COMPACT SYMPLECTIC MANIFOLDS OF LOW COHOMOGENEITY

Lê Hông Vân

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-97

Dedicated to Victor Martínez Olivé (1961-1995).

COMPACT SYMPLECTIC MANIFOLDS OF LOW COHOMOGENEITY

LÊ HÔNG VÂN *
MAX-PLANCK-INSTITUT FÜR MATHEMATIK
GOTTFRIED-CLAREN-STR. 26
53225 BONN

Abstract. In this note we classify compact symplectic manifolds M which admit a Hamiltonian action of a compact Lie group G such that the quotient space M/G has dimension 1. For a class of these manifolds we compute their small quantum cohomology algebra. We also construct some symplectic manifolds of cohomogeneity 2.

1. Introduction .

An action of a Lie group G on a manifold M is called of **cohomogeneity** k if the regular (principal) G-orbits have codimension k in M. In other words the orbit space M/G has dimension k. It is well-known (see e.g. [Kir]) that homogeneous symplectic manifolds are locally symplectomorphic to coadjoint orbits of Lie groups whose symplectic geometry can be investigated in many aspects [Gr, H-V, G-K]. Our motivation is to find a wider class of symplectic manifolds via group approach, so that they could serve as test examples for many questions in symplectic geometry (and symplectic topology). In this note we describe all compact symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of a compact Lie group. We always assume that the action is effective. We also remark that 4-manifolds admitting symplectic group actions (of cohomogeneity 1 or of S^1-action) have been studied intensively by many authors, see [Au] for references.

* supported by DFG/Heisenberg-Programm
Email: lehong@mpim-bonn.mpg.de.

Let us recall that if an action of a Lie group G on (M,ω) preserves the symplectic form ω then there is a Lie algebra homomorphism

$$g = Lie\,G \xrightarrow{\mathcal{F}_*} Vect_\omega(M), \tag{1.1}$$

where $Vect_\omega(M)$ denotes the Lie algebra of symplectic vector fields. The action of G is said to be **almost Hamiltonian** if the image of $\mathcal{F}_*$ lies in the subalgebra $Vect_{Ham}(M)$ of Hamiltonian vector fields. Finally, if the map $\mathcal{F}_*$ can be lifted to a homomorphism $g \xrightarrow{\mathcal{F}} C^\infty(M,\mathbf{R})$ (i.e. $\mathcal{F}_* v = sgrad\,\mathcal{F}_v$) then the action of G is called **Hamiltonian**. In this note we shall prove the following theorem.

Theorem 1. *Suppose that a compact symplectic manifold (M,ω) is provided with a Hamiltonian action of a compact Lie group G such that dim M/G =1. Then M is diffeomorphic to a $\mathbf{C}P^n$-bundle over a coadjoint orbit of G.*

In other words, up to diffeomorphism, the only primitive compact symplectic manifolds of cohomogeneity 1 are $\mathbf{C}P^n$. The complete classification up to equivariant symplectomorphism of these symplectic spaces shall be shown in section 2.

In section 3 we give a computation of (small) quantum cohomology ring of some spaces admitting a Hamiltonian U_n-action with cohomogeneity 1 and discuss its corollaries.

We also consider the case of a symplectic action of cohomogeneity 2. In particular we get

Theorem 2. *Suppose that a compact symplectic manifold M is provided with a Hamiltonian action of a compact Lie group G such that dim M/G=2. Then all the principal orbits of G must be either (simultaneously) coisotropic or (simultaneously) symplectic. Thus a principal orbit of G is either diffeomorphic to a T^2-bundle over a coadjoint orbit of G (in the first case) or diffeomorphic to a coadjoint orbit of G (in the second case).*

At the end of our note we collect in Appendix some useful facts of the symplectic structures on the coadjoint orbits of compact Lie groups.

Acknowledgement. I would like to express my sincere thanks to Dmitri Alekseevski, Michael Grossberg, Yuri Manin and Tien Zung Nguyen for stimulating and helpful discussions.

2. Classification of compact symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of a compact Lie group.

It is known [B] that if an action of a compact Lie group G on a compact oriented manifold M has cohomogeneity 1 (i.e. dim M/G =1) then the topological space $Q = M/G = \pi(M)$ must be either diffeomorphic to the interval $[0,1]$ or a circle S^1. The slice theorem gives us immediately that $G(m)$ is a principal orbit if and only if the image $\pi(G(m))$ in Q is a interior point. In what follows we assume that (M,ω) is symplectic and the action of G on M is Hamiltonian. Under this assumption the quotient Q is [0,1] (see the proof below).

Proposition 2.1. *Let $G(m)$ be a principal orbit of a Hamiltonian G-action on (M^{2n},ω). Then $G(m)$ is a S^1-bundle over a coadjoint orbit of G.*

Proof. In this case there exists a moment map

$$M^{2n} \xrightarrow{\phi} g^* : < \phi(m), v >= \mathcal{F}_v(m). \tag{2.1}$$

For a vector $V \in T_*G(m)$ there is a vector $v \in g$ such that $V = sgrad\mathcal{F}_v = \frac{d}{dt}_{t=0}(exp\, tv)$. Hence we get

$$< \phi_*(V), w >= d\mathcal{F}_w(V) =< [w,v], \phi(m) > \tag{2.2}$$

which implies that ϕ is an equivariant map. Therefore the image $\phi(G(m))$ of any orbit $G(m)$ on M is an adjoint orbit $G(\phi(m)) \subset g^*$.

By the very definition the Hamiltonian vector field $sgrad\mathcal{F}_v$ on M^{2n} associated to $\mathcal{F}_v$ also generates the action of the subgroup $exp(tv) \subset G$ on M.

Lemma 2.2. *The preimage $\phi^{-1}\{\phi(m)\}$ is a closed submanifold of dimension at most one in M. If the preimage has dimension one then it is a orbit of a connected subgroup $S^1_m \subset G$.*

Proof. Clearly the preimage is a closed subset. We shall show that its dimension is at most 1. Let V be a non-zero tangent vector to the preimage $\phi^{-1}\{\phi(m)\}$ at x. Then V is also a tangent vector to $G(m)$, so we can assume that $V = sgrad\mathcal{F}_v(m)$ for some $v \in g$. Clearly $\phi_*(V) = 0$. Then for any $a \in g$ the following identities hold

$$0 =< \phi_*(V), a > \stackrel{2.1}{=} < [v,a], \phi(m) >= d\mathcal{F}_a(V). \tag{2.3}$$

From (2.3) we conclude that V annihilates the space $span\,\{d\mathcal{F}_a\,|a \in g\}$ which has codimension 1 in T^*M. Our claim on the manifold structure follows from the fact that $exp_x tV \subset \phi^{-1}\{\phi(m)\}$. This fact also yields the last statement on the orbit structure of the preimage. Finally, the preimage is connected because the quotient $\phi(G(m)) = G(m)/\{\phi^{-1}\}$ is simply-connected (see Appendix) and $G(m)$ is connected. □

G is compact we can identify the co-algebra g^* with g via the Killing form. From the well-known convexity property of the moment map (see e.g. [Kiw]) we see that the quotient $Q = M/G$ is canonically isomorphic to the intersection of the image of the moment map $\phi(M)$ with a Weyl chamber W in a Cartan subalgebra $Lie\,T \subset g$ (alternatively we can use Lemma 2.2 to get the isomorphism between two quotient spaces: M/G and $\phi(M)/G$).

Proposition 2.3. *There is a Hamiltonian S^1-action on M such that $G(\phi(m))$ is a symplectic reduction $M//S^1$.*

Proof. First we shall show that if $G(m)$ is a principal orbit then the image $\phi(G(m))$ is a topological S^1-quotient of the orbit $G(m) \subset M$. It suffices to show that there is a Hamiltonian S^1-action on M which preserves the orbit $G(m)$ invariant and moreover the S^1-orbit through $m \in M$ coincides with the preimage $\phi^{-1}\{\phi(m)\}$. To find a Hamiltonian function H_0 which generates this S^1-action we use the following simple lemma which is a consequence of the fact that the orbits of Weyl group action meet each Weyl chamber at precisely one point.

Lemma 2.4. *For a linear interval I on the Weyl chamber $W \subset Lie\,T$, where $Lie\,T$ is a Cartan subalgebra of g, there exists a Weyl-invariant smooth function θ_0 on lT such that $d\theta_0(v) \neq 0$ for any $v \in T_*\{I \setminus \partial I\}$ and moreover $d\theta_0(T_{\partial I}I) = 0$.*

Since $sgrad\,H_0$ annihilates the space $\{d\mathcal{F}_a \mid a \in g\}$ the characteristic flow $sgrad\,H_0$ through m lies in the preimage $\phi^{-1}\{\phi(m)\}$. Clearly the symplectic form on M descends to a symplectic form on the quotient $G(m)/S^1 = \phi(G(m))$ of $G(m)$. We also note that the stabilizer St of this coadjoint orbit $\phi(G(m))$ is the product $G_m \times S^1_m$, where G_m is the stabilizer of the orbit $G(m) \subset M$. Finally to show that the $\mathbf{R}$-action generated by the Hamiltonian H_0 descends to a S^1-action we can use the following simple lemma, a proof of it can be found in [McD-S].

Lemma 2.5. *There is a compatible to ω almost complex structure J on M which is G-invariant.*

So all the closed characteristic leaves on the same orbit $G(m)$ have the same length, henceforth we can easily construct a S^1-action on M which lifts to the above Hamiltonian $\mathbf{R}$-action generated by H_0. □

Thus if an action of G on (M,ω) is Hamiltonian with cohomogeneity 1 then the quotient space M/G can be identified with the intersection of $\phi(M)$ with a Weyl chamber. Hence G acts on the image $\phi(M)$ of M

with only at most three orbit types: a regular one $G/Z(v)$ is the image of a regular orbit $G(m) \subset M$ and possibly other two orbit types G/Z_{min} and G/Z_{max} which are also coadjoint orbits of G. By the Duistermaat-Heckman theorem [D-H] (see also [A-B]) the induced symplectic forms on the reduced spaces $G/Z(v)$ change linearly on t:

$$\omega_t = \omega_0 + te. \tag{2.6}$$

Here t is the linear parameter on $\phi(M) \cap W = Q = [0,1]$, and e is the Euler class of the S^1-bundle $G/G_m \to G/Z(v)$. Since the volume of orbit $G(\phi(m))$ tends to zero when $\phi(m)$ tends to a point in a singular (coadjoint) orbit, from (2.6) we conclude that the action of G on $\phi(M)$ has no more than one singular orbit. Note that the adjoint action has no exceptional orbit on g. Henceforth we get

Lemma 2.6. *There are only three posibilites:*
I) $Z(v) \cong Z_{min} \cong Z_{max}$
II) $Z(v) \cong Z_{min} \subset Z_{max}$
III) $Z(v) \cong Z_{max} \subset Z_{min}$.
(Here "$\cong$ " stands for conjugacy.)

Now we shall describe M according to three cases described in Lemma 2.6.

CASE I: all symplectic quotients $G(m)/S^1$ are G-diffeomorphic. In this case by dimension reason and the fact that $G/Z(v)$ is simply-connected, we see immediately that a singular orbit $G(m')$ is G-diffeomorphic to its image $\phi(G(m')) = G/Z(v)$. To specify the G-diffeomorphism type of M it is useful to use the notion of segment [A-A]. In our case we just consider the gradient flow of the function H_0 on M. After a completion and a reparametrization we get a segment $[s(t)]$, $t \in [0,1]$, in M such that the stabilizer of all the interior point $s(t), t \in (0,1)$ coincide with, say, G_m and the stabilizers G_0 and G_1 at singular points $s(0)$ and $s(1)$ are $G_m \times S^1_0$ and $G_m \times S^1_1$.

Lemma 2.7. *There is an element $g \in G$ such that $Ad_g(G_m) = G_m$ and $Ad_g(S^1_0) = S^1_1$.*

Proof. Let v_i be a generator of the Lie algebra of S^1_i such that $||v_i|| = 1$. Denote by T_m the maximal torus in G_m and by $Lie\,T_m$ the Cartan algebra of G_m. Then $K_0 = Lie\,T_m \oplus v_0$ is a Cartan algebra of G. Since v_1 also commutes with $Lie\,T_m$ there is a root α (respect to K_0) such that $\alpha_{|Lie\,G_m} = 0$ and if $\alpha = 0$ then v_1 is also in K_0 and if $\alpha \neq 0$ then v_1 is in the Lie subalgebra $L(\alpha)$ generated by α. Thus if $\alpha = 0$ then $v_0 = \pm v_1$ since v_0 is orthogonal to $Lie\,G_m$ and we can take g as Id. If $\alpha \neq 0$ we have the inclusion $v_0 \in L(\alpha)$ and we take g as an element in the subgroup $exp\,L(\alpha)$ such that $Ad_g(v_0) = v_1$. □

By the slice theorem a neighborhood of the singular orbits in M are diffeomorphic to $G \times_{(G_0,\phi_0)} D^2$ and $G \times_{(G_1,\phi_1)} D^2$ respectively, where ϕ_0, ϕ_1 are the slice representations.

Proposition 2.8. *In the case (I) M is G-diffeomorphic to $G \times_{(G_0,\phi_0)} D^2 \#_\chi G \times_{(G_1,\phi_1)} D^2$, where χ is G-equivariant diffeomorphism. In particular M is diffeomorphic to $G \times_{G_0} S^2$, where $G_0 = G_m \times S^1$ and the left action of G_0 on S^2 is obtained via the composition of the projection $G_0 \to S^1$ with a Hamiltonian action of S^1 on T^2. Here the action of S^1 on S^2 is free outside two fixed points.*

Proof. We define the gluing map χ with helps of the segment $s(t)$. Namely $s(1/2)$ gives us G-equivariant map between $G \times_{G_0} S^1$ and $G \times_{G_1} S^1$. Hence follows the first statement.

To prove the second statement we first note that $G \times_{(G_1,\phi_1)} D^2$ is diffeomorphic to $G \times_{(G_0,\phi_2)} D^2$, where the action ϕ_2 of G_0 on D^2 is induced from the slice representation ϕ_1 of G_1. According to Lemma 2.7 the action ϕ_0 and ϕ_2 are conjugate through Ad_g: $\phi_2 = \phi_0 \circ Ad_g$ thus we can write $M = G \times_{(G_0,\phi_0)} D^2 \#_{\chi_1} G \times_{(G_0,\phi_0)} D^2$, where χ_1 is conjugate with χ through the above identification (and using the fact that two actions of S^1 on D^2 with only one fixed point at the origin of D^2 are equivalent). The gluing map $\chi_1 : G \times_{G_0} S^1 \to G \times G_0 S^1$ is nothing else but the restriction of the action g_0. Now it is obvious that M is diffeomorphic to $G \times_{(G_0,\phi_0)} S^2$ as a S^2-fibration over G/G_0, because the gluing map χ can be extended to a diffeomorphism of the whole manifold $G \times_{G_0} D^2$ as the same action g_0.

To show that the action of S^1 on S^2 is free outside two fixed points we use the fact that the stabilizer subgroup G_m is connected. □

Now let us compute the cohomology ring $H^*(M, \mathbf{R})$ (for M in the case I). Once we fix a Weyl chamber we get a canonical G-invariant projection $\Pi_\phi\ \phi(M) \to \phi(G(m_0))$, where $G(m_0)$ is a singular orbit in M. Let $j; = \Pi_\phi \circ \phi$ denote the projection $M \to \phi(G(m_0)) \cong G(m_0)$. Let $\{x_i, R_1\}$ denote the set of generators and relations in cohomology ring $H^*(\phi(G(m_0)), \mathbf{R})$ (see [Bo], correspondingly Proposition A.4 in our Appendix). Note that $G(m_0)$ is the image of a section $s : \phi(G(m_0)) \to M$ of our S^2-bundle and in what follows we shall identify the base $\phi(G(m_0)$ with ist section $G(m_0)$. Let f denote the Poincare dual of the cohomology class $[G(m_0)]$. Let $x_0 \in H^2(\phi(G(m_0)), \mathbf{R})$ be the image of the Chern class of the S^1-bundle $G(m) \to G(m_0)$, where $G(m)$ is a regular orbit G/G_m.

Proposition 2.9. *We have the following isomorphism of additive groups*

$$H^*(G \times_{Z(v)} S^2, \mathbf{R}) = H^*(G/Z(v), \mathbf{R}) \otimes H^*(S^2, \mathbf{R}) \qquad (2.11).$$

The only non-trivial relation in the algebra $H^*(M, \mathbf{R})$ are $R1$, $R2$, with

$$f(f - x_0) = 0. \qquad (R2)$$

Proof. The statement on the additive structure of $H^*(M)$ follows from the triviality of the cohomology spectral sequence of our S^2-bundle. The relation (R.2) follows from the fact that the restriction of $(f - x_0)$ on the singular orbit $G(m_0)$ is trivial. Indeed from the identity

$$(PD_M([G(m_0)]))^2 = PD_M([G(m_0) \cap G(m_0)]) = PD_M[PD_B(x_0)]$$

we get that $f^2 = PD_M[PD_B(f_0)]$, which implies that the restriction of f^2 on $G(m_0)$ coincides with x_0. □

Remark 2.10. (i) If we take the other singular orbit $G(m_1) = G/G_1$ then the Chern class of the S^1-bundle of $G(m) \to G(m_1)$ is $-x_0$ (after an obvious identification $G(m_0)$ with $G(m_1)$ since $G(m_1)$ can be considered as another section (at infinity) of our S^2-bundle. It is also easy to see that the restriction of f on $G(m_1)$ is zero since $G(m_0)$ has no common point with $G(m_1)$.

(ii) Given a S^1-action on S^2 which is free outside two fixed points, the space of S^1-invariant measure on S^2 is parametrized by the space of positive functions ρ on interval $[0, 1]$. Given any S^1-invariant metric Met_G on S^2 there is unique S^1-invariant symplectic form on S^2 in a given positive class $[\omega] \in H^2(S^2, \mathbf{R})$: $[\omega] = \lambda[\omega_{stand}]$, $\lambda > 0$ such that ω is compatible to Met_G. We can write $\omega = \lambda\rho(x)\omega_{stand}$, where $\rho(x)$ is a positive S^1-invariant function on S^2 which is the weight of the measure defined by Met_G and λ' is a positive constant on S^2.

(iii) Any G-invariant metric on $M = G \times_{G_0} D^2 \# G \times_{G_1} D^2$ can be parametrized by a 1-parameter family of G-invariant metrics on $G/Z(v)$ and a S^1-invariant metric on the (G-invariant) fiber S^2. In other words the space of G-invariant metrics on M is 1-1 corresponding to the space of G-invariant metrics in the image of M under the moment map ϕ and the length of the preimage $\phi^{-1}\{\phi(m)\}$.

Proposition 2.11. *Let M^{2n} be in the case I of Lemma 2.6 and let us keep the notation of Proposition 2.8 for M . Then M^{2n} admits a G-equivariant symplectic form ω in a class $[\omega] \in H^2(M^{2n}, \mathbf{R})$ if and* *only if* $[\omega] = j^*(x) + \alpha \cdot f$ *with* $\alpha > 0$, $x^{n-1} > 0$ *and* $(x + x_0)^{n-1} > 0$. *In particular* M^{2n} *always admits a* G*-invariant symplectic structure.*

Proof. Let $[\omega] = j^*(x) + \alpha f$ with $x \in H^2(G/G_0, \mathbf{R})$. Since the restriction of f on $G(m_1)$ is trivial we have $[\omega]_{|G(m_1)} = j^*(x)_{|G(m_1)}$. Since ω is a symplectic form on $G(m_1)$ we get that $(j^*x)^{n-1} > 0$. Considering the restriction of ω to the singular orbit $G(m_0)$ yields the inequality $(x + x_0)^{n-1} > 0$. The condition that $\alpha > 0$ follows from the fact that the restriction of ω on each fiber S^2 is positive. (Here we assume that the orientation of M agrees with that of $G(m)$ and the frame $(sgrad\, H, gradH)$. This proves the "only if" statement.

Now let us assume that the class $[\omega]$ satisfies the condition in Proposition 2.11. We know ([D-H]) that the reduced symplectic form ω_{red} on on $\phi(G(m_t))$ is $(x + (1-t)x_0)$. Clearly all these cohomology classes $(x + tx_0)$, $0 \le t \le 1$ are realized by G-invariant symplectic forms by our condition (see also Remark A.5). We fix a G-invariant metric on $\phi(M)$ which is compatible with these symplectic forms. According to Remark 2.10 (iii) we can construct a G-invariant metric on M which compatible with the G-invariant metric on $\phi(M)$. Lifting on M we can define the restriction $\bar{\omega}$ of ω to each orbit $G(m)$. From the formula of [D-H] (or see [A-B], [Au]) we see that $\bar{\omega}$ is a closed form on M which reprensents the class $[j^*(x)]$ and by the construction, its restriction to each orbit $G(m)$ has maximal rank. Next we note that the restriction of ω to each fiber S^2 represents a positive class $\alpha \cdot f$ and hence can be represented by the unique harmonic symplectic form $\mathcal{H}_M \alpha \cdot f$. Using this uniqueness (or using the fact that $\mathcal{H}_M \alpha \cdot f$ minimizers the L^2-energy in the class of closed forms representing the same cohomology) we easily get that $\mathcal{H}_M \alpha \cdot f$ has rank 2. Clearly the form $\omega = \bar{\omega} + \mathcal{H}_M \alpha \cdot f$ is the required G-invariant symplectic form in the class $[\omega]$.

The statement on the existence of a symplectic structure follows from the fact that G/G_0 always admits such a class x. Since we can multiply x with a big positive constant λ the class $(x + tx_0)^{n-1}$ is also positive. □

Proposition 2.12. *Each* G*-invariant symplectic form on* $G \times_{Z(v)} S^2$ *is Kähler and each* G*-invariant symplectic form on* $G \times_{Z_1} D^2 \# G \times_{Z_2} D^2$ *is deformation equivalent to a monotone* G*-invariant symplectic form.*

Proof. The statement on the existence of a Kähler structure follows from the fact that $G \times_{Z(v)} S^2$ is the projectivization of the holomorphic vector bundle $\mathbf{C}_1 \oplus \mathbf{C}_2$, where $Z(v)$ acts on $\mathbf{C}_1$ via the slice representation and $\mathbf{C}_2$ is the trivial line bundle. Clearly the first Chern class $c_1 \in H^2(M, \mathbf{R})$ of the tangent bundle TM is the sum of the two

Chern classes: the pull-back from the base $G/Z(v)$ and the other from the fiber S^2. It is well-known that any G-invariant symplectic form on the base $G/Z(v)$ is deformation equivalent to a monotone one (see also Remark A.5). Using the same construction as in the proof of the Proposition 2.9 we see that any G-equivariant symplectic form on M is deformation equivalent to a monotone one. □

CASES (II) and (III) (in Lemma 2.5). Clearly these cases are equivalent by changing the sign of the function H. Thus we shall consider the case (II): $Z(v) = Z_{min}$. In this case as before, by dimension reason, the preimage $\phi^{-1}\{\phi(m)\}$, where $m \in G/G_{min}$, consists of only one point. Hence we have $G_{min} = Z_{min}$. There is two possibilities for G_{max}

(IIa): $G_{max} \cong Z_{max}$. In other words the preimage of $\phi^{-1}\{\phi(m)\}$, where $m \in G/G_{max}$, consists of one point. Note that $G_{max}/G_{reg} = S^k$ by the slice theorem. On the other hand we have $Z(v) = G_{reg} \times S^1$. Because $Z_{max}/Z(v)$ is always of even dimension we have $Z_{max}/Z(v) = \mathbf{C}P^{\frac{k-1}{2}}$.

(II b): $Z_{max} = G_{max} \times S^1$. In other words the preimage $\phi^{-1}\{\phi(m)\}$, where $m \in G/G_{max}$, consists of a circle. But in this case $Z_{max}/Z(v) = (G_{max} \times S^1)/(G_{reg} \times S^1) = S^n$ by the slice theorem. Thus $Z_{max}/Z(v) = S^n$. On the other hand since $Z(v)$ is a subgroup of maximal rank in Z_{max} the second homology of $Z_{max}/Z(v)$ is non-trivial. Hence n must be 2. Now let us consider the projection Π from M to G/Z_{max}: $x \mapsto \pi \circ \phi(x)$. Here ϕ is a moment map and π is the projection of $G/Z(v) \to G/Z_{max}$. The fiber Π^{-1} is the sum $D^3 \times S^1 \cup S^2 \times D^2 = S^4$ (using the constrains on $G(m)$ we easily see that $(\Pi^{-1}(\Pi(m))) \cap G(m) = S^2 \times S^1$). Using the cohomology spectral sequence for a S^4-bundle we see immediately that M admits no symplectic structure. So this case (IIb) actually never happens.

Lemma 2.13. *In the case IIa we have the following decompositions: $G_{max} = SU_{l+1} \times G_0$ and $Z_v = S(U_l \times U_1) \times G_0$, where the inclusion of $Z_v \to G_{max}$ is standard.*

Proof. By checking the table A.3 of possible coadjoint orbit types we see that the pair $(Z(v), Z_{max} \cong G_{max})$ in case (II a) can be only:

Serie A. $Z_{max} = S(U_{l+1} \times \cdots \times U_{n_k})$. Then $Z(v) = S(U_l \times S^1 \times \cdots \times U_{n_k})$ and $G_{reg} = S(U_l \times \cdots U_{n_k})$.

Serie B, (D). $Z_{max} = U_{l+1} \times \cdots \times SO_{2n_k+(1)}$, $Z(v) = U_l \times U_1 \cdots \times SO_{2n_k+(1)}$ and $G_{reg} = U_n \times \cdots \times SO_{2h_k+(1)}$.

Serie C. Analogous to B and D.

Exceptional case: the same (see Table A.3 in Appendix).

If G is a product of compact Lie groups then its coadjoint orbits are product of coadjoint orbits of each factor. Thus to prove Lemma 2.13 in general case it suffices to consider the above cases. □

The following Lemma is an analog of Proposition 2.8 and Propositions 2.9 and 2.11.

Lemma 2.14. *Let M be in case IIa. Then M is G-equivariantly diffeomorphic to $G \times_{G_{min}} D^2 \# G \times_{G_{max}} D^{2(l+1)}$. Thus M is a $\mathbf{C}P^{l+1}$-bundle over G/G_{max}. M always admits a G-invariant symplectic structure. Any symplectic form is deformation equivalent to a monotone G-equivariant symplectic form.*

Proof. The first statement follows from the existence of the slice representation nearby the singular orbits. To prove the second statement we consider the projection $M \to G/G_{max} : x \mapsto \phi(x) \mapsto \Pi(\phi(x))$, where Π is a canonical projection from $\phi(M)$ to the singular coadjoint orbit G/G_{max}. We recall that this canonical projection can be chosen using the intersection of $\phi(M)$ with a Weyl chamber (see [Kir]). Clearly the fiber of this projection is diffeomorphic to $\mathbf{C}P^l$. It is also easy to describe the cohomology algebra of M by the method in Proposition 2.9. Namely we take f the Poincare dual to the singular orbit G/G_{min} of codimension 2 in M. Clearly the restriction of f on the fiber $\mathbf{C}P^n$ is the generator of the cohomology group $H^2(\mathbf{C}P^n, \mathbf{R})$. Henceforth the ring $H^*(M, \mathbf{R})$ is generated by $\{f, x_i\}$, where x_i are the pull-back of the generators of the ring $H^*(G/G_{max}, \mathbf{R})$. The relations in $H^*(M, \mathbf{R})$ are those coming from the base $G/Z(v)$ and from the extra relation $f^n(f - x_0) = 0$. Here x_0 is the first Chern class of the complex vector bundle associated with $\mathbf{C}P^n$ over G/G_{max}. To investigate the existence of a G-invariant symplectic structure on M we use the Duistermaat-Heckman theorem as in the proof of Proposition 2.11. Using the same line of Proof of Proposition 2.12 we get the last statement of Lemma 2.14. □

Summarizing we get

Proposition 2.15. *Suppose that M is provided with a Hamiltonian action of a connected compact Lie group G such that dim M/G=1. Then M is diffeomorphic to a $\mathbf{C}P^n$-bundle over a coadjoint orbit of G. Conversely a $\mathbf{C}P^n$-bundle over a coadjoint orbit of G can be equipped with a Hamiltonian action of G with dim M/G=1 if and only if M is G-diffeomorphic to one of manifolds described in Proposition 2.8 and Lemma 2.12.*

Remark 2.16. The case of a non-Hamiltonian symplectic action with cohomogeneity 1 of a compact Lie group is a bit more combinatorially complicated. The main observation in this case is the fact, analogous to Proposition 2.1, namely any principal orbit of such an action is a S^1-bundle over a homogeneous symplectic action. A simple case when the quotient space $Q = M/G$ is isomorphic to S^1 can be done easily because in this case, according to Alekseevskis'theorem [A-A, Proposition 4.4] M must be an extension of a primitive manifold T^{l+1} with a free action of $\mathbf{Z}_k \times T^l$ by means of group G and an epimorphism ϕ from a subgroup $H \subset G$ $\mathbf{Z}_k \times T^l$, where $\mathbf{Z}_k \times T^l$ acts freely on T^{l+1}.

3. Small quantum cohomology of some symplectic manifolds admitting a Hamiltonian action with cohomogeneity 1 of U_n .

Small quantum cohomology[1] (or more precisely the quantum cup-product deformed at $H^2(M, \mathbf{C}) \in H^*(M, \mathbf{C})$) was first suggested by Witten in context of quantum field theory and then has been defined mathematically rigorous for semi-positive (weakly monotone) symplectic manifolds by Ruan-Tian [R-T] (see also [M-S]) and recently for all compact symplectic manifolds by [F-O]. We do not discuss its physical meaning just emphasize that this quantum product structure is a deformation invariant of symplectic manifolds and recently M. Schwarz has derived a symplectic fixed points estimate in terms of quantum cup-length [Sch]. Nevertheless there are not so much examples of symplectic manifolds whose quantum cohomology can be computed (see [CF], [D], [FGP], [G-K], [S-T], [R-T], [W]). Donaldson's computations are based on Salamon's theorem on the isomorphism between the instanton Floer cohomology ring and the symplectic Floer cohomology ring. The main difficult in computation is that it is not easy to "see" all the holomorphic spheres realizing some given homology class in $H_2(M, \mathbf{Z})$. In this section we consider only the case of M being a $\mathbf{C}P^k$-bundle over Grassmannian $Gr_k(N)$ of k-planes in $\mathbf{C}^N$: $M = U(N) \times_{(U(k)\times U(N-k),\phi)} \mathbf{C}P^k$, where ϕ acts on $\mathbf{C}P^k$ through the embedding $U(k) \to U(k+1)$. Thus the generic orbit of G-action on M is $U(N)/(U(k-1) \times U(N-k))$ and its image under the moment map is symplectomorphic to the flag manifold $U(N)/(U(1) \times U(k-1) \times U(N-k))$. With respect to Lemma 2.9 we see that M belongs to the case (I) if and only if $k = 1$, in this case M is a toric manifold. We can also consider M as the projectivization

[1]for a definition and a formal construction of full quantum cohomology see[K-M]

of the rank $(k+1)$ complex vector bundle over $Gr_k(N)$ which is the sum of the tautological $\mathbf{C}^k$-bundle and the trivial bundle $\mathbf{C}$. A special case of such M is $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ whose quantum cohomology is computed in [R-T (example 8.6)] (see also [K-M]).

To compute the small quantum cohomology algebra of these spaces we use several tricks well-known before [S-T], [R-T], [W] and the fact that the projection to the base $Gr_k(N)$ of a holomorphic sphere in M is also a holomorphic sphere in $Gr_k(N)$. Combining with some techniques of the theory of area-minimizing surfaces (note that all holomorphic spheres are area-minimizing surfaces thus being holomorphic and being globally minimal in a given homological class impose some extra conditions on these spheres) we can solve this question in our cases positively. It seems that by the same way we can give a recursive rigorous computation of small quantum cohomology ring of full or partial flag varieties since any k-flag manifold is a Grassmannian bundle over a $(k-1)$-flag manifold (see also [G-K], [CF], [FGP] on other approachs to this problem).

Recall that [Bo] the cohomology algebra $H^*(Gr_k(N), \mathbf{C})$ is isomorphic to the factor-algebra of the algebra $\mathbf{C}[x_1, \cdots, x_k] \otimes \mathbf{C}[y_1, \cdots, y_{N-k}]$ over the ideal generated by $S^+_{U(N)}(x_1, \cdots, y_{N-k})$ (see Prop. A.4). Geometrically x_i is i-th Chern class of the dual bundle of the tautological $\mathbf{C}^k$-vector bundle over $Gr_k(N)$, and y_i is i-th Chern class of the dual bundle of the other complementary $\mathbf{C}^{N-k}$-vector bundle over $Gr_k(N)$. Another description of $H^*(Gr_k(N), \mathbf{R})$ uses Schubert cells which form an additive basis, the Schubert classes, in $H^*(Gr_k(N), \mathbf{R})$ (see e.g. [FGP] and the references therein for the relation between two approaches). Summarizing we have (see e.g [S-T], [M-S])

$$H^*(Gr_k(N), \mathbf{C}) = \frac{\mathbf{C}[x_1, \cdot, x_k]}{< y_{N-k+1}, \cdots, y_N >}$$

where $y_{N-k+j} := -\sum_{i=0}^{N-k+j} x_i y_{N-k+j-i}$ (are defined inductively). The first Chern class of $T_*Gr_k(N)$ is Nx_1.

The quantum cohomology of $Gr_k(N)$ was computed in [S-T] and [W]. Now let us compute the quantum cohomology algebra $QH^*(M, \mathbf{C})$. Denote by f the Poincare dual of the singular orbit $U(N)/(U(1) \times U(k-1) \times U(N-k)$ in M. Let $x_1, \cdots, x_k$ be generators of $GR_k(N)$ as above. It is easy to see that the first Chern class of T_*M is $(N-1)x_1 + (k+1)f'$, where $f' = f + x_1$. Then the minimal Chern number of T_*M is GCD $(N-1, k+1)$. It is easy to get (see the previous section)

$$H^*(M, \mathbf{C}) = \frac{\mathbf{C}[f, x_1, \cdots, x_k]}{< f^k \cdot f', y_{N-k+1}, \cdots, y_N >}$$

According to a general principle for computing the small quantum cohomology ring of a monotone symplectic manifold (M, ω) we need to compute only the quantum relations ([S-T], [W]). More precisely, let $g_i(z_1, \cdots, z_m)$ be polynomials generating the relations ideal of the cohomology algebra $H^*(M, \mathbf{C})$ generated by $\{z_i\}$. Then z_i are also generators of the small quantum algebra $QH^*(M, \mathbf{C}) = H^*(M, \mathbf{C}) \otimes \mathbf{Z}[q]$ with the new relations $\hat{g}_i(z_i) = qP_i(z_i, q)$. Here q is the quantum variable, $\hat{g}_i$ is the polynomial defined by g_i with respect to quantum product in $QH^*(M, \mathbf{C})$. Denote the quantum product by $\star$. There are several equivalent approachs to small quantum cohomology but we use notations (and formalism) in [M-S].

Theorem 3.1. *Let M satisfies the condition $2(k+1) = N-1$. Then its small quantum cohomology ring is isomorphic to*

$$QH^*(M) = \frac{C[f, x_1, \cdots, x_k, q]}{< f^k \star f' = q, y_{N-k+1}, \cdots, y_{N-1}, y_N = (-1)^{k+1} q^2 f >}$$

Proof. By degree (dimension) argument we see that the only non-trivial contributions to the quantum relations come from the moduli spaces of holomorphic curves realizing the following elements in $H_2(M, \mathbf{Z})$. (3.1) : the homology classes $[u]$ generating the homology group $H_2(\mathbf{C}P^k, \mathbf{Z}) = \mathbf{Z}$ of the fiber $\mathbf{C}P^k$; (3.2) : class $2[u]$; (3.3) : class $[v]$ which can be realized as a the holomorphic sphere on one singular orbit $G(m_s)$ which is diffeomorphic to $Gr_k(N)$ (see also the previous section); finally the (exceptional classes) (3.4) : $[v] - [u]$, and (3.5) : $2([v] - [u])$. Note that $[u]$ and $[v]$ are the generators of $H_2(M, \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$.

Let us consider the moduli space of holomorphic spheres in class $[u]$. It is easy to see that with respect to the standard integrable complex structure J on M the J-holomorphic spheres realizing this class $[u]$ are exactly the complex lines of the fiber $\mathbf{C}P^k$. One way to show it is to use calibration associated to $U(N)$-invariant harmonic 2-form in the class f. It is easy to see (by the same argument as in the previous section) that the integral surfaces corresponding to this calibration should be in the fiber $\mathbf{C}P^n$. Hence they must be complex line in $\mathbf{C}P^n$. We can also use the curvature estimate in [L 1] to show that the minimal sectional curvature distribution in M consists of 2-planes in the tangent space of the fiber $\mathbf{C}P^k$. Using the same curvature estimate we have characterized the space of holomorphic spheres of minimal degree in complex Grassmanian and other complex symmetric spaces [L 1] as the space of Helgason spheres.) But the simplest way to see it is to look at the projection of these holomorphic spheres on the base $Gr_k(N)$. A simple computation shows that the virtual dimension of the moduli space $\mathcal{M}_u(\mathbf{C}P^1, M)$ of J-holomorphic spheres realizing $[u]$ equals the real dimension of this space and equals $2(k+1) + 2k + 2N(N-k)$. We can also apply the regularity criterion $H^1(\mathbf{C}P^1, f^*(T_*M)) = H^1(\mathbf{C}P^1, \bar{f}^*(T_*(\mathbf{C}P^n)) = 0$. Here f is a J-holomorphic map $\mathbf{C}P^1 \to M$ and $\bar{f}$ is its restriction on the fiber $\mathbf{C}P^n$. Now let us compute the new quantum relation related to class $[u]$. First we note by dimension reason that in the new quantum relation $f \star_{(k \text{ times})} \star f \star f'$ there is possibly only one non-trivial contribution from $\mathcal{M}_u(\mathbf{C}P^1, M)$. Secondly by the same dimension argument we see that $f \star_{(k \text{ times})} \star f = f^k$. Thus to prove the first quantum relation in Theorem 3.1 it suffices to show the following relation between two Gromov-Witten invariants (for definition of $\Phi_{[u]}$ see [M-S]).

$$\Phi_{[u]}(PD_M(f^k), PD_M(f), pt) = -\Phi_u(PD_M(f^k), PD_M(x_0), pt) + 1 \tag{3.1.1}$$

To prove (3.1.1) we first note that the RHS of (3.1.1) is 1 since $PD_M(x_0) = j^{-1}(PD_B(x_0))$, where as in the previous section we denote by j the projection of M to $Gr_k(N)$. Hence, taking into account that u is a "fiber" class we see immediately that there is no holomorphic curve in class u which intersects $j^{-1}(PD_B(x_0)$ and goes through a given (arbitrary) point.

To see the meaning of LHS of (3.1.1) first we fix a fiber $\mathbf{C}P^k$ which contains the given point pt. We observe that $PD_M(f)$ intersects with each fiber $\mathbf{C}P^k$ at a divisor $\mathbf{C}P^{k-1}$. Finally we note that $PD_M(f^k)$ intersects with the fixed $\mathbf{C}P^k$ at one point because $f^k([\mathbf{C}P^{k-1}]) = 1$. Since there is exactly one complex line through the given two points in $\mathbf{C}P^n$ (and this line always intersects the divisor $\mathbf{C}P^{k-1} \subset \mathbf{C}P^k$) we deduce that the LHS of (3.1.1) is 1.

Next we shall show that

$$\Phi_{[u]}(PD_M(x_p), PD_M(y_{j-p}), PD_M[w]) = 0 \tag{3.1.2}$$

Here the degree of w must be $dim\, M + 2(k+1) - 2j$. Using the formula $PD_M[j^*(y)] = j^{-1}PD_B[y]$ for the Poincare dual of a pull-back cohomology class of the base of a fiber bundle we observe that if (3.1) is not zero then $PD_M[w]) \cap PD_M(x_p) \cap PD_M(y_{j-p}) \neq \emptyset$. But it is impossible by the dimension reason.

Thus there remain possibly four other non-trivial contributions to the quantum relations. The first one is related to the Gromov-Witten invariants

$$\Phi_{[2u]}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w)), \tag{3.2}$$

the second to the Gromov-Witten invariants

$$\Phi_{[v]}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w)), \tag{3.3}$$

and the two other Gromov-Witten invariants related to the (exceptional) classes

$$\Phi_{[v]-[u]}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w)), \tag{3.4}$$

$$\Phi_{2([v]-[u])}(PD_M(x_p), PD_M(y_{j-p}), PD_M(w). \tag{3.5}$$

Here in the cases (3.2), (3.3) and (3.5) the degree of w must be dim $M + 4(k+1) - 2j$ and in case (3.4) the degree of w must be dim $M + 2(k+1) - 2j$.

To compute (3.2) we use a generic almost complex structure J_{reg} nearby the integrable one. Thus the image of J_{reg}-holomorphic spheres in class $2[u]$ must in a (arbitrary) small neighborhood of a complex line in the fiber $\mathbf{C}P^k$, that is the projection of a J_{reg}-holomorphic sphere in class $[u]$ must be in a ball of radius $\varepsilon/2$. Now we can use the same argument as before. Since $PD_M(x_p) \cap PD_M(y_{j-p} \cap PD_M(w) = \emptyset$ there exists a positive ε such that ε-neighborhood of these cycles also do not have a common point. Now looking at the projection of these cycles on the base $Gr_k(N)$ we conclude that the contribution (3.2) is zero.

In order to compute the contribution (3.3) we have to know the moduli space of the holomorphic spheres in class $[v]$ whose dimension is $\dim M + 4(k+1) = \dim Gr_k(N) + 6k+4 = \dim Gr_k(N) + 2N+2(k-1)$. All these holomorphic spheres can be realized as sections of $\mathbf{C}P^k$-bundle over $\mathbf{C}P^1_v$ where $\mathbf{C}P^1_{[v]}$ is a holomorphic sphere of minimal degree in $Gr_k(N)$. Over this $\mathbf{C}P^1$ the bundle $\mathbf{C}P^k$ is the projectivization of sum of $(k+1)$ holomorphic line bundle with Chern numbers being 0 and (-1). We can check easily the dimension of this real space and the Grothendieck regularity criterion. To show that they exhaust all the holomorphic spheres in the class $[v]$ we look at their projection on the base $Gr_k(N)$, or alternatively, use the calibration defined by the pull-back of the $U(N)$-invariant 2-form realizing class $x_0 \in H^2(Gr_k(N), \mathbf{R})$. Now let us to compute (3.3) with $j = N-1$ or $j = N$ (that are the only cases which may enter in the quantum relations).

If $j = N-1$ then the contribution in (3.3) must be 0 since we know that on the base $B = Gr_k(N)$ there is no holomorphic curve of minimal degree which go through the cycle $PD_B(x_p)$ and $PD_B(y_{N-p-1})$ (by dimension reason).

If $j = N$ then there are two possibilities for $PD_M(w)$, namely they are $[u]$ and $[v]$ - the generators of $H^2(M, \mathbf{Z})$.

If $PD_M(w)$ is a fiber u then the induction argument on $Gr_k(N)$ ([S-T], [W]) shows that p in (3.3) must be k and there is unique (up to projection j holomorphic sphere in class $[v]$ which intersects with $PD_M(x_k)$ and $PD_M(y_{N-k})$ and goes through the fixed point $j(u)$, which is the image of the cycle u. Hence we can reduce our computation in the $\mathbf{C}P^k$-bundle over $\mathbf{C}P^1_{[v]}$ and get the number $(-1)^{k+1}$.

If $PD_M(w)$ is a class $[v]$ then using the fact that any two section in the $\mathbf{C}P^k$-bundle over $\mathbf{C}P^1$ has no intersection we also get the contribution number equal $(-1)^{k+1}$. Thus the only non-trivial Gromov-Witten invariants related to $[v]$ are

$$\Phi_{[v]}(PD_M(x_k), PD_M(y_{N-k}), [v]) = \Phi_{[v]}(PD_M(x_k), PD_M(y_{N-k}), [u]) = (-1)^{k+1} \tag{3.3.1}$$

Let us go to the moduli space of holomorphic spheres in the classes $[v] - [u]$. It is easy to see that there is no J-holomorphic sphere in this class because otherwise its area equals the value $\omega([v] - [u])$ and hence is the same as the area of the complex line in the fiber. But our maximal sectional curvature distribution shows that it is impossible. Since the class $[u] - [v]$ is indecomposable in the Gromov sense it follows from the Gromov compactness theorem that for nearby generic almost complex structure J'_{reg} there is also no J'_{reg}-holomorphic sphere. Thus the contribution in (3.4) is zero.

Finally we consider the contribution related to the class $2([v] - [u])$. This space is empty by the following reason. Suppose there is a J-holomorphic sphere in the class $2([u] - [v])$. Then its projection to the Grassmannian $Gr_k(N)$ realizes the class $2[v]$. Hence its area must be bigger than $[v]$. Finally by using the Gromov compactness theorem we can show the existence of a regular almost complex structure J_{reg} nearby J such that there is no J_{reg}-holomorphic sphere. (We can use the same argument for holomorphic spheres in the class $([v] - [u])$.)

Summarizing we get that the only new quantum relations are those involving (3.1.1) and (3.3.1). Note that f is defined uniquely by the condition $f(u) = 1 = f(v)$. This completes the proof of Theorem 3.1. □

Remark 3.2. Since the rank of $H_2(M)$ is 2 it is more convenient to take 2 quantum variables q_1, q_2. In this case our computations give a slightly different (formal) answer, namely $f^k \star f' = q_1$ and $y_N = (-1)^{k+1}(q_1^2 f_1 + q_2^2 f_2)$. Here f_1 and f_2 are the basis of

$Hom\,(H_2(M,\mathbf{C}),\mathbf{C}) = H^2(M,\mathbf{C})$ which is dula to the basis $([u],[v]) \in H_2(M,\mathbf{C})$.

Remark 3.3. Let M be a symplectic manifold as in Theorem 3.1.

(i) It follows immediately from Theorem 3.1 and Schwarz's result [Sch] that the any exact symplectomorphism on M has at least $k+1$ fixed points.

(ii) It seems that after a little work we can apply the result in [H-V] to show that the Weinstein conjecture also holds for those M.

4. Compact symplectic manifolds admitting symplectic action of cohomogeneity 2

A direct product of (M_1,ω_1) and (M_2,ω_2) is a symplectic manifold which admits a symplectic action of cohomogeneity 2 provided that either both (M_i,ω_i) admit symplectic action of cohomogeneity 1 or (M_1,ω_1) is a homogeneous symplectic manifold and (M^2,ω_2) has dimension 2. These examples are extremally opposite in a sense that, in the first case the normal bundle of any regular orbit is isotropic, and in the second case the normal bundle is symplectic.

Proposition 4.1. *Suppose that an action of G on (M,ω) is Hamiltonian and $dim\, M/G = 2$. Then either all the principal orbits of G are symplectic (simultaneously), or all the principal orbits of G are coisotropic (simultaneously). In the last case a principal orbit must be a T^2-bundle over a coadjoint orbit.*

Proof. We consider the moment map $\phi : M \to g^* = g$. By Kirwan convexity theorem the intersection of $\phi(M)$ with a Weyl chamber W is a k-dimensional convex set P, where $0 \le k \le 2$. Denote by $\phi\circ\pi$ the map $M \to \phi(M)/G$ which factors through M/G and P° the interior of P. Clearly if $y \in P^\circ$ then $[\phi\circ\pi]^{-1}(y)$ contains only regular points $x \in M$, i.e. the orbit $G(x)$ is principal. It follows that the preimage $\phi^{-1}\{\phi(m)\}$ for all points $m \in [\phi\circ\pi]^{-1}(P^\circ)$ has a constant dimension d. Since dimensions of $G(m)$ and $\phi(G(m))$ are even d must be either 0 or 2. First we suppose that $d = 0$. Since G is connected all the other principal orbit $G(m')$ in M also must diffeomorphic to $\phi(G(m))$ by dimension argument and Corollary A.2, hence all the principal orbits are symplectic. Thus the first statement is proved. Now let us assume that the "generic" dimension d of $\phi^{-1}\{\phi(m)\}$ is 2. It is easy to see that any tangent vector to $\phi^{-1}\{\phi(m)\}$ is in kernel of the restriction of ω to $G(m)$. Arguing as in the proof of Proposition 2.1 we see that its preimage admits 2 commutative nowhere zero vector fields $sgrad\,\mathcal{F}_{v_1}$ and $sgrad\,\mathcal{F}_{v_2}$. Thus it must be an isotropic torus. □

Remark 4.2. *If the action of G is Hamiltonian and the principal orbit is symplectic then the condition that $\phi(M)/G$ is zero dimensional is equivalent to that fact that M is a direct product of a coadjoint orbit and a 2-dimensional symplectic surface.*

More generally we have

Proposition 4.3. *Suppose that the action of G is Hamiltonian and the principal orbit is symplectic and moreover the action of G on $\phi(M)$ has only one orbit type. Then M is G-diffeomorphic to a coadjoint orbit of G bundle over a 2-dimensional surface Σ.*

The proof of Proposition 4.3 is very simple since in this case there is also only one orbit type of G-action on M. Note that such a bundle always admits a G-invariant symplectic structure. To prove the only non-trivial statement in Remark 4.2 we consider the moment map $M \to \phi(M)$, where $\phi(M)$ is a coadjoint orbit. The projection $M \to \Sigma$ and the moment map ϕ defines a product structure on M.

If the principal orbits of G in M are coisotropic then $P = \phi(M)/G$ is always a 2-dimensional convex polytop.

Proposition 4.4. *If the action of G on M is Hamiltonian and the principal orbit of G is coisotropic then M is diffeomorphic to the bundle of ruled surface over a coadjoint orbit of G provided that the action of G on $\phi(M)$ has only one orbit type.*

Proof. In this case M admits a projection π over a coadjoint orbit $\phi(G(m))$ with fiber π^{-1} being a symplectic 4-manifold. This symplectic 4-manifold admits a T^2-Hamiltonian action. Hence it must be a rational or ruled surface (see[Au]). □

Appendix. Homogeneous symplectic spaces of compact Lie groups.

First we recall a theorem of Kirillov-Kostant-Sourrie (see e.g. [Kir]).

Theorem A.1. *A symplectic manifold admitting a Hamiltonian homogeneous action of a connected Lie group G is isomorphic to a covering of a coadjoint orbit of G.*

If G is a connected compact Lie group then all its coadjoint orbits are simply-connected. Thus in this case we have the following simple

Corollary A.2. *A symplectic manifold admitting a Hamiltonian homogeneous action of a connected compact Lie group G is a coadjoint orbit of G.*

Table A.3. We present here a list of all coadjoint orbits of simple compact Lie groups. Recall that a coadjoint orbit through $v \in g$ can be identified with the homogeneous space $G/Z(v)$ with $Z(v)$ being the centralizer of v in G. Element v in a Cartan algebra $Lie\, T^k \subset g$ is regular iff for all root α of g we have $\alpha(v) \neq 0$. In this case $Z(v) = T^k$ - the maximal torus of G. If v is a singular element with $\alpha_i(v) = 0$ then $Lie\, Z(v)$ is a direct sum of the subalgebra in g generated by the roots α_i and $Lie\, T^k$. Looking at tables of roots of simple Lie algebra [V-O] and Dinkin schemes we get easily the following list (which perhaps could be found somewhere else)

(A). If $G = SU_{n+1}$ then $Z(v) = S(U_{n_i} \times \cdots \times U_{n_k})$, $\sum n_i = n + 1$.

(B,C,D). If G is in B_n, Z_n or D_n then $Z(v)$ is a direct product $U_{n_1} \times \cdots U_{n_k} \times G_p$ with $rkG_p + \sum n_i = rkG$, and G_p and G must be from the same series B, C, D.

Analogously but more combinatorically complicated are the types of $Z(v)$ in the exceptional series. Note that all the listed below groups are simply connected.

(E_6). Except the regular orbits with $Z(v) = T^6$ we also have other possible singular orbits with $Z(v) = S(U_{k_1} \times \cdots \times U_{k_n})$ with $n \geq 2$, $\sum k_i = 7$ and $T^k \times Spin_{6-k}$ with $k = 1, 2$.

(E_7). Analogously. Possible are also $Z(v) = T^1 \times SU_2 \times Spin_{10}$ and $Z(v) = T^1 \times E_6$.

(E_8). Analogously. (Possible are also $T^1 \times E_7$ and $T^1 \times SU_2 \times E_6$).

(F_4). Singular orbits can have $Z(v)$ being $T^2 \times SU_2 \times SU_2$ or $T^1 \times Spin_7$ and $T^1 \times Sp_3$.

(G_2) Except the regular orbit G_2/T^2 there are also singular orbit $G_2/SU_2 \times T^1$.

To compute the cohomology ring of $G/Z(v)$ we use

Proposition A.4. [Bo, Theorem 26.1]. *The cohomology algebra $H(G/Z(v), \mathbf{R})$ is a factor-algebra $S_{Z(v)}$ over the ideal generated by $\rho^*_R(S^+_G)$ which equals the characteristic subalgebra.*

(ii) Let $s_1 - 1, \cdots, s_l - 1$ and correspondingly, $r_1 - 1, \cdots, r_l - 1$ be degree of the generators in $H^*(G)$ and $H^*(Z(v))$. Then the Poincare polynom of $G/Z(v)$ equals

$$\frac{(1 - t^{s_1}) \cdots (1 - t^{s_l})}{(1 - t^{r_1}) \cdots (1 - t^{r_l})}$$

Here S_G^+ is the subalgebra of G-invariant polynomials in g which is generated by monomials of positive degree.

Remark A 5. All the G-invariant symplectic form on $G/Z(v)$ are compatible with the (obvious) G-invariant complex structure. Thus all of them are deformation equivalent to a monotone symplectic form.

Remark A 6. For any symplectic form ω on a homogeneous space M^{2n} of a compact Lie group G the averaged form ω^G is a G-invariant symplectic form in the same cohomology class with $[\omega]$. Thus the necessary and sufficient condition for the existence of a symplectic form in a cohomology class $[\omega] \in H^2(M^{2n}, \mathbf{R})$ is that $[\omega]^n > 0$. As another consequence we see that any homogeneous space of a compact Lie group which admits a symplectic structure is diffeomorphic to a homogeneous symplectic manifold. But it is not true for a compact manifold of cohomogeneity 1, (or higher cohomogeneity). Example: $\mathbf{C}P^2\#\overline{\mathbf{C}P^2}$ admist a $SU(3)$ action of cohomogeneity 1 (with no fixed point) but no symplectic form invariant under this action.

REFERENCES

[A-A] A. V. Alekseevskii and D. V. Alekseevskii, *G-manifolds with 1-dimensional orbits spaces*, Adv. in Sov. Math. N8 (1992), 1-31.

[Au] M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*, Birkhäuse 1991.

[A-B] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology, 23, N1, (1984), 1-28.

[Bo] A. Borel, *Sur la cohomologie des espaces fibres principaux et des espaces homogenes de groupes de Lie compacts*, Ann. Math., 57 (1953), 115-207.

[Br].G. E. Bredon, *Introduction to compact transformation groups,* Academy Press, 1972.

[CF] I. Ciocan-Fontanine, *Quantum cohomology of flag variety*, Inter. Math. Research Notes, 6, (1995), 263-277.

[FGP] S. Fomin, S. Gelfand and A. Posnikov, *Quantum Schubert Polynomials*, preprint 5/1996.

[D-H] J. J. Duistermaat and G.J. Heckman, *On the variation in the cohomology of the symplectic form of the reduced phase space* and *Addendum*, Inv. Math. 69 (1982), 259-269, and 72 (1983), 153-158.

[F-O] K. Fukaya and K. Ono, *Arnold conjecture and Gromov-Witten invariant for general symplectic manifolds*, preprint 2/1996.

[G-K] A. Givental and B. Kim, *Quantum cohomology of flag manifolds and Toda Lattices*, Comm. Math. Physics, 168 (1995), 609-641.

[Gr] M. Gromov, *Pseudo-holomorphic curves in symplectic geometry*, Inv. Math. 82 (1985), 307-247.

[H-V]. H. Hofer and C. Viterbo, *The Weinstein conjecture in the presence of holomorphic sphere,* Comm. Pure App. Math. 45(5), 1992), 583-622.

[Kir] A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag 1976

[Kiw] F. Kirwan, *Convexity of the moment mapping, III*, Inv. Math., 77 (1984), 547-552.

[K-M] M. Kontsevich and Iu. Manin, *Gromov-Witten classes, quantum cohomology and enumerative geometry*, Comm. Math. Physics. 164 (1994), 525-562.

[L] Lê Hông Vân, *Curvature estimate for the volume growth of globally minimal submanifolds*, Math. Ann., 1993, v.296, N1, 103-118.

[M-S] D. McDuff and D. Salamon *Lectures on J-holomorphic curves and quantum cohomology*, University Lect. Serie , 6, AMS, 1994.

[O] V. M. Olivé, *El método WKB complejo.* Aportaciones Matemáticas, Ser. Comunicaciones, **14**, 1994, pp. 421-436.

[O-V] A. L. Onhischchik, E. B. Vinberg, *Lie groups and algebraic groups*, Berlin, Springer, 1990.

[Sch] M. Schwarz, *A quantum cup-length estimate for symplectic fixed points*, preprint 5/1996.

[S-T] B. Sieberg and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, preprint 1994

[R-T] Ruan and G. Tian, *Mathematical Theory of quantum cohomology*, J. Diff. Geom. 42(1995), 2, 259-367.

[W] E. Witten, *The Verlinde algebra and the cohomology of the Grassmanian*, hep-th 9312104.