The Generalized Double–point formula of curves

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§ 1. Introduction

For a morphism $\varphi : X \longrightarrow Y$ of smooth varieties, if X is birationally isomorphic to $\varphi(X)$ via φ, there is a so–called double–point locus of φ consisting of those points of X such that for each of them there exists an another point having the same image as its. Kleiman and others [4] [5] [6] gave a formula for expressing the locus by Chern classes of X and Y under an assumption of genericness.

Now we assume that X is a curve and φ is determined by a linear system $|D|$ which may have some base points and in the same time may be φ not birational. Naturally in this case we can factor φ as the composition of a birational morphism and a finite morphism, but as a double point how to distinguish the one which is caused by the "finite covering" from the one caused by birational morphism?

In fact we can associate every point x with a unique codimension 1 subspace $U(x)$ of $H^0(X,L)$ even $|L|$ has a base point, it turns out to be the image $\varphi(x) \in \mathbb{P}(H^0(X,L)^\vee)$. We lift this family of subspaces to $X \times X$; on each fiber the base point y of $U(x)$ simply means $U(y) = U(x)$. Therefore, on $X \times X$ the locus $\{(x,y) \mid U(x) = U(y)\}$ will be splitted into two parts: the discrete part and the 1–dimensional part. The later one is taken as the locus of "finite covering" and the former one is the locus of double points. Using this machinery we could discuss the locus of Weierstrass points of this family. We have not gone further on this direction.

Unfortunately we could not give an explicit expression of the double–point formula for arbitrary linear system and arbitrary curve because of the technical sake. Nevertheless, for φ being birational we recover the classical Plücker formula.

§ 2. Double point

Let C be a smooth curve over an algebraically closed field k of arbitrary characteristic, D an effective divisor on C with $L = \mathcal{O}(D)$ and $V \subset H^0(C,L)$ a linear system with $\dim_k V = r+1$. Let B be the locus of base–point of V, then V determines a morphism $\varphi : C \longrightarrow \mathbb{P}^r$. We assume that φ is not a constant morphism and hence $\mu = [k(C) : k(\varphi(C))] < \infty$, where $k(C)$ and $k(\varphi(C))$ are the rational fields of C and $\varphi(C)$ respectively.

Let p, q be the first and the second projection from $C \times C$ to C respectively. We know from [7] there is a canonical homomorphism

$$a_0 : V_C = V \otimes_k \mathcal{O}_C \hookrightarrow p_* q^* L = H^0(C,L)_C \longrightarrow L \ .$$

Denoting the kernel and the image of a_0 by E and B_0 respectively we have the following exact sequence

$$0 \longrightarrow E \longrightarrow V_C \xrightarrow{a_0} B_0 \longrightarrow 0 \qquad (1)$$

Suppose $x \in C$ and $0 \leq \mu_0 < \mu_1 < \dots < \mu_r \leq d = \deg D$ is the Schubert sequence of V at x ; that means there exists a basis of V , denoted by $f_0,\dots,f_r$, such that $f_0 = t^{\mu_0} + \dots, \dots, f_r = t^{\mu_r} + \dots$, if we restrict V at x and identify L_x with $\mathcal{O}_{x,C}$ and where t is a uniform parameter of $\mathcal{O}_x$.

We write $f_i = t^{\mu_i} h_i$, then $h_i \in \mathcal{O}_x$ is a unit. By (1) we see that E_x is generated by

$$
\begin{aligned}
e_1 &= f_1 - t^{\mu_1-\mu_0} h_1(t) h_0(t)^{-1} f_0 \ , \\
&\dots\dots \\
e_r &= f_r - t^{\mu_r-\mu_0} h_r(t) h_0(t)^{-1} f_0 \ .
\end{aligned}
$$

Now let us consider the diagram

$$
\begin{tikzcd}
 & (C\times C) \times_C (C\times C) \arrow[dl, "P"'] \arrow[dr, "Q"] & \\
C\times C \arrow[dr, "p"'] & & C\times C \arrow[dl, "p"] \\
 & C &
\end{tikzcd}
$$

We denote $C\times C$ by Y and take it as a variety over C by p .

We have [2] the canonical morphism

$$
b_m : P_* Q^*(q^* L) \longrightarrow \mathscr{P}^m_{Y/C}(q^* L) \ ,
$$

where $\mathscr{P}^m_{Y/C}(q^* L) = P_*(\mathcal{O}_{Y\times_C Y}/I^{m+1}_{\Delta_{Y/C}} \otimes Q^*(q^* L))$ is the sheaf of the relative m–principal of $q^* L$ and $I_{\Delta_{Y/C}}$ is the ideal for defining diagonal of $Y\times_C Y$.

By the standard argument of Base Change Theorem (see e.g. [3]) we have

$$P_*Q^*(q^*L) \simeq p^*p_*L \ .$$

Therefore by (1) we have an injective homomorphism

$$p^*E \longrightarrow p^*p_*q^*L \ ,$$

and then by compositing we have

$$A_m : p^*E \longrightarrow \mathscr{P}^m_{Y/C}(q^*L) \ ,$$

in particular, $A_0 : p^*E \longrightarrow q^*L$.

Locally at a point $(x,y) \in C\times C$, $e_1,\dots,e_r$ span $(p^*E)_{(x,y)} = E_x$ over $\mathscr{O}_{(x,y)}$ and A_0 carries $e_i = f_i - t^{\mu_i-\mu_0}h_i(t)h_0^{-1}(t)f_0$ into $f_i(s) - t^{\mu_i-\mu_0}h_i(t)h_0^{-1}(t)f_0(s) \in \mathscr{O}_{(x,y)}$, where s is a uniform parameter of $\mathscr{O}_{y,C}$ and hence (t,s) is the system of local parameter of $\mathscr{O}_{(x,y)}$.

For any m, A_m is simply the partial m–truncated Taylor sery of e_i with respect to s. When restricting A_m to the fiber $p^{-1}(x) \simeq C$, this is the canonical homomorphism for linear systems spanned by $f_1,\dots,f_r$, i.e.

$$\{f_1,\dots,f_r\} \otimes_k \mathscr{O}_C \longrightarrow \mathscr{P}^m_C(L) \ .$$

So actually A_m is a family of the canonical homomorphism which corresponds to sub–linear system passing through point $x \in C$ (even x is a base point of V, there still exists unique such a sub–linear system.) Therefore the degeneracy of various A_m will give a family of some kind of Weierstrass points.

We shall investigate A_0.

Let the image, the kernel and cokernel of A_0 be G_0, H_0 and W_0 respectively. The Fitting ideal $F^0 = F^0(W_0)$ defines a subscheme on $C \times C$. As a scheme–theoretic union $Z = Z_0 \cup Z_1$, where Z_1 is a divisor contained in Z and Z_0 is a zero dimensional scheme which is the residual scheme of Z_1 in Z ([1]).

Definition (1) Z_1 is the finite covering part of the double points of V.

(2) Z_0 is the double–point part of V.

The reason for so naming Z_1 is from the following proposition.

Proposition 2.1 $$p_*[Z_1] = \mu[C] .$$

Proof. Since $\mu = [k(C) : k(\varphi(C))]$, so for generic point $y \in \varphi(C)$ $\# \varphi^{-1}(y) = \mu$. This means every point of $\varphi^{-1}(y)$ determines a same hyperplane in V as others. Then there is a basis of the hyperplane $f_1,\dots,f_r$ such that $f_i(x_j) = 0$, where $\varphi^{-1}(y) = \{x_1,\dots,x_\mu\}$ and $1 \le i \le r$, $1 \le j \le \mu$, and there is no point other than x_j making f_i vanish for all i. Therefore $\#(p^{-1}(y) \cap Z_1) = \mu$ and the assertion follows.

It is worth noting that Z_1 and Z_0 both are symmetric respect to diagonal, that means if $(x,y) \in Z_i$ then $(y,x) \in Z_i$. Besides, we have $\Delta_C \subset Z_1$.

§ 3 Formula

We intend to compute Z_0 and try to express it by Chern classes of C and L. But it seems difficult for a general case. For the time being we restrict our attention to the case $\dim V = 3$, namely φ is a morphism from C to $\mathbb{P}^2$.

Theorem 3.1.

$$[Z_0] = C_1(p^*B_0)C_1(q^*L) - (C_1(p^*B_0) + 2C_1(q^*L)) \cap [Z_1] - [Z_1]^2 ,$$

or

$$p_*[Z_0] = (d-\mu)(D-B) - 2p_*(C_1(q^*L) \cap [Z_1]) - p_*[Z_1]^2 .$$

Remark. If φ is birational and V has not a base point, then $Z_1 = \Delta$ and $\Delta^2 = K_{C|\Delta}$. Therefore $\# \, p_*[Z_0] = (d-1)(d-2)-2g$. When $\varphi(C)$ has only nodes, $p_*[Z_0] = 2\cdot\kappa$ where κ denotes the number of nodes. So we have

$$\kappa = \frac{(d-1)(d-2)}{2} - g .$$

It turns out to be the classical Plücker formula.

Proof. In the present case $\mathrm{rk}\, E = 2$, and we have $A_0 : p^*E \longrightarrow q^*L \longrightarrow W_0$. But Z defined by $F^0(W_0)$ is the zero locus of a section $s : \mathcal{O}_{C\times C} \longrightarrow p^*E \otimes (q^*L)^{-1}$. Since Z_0 is the residual scheme of Z_1 in Z , then we have a diagram

$$\begin{tikzcd} & Z_1 \arrow[d] & \\ Z_0 \arrow[r, hook] & Z \arrow[r] \arrow[d] & C\times C \arrow[d, "s"] \\ & C\times C \arrow[r, "S_0"] & p^*E\otimes q^*L^{-1} \end{tikzcd} ,$$

where S_0 is the 0–section of $p^*E \otimes q^*L^{-1}$.

Therefore by the residual intersection formula [1] we have

$$C_2(p^*E \otimes q^*L^{-1}) = [Z_0] - C_1(p^*E \otimes q^*L^{-1}) \cap [Z_1] + [Z_1]^2 \ .$$

But $C(p^*E) \cdot C(p^*B_0) = 1$, then the formula follows.

Now we would like to give $[Z_1]$ an explanation. We have an exact sequence

$$0 \longrightarrow H_0 \longrightarrow p^*E \longrightarrow G_0 \longrightarrow 0 \ .$$

G_0, as a subsheaf of q^*L, is torsion–free and $\mathrm{rk}\, G_0 = 1$. H_0 has a resolution with length 1, i.e.

$$0 \longrightarrow H_0 \longrightarrow p^*E \longrightarrow q^*L \ ,$$

then the set where H_0 is not locally free has codimension ≥ 3, and hence H_0 is locally free.

Lemma 3.2. $$C_1(G_0 \otimes q^*L^{-1}) \cap [C \times C] = -[Z_1]$$

Proof. The proof is standard.

Since $F^0(W_0) = G_0 \otimes q^*L^{-1}$, so it is the ideal sheaf for defining $Z = Z_0 \cup Z_1$. Let $U = C \times C - Z_0$. $G_0 \otimes q^*L^{-1}$ has a locally free resolution on $C \times C$, which is assumed to be

$$0 \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_0 \longrightarrow G_0 \otimes q^*L^{-1} \longrightarrow 0 \ .$$

Then on U,

$C_1(G_0 \otimes q^* L^{-1}) \cap [C\times C] = (C_1(F_0) - C_1(F_1) + \ldots + (-1)^n C_1(F_n)) \cap [C \times C]$. But F_i is locally free on whole $C\times C$, then the right hand of the equality is $-[Z_1]$ on $C\times C$ and so does $C_1(G_0 \otimes q^* 1^{-1})$.

Since

$C_1(G_0 \otimes q^* L^{-1}) = C_1(p^* E) - C_1(H_0) - C_1(q^* L^{-1}) = -C_1(p^* B_0) - C_1(H_0) - C_1(q^* L^{-1})$, by Lemma $[Z_1] = (C_1(p^* B_0) + C_1(H_0) - C_1(q^* L)) \cap [C\times C]$.

Now suppose the gap sequence for V is $0, b_1, b_2$ [7].

Lemma 3.3. For the generic 2–subspace of V, the gap sequence is $0, b_1$.

Proof. As a generic point of C we can choose a basis for V such that these elements of basis have the following Taylor series:

$$f_0 = 1 + \ldots$$
$$f_1 = C_1(dt)^{b_1} + \ldots$$
$$f_2 = C_2(dt)^{b_2} + \ldots$$

Then the generic 2–space generated by the linear combination of the basis has the property we expected.

Proposition 3.4. The morphism $A_{b_1} : p^* E \longrightarrow P^{b_1}_{Y/C}(q^* L)$ is injective and the image of A_i, $i < b_1$ is isomorphic to G_0.

Proof. Since for the generic fiber $p^{-1}(x)(p^* E)\Big|_{p^{-1}(x)} = E_x \otimes \mathcal{O}_C$, and E_x is a generic 2–space of V, then by Lemma 3.3 $A_{b_1}\big|_{p^{-1}(x)} : E_x \otimes \mathcal{O}_C \longrightarrow \mathcal{P}_C^{b_1}(L)$ is injective. Therefore as a torsion–free sheaf the image of A_{b_1} has rank 2 and hence it is isomorphic to p^*E.

By the same reason as above the rank of image of A_i $i < b_1$ is 1 and the natural projection from $\operatorname{Im} A_i$ to G_0 is surjective, so that the kernel of the projection is torsion and hence zero.

We now have the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & H_0 \arrow[r, "j"] \arrow[d] & \Omega_{Y/C}^{\otimes b_1}(q^*L) \arrow[d] \\
0 \arrow[r] & p^*E \arrow[r] \arrow[d] & \mathcal{P}_{Y/C}^{b_1}(q^*L) \arrow[d] \\
0 \arrow[r] & G_0 \arrow[r] \arrow[d] & \mathcal{P}_{Y/C}^{b_1-1}(q^*L) \\
 & 0 &
\end{tikzcd}$$

On each fiber $p^{-1}(x)$, the degeneracy of j is the first part of Weierstrass point for E_x [8]. Then the degeneracy of j, which is rationally equivalent to $(b_1C_1(q^*\Omega_C) + C_1(q^*L) – C_1(q^*L) - C_1(H_0)) \cap [C\times C]$, is simply the divisor of the first part of Weierstrass points for the family E, denoted by W. In summary, we have

<u>Proposition 3.5</u>. $[Z_1] + [W] + [p^*B] = b_1q^*[K_C] + q^*D$, where B is the

divisor of base point, $[K_C]$ is the canonical divisor of C .

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