On an Explicit Formula for Whittaker-Shintani Functions on Sp_2

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The boundary of the Eisenstein symbol

by

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Introduction

The determination of an explicit formula for local Whittaker functions has been done by several authors (see [Shi], [K], [C-S], [B-F-H]; for various applications of this formula to the theory of automorphic L-functions, see [Bu]).

In [M-S], we studied the Whittaker-Shintani function on Sp_n that is one of variants of the Whittaker function first introduced by Shintani; we proved the uniqueness of local Whittaker-Shintani functions (see Theorem 1.1 in §1) and showed that a certain integral of the (global) Whittaker-Shintani function over a one-dimensional torus is expressed as a quotient of the L-functions attached to a Siegel modular form and a Jacobi form. This result is, in fact, essentially the same as an explicit formula for the local Whittaker-Shintani function on the torus (see Theorem 1.2).

In this short note, we present an explicit formula for the whole values of local Whittaker-Shintani functions on Sp_2 (Main Theorem in §2). It is noted that the general form of our explicit formula is rather complicated than that for the usual Whittaker functions.

The content of this paper is as follows. In §1, we recall the definition of Whittaker-Shintani functions and summarize several results of [M-S]. The main result of this paper is stated in §2. Our explicit formula is described in terms of irreducible characters of $SO(5, \mathbf{C})$ (= the dual group of Sp_2). The last section is devoted to proof of the Main Theorem. We prove the theorem by solving a system of difference equations satisfied by various values of a Whittaker-Shintani function.

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§1. Whittaker-Shintani functions

1.1 Let n be a positive integer and let $G_{n+1} = Sp_{n+1}$ be the symplectic group of degree $(n+1)$:

$$G_{n+1} = \{g \in GL_{2n+2} \mid {}^t g\, J_{n+1}\, g = J_{n+1}\} \quad \left(J_{n+1} = \begin{bmatrix} 1_{n+1} & 0 \\ 0 & 1_{n+1} \end{bmatrix} \right).$$

The Jacobi group $\mathbf{G}_n$ of degree n is a subgroup of G_{n+1} consisting of elements

$$(\lambda, \mu, \kappa)\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{bmatrix} 1 & 0 & \kappa & \mu \\ 0 & 1_n & {}^t\mu & 0 \\ & & 1 & 0 \\ & & 0 & 1_n \end{bmatrix} \begin{bmatrix} 1 & \lambda & & \\ 0 & 1_n & & \\ & & 1 & 0 \\ & & -{}^t\lambda & 1_n \end{bmatrix} \begin{bmatrix} 1 & & & \\ & a & & b \\ & & 1 & \\ & c & & d \end{bmatrix}$$

where λ, μ are n-row vectors, κ is a scalar and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$. The center of $\mathbf{G}_n$ is $Z_n = \{(0, 0, \kappa)\}$.

1.2. Let F be a nonarchimedian local field and $\mathfrak{o} = \mathfrak{o}_F$ be the ring of integers of F. In what follows, we fix a prime element π of F and a nontrivial additive character ψ of F with conductor $\mathfrak{o}$. We denote by q the cardinality of $\mathfrak{o}/\pi\mathfrak{o}$. We use the same letter X to denote the group of F-rational points of a linear algebraic group X over F if there is no fear of confusion. Put $K_{n+1} = G_{n+1}(\mathfrak{o})$ and $\mathbf{K}_n = \mathbf{G}_n(\mathfrak{o})$.

The Hecke algebras H_{n+1} and $\mathbf{H}_n$ of (G_{n+1}, K_{n+1}) and $(\mathbf{G}_n, \mathbf{K}_n; \psi)$ respectively, are defined as follows:

$H_{n+1} = \{\Phi\colon G_{n+1} \to \mathbf{C} \mid$

(i) $\Phi(kgk') = \Phi(g)$ $(g \in G_{n+1},\ k, k' \in K_{n+1})$

(ii) Φ is compactly supported$\}$,

$\mathbf{H}_n = \{\varphi\colon \mathbf{G}_n \to \mathbf{C} \mid$

(i) $\varphi((0, 0, \kappa)\mathbf{k}\mathbf{g}\mathbf{k}') = \psi(\kappa)\cdot\varphi(\mathbf{g})$ $(\mathbf{g} \in \mathbf{G}_n,\ \mathbf{k}, \mathbf{k}' \in \mathbf{K}_n,\ \kappa \in F)$

(ii) φ is compactly supported modulo $Z_n\}$.

The multiplications of H_{n+1} and $\mathbf{H}_n$ are defined by

$$(\Phi_1 * \Phi_2)(g) = \int_{G_{n+1}} \Phi_1(gx^{-1})\, \Phi_2(x)\, dx,$$

$$(\varphi_1 * \varphi_2)(\mathbf{g}) = \int_{Z_n\backslash \mathbf{G}_n} \varphi_1(\mathbf{g}\mathbf{x}^{-1})\, \varphi_2(\mathbf{x})\, d\mathbf{x},$$

where dx (resp. $d\mathbf{x}$) is the Haar measure on G_{n+1} (resp. $Z_n\backslash \mathbf{G}_n$) normalized by $\int_{K_{n+1}} dx = 1$ (resp. $\int_{Z_n\mathbf{K}_n\backslash \mathbf{K}_n} d\mathbf{x} = 1$).

There exist canonical isomorphisms (Satake isomorphisms) $\Phi \to F_\Phi$ and $\varphi \to f_\varphi$ of H_{n+1} onto $\mathbf{C}[T_1^{\pm 1}, \cdots, T_{n+1}^{\pm 1}]^{W_{n+1}}$ and of $\mathbf{H}_n$ onto $\mathbf{C}[T_1^{\pm 1}, \cdots, T_n^{\pm 1}]^{W_n}$ respectively, where $\mathbf{C}[T_1^{\pm 1}, \cdots, T_r^{\pm 1}]^{W_r}$ denotes the algebra of polynomials in $T_1^{\pm 1}, \cdots, T_r^{\pm 1}$ invariant under the automorphism group W_r of $\mathbf{C}[T_1^{\pm}, \cdots, T_r^{\pm 1}]$ generated by the permutations of $T_1, \cdots, T_r$ and the involutions $T_i \to T_i^{-1}$ $(1 \le i \le r)$ (these isomorphisms are due to Satake and Shintani; see [Sa] and [M]). It follows that the $\mathbf{C}$-algebra homomorphisms of H_{n+1} (resp. $\mathbf{H}_n$) to $\mathbf{C}$ are parametrized by $\chi = (\chi_1, \cdots, \chi_{n+1}) \in (\mathbf{C}^\times)^{n+1}/W_{n+1}$ (resp. $\xi = (\xi_1, \cdots, \xi_n) \in (\mathbf{C}^\times)^n/W_n$) in the following manner:

(1.1) $\Phi \to \chi^\wedge(\Phi) := F_\Phi(\chi_1, \cdots, \chi_{n+1}) \qquad (\Phi \in H_{n+1})$

(1.2) $\varphi \to \xi^\wedge(\varphi) := f_\varphi(\xi_1, \cdots, \xi_n) \qquad (\varphi \in \mathbf{H}_n)$.

1.3. For $\chi \in (\mathbf{C}^\times)^{n+1}/W_{n+1}$ and $\xi \in (\mathbf{C}^\times)^n/W_n$, let $WS(\chi, \xi)$ be the space of $W: G_{n+1} \to \mathbf{C}$ satisfying

(1.3) $W((0, 0, \kappa)\mathbf{k}gk) = \psi(\kappa)W(g) \quad (g \in G_{n+1}, \mathbf{k} \in \mathbf{K}_n, k \in K_{n+1}, \kappa \in F)$

$$(1.4) \quad (\varphi * W * \Phi)(g) := \int_{Z_n\backslash \mathbf{G}_n} d\mathbf{x} \int_{G_{n+1}} dy \, \overline{\varphi(\mathbf{x})} \, W(\mathbf{x}gy^{-1}) \, \Phi(y)$$

$$= \overline{\xi^\wedge(\varphi)} \, \chi^\wedge(\Phi) \, W(g) \qquad (\varphi \in \mathbf{H}_n, \Phi \in H_{n+1}).$$

We call each element of $WS(\chi, \xi)$ a *Whittaker-Shintani function* attached to (χ, ξ). In [M-S], we proved the following uniqueness theorem.

Theorem 1.1 ([M-S], Theorem 1.2, Corollary 3.2)

(i) $\dim_{\mathbf{C}} WS(\chi, \xi) \le 1$.

(ii) *If* $W \in WS(\chi, \xi)$ *is not identically equal to zero, then* $W(e) \ne 0$, *where* e *denotes the identity element of* G_{n+1}.

1.4. We recall another result of [M-S]. For $\chi \in (\mathbf{C}^\times)^{n+1}/W_{n+1}$ and $\xi \in (\mathbf{C}^\times)^n/W_n$, define $Y_{\chi,\xi}(f) \in \mathbf{C}$ $(f \ge 0)$ by

$$(1.5) \quad \sum_{f \ge 0} Y_{\chi,\xi}(f)\, t^f$$

$$= \frac{(1+t)\prod_{i=1}^{n}(1 - q^{-1/2}\overline{\xi_i}\,t)(1 - q^{-1/2}\overline{\xi_i}^{-1}t)}{\prod_{i=1}^{n+1}(1-\chi_i t)(1-\chi_i^{-1}t)}.$$

Then Theorem 6.1 in [M-S] implies the following:

Theorem 1.2 *Let* $W \in WS(\chi, \xi)$. *Then*

$$W\left(\begin{bmatrix} \pi^f & & & \\ & 1_n & & \\ & & \pi^{-f} & \\ & & & 1_n \end{bmatrix}\right) = q^{-(n+1)f}\, Y_{\chi,\xi}(f)\cdot W(e) \qquad (f \geq 0). \tag{1.6}$$

§2. Main result

2.1. In the remaining part of the paper, we only deal with the case $n = 1$ and write G, K, $\mathbf{G}$, $\mathbf{K}$, Z for G_2, K_2, $\mathbf{G}_1$, $\mathbf{K}_1$, Z_1.

Lemma 2.1. ([M-S], Lemma 2.1, Proposition 2.2)

(i) *For* $W \in WS(\chi, \xi)$, $W((\lambda, \mu, \kappa)\begin{bmatrix} \pi^f & & & \\ & \pi^m & & \\ & & \pi^{-f} & \\ & & & \pi^{-m} \end{bmatrix}) = 0$ *if* $f, m \geq 0$ *and* $\mu \notin \mathfrak{o}$.

(ii) *The support of* $W \in WS(\chi, \xi)$ *is contained in*

$$\bigcup_{f,m\geq 0,\ 0\leq r\leq m} Z\mathbf{K}(\pi^{-r}, 0, 0)\begin{bmatrix} \pi^f & & & \\ & \pi^m & & \\ & & \pi^{-f} & \\ & & & \pi^{-m} \end{bmatrix}K.$$

We denote by $W(f, m; r)$ the value of $W \in WS(\chi, \xi)$ at

$$(\pi^{-r}, 0, 0)\begin{bmatrix} \pi^{f} & & & \\ & \pi^{m} & & \\ & & \pi^{-f} & \\ & & & \pi^{-m} \end{bmatrix}.$$

For simplicity, we write $W(f, m)$ for $W(f, m; 0)$. Note that $W(f, m; r) = W(f, m)$ if $r \le m - f$.

2.2. Let $SO(Q, \mathbf{C})$ be the special orthogonal group of

$$Q = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

Let λ_i be the character of a maximal torus $T = \{ x = \mathrm{diag}(x_1, x_2, 1, x_2^{-1}, x_1^{-1}) \}$ of $SO(Q, \mathbf{C})$ given by $\lambda_i(x) = x_i$ $(i = 1, 2)$.

For $f_1 \ge f_2 \ge 0$, let $X(f_1, f_2)$ be the irreducible character of $SO(Q, \mathbf{C})$ with highest weight $f_1\lambda_1 + f_2\lambda_2$. To give an explicit form of $X(f_1, f_2)$, define $X_f(x) \in \mathbf{C}$ for $f \in \mathbf{Z}$, $f \ge 0$ and $x = \mathrm{diag}(x_1, x_2, 1, x_2^{-1}, x_1^{-1}) \in T$ by

$$(2.1) \qquad \sum_{f=0}^{\infty} X_f(x)\, t^f = \frac{1 + t}{(1 - x_1 t)(1 - x_1^{-1} t)(1 - x_2 t)(1 - x_2^{-1} t)}.$$

We put $X_f(x) = 0$ if $f < 0$. Then $X(f_1, f_2)$ is given by

$$(2.2) \qquad X(f_1, f_2)(x) = \det \begin{pmatrix} X_{f_1}(x) & X_{f_1-1}(x) + X_{f_1+1}(x) \\ X_{f_2-1}(x) & X_{f_2-2}(x) + X_{f_2}(x) \end{pmatrix} \quad (x \in T).$$

2.3. In what follows, we fix $\chi = (\chi_1, \chi_2) \in (\mathbf{C}^{\times})^2/W_2$ and $\xi \in \mathbf{C}^{\times}/W_1$. Without loss of generality, we may assume $W(e) = 1$ if W

$\in WS(\chi, \xi)$ is not identically equal to zero (see Theorem 1.1).

To simplify notation, for $(f, m) \in \mathbf{Z}^2$, we set

$$(2.3)\qquad \{f, m\} = \begin{cases} X(f, m)(x_\chi) & \text{if } f \geq m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $x_\chi = \mathrm{diag}(\chi_1, \chi_2, 1, \chi_2^{-1}, \chi_1^{-1}) \in T$. Furthermore we set

$$(2.4)\qquad Y(f, m) = \{f, m\} - q^{-1/2}\,\Xi\,\{f-1, m\} + q^{-1}\{f-2, m\}$$

for $f, m \in \mathbf{Z}$, where

$$(2.5)\qquad \Xi = \overline{(\xi + \xi^{-1})}\ .$$

We denote by $\delta_{f,f'}$ the Kronecker symbol: $\delta_{f,f'} = \begin{cases} 1 & \text{if } f = f' \\ 0 & \text{otherwise} \end{cases}$.

Main Theorem *Let* $W \in WS(\chi, \xi)$ *and assume that* $W(e) = 1$. *Then the values* $W(f, m; r)$ $(f, m \geq 0, 0 \leq r \leq m)$ *are given as follows*:

(a) *For* $f \geq 0$, $q^{2f}W(f, 0) = Y(f, 0)$.

(b) *For* $f \geq m > 0$,

$$q^{2f+m}(q+1)\,W(f, m) = q\cdot Y(f, m) - Y(f, m-1) - \delta_{f,m}Y(f-1, f-1).$$

(c) *For* $m > 0$,

$$q^m(q+1)\,W(0, m) = q^{1-m/2}\,\overline{\{\xi^m + \xi^{-m} + (1-q^{-1})\sum_{j=1}^{m-1}\xi^{2j-m}\}}\ .$$

(d) *For* $f \geq 1$, *put* $A_f(t) = \sum_{j=0}^{\infty} q^{3f+j}(q+1)W(f, f+j)\,t^j$. *Then*

$$A_f(t) = \frac{\alpha_1 t + \alpha_0}{t^2 - q^{1/2}\Xi t + q}$$

where

$$\alpha_1 = -\, q\cdot Y(f,\ f) + (q^{1/2}\Xi - q)\ Y(f,\ f-1) + Y(f-1,\ f-1),$$

$$\alpha_0 = q^2 Y(f,\ f) - q\cdot Y(f,\ f-1) - q\cdot Y(f-1,\ f-1).$$

(e) *For* $f > m \geq 0$, *put* $B_{f,m}(t) = \sum_{j=0}^{\infty} q^{2f+m+j}(q^2-1)W(f,\ m+j;\ j)\ t^j$.

Then

$$B_{f,m}(t) = \frac{\beta_2 t^2 + \beta_1 t + \beta_0}{t^2 - q^{1/2}\ \Xi t + q}.$$

Here $\beta_i \in \mathbf{C}$ $(i = 1, 2, 3)$ *are given as follows*:

If $m = 0$,

$$\beta_2 = q\cdot Y(f,\ 1) - (q^{1/2}\ \Xi + 1)Y(f,\ 0) - \delta_{f,1},$$

$$\beta_1 = -\, q^2 Y(f,\ 1) + (q + q^{1/2}\ \Xi)\ Y(f,\ 0) + \delta_{f,1}\ q,$$

$$\beta_0 = q(q^2 - 1)\cdot Y(f,\ 0).$$

If $m \geq 1$,

$$\beta_2 = q\cdot Y(f,\ m+1) - q^{1/2}\ \Xi\cdot Y(f,\ m) + Y(f,\ m-1) - \delta_{f,m+1}Y(f-1,\ f-1),$$

$$\beta_1 = -\, q^2 Y(f,\ m+1) + q\ (q^{1/2}\ \Xi - q + 1)\ Y(f,\ m) + [q^{1/2}(q-1)\ \Xi - q^2]\ Y(f,\ m-1) + \delta_{f,m+1}\cdot q\ Y(f-1,\ f-1),$$

$$\beta_0 = q\ (q-1)\cdot[q\cdot Y(f,\ m) - Y(f,\ m-1)].$$

§3. Proof of Main Theorem.

3.1. To prove the theorem, we derive a system of difference equations satisfied by $W(f, m; r)$. Let $\Phi_1 \in H = H_2$ be the characteristic function of $K \begin{bmatrix} \pi & & & \\ & 1 & & \\ & & \pi^{-1} & \\ & & & 1 \end{bmatrix} K$. For $r \geq 0$, denote by $U_r = \{\varepsilon \in o^{\times} \mid \varepsilon \equiv 1 \bmod \pi^r\}$. The following two results follow from the left K-coset decomposition of $K \begin{bmatrix} \pi & & & \\ & 1 & & \\ & & \pi^{-1} & \\ & & & 1 \end{bmatrix} K$.

Lemma 3.1. *Let* W *be a function on* G *satisfying* (1.3). *Then, for* $g \in G$, *we have*

$$(W * \Phi_1)(g) := \int_G W(gy^{-1})\, \Phi(y)\, dy$$

$$= W\left(g \begin{bmatrix} \pi^{-1} & & & \\ & 1 & & \\ & & \pi & \\ & & & 1 \end{bmatrix}\right)$$

$$+ \sum_{\lambda \in \pi^{-1}o/o} W\left(g \begin{bmatrix} 1 & & & \\ & \pi^{-1} & & \\ & & 1 & \\ & & & \pi \end{bmatrix} (\lambda, 0, 0)\right)$$

$$+ \sum_{\kappa \in U_0/U_1} W(g\ (0,\ 0,\ \pi^{-1}\kappa))$$

$$+ \sum_{\substack{x \in U_0/U_1,\ \mu,\kappa \in \pi^{-1}o/o \\ \pi\kappa x \equiv (\pi\mu)^2 \bmod \pi}} W\left(g \begin{bmatrix} 1 & & & \\ & 1 & \pi^{-1}x & \\ & & 1 & \\ & & & 1 \end{bmatrix} (0,\ \mu,\ \kappa)\right)$$

$$+ \sum_{x \in o/\pi^2 o,\ \mu \in \pi^{-1}o/o} W\left(g \begin{bmatrix} 1 & & & \\ & \pi & \pi^{-1}x & \\ & & 1 & \\ & & & \pi^{-1} \end{bmatrix} (0,\ \mu,\ 0)\right)$$

$$+ \sum_{\lambda,\ \mu \in \pi^{-1}o/o,\ \kappa \in \pi^{-2}o/o} W\left(g \begin{bmatrix} \pi & & & \\ & 1 & & \\ & & \pi^{-1} & \\ & & & 1 \end{bmatrix} (\lambda,\ \mu,\ \kappa)\right).$$

Lemma 3.2. $\chi^\wedge(\Phi_1) = q^2\{1,\ 0\} - 1.$

We next consider the action of $\mathbf{H} = \mathbf{H}_1$ on $WS(\chi, \xi)$. Let φ_m $(m \geq 0)$ denote the element of $\mathbf{H}$ with support $\mathbf{ZK}\begin{pmatrix} \pi^m & \\ & \pi^{-m} \end{pmatrix}\mathbf{K}$ and satisfying $\varphi_m(\begin{pmatrix} \pi^m & \\ & \pi^{-m} \end{pmatrix}) = 1$. By the left $\mathbf{ZK}$-coset decomposition of $\mathbf{ZK}\begin{pmatrix} \pi^m & \\ & \pi^{-m} \end{pmatrix}\mathbf{K}$, we obtain

Lemma 3.3. *Under the same assumption of Lemma 3.1, we have*

$$(\varphi_m * W)(g) := \int_{Z\backslash \mathbf{G}} \overline{\varphi(\mathbf{x})}\ W(\mathbf{x}g)\ d\mathbf{x}$$

$$= \sum_{\lambda \in \pi^{-m}o/o} W((\lambda,\ 0,\ 0)\begin{pmatrix} \pi^m & \\ & \pi^{-m} \end{pmatrix} g)$$

$$+ \sum_{\substack{-(m-1)\le i\le m-1 \\ \mu\in\pi^{-m}o/o,\ x\in U_0/U_{m-i}}} \psi(-x^{-1}\mu^2\pi^{m-i})\cdot W((-\pi^{m-i}x^{-1}\mu,\ \mu, 0)\begin{pmatrix} \pi^i & \pi^{-m}x \\ 0 & \pi^{-i} \end{pmatrix} g)$$

$$+ \sum_{x\in o/\pi^{2m}o,\ \mu\in\pi^{-m}o/o} W((0,\ \mu,\ 0)\begin{pmatrix} \pi^{-m} & \pi^{-m}x \\ 0 & \pi^{m} \end{pmatrix} g).$$

Lemma 3.4.

$$\xi^\wedge(\varphi_m) = q^{3m/2}\ \{\ \xi^m + \xi^{-m} + (1-q^{-1}) \sum_{j=1}^{m-1} \xi^{2j-m}\ \}.$$

(*In particular,* $\xi^\wedge(\varphi_1) = q^{3/2}\ \Xi$.)

3.2 We now present a system of difference equations that will be used in proof of the theorem.

Proposition 3.5. *Let* $W\in WS(\chi, \xi)$.

(3.1) *For* $f\ge 1$,

$$q^2(q+1)\ W(f,\ 1) = -\ W(f-1,\ 0) + [q^2\{1,\ 0\} - q\}\ W(f,\ 0) - q^4\ W(f+1,\ 0).$$

(3.2) *For* $f\ge m\ge 1$,

$$q^3W(f,\ m+1) = -\ q^4W(f+1,\ m) + q^2[\{1,\ 0\} - 1]\ W(f,\ m) - W(f-1,\ m) - q\ W(f,\ m-1).$$

(3.3) *For* $f\ge 1$,

$$q(q^2-1)\ W(f,\ 1;\ 1) = -\ q(q+1)\ W(f,\ 1) + q^{3/2}\ \Xi\ W(f,\ 0).$$

(3.4) *For* f, m ≥ 1,

$$q^2(q-1)\, W(f, m+1; 1) = (q-1)\, W(f, m; 1) - q^2\, W(f, m+1) + (q^{3/2}\,\Xi - q + 1)\, W(f, m) - W(f, m-1).$$

(3.5) *For* m > f ≥ 1,

$$q^3\, W(f, m+1) = q^{3/2}\,\Xi\, W(f, m) - W(f, m-1).$$

(*This is a special case of (iv).*)

(3.6) *For* f, m ≥ 1 *and* 1 ≤ r ≤ m,

$$q^3\, W(f, m+1; r+1)$$

$$= q^{3/2}\,\Xi\, W(f, m; r) - W(f, m-1; r-1) - \begin{cases} W(f, m; 1) - W(f, m) & \text{if } r = 1 \\ 0 & \text{if } r \geq 2 \end{cases}.$$

Proof: These follow from the definition of W, Lemma 2.1 (i) and Lemmas 3.1–3.4. q.e.d.

Proof of Main Theorem: The statement (a) is a special case of Theorem 1.2 (Note that $Y_{\chi,\xi}(f)$ defined by (1.5) is equal to Y(f, 0)). By the well-known formula (see [Bo], Ch. VIII, §9, Proposition 2)

$$\{f, m\} \cdot \{1, 0\} = \{f+1, m\} + \{f, m+1\} + \{f-1, m\} + \{f, m-1\} + \begin{cases} \{f, m\} & \text{if } m \geq 1 \\ 0 & \text{if } m = 0 \end{cases},$$

we have

$$(3.7) \qquad [\{1, 0\} - 1]\cdot Y(f, m) = Y(f + 1, m) + Y(f, m + 1) + Y(f - 1, m)$$

$$+ Y(f, m - 1) - q^{-1}\delta_{f,m+1}\,[Y(f - 1, f - 1) + Y(f - 2, f - 2)]$$

$$- \delta_{m,0}\, Y(f, 0)$$

for $m \geq 0$ and $f \geq m + 1$. Then we can prove (b) by induction on m using (3.1), (3.2) and (3.7). Since

$$(\varphi_m * W)(e) = q^{3m-1}(q + 1)\, W(0, m),$$

we obtain (c) by Lemma 3.4. To prove (d) and (e), we first see

$$(3.8) \qquad q^{2f+1}(q^2 - 1)\, W(f, 1; 1)$$

$$= - q\, Y(f, 1) + (q^{3/2}\,\Xi + 1)Y(f, 0) + \delta_{f,1}\cdot 1$$

for $f \geq 1$ by (3.3). We next observe

$$(3.9) \qquad q^{2f+m}(q^2-1)\, W(f, m; 1)$$

$$= - q\cdot Y(f, m) + (q^{3/2}\,\Xi - q + 1)Y(f, m - 1) - q\cdot Y(f, m - 2)$$

$$+ \delta_{f,m} Y(f - 1, f - 1)$$

for $f \geq m \geq 2$. This is proved by induction on m (we use the equation (3.4) and the formula (b)). We again apply (3.4) for $m = f$ and use (3.9) and (b) to get

$$(3.10) \qquad q^{3f+1}(q+1)\, W(f, f + 1) = (q^{1/2}\,\Xi - 1)\, Y(f, f) - Y(f, f - 1)$$
$$- q^{-1}(q^{1/2}\,\Xi - 1)\, Y(f - 1, f - 1)$$

for $f \geq 2$. Then (d) is a direct consequence of (b), (3.10) and (3.5). The last statement (e) follows from the equation (3.6) and the formulas (b), (3.8) and (3.9). q.e.d.

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