. In this section we prove the following

Theorem 4.1. *The field extension $F(SB(A))/F$ is 5-excellent for any CS algebra A over F.*

The following lemma is obvious.

Lemma–definition 4.2. *Let A be a CS algebra. Let us construct an algebra $A_{(2)}$ in the following way. We set $A_{(2)} = F$ if $\exp(A)$ is odd. If $\exp(A)$ is even we let $A_{(2)}$ be a division algebra such that $[A_{(2)}] = \frac{\exp(A)}{2}[A]$.*

The algebra $A_{(2)}$ is subject to the following properties:

1) $[A_{(2)}] \in \mathrm{Br}_2(F)$,
2) *For any $m \in \mathbb{Z}$ such that $m[A] \in \mathrm{Br}_2(F)$ we have $m[A] = [A_{(2)}]$ or $m[A] = 0$.*
3) *If $m \in \mathbb{Z}$ is a minimal positive integer such that $m[A] \in \mathrm{Br}_2(F)$ then $m[A] = [A_{(2)}]$.* □

Lemma 4.3. *Let q be an anisotropic Albert form and A be a CS algebra. Let $E = SB(A)$. Suppose that q_E is isotropic. Then there is $\pi \in P_2(F)$ such that $[A_{(2)}] = c(\pi) + c(q)$. Moreover if $c(q) = [A_{(2)}]$ then q_E is hyperbolic. If $c(q) \neq [A_{(2)}]$, then $\dim(q_E)_{\text{an}} = 4$, and for any $s \in D_E((q_E)_{\text{an}})$ we have $(q_E)_{\text{an}} = s\pi_E$.*

Proof. Since q_E is isotropic, we have $\operatorname{ind}(C(q_E)) \leqslant 2$. By the Schofield–Van den Bergh–Blanchet index reduction formula (see [Bla], [SV], or [MPW]) we have

$$\operatorname{ind}(C(q_E)) = \min\{\operatorname{ind}(C(q) \otimes A^{\otimes m}) \mid m \in \mathbb{Z}\}.$$

Hence there exists m such that $\operatorname{ind}(C(q) \otimes A^{\otimes m}) \leqslant 2$. Therefore there exists $\pi \in P_2(F)$ such that $c(q) + m[A] = c(\pi)$. Hence $m[A] = c(q) + c(\pi) \in \mathrm{Br}_2(F)$. By Lemma 4.2, we have $m[A] = [A_{(2)}]$ or $m[A] = 0$.

We claim that $m[A] = [A_{(2)}]$. Indeed, otherwise $m[A] = 0$, and hence $c(\pi) = c(q) + m[A] = c(q)$. However $\operatorname{ind}(C(\pi)) \leqslant 2$ and $\operatorname{ind}(C(q)) = 4$, a contradiction.

It follows from $m[A] = [A_{(2)}]$ that $[A_{(2)}] = c(q) + c(\pi)$. Since $[A_E] = 0$, we have $c(q_E) = c(\pi_E) + m[A_E] = c(\pi_E)$.

Case 1. $c(q) = [A_{(2)}]$: we have $c(\pi) = c(q) + [A_{(2)}] = 0$. Hence $c(q_E) = c(\pi_E) = 0$, i.e., q_E is hyperbolic.
Case 2. $c(q) \neq [A_{(2)}]$: It follows from Lemma 4.2 that $c(q) \neq m[A]$ for any $m \in \mathbb{Z}$. Therefore $c(q) \notin \{m[A] \mid m \in \mathbb{Z}\} = \mathrm{Br}(E/F)$, i.e., q_E is not hyperbolic. Thus $\dim(q_E)_{\mathrm{an}} = 4$. Since $c(q_E) = c(\pi_E)$, it follows that $(q_E)_{\mathrm{an}} \equiv \pi_E \pmod{I^3(F)}$. By APH' we have $(q_E)_{\mathrm{an}} \cong s\pi_E$ for any $s \in D_E((q_E)_{\mathrm{an}})$. □

Lemma 4.4. *Let ϕ be an anisotropic 5-dimensional quadratic form and A be a CS algebra over F. That $(\phi_{F(SB(A))})_{\mathrm{an}}$ is defined over F*

Proof. Let $E = F(SB(A))$. We can suppose that ϕ_E is isotropic. Let $s = -\det\phi$ and $q = \phi \perp \langle s\rangle$. If q is isotropic, then ϕ is a 5-dimensional Pfister neighbor. In this case ϕ is an excellent form (see [Kn2]). Then $(\phi_E)_{\mathrm{an}}$ is defined over F. So we can suppose that q is an anisotropic Albert form. Then the conditions of Lemma 4.3 hold. Let $\pi \in P_2(F)$ be as in Lemma 4.3.

If $c(q) = [A_{(2)}]$, then q_E is hyperbolic and hence $[\phi_E] = [q_E] - [\langle s\rangle] = [\langle -s\rangle]$. Then $(\phi_E)_{\mathrm{an}} = \langle -s\rangle$. Therefore $(\phi_E)_{\mathrm{an}}$ is defined over F.

If $c(q) \neq [A_{(2)}]$, then $\dim(q_E)_{\mathrm{an}} = 4$. Therefore $\dim(\phi_E)_{\mathrm{an}} \geqslant \dim(q_E)_{\mathrm{an}} - 1 = 3$. Since ϕ_E is isotropic we have $\dim(\phi_E)_{\mathrm{an}} = 3$. Therefore $(q_E)_{\mathrm{an}} = (\phi_E)_{\mathrm{an}} \perp \langle s\rangle$. Hence $s \in D_E((q_E)_{\mathrm{an}})$. By Lemma 4.3, we have $(q_E)_{\mathrm{an}} = s\pi_E$. Let π' be a pure subform of π. Since $(\phi_E)_{\mathrm{an}} \perp \langle s\rangle = (q_E)_{\mathrm{an}} = s\pi_E = s\pi'_E \perp . \langle s\rangle$, we get $(\phi_E)_{\mathrm{an}} = (s\pi')_E$. Hence $(\phi_E)_{\mathrm{an}}$ is defined over F. □

Proof of Theorem 4.1. Let $E = F(SB(A))$ and let τ be a quadratic form of dimension $\leqslant 5$ over F. We need to verify that τ_E is defined over F. In view of Lemma 4.4, we can assume that $\dim\tau \leqslant 4$. Since all forms of dimension < 4 are excellent, we can suppose that $\dim\tau = 4$.

Let $\phi = \tau_{F(t)} \perp \langle t\rangle$ and $\xi = (\tau_E)_{\mathrm{an}}$. We have $\xi_{E(t)} \perp \langle t\rangle = (\tau_{E(t)})_{\mathrm{an}} \perp \langle t\rangle \cong (\phi_{E(t)})_{\mathrm{an}} = (\phi_{F(t)(SB(A))})_{\mathrm{an}}$. By Lemma 4.4, $(\phi_{F(t)(SB(A))})_{\mathrm{an}}$ is defined over $F(t)$. Hence $\xi_{E(t)} \perp \langle t\rangle$ is defined over $F(t)$. It follows from Lemma 4.5 bellow that $\xi = (\tau_E)_{\mathrm{an}}$ is defined over F. □

Lemma 4.5. *Let E/F be a field extension and ξ be a quadratic form over E. Suppose that $\xi_{E(t)} \perp \langle t\rangle$ is defined over $F(t)$. Then ξ is defined over F.*

Proof. Let γ be a quadratic form over $F(t)$ such that $\xi_{E(t)} \perp \langle t\rangle \cong \gamma_{E(t)}$. We can write $\gamma_{F((t))}$ in the form $\gamma_{F((t))} \cong \lambda_{F((t))} \perp t\lambda'_{F((t))}$ where λ and λ' are quadratic forms over F. Obviously $\xi_{E((t))} \perp t\langle 1\rangle \cong \lambda_{E((t))} \perp t\lambda'_{E((t))}$. Since ξ and $\langle 1\rangle$ are anisotropic, we have $\xi = \lambda_E$, $\langle 1\rangle = \lambda'_E$. Hence ξ is defined over F. □

Theorem 4.6. *Let A be a CS algebra over F. If $u(F) \leqslant 6$, then the field extension $F(SB(A))/F$ is excellent.*

Proof. Let $E = F(SB(A))$. Let q be an anisotropic quadratic form over F. We need to prove that $(q_E)_{\mathrm{an}}$ is defined over F. By Theorem 4.1, we can assume that $\dim q > 5$. Since $u(F) \leqslant 6$, we conclude that q is an anisotropic Albert form. Therefore the conditions of Lemma 4.3 hold. Let $\gamma \in I^2(F)$ be an anisotropic form such that $c(\gamma) = [A_{(2)}]$. Then $c(\gamma_E) = 0$ and hence $\gamma_E \in I^3(E)$. Since $u(F) \leqslant 6$, we have $\dim\gamma \leqslant 6$. By APH, $[\gamma_E] = 0$.

It follows from Lemma 4.3 that $c(\pi) + c(q) = [A_{(2)}] = c(\gamma)$. Hence $[q] \equiv [\pi] + [\gamma] \pmod{I^3(F)}$. Since $u(F) \leqslant 6$, we have $I^3(F) = 0$. Hence $[q] = [\pi] + [\gamma]$. Therefore

$[q_E] = [\pi_E] + [\gamma_E] = [\pi_E]$. Hence $(q_E)_{\mathrm{an}} = (\pi_E)_{\mathrm{an}}$. Since π is a Pfister form, we see that $(q_E)_{\mathrm{an}} = (\pi_E)_{\mathrm{an}}$ is defined over F. □

Corollary 4.7. *Let A be a biquaternion division algebra over F. Then there is a field extension E/F such that A_E is a division algebra and the field extension $E(SB(A))/E$ is excellent.* □

Proof. By [Me2] there is a field extension E/F such that $u(E) = 6$ and A_E is a division algebra. □

Corollary 4.8. *There exist a field F and a biquaternion division algebra A over F such that the field extension $F(SB(A))/F$ is excellent.* □

§5. Examples of nonexcellent field extensions $F(SB(A))/F$

In this section we give some explicit examples of nonexcellent field extensions $F(SB(A))/F$. The main tool for constructing these examples is the following assertion.

Lemma 5.1. *Let $\mu_1, \mu_2, \mu_3, \mu_1', \mu_2', \mu_3'$ be anisotropic 2-dimensional quadratic forms over K. Let $\pi \in GP_2(K)$. Suppose that $\pi_{K(\mu_i)}$ is anisotropic for all $i = 1, 2, 3$. Let $\widehat{K} = K((x))((y))$ and $k, k' \in \widehat{K}^*$. Then*

$$k(\mu_1 \perp x\mu_2 \perp y\mu_3) \perp k'(\mu_1' \perp x\mu_2' \perp y\mu_3') \perp \pi_{\widehat{K}} \notin I^4(\widehat{K}).$$

Proof. In view of Srpinger's theorem we can identify $W(\widehat{K})$ with the direct sum $W(K) \oplus xW(K) \oplus yW(K) \oplus xyW(K)$. Moreover we can regard $W(K)$ as a subring of $W(\widehat{K})$.

Let $\phi = k(\mu_1 \perp x\mu_2 \perp y\mu_3) \perp k'(\mu_1' \perp x\mu_2' \perp y\mu_3')$. Suppose at the moment that $\phi \perp \pi_{\widehat{K}} \in I^4(\widehat{K})$. Then $\phi \perp \pi_{\widehat{K}} \in GP_4(\widehat{K})$. Since $(\phi \perp \pi_{\widehat{K}})_{\widehat{K}(\pi)}$ is isotropic, it is hyperbolic. Hence $\phi_{\widehat{K}(\pi)}$ is hyperbolic. Therefore $\phi \in [\pi_{\widehat{K}}]W(\widehat{K})$.

Since $W(\widehat{K}) = W(K) \oplus xW(K) \oplus yW(K) \oplus xyW(K)$, we can write $[\phi]$ in the form $[\phi] = [\tau_1] + x[\tau_2] + y[\tau_3] + xy[\tau_4]$ where τ_i $(i = 1, 2, 3, 4)$ are defined over K. Since all the forms μ_i, μ_i' $(i = 1, 2, 3)$ have dimension 2, we have $\dim \tau_i \leqslant 4$ $(i = 1, \dots, 4)$. Since

$$[\phi] \in [\pi_{\widehat{K}}]W(\widehat{K}) \cong [\pi]W(K) \oplus x[\pi]W(K) \oplus y[\pi]W(K) \oplus xy[\pi]W(K)$$

we have $\tau_1, \tau_2, \tau_3, \tau_4 \in [\pi]W(K)$.

Suppose that there exists j such that $[\tau_j] \neq 0$. Since $\dim \tau_j \leqslant 4$ and $\tau_j \in [\pi]W(K)$, we see that $\tau_j \sim \pi$. By the definition of ϕ, there exists i $(1 \leqslant i \leqslant 3)$ such that μ_i is similar to a subform in τ_j. Therefore μ_i is similar to a subform in π and hence the form $\pi_{K(\mu_i)}$ is isotropic, which yields a contradiction (see the assumptions of the lemma).

Therefore $[\tau_i] = 0$ for all $i = 1, 2, 3, 4$. Then $[\phi] = 0$. It follows from $\phi \perp \pi_{\widehat{K}} \in I^4(\widehat{K})$ that $[\pi_{\widehat{K}}] \in I^4(\widehat{K})$. Hence $[\pi] \in I^4(K)$. By APH the form π is isotropic, a contradiction. □

Corollary 5.2. *Let r, s, u, v be elements of a field K and let $\pi \in P_2(K)$ satisfy the properties:*

1) $c(\pi) = (r, u) + (s, v)$,
2) *π is anisotropic over the fields $K(\sqrt{u})$, $K(\sqrt{v})$, and $K(\sqrt{uv})$.*

Let $q_1 = \langle\langle uv\rangle\rangle \perp -x\langle\langle u\rangle\rangle \perp -y\langle\langle v\rangle\rangle$ and $q_2 = \langle\langle uv\rangle\rangle \perp -xr\langle\langle u\rangle\rangle \perp -ys\langle\langle v\rangle\rangle$ be quadratic forms over $\widetilde{K} = K(x, y)$. Then $(q_1, q_2, \pi_{\widetilde{K}})$ is a special triple over $\widetilde{K}$ and $\Gamma(\widetilde{K}; q_1, q_2, \pi) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Obviously q_1 and q_2 are Albert forms. Since $c(q_1 \perp q_2 \perp \pi) = c(-q_1 \perp q_2 \perp \pi) = c(x\langle\langle u, r\rangle\rangle \perp y\langle\langle s, v\rangle\rangle \perp \pi) = (u, r) + (s, v) + c(\pi) = 0$, the triple $(q_1, q_2, \pi_{\widetilde{K}})$ is special. The quadratic forms $\mu_1 = \langle\langle uv\rangle\rangle$, $\mu_2 = -\langle\langle u\rangle\rangle$, $\mu_3 = -\langle\langle v\rangle\rangle$, $\mu_1' = \langle\langle uv\rangle\rangle$, $\mu_2' = -s\langle\langle u\rangle\rangle$, $\mu_3' = -r\langle\langle v\rangle\rangle$, and π satisfy all the conditions of Lemma 5.1. Hence for any $k_1, k_2 \in \widehat{K} = K((x))((y))$ we have $k_1(q_1)_{\widehat{K}} \perp k_2(q_2)_{\widehat{K}} \perp \pi_{\widehat{K}} \notin I^4(\widehat{K})$. Therefore for any $k_1, k_2 \in \widetilde{K} = K(x, y)$ we have $k_1 q_1 \perp k_2 q_2 \perp \pi_{\widetilde{K}} \notin I^4(\widetilde{K})$. It follows from Lemma 3.1, that $\Gamma(\widetilde{K}; q_1, q_2, \pi_{\widetilde{K}}) = \mathbb{Z}/2\mathbb{Z}$. □

Remark **5.3.** Under the assumptions of Lemma 5.2, we have $c(q_1) = (x, y) + (xw_2, yw_1)$ and $c(q_2) = (rx, sy) + (rxw_2, syw_1)$.

Lemma 5.4. *Let $w_1, w_2 \in F^*$ be such that $w_1, w_2, w_2w_2 \notin F^{*2}$. Let $K = F(t)$ be the field of rational functions in one variable. Let*

$$r = -tw_1, \quad s = -tw_2, \quad u = t + w_1, \quad v = t + w_2, \quad \text{and} \quad \pi = \langle\langle t, w_1w_2\rangle\rangle.$$

Then $r, s, u, v \in K^*$ and $\pi \in P_2(K)$ satisfy all the conditions of Corollary 5.2.

Proof. 1) We have $(r, u) + (s, v) = (-tw_1, t + w_1) + (-tw_2, t + w_2) = (t, w_1) + (t, w_2) = (t, w_1w_2) = c(\pi)$.

2) Let $p(t)$ be equal to one of the polynomials $u = t + w_1$, $v = t + w_2$, or $uv = t^2 + (w_1 + w_2)t + w_1w_2$. We need to verify that π is anisotropic over the field $K(\sqrt{p(t)})$. Suppose that $\pi_{K(\sqrt{p(t)})}$ is isotropic. Then $p(t) \in D_F(-\pi')$ where $\pi' = \langle -t, -w_1w_2, tw_1w_2\rangle$ is the pure subform of π (see [Sch, Ch. 4, Th. 5.4(ii)]). Therefore $p(t) \in D_{F(t)}(\langle t, w_1w_2, -tw_1w_2\rangle)$. By Cassels–Pfister theorem[4] there are polynomials $p_1(t), p_2(t), p_3(t) \in F[t]$ such that

$$\begin{aligned} p(t) &= tp_1^2(t) + w_1w_2p_2^2(t) - tw_1w_2p_3^2(t) \\ &= t(p_1^2(t) - w_1w_2p_3^2(t)) + w_1w_2p_2^2(t). \end{aligned} \tag{5.5}$$

If $p(t) = t + w_1$, we have $w_1 = p(0) = w_1w_2p_2^2(0) \in w_1w_2F^{*2}$. Therefore $w_2 \in F^{*2}$, a contradiction. If $p(t) = t + w_2$, then $w_2 = p(0) = w_1w_2p_2^2(0) \in w_1w_2F^{*2}$. Then $w_2 \in F^{*2}$, a contradiction.

Let now $p(t) = t^2 + (w_1 + w_2)t + w_1w_2$. Since $w_1w_2 \notin F^{*2}$, it follows that $\deg(t(p_1^2(t) - w_1w_2p_3^2(t)))$ is odd and $\deg(p(t) - w_1w_2p_2^2(t))$ is even. We get a contradiction to the equation (5.5). □

[4]Note that the strong version of the Cassels–Pfister theorem assumes that all the coefficient of a quadratic form are polynomials of degree $\leqslant 1$. In the books of Lam [Lam] and Scharlau [Sch] a slightly relaxed version of the Cassels-Pfister theorem is adduced, in which all the coefficients of a quadratic form belong to F.

Corollary 5.6. *Let $w_1, w_2 \in F^*$ and assume that $w_1, w_2, w_2 w_2 \notin F^{*2}$. Let $E = F(t,x,y)$ be the field of rational functions in 3 variables. Consider the quadratic forms*

$$\begin{aligned} q_1 &= \langle\langle (t+w_1)(t+w_2)\rangle\rangle \perp -x\langle\langle t+w_1\rangle\rangle \perp -y\langle\langle t+w_2\rangle\rangle, \\ q_1 &= \langle\langle (t+w_1)(t+w_2)\rangle\rangle \perp xtw_1\langle\langle t+w_1\rangle\rangle \perp ytw_2\langle\langle t+w_2\rangle\rangle, \\ \pi &= \langle\langle t, w_1 w_2\rangle\rangle \end{aligned}$$

and algebras

$$\begin{aligned} A_1 &= (x,y) \otimes (x(t+w_2), y(t+w_1)), \\ A_2 &= (-xtw_1, -ytw_2) \otimes (-xtw_1(t+w_2), -ytw_2(t+w_1)), \\ B &= (t, w_1 w_2) \end{aligned}$$

over E. Then (q_1, q_2, π) is a special triple (and (A_1, A_2, B) is the corresponding special triple of algebras), and $\Gamma(E; A_1, A_2, B) = \Gamma(E; q_1, q_2, \pi) = \mathbb{Z}/2\mathbb{Z}$. □

Corollary 5.7. *Let F be a field such that $|F^*/F^{*2}| \geqslant 4$. Let $E = F(x,y,t)$ be the field of rational functions in 3 variables. Then there is a biquaternion algebra A over E such that the field extension $E(SB(A))/E$ is not excellent.*

Proof. Since $|F^*/F^{*2}| \geqslant 4$, it follows that there are $w_1, w_2 \in F^*$ such that w_1, w_2, $w_1 w_2 \notin F^{*2}$. Now it suffices to set $A = (x,y) \otimes (x(t+w_2), y(t+w_1))$. □

Lemma 5.8. *Suppose that a field F satisfies the following condition: there exists $w \in F^*$ such that $w, w+1, w(w+1) \notin F^{*2}$. Let $E = F(a,b,c)$ be the field of rational functions in 3 variables and define a biquaternion algebra A over E as $A = (a,b) \otimes (a+1, c)$. Then the field extension $E(SB(A))/E$ is not excellent.*

Proof. Let $E' = F(t,x,y)$ be the field of rational functions in 3 variables. Let $w_1 = w$, $w_2 = w+1$. Let $A' = (x,y) \otimes (x(t+w_1), y(t+w_2)) = (x,y) \otimes (x(t+w), y(t+w+1))$. All the conditions of Corollary 3.7 hold. Therefore the field extension $E'(SB(A'))/E'$ is not excellent. Let us identify the fields $E' = F(t,x,y)$ and $E = F(a,b,c)$ by menas of the birational isomorphism $t \mapsto (a-w)$, $x \mapsto ac$, $y \mapsto b$. We have

$$\begin{aligned} [A'] =& (x,y) + (x(t+w), y(t+w+1)) \mapsto \\ &\mapsto (ac, b) + (ac(a-w+w), b(a-w+w+1)) = \\ &= (ac, b) + (c, b(a+1)) = (a,b) + (a+1, c) = [A]. \end{aligned}$$

Since the algebra A' maps to A, it follows that $E(SB(A))/E$ is not universally excellent. □

Example 5.9. Let $E = \mathbb{Q}(a,b,c)$ be the field of rational function in 3 variables over $\mathbb{Q}$. Let $A = (a,b) \otimes (a+1, c)$. Then the field extension $E(SB(A))/E$ is not excellent.

Proof. It is sufficient to let $w = 2$ in Lemma 5.8. □

Proposition 5.10. *Let $E = F(a,b,c,d)$ be the field of rational functions in 4 variables. Then there is a special triple (A_1, A_2, B) over E such that $A_1 = (a,b) \otimes (c,d)$ and $\Gamma(E; A_1, A_2, B) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $F' = F(z)$ and $E' = F(x,y,t,z)$ be fields of rational function in 1 and 4 variables correspondingly. Let $w_1 = 1-z$ and $w_2 = 1+z$. Obviously $w_1, w_2, w_1w_2 \notin (F')^{*2}$. It follows from Corollary 5.6 that there is a special triple (A_1', A_2', B') over E' so that $A_1' = (x,y)\otimes(x(t+1+z), y(t+1-z))$ and $\Gamma(E'; A_1', A_2', B') \cong \mathbb{Z}/2\mathbb{Z}$. Now it is sufficient to identify the fields $E = F(a,b,c,d)$ and $E' = F(x,y,t,z)$ by means of F-birational isomorphism: $a \mapsto x$, $b \mapsto y$, $c \mapsto x(t+1+z)$, $d \mapsto y(t+1-z)$. □

Corollary 5.11. *Let $E = F(a,b,c,d)$ be the field of rational functions in 4 variables and $A = (a,b) \otimes (c,d)$ be a biquaternion algebra over E. The field extension $E(SB(A))/E$ is not excellent.* □

Corollary 5.12. *For any field F there exist a field extension E/F and a special triple of quadratic forms (q_1, q_2, π) over E such that $\Gamma(E; q_1, q_2, \pi) = \mathbb{Z}/2\mathbb{Z}$.* □

Example 5.13. 1) Let $E = \mathbb{R}(a,b,c,d)$ be the field of rational functions in 4 variables over $\mathbb{R}$. Let $D = (a,b)\otimes_E(c,d)$ be a biquaternion algebra over E. Then the anisotropic part of the quadratic form $\langle -a, b, -ab, c, d(a-1), -cd(a-1)\rangle_{E(SB(D))}$ is not defined over E. *Sketch of the proof*: let $K = F(u,v)$ and $r = -1$, $s = u-1$, $\pi = (u-1, uv)$. All the conditions of Corollary 5.2 hold. Let us identify the fields $F(u,v,x,y)$ and $F(a,b,c,d)$ by the rool $u \mapsto a$, $v \mapsto c$, $x \mapsto bc$, $y \mapsto d$. One can verify that $c(q_1) \mapsto (a,b)+(c,d)$ and $c(q_2) \mapsto c(\langle -a, b, -ab, c, d(a-1), -cd(a-1)\rangle)$.

2) Let K be an arbitrary finite generated field extension of the field $\mathbb{Q}$ and let $E = K(a,b,c)$ be the field of rational functions in 4 variables over K. Let $D = (a,b) \otimes_E (a+1, c)$ be a biquaternion algebra over E. Then the field extension $E(SB(D))/E$ is not excellent. (*Sketch of the proof*: By Lemma 5.8 it is sufficient to find $w \in K$ such that $w, w+1, w(w+1) \notin K^{*2}$.)

Appendix A. Surjectivity of $\bar{\varepsilon}_2$: $H^3(F(X)/F, \mu_2^{\otimes 2}) \to \operatorname{Tor}_2 CH^2(X)$ for certain homogeneous varieties

The main goal of this Appendix is to prove the following theorem.

Theorem A.1. *Let A and B be CS algebras of exponent 2 over a field F of characteristic $\neq 2$. Then the homomorphism $\bar{\varepsilon}_2$*

$$\frac{\ker\left(H^3(F) \to H^3(F(SB(A) \times SB(B)))\right)}{[A]H^1(F) + [B]H^1(F)} \to \operatorname{Tor}_2 CH^2(SB(A) \times SB(B))$$

is an isomorphism. Here $H^i(F)$ denotes $H^i(F, \mathbb{Z}/2\mathbb{Z})$.

In this section we will use the following notation and agreements.

- We identify the group $H^3(F, \mu_m^{\otimes 2})$ with the m-torsion subgroup of the group $H^3(F, \mathbb{Q}/Z(2))$.
- For any field extension E/F we set $H^i(E/F, \mathbb{Q}/\mathbb{Z}(j)) = \ker(H^i(F, \mathbb{Q}/\mathbb{Z}(j)) \to H^i(E, \mathbb{Q}/\mathbb{Z}(j)))$ and $H^i(E/F, \mu_m^{\otimes i}) = \ker(H^i(F, \mu_m^{\otimes i}) \to H^i(E, \mu_m^{\otimes i}))$.
- Recall that $H^i(F) = H^i(F, \mathbb{Z}/2\mathbb{Z})$. For any field extension E/F we let $H^i(E/F) = \ker(H^i(F) \to H^i(E))$.

The proof of the following lemma is standard and we omit it.

Lemma A.2. *Let X be a variety over F and let L/F be a finite field extension of degree m such that X_L is unirational. Then*

1) $H^i(F(X)/F, \mathbb{Q}/Z(j)) \subset H^i(L/F, \mathbb{Q}/\mathbb{Z}(j))$,
2) $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F, \mu_m^{\otimes 2})$. □

Theorem A.3. (see [Ar1]). *Let q be an Albert form over F. Then the homomorphism $H^3(F) \to H^3(F(q))$ is injective.* □

Corollary A.4. *Let q be an Albert form over F. Then the homomorphism*

$$H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(F(q), \mathbb{Q}/\mathbb{Z}(2))$$

is injective.

Proof. Let X_q be the projective quadric hyper-surface defined by the equation $q = 0$. Let L/F be a quadratic field extension such that q_L is isotropic. Then the variety X_q is rational. It follows from Lemma A.2 that $H^3(F(X_q)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X_q)/F, \mu_2^{\otimes 2}) = H^3(F(q)/F)$. By Theorem A.3, we have $H^3(F(q)/F) = 0$. Hence $H^3(F(q)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$. □

We recall that a field F is said to be linked [Elm], [EL] if the following equivalent conditions hold.

(a) The classes of quaternion algebras form a subgroup in the Brauer group $\mathrm{Br}(F)$.
(b) All the algebras of exponent 2 have index $\leqslant 2$.
(c) All the Albert forms over F are isotropic.

Lemma A.5. *For any field F there exists a field extension E/F with the following properties:*

1) *The homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective,*
2) *The field E is linked.*

Proof. Let us define the fields $F_0 = F, F_1, F_2, \dots$ recursively. We set F_i to be the free composite of all the fields of the form $F_{i-1}(q)$ where q runs over all Albert forms over F_{i-1}. Further we let $E = \cup_{i=1}^{\infty} F_i$. By Corollary A.4, the homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective. By the construction, all Albert forms over E are isotropic. Hence the field E is linked. □

Proposition A.6. (cf. [Pe, Lemma 5.3]). *Let A_1, A_2 be two F-algebras of index $\leqslant 2$ and let $X = SB(A_1) \times SB(A_2)$. Then*

$$H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = [A_1]H^1(F, \mathbb{Q}/\mathbb{Z}(1)) + [A_2]H^1(F, \mathbb{Q}/\mathbb{Z}(1)).$$

Proof. By [Kar2], the group $\mathrm{Tor}\, CH^2(X)$ is trivial. Now it is sufficient to apply Theorem 1.8. □

Corollary A.7. *Let A_1, A_2 be F-algebras of index $\leqslant 2$ and let $X = SB(A_1) \times SB(A_2)$. Then $2H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$.* □

Lemma A.8. *Let A_1 and A_2 be algebras of exponent 2 and let $X = SB(A_1) \times SB(A_k)$. Then $2H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = 0$.*

Proof. Let E/F be the field extension constructed in Lemma A.5. Since the homomorphism $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E, \mathbb{Q}/\mathbb{Z}(2))$ is injective, the homomorphism $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(E(X)/E, \mathbb{Q}/\mathbb{Z}(2))$ is injective too. Therefore it is sufficient to prove that $2H^3(E(X)/E, \mathbb{Q}/\mathbb{Z}(2)) = 0$. This assertion follows immediately from Corollary A.7 since any algebra over a linked field has index $\leqslant 2$. $\square$

Proof of Theorem A.1. By Theorem 1.8 it is sufficient to verify surjectivity of $\bar{\varepsilon}_2 \colon H^3(F(X)/F) \to \operatorname{Tor}_2 CH^2(X)$. By Lemma A.8, we have $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) \subset \operatorname{Tor}_2 H^3(F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F)$. Hence $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$. By Peyre's Theorem 1.8, the homomorphism $\varepsilon \colon H^3(F(X)/F, \mathbb{Q}/Z(2)) \to \operatorname{Tor} CH^2(F)$ is surjective. Since $H^3(F(X)/F, \mathbb{Q}/\mathbb{Z}(2)) = H^3(F(X)/F)$ it follows that the homomorphism $\varepsilon_2 \colon H^3(F(X)/F) \to \operatorname{Tor} CH^2(F)$ is surjective too. Hence $\bar{\varepsilon}_2$ is surjective. $\square$

Corollary A.9. *For any F-algebra A of exponent 2 the homomorphism $\bar{\varepsilon}_2$*

$$\frac{\ker\left(H^3(F) \to H^3(F(SB(A)))\right)}{[A]H^1(F)} \to \operatorname{Tor}_2 CH^2(SB(A))$$

is an isomorphism $\square$

Remark **A.10.** The analog of Corollary A.9 for algebras of prime exponent p is proved in [Izh2].

Corollary A.11. *Let A, B and C be algebras of exponent 2 over F such that $[A] + [B] + [C] = 0 \in \operatorname{Br}_2(F)$. Let $X = SB(A) \times SB(B) \times SB(C)$. Then the homomorphism $\bar{\varepsilon}_2$*

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A]H^1(F) + [B]H^1(F) + [C]H^1(F)} \to \operatorname{Tor}_2 CH^2(X)$$

is an isomorphism.

Proof. Let $Y = SB(A) \times SB(B)$. The vertical arrows in the commutative diagram

$$\begin{array}{ccc} H^3(F(Y)/F) & \xrightarrow{\varepsilon_{Y,2}} & \operatorname{Tor}_2 CH^2(Y) \\ \downarrow & & \downarrow \\ H^3(F(X)/F) & \xrightarrow{\varepsilon_{X,2}} & \operatorname{Tor}_2 CH^2(X) \end{array}$$

are isomorphisms (see §1), hence we are done. $\square$

Remark **A.12.** Let $A_1, \dots, A_k$ be F-algebras of exponent 2. Let $X = SB(A_1) \times \cdots \times SB(A_k)$. It is not true that the homomorphism

$$\frac{\ker\left(H^3(F) \to H^3(F(X))\right)}{[A_1]H^1(F) + \cdots + [A_k]H^1(F)} \xrightarrow{\bar{\varepsilon}_2} \operatorname{Tor}_2 CH^2(X). \tag{A.13}$$

is bijective for an arbitrary collection of algebras $A_1, \dots, A_k$ of exponent 2. The following counterexample was constructed by E. Peyre.

Example A.14. (see Remark 4.1 and Proposition 6.3 in [Pe]). Consider an arbitrary field F such that $H^3(F) \neq 0$ and $\mu_4 \in F^*$. Let $(a,b,c) \in H^3(F)$ be an arbitrary nontrivial symbol. Then the quaternion algebras $A_1 = (a,b)$, $A_2 = (b,c)$, $A_3 = (c,a)$ yield the required counterexample, i.e., the homomorphism $\bar{\varepsilon}_2$ is not surjective.

Sketch of the proof. Applying Theorem 1.8, one shows easily that the homomorphism (A.13) is not surjective if there exists an element $u \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ with the following properties: $u_{F(X)} = 0$, $2u \neq 0$, and $2u \in [A_1]H^1(F) + \cdots + [A_k]H^1(F)$ (one can verify that in this case $\varepsilon(u) \in \operatorname{Tor}_2 CH^2(X)$ but $\varepsilon(u) \notin \operatorname{im} \varepsilon_2$). To complete the proof it is sufficient to define $u \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ as the image of the element $\{a,b,c\}$ by means of the following homomorphism

$$K_3^M(F)/4K_3^M(F) \xrightarrow{h_{3,4,F}} H^3(F, \mu_4^{\otimes 3}) \cong H^3(F, \mu_4^{\otimes 2}) \hookrightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2)).$$

Here $h_{3,4,F}$ is the *norm residue homomorphism*. □

Appendix B. A criterion of universal excellence for generic spitting fields of quadratic forms.

Definition B.1. Let E/F be a finitely generated field extension. We say that E/F is *universally excellent* if for any field extension K/F and for any free composite EK of E and K over F, the field extension EK/K is excellent.

Remarks. 1) By a free composite of K and E over F we mean the field of fractions of the factor ring $(K \otimes_F E)/\mathcal{P}$, where $\mathcal{P}$ is a minimal prime ideal in $K \otimes_F E$. 2) In the case where X is a geometrically integral variety over F and $E = F(X)$, a free composite EK is uniquely defined and coincides with $K(X)$.

Let ϕ be a nonhyperbolic quadratic form over F. Put $F_0 = F$ and $\phi_0 = \phi_{\mathrm{an}}$. For $i \geqslant 1$ let $F_i = F_{i-1}(\phi_{i-1})$ and $\phi_i = ((\phi_{i-1})_{F_i})_{\mathrm{an}}$. The smallest h such that $\dim \phi_h \leqslant 1$ is called the *height* of ϕ. The *degree* of ϕ is defined to be zero if $\dim \phi$ is odd. If $\dim \phi$ is even then there is m such that $\phi_{h-1} \in GP_m(F_{h-1})$. In this case we set $\deg \phi = m$.

The maint goal of this Appendix is to prove the following

Theorem B.2. *Let ϕ be an anisotropic quadratic form over F and $F_0, F_1, \ldots, F_h$ be a generic splitting tower of ϕ. Let s be a positive integer such that $s \leqslant h$. Then*

1) *If the field extension F_s/F is universally excellent then $s = h$.*
2) *The field extension F_h/F is universally excellent if and only if one of the following conditions holds:*
   (a) *ϕ has the form $\langle\langle a,b\rangle\rangle\gamma$, where γ is an odd-dimensional quadratic form,*
   (b) *$\phi \perp \langle -\det_{\pm} \phi\rangle$ has the form $\langle\langle a,b\rangle\rangle\gamma$, where γ is an odd-dimensional quadratic form,*
   (c) *ϕ has the form $\langle\langle a\rangle\rangle\gamma$ where γ is an odd-dimensional quadratic form,*
   (d) *there exist $d \notin F^{*2}$, $\pi \in P_2(F)$ and two odd-dimensional quadratic forms γ_1 and γ_2 such that the following conditions hold: $\pi_{F(\sqrt{d})}$ is anisotropic, the field extension $F(\pi, \sqrt{d})/F$ is universally excellent, and $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle\gamma_2]$. In this case $\dim \phi$ is even and $\det_{\pm} \phi = d \notin F^{*2}$.*

Remark B.3. We do not know whether there exist d and π (and hence the quadratic form ϕ) as in item (d) of Theorem B.2.

Definition B.4. Let q be a quadratic form and $k \geqslant 0$. We say that a field extension E/F is *universal in the class of the field extensions over which the Witt index of* ϕ *is greater or equal to* k (for short (ϕ, k)-universal) if the following conditions hold:

1) $i_W(\phi_E) \geqslant k$,
2) For any field extension K/F with $i_W(\phi_K)_{\mathrm{an}} \geqslant k$ and for any free composite EK of the fields E and K over F, the field extension KE/K is purely transcendental.

Lemma B.5. *Let q be a quadratic form and k be a positive integer. Let E_1/F and E_2/F be (ϕ, k)-universal field extensions. Then $E_1/F \overset{\mathrm{st}}{\sim} E_2/F$.*

Proof. By Definition B.4, E_1E_2/E_1 and E_1E_2/E_2 are purely transcendental. Hence $E_1/F \overset{\mathrm{st}}{\sim} E_2/F$. $\square$

Proposition B.6. (see [Kn1, Cor. 3,9 and Prop. 5.13]). *Let ϕ be a quadratic form over F. Let $F_0, F_1, \dots, F_h$ be a generic splitting tower of ϕ. Let $k_s = i_W(\phi_{F_s})$ $(0 \leqslant s \leqslant h)$. Then the field extension F_s/F is a (ϕ, k_s)-universal.*

Theorem B.7. (see [Izhl, Th. 1.1]). *Let ϕ be an anisotropic form over F. The field extension $F(\phi)/F$ is universally excellent if and only if* $\dim \phi \leqslant 3$ *or* $\phi \in GP_2(F)$.

Lemma B.8. *Let ϕ be a non hyperbolic quadratic form over F and $F_0, F_1, \dots, F_h$ be a generic splitting tower of ϕ. Let r be an integer such that $0 < r \leqslant h = h(\phi)$. Suppose that the field extension F_r/F is universally excellent. Then*

1) *For any s with $0 \leqslant s \leqslant r$, the field extension F_r/F_s is universally excellent.*
2) *$r = h$ and* $\deg \phi \leqslant 2$.

Proof. 1) Let F'_s and F'_r be "second copies" of the fields F_s and F_r. Let $k = i_W(\phi_{F_r})$. By Proposition B.6, both field extensions F'_rF_s/F_s and F_r/F_s are (ϕ_{F_s}, k)-universal. By Lemma B.5, we have $F'_rF_s/F_s \overset{\mathrm{st}}{\sim} F_r/F_s$.

Since F_r/F is universally excellent and $F'_r/F \cong F_r/F$, it follows that F'_r/F is universally excellent too. Hence F'_rF_s/F_s is universally excellent. Since $F'_rF_s/F_s \overset{\mathrm{st}}{\sim} F_r/F_s$ it follows that F_r/F_s is universally excellent.

2) Since F_r/F is universally excellent, it follows that F_r/F_{r-1} is universally excellent. Let $\phi_{r-1} = (\phi_{F_{r-1}})_{\mathrm{an}}$. We see that $F_{r-1}(\phi_{r-1})/F_{r-1}$ is universally excellent. It follows from Theorem B.7, that either $\dim \phi_{r-1} \leqslant 3$ or $\phi_{r-1} \in GP_2(F_{r-1})$. In both cases $\dim \phi_r \leqslant 1$, i.e., $r = h(\phi)$. Since $\dim \phi_{h-1} = \dim \phi_{r-1} \leqslant 4$, it follows that $\deg \phi \leqslant 2$. $\square$

Notation B.9. Let ϕ be a quadratic form over F and $F_0, F_1, \dots, F_h$ be a generic splitting tower of ϕ. We denote by F_ϕ the field $F_h = F_{h(\phi)}$. For any field extension E/F, we let $E_\phi \overset{\mathrm{def}}{=} E_{\phi_E}$.

Lemma B.10. *Let ϕ be a quadratic form over F and E/F be a field extension. Then $EF_\phi/E \overset{st}{\sim} E_\phi/E$.*

Proof. Let $k = [\dim\phi/2]$. The field extensions EF_ϕ/E and E_ϕ/E are (ϕ_E, k)-universal. By Lemma B.5, the proof is complete. □

Corollary B.11. *Let ϕ be a quadratic form over F and E/F be a field extension. Suppose that the field extension F_ϕ/F is universally excellent. Then E_ϕ/E is universally excellent.* □

Corollary B.12. *Let $\phi \in I^3(F)$ a quadratic form such that the field extension F_h/F is universally excellent. Then ϕ is hyperbolic.*

Proof. Suppose that ϕ is not hyperbolic. Since $\phi \in I^3(F)$, we have $\deg(\phi) \geqslant 3$. This contradicts to Lemma B.8. □

Corollary B.13. *Let ϕ be a quadratic form over F and E/F be a field extension such that F_ϕ/F is universally excellent. Then for any field extension E/F the condition $\phi_E \in I^3(E)$ implies that ϕ_E is hyperbolic.* □

Lemma B.14. *Let ϕ and ψ be quadratic forms over F. The following conditions are equivalent:* 1) $F_\phi \overset{st}{\sim} F_\psi$; 2) $\dim(\phi_{F_\psi}) \leqslant 1$ *and* $\dim(\psi_{F_\phi}) \leqslant 1$.

Proof. 1)⇒2). Obvious; 2)⇒1). It follows from Proposition B.6 and Definition B.4 that the field extensions $F_\phi F_\psi/F_\psi$ and $F_\phi F_\psi/F_\phi$ are purely transcendental. Hence $F_\phi \overset{\mathrm{st}}{\sim} F_\psi$. □

Examples B.15. 1) *Let ϕ be an odd-dimensional quadratic form. Let $\psi = \phi \perp \langle -\det_\pm \phi\rangle$. Then $F_\phi/F \overset{st}{\sim} F_\psi/F$.*

2) *Let π_i be anisotropic m_i-fold Pfister forms ($m_1 < m_2 < \cdots < m_n$). Let $\gamma_1, \ldots, \gamma_n$ be anisotropic odd-dimensional quadratic forms. Let ϕ be quadratic form such that $[\phi] = [\pi_1\gamma_1] + \cdots + [\pi_n\gamma_n]$. Then $F_\phi/F \overset{st}{\sim} F(\pi_1, \ldots, \pi_n)/F$.*

3) *Let $\pi \in GP_n(F)$ and let γ be an odd-dimensional quadratic form. Let $\phi = \tau\gamma$. Then $F_\phi/F \overset{st}{\sim} F_\pi/F$.*

Proof. 1) Since $\psi \in I(F)$, it follows that ψ_{F_ψ} is hyperbolic. Hence $\dim(\phi_{F_\psi})_{\mathrm{an}} = 1$. Since $\dim(\psi_{F_\psi})_{\mathrm{an}} = 1$, we have $\dim(\phi_{F_\psi})_{\mathrm{an}} \leqslant 2$. It follows from $\psi \in I^2(F)$ that $\dim(\phi_{F_\psi})_{\mathrm{an}} = 0$. By Lemma B.14, we have $F_\phi/F \overset{\mathrm{st}}{\sim} F_\psi/F$.

2). Obviously $\phi_{F(\pi_1,\ldots,\pi_n)}$ is hyperbolic. Let $E = F_\phi$. It is sufficient to verify that $(\pi_1)_E, \ldots, (\pi_n)_E$ are hyperbolic. Suppose that there is i such that $[(\pi_i)_E] \neq 0$. Let i be the minimal integer such that $[(\pi_i)_E] \neq 0$. Obviously, $[(\pi_i\gamma_i)_E] \equiv [\phi_E] \equiv 0 \pmod{I^{m_i+1}(F)}$. Since $\dim\gamma$ is odd, we have $[(\pi_i)_E] \equiv [(\pi_i\gamma_i)_E] \equiv 0 \pmod{I^{m_i+1}(F)}$. By APH, we have $[(\pi_i)_E] = 0$, a contradiction.

3) It is sufficient to set $n = 1$ in previous example 2). □

The following lemma is a consequence of the index reduction formula [Me1].

Lemma B.16. (see [HR, Th. 1.6] or [Ho1, Prop 2.1].) Let $\phi \in I^2(F)$ be a quadratic form with $\mathrm{ind}(C(\phi)) \geqslant 2^r$. Then there is s $(0 \leqslant s \leqslant h(\phi))$ such that $\dim\phi_s = 2r + 2$ and $\mathrm{ind}\, C(\phi_s) = 2^r$. □

Lemma B.17. *Let $\phi \in I^2(F)$ be a nonhyperbolic quadratic form such that the field F_ϕ is universally excellent. Then* $\operatorname{ind} C(\phi) = 2$.

Proof. By Corollary B.12, we have $\phi \notin I^3(F)$. Hence $\operatorname{ind} C(\phi) \geqslant 2$. Suppose that $\operatorname{ind} \phi \geqslant 4$. By Lemma B.16, there is s such that $\dim \phi_s = 6$. Therefore ϕ_s is an anisotropic Albert form. By Lemma B.8, the field extension F_s/F_h is universally excellent. Replacing F and ϕ by F_s and ϕ_s, we can suppose that ϕ is an anisotropic Albert form. Let $A = C(\phi)$. Clearly $F_\phi/F \overset{\text{st}}{\sim} F(SB(A))/F$. By Theorem 3.3, the field extension $F(SB(A))/F$ is not universally excellent, a contradiction. □

Proposition B.18. *Let $\phi \in I^2(F)$ be an anisotropic quadratic form. Then the following conditions are equivalent:*

1) *The field extension F_ϕ/F is universally excellent,*
2) *ϕ has the form $\langle\langle a, b\rangle\rangle\mu$, where μ is an odd-dimensional form.*

Proof. 1)⇒2). Suppose that the field extension F_ϕ/F is universally excellent. By Lemma B.17, we have $\operatorname{ind} C(\phi) = 2$. Therefore there exists an anisotropic 2-fold Pfister form $\pi = \langle\langle a, b\rangle\rangle$ such that $[c(\phi)] = [c(\pi)]$. Let $E = F(\pi)$. Obviously $\phi_E \in I^3(E)$. By Corollary B.13, ϕ_E is hyperbolic. Hence there is γ such that $\phi = \langle\langle a, b\rangle\rangle\gamma$. Since $\phi \notin I^3(F)$, $\dim \gamma$ is odd.

2)⇒1). Suppose that $\phi \cong \langle\langle a, b\rangle\rangle\gamma$, where γ is an odd-dimensional quadratic form. Let $\pi = \langle\langle a, b\rangle\rangle$. By Example B.15, we have $F_\phi/F \overset{\text{st}}{\sim} F_\pi/F$. By Arason's theorem, the field extension F_π/F is universally excellent. Hence F_ϕ/F is universally excellent. □

Proposition B.19. *Let ϕ be an odd-dimensional anisotropic quadratic form. Then the following conditions are equivalent:*

1) *The field extension F_ϕ/F is universally excellent,*
2) *$\phi \perp \langle -\det_\pm \phi\rangle$ has the form $\langle\langle a, b\rangle\rangle\mu$, where μ is an odd-dimensional form.*

Proof. Obvious by virtue of Proposition B.18 and Example B.15. □

Proposition B.20. *Let ϕ be an even-dimensional anisotropic quadratic form with $d = \det_\pm(\phi) \neq 1 \in F^*/F^{*2}$. Then the following conditions are equivalent:*

1) *The field extension F_ϕ/F is universally excellent.*
2) *There exist $\pi \in GP_2(F)$ and odd-dimensional quadratic forms γ_1, γ_2 such that $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle\gamma_2]$ and the field extension $F(\pi, \sqrt{d})/F$ is universally excellent.*

Proof. 1)⇒2). Let $L = F(\sqrt{d})$. Since F_ϕ/F is universally excellent, it follows that L_ϕ/L is universally excellent. If ϕ_L is hyperbolic, we set $\pi = 2\mathbb{H}$, which completes the proof. Suppose now that ϕ_L is not hyperbolic. By Lemma B.17, $\operatorname{ind}(C(\phi_L)) = 2$. Since $C(\phi_L)$ is defined over F, it follows that there is $\pi \in GP_2(F)$ such that $C(\phi_L) = C(\pi_L)$. Let $E = L(\pi) = F(\pi, \sqrt{d})$. Since F_ϕ/F is universally excellent, it follows that E_ϕ/E is universally excellent. We have $C(\phi_E) = C(\pi_E) = 0$. Hence $\phi_E \in I^3(E)$. It follows from Corollary B.13 that ϕ_E is hyperbolic. Therefore $[\phi] \in W(E/F) = [\pi]W(F) + [\langle\langle d\rangle\rangle]W(F)$. Choose γ_1 and γ_2 such that $[\phi] = [\pi\gamma_1] + [\langle\langle d\rangle\rangle\gamma_2]$. Since $\phi \notin I^2(F)$, the dimension of γ_2 is odd. Since $\deg C(\phi_L) = 2$, the dimension of γ_1 is odd. By Example B.15, we have $F_\phi/F \overset{\text{st}}{\sim} E/F$. Therefore the field extension $E/F = F(\pi, \sqrt{d})/F$ is universally excellent.

2)$\Rightarrow$1). Obvious in view of Example B.15. □

Theorem B.2 is now an obvious consequence of Lemma B.8 and Propositions B.18, B.19, and B.20. □

Let ϕ be a non-degenerate quadratic form on an F-vector space V and k be a positive integer such that $k \leqslant \frac{1}{2}\dim V = \frac{1}{2}\dim\phi$. Let $X(\phi, k)$ be the variety of totally isotropic subspaces of dimension k. It is well known that $X(\phi, k)$ is geometrically integral if and only if $k = \frac{1}{2}\dim\phi$.

Suppose now that $k < \frac{1}{2}\dim\phi$. Clearly, the field extension $F(X(\phi, k))/F$ is a (ϕ, k)-universal. Therefore there exists r $(0 \leqslant r \leqslant h = h(\phi))$ such that the field extension $F(X(\phi, k))/F$ is stable isomorphic to F_r/F. Obviously $r = 0$ if and only if $k \leqslant i_W(\phi)$. In the case where $k > i_W(\phi)$, the integer r is defined by the condition $\dim(\phi_{r-1})_{\text{an}} - 2 \geqslant \dim\phi - 2k \geqslant \dim(\phi_r)_{\text{an}}$.

Theorem B.21. *Let q be a quadratic form over F and $X(\phi, k)$ be the variety of totally isotropic subspaces of dimension k $(k < \frac{1}{2}\dim\phi)$. The field extension $F(X(\phi, k))/F$ is universally excellent if and only if one of the following conditions holds:*

1) $k \leqslant i_W(\phi)$
2) ϕ_{an} *has the form* $\langle\langle a, b\rangle\rangle\gamma$, *where* γ *is an odd-dimensional quadratic form and* $k = \frac{1}{2}\dim\phi - 1$,
3) $\phi_{\text{an}} \perp \langle -\det_{\pm}\phi\rangle$ *has the form* $\langle\langle a, b\rangle\rangle\gamma$, *where* γ *is an odd-dimensional quadratic form, and* $k = \frac{1}{2}\dim(\phi - 1)$,

Proof. Let r be such that $F(X(\phi, k)) \overset{\text{st}}{\sim} F_r/F$. If $r = 0$ then $k \leqslant i_W(\phi)$ and the proof is complete. Suppose now that $r > 0$. By Lemma B.8, we have $r = h = h(\phi)$ and $\deg(\phi) \leqslant 2$. Therefore $\dim\phi - 2k \leqslant \dim(\phi_{h-1}) - 2 \leqslant 2^{\deg\phi} - 2 \leqslant 2$. By the assumption of the theorem, we have $\dim\phi - 2k > 0$. Therefore $k = \frac{1}{2}\dim\phi - 1$ or $k = \frac{1}{2}(\dim\phi - 1)$. Since $\dim\phi_{h-1} \geqslant 2 + (\dim\phi - 2k) \geqslant 3$, it follows that either $\phi \in I^2(F)$, or $\dim\phi$ is odd. To complete the proof it is sufficient to apply Theorem B.2. □

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