# Existence of stable 2-vector bundles over ruled surfaces

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## Introduction

Let $\pi : X \to C$ be a ruled surface over a smooth algebraic curve $C$, defined over the complex number field $\mathbb{C}$. Let $c_1 \in \mathrm{Num}(X)$ and $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ be fixed. For any polarization $H$, denote the moduli space of rank-2 vector bundles stable with respect to $H$ in the sense of Mumford–Takemoto by $\mathcal{M}_H(c_1, c_2)$. Stable 2-vector bundles over a ruled surface have been investigated by many authors; see, for example [T1], [T2], [H-S], [Q2], [F]. In this paper we shall study the non-emptiness of the moduli spaces $\mathcal{M}_H(c_1, c_2)$.

For an algebraic 2-vector bundle over a ruled surface $X$ one introduced two numerical invariants $d$ and $r$ and one defined the set $M(c_1, c_2, d, r)$ of isomorphism classes of bundles with fixed invariants $c_1$, $c_2$, $d$, $r$; see [B], [B-St1], [B-St2]. The integer $d$ is given by the splitting of the bundle on the general fibre and the integer $r$ is given by some normalization of the bundle. The moduli spaces $M(c_1, c_2, d, r)$ are defined independent of any ample divisor (line bundle) on $X$; see also [Br1], [Br2], [W]. In [A-B2] we obtained necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we applied this result to some moduli spaces $\mathcal{M}_H(c_1, c_2)$ (see, also [A-B1]).

In section 1 we give necessary and sufficient conditions for a 2-vector bundle $E \in M(c_1, c_2, d, r)$ to be $H$-stable for some ample line bundle $H$. By using this result, the results in [A-B2] and some results of Qin in [Q1], [Q2], [Q3], we solve in section 2 the problem of non-emptiness of moduli spaces $\mathcal{M}_H(c_1, c_2)$ of stable 2-vector bundles in almost all cases.

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# 1 Stability of vector bundles in $M(c_1, c_2, d, r)$

We recall from [B], [B-St1], [B-St2], [A-B2] some basic notions and facts.

The notations and the terminology are those of Hartshorne's book [Ha]. Let $C$ be a nonsingular curve of genus $g$ over the complex number field and let $\pi : X \to C$ be a ruled surface over $C$. We shall write $X \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is normalized. Let us denote by $\mathbf{e}$ the divisor on $C$ corresponding to $\bigwedge^2 \mathcal{E}$ and by $e = -\deg(\mathbf{e})$. We fix a point $p_0 \in C$ and a fibre $f_0 = \pi^{-1}(p_0)$ of $X$. Let $C_0$ be a section of $\pi$ such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.

Any element of the group of divisors on $X$ modulo numerical equivalence $\mathrm{Num}(X) \cong H^2(X, \mathbb{Z})$ can be written $aC_0 + bf_0$ with $a,\ b \in \mathbb{Z}$. We shall denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the divisor $p_0$ on $C$. If $L$ is an element of $\mathrm{Pic}(C)$ we shall write $L = \mathcal{O}_C(k) \otimes L_0$, where $L_0 \in \mathrm{Pic}_0(C)$ and $k = \deg(L)$. We also denote by $F(aC_0 + bf_0) = F \otimes \mathcal{O}_X(a) \otimes \pi^*\mathcal{O}_C(b)$ for any sheaf $F$ on $X$ and any $a, b \in \mathbb{Z}$.

Let $E$ be an algebraic rank-2 vector bundle on $X$ with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, c_2 \in \mathbb{Z}$.

Since the fibres $f$ of $\pi$ are isomorphic to $\mathbb{P}^1$ we can speak about the generic splitting type of $E$ and we have $E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$ for a general fibre $f$, where $d' \leq d$, $d + d' = \alpha$. The integer $d$ is the first numerical invariant of $E$.

The second numerical invariant is obtained by the following normalization:

$$-r = \inf\{l \mid \exists L \in \mathrm{Pic}(C), \deg(L) = l,\ s.t.\ H^0(X, E(-dC_0) \otimes \pi^* L) \neq \{0\}\}.$$

We shall denote by $M(c_1, c_2, d, r)$ the set of isomorphism classes of algebraic rank-2 vector bundles on $X$ with fixed Chern classes $c_1$, $c_2$ and invariants $d$ and $r$.

With these notations we have the following result (see [B]):

**Theorem 1** *For every vector bundle $E \in M(c_1, c_2, d, r)$ there exist $L_1$, $L_2 \in \mathrm{Pic}_0(C)$ and $Y \subset X$ a locally complete intersection of codimension 2 in $X$,* *or the empty set, such that E is given by an extension*

$$0\to\mathcal{O}_X(dC_0+rf_0)\otimes\pi^*L_2\to E\to\mathcal{O}_X(d'C_0+sf_0)\otimes\pi^*L_1\otimes I_Y\to 0, \qquad (1)$$

*where* $c_1 = (\alpha,\beta) \in \mathbb{Z}\times\mathbb{Z}$, $c_2 \in \mathbb{Z}$, $d+d' = \alpha$, $d \geq d'$, $r+s = \beta$, $l(c_1,c_2,d,r) := c_2+\alpha(de-r)-\beta d+2dr-d^2e = \deg(Y) \geq 0$.

For the following result see [A-B2]:

**Theorem 2** *$M(c_1,c_2,d,r)$ is non-empty if and only if $l := l(c_1,c_2,d,r) \geq 0$ and one of the following conditions holds:*
*(I)* $2d > \alpha$ *or,*
*(II)* $2d = \alpha$, $\beta - 2r \leq g + l$.

Let $\mathbf{C}_X$ be the ample cone in $\mathrm{Num}(X)\otimes\mathbb{R}$ generated by ample divisors. We fix the Chern classes $\tilde{c}_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. We shall use (see, for example [Q2]) the following definitions:

**Definition 3** (i) For $\zeta \in \mathrm{Num}(X)\otimes\mathbb{R}$, we define

$$W^\zeta := \mathbf{C}_X \cap \{x \in \mathrm{Num}(X)\otimes\mathbb{R} \mid x.\zeta = 0\};$$

(ii) We define $\mathcal{W}(\tilde{c}_1,c_2)$ to be the union of $W^\zeta$, where $\zeta$ is the numerical equivalence class of $(2F-\tilde{c}_1)$ for some divisor $F$, and which satisfies the conditions:

$$-(4c_2-\tilde{c}_1^2) \leq \zeta^2 < 0 \,;$$

(iii) A *wall* of type $(\tilde{c}_1,c_2)$ is an element $W^\zeta$, where $\zeta$ satisfies the conditions in (ii). A *chamber* of type $(\tilde{c}_1,c_2)$ is a connected component of the set $\mathbf{C}_X \setminus \mathcal{W}(\tilde{c}_1,c_2)$;
(iv) A numerical equivalence class $\zeta$ which represents a nonempty wall $W^\zeta$ is *normalized* if the integer $(\zeta.f)$ is positive.
(v) Let $W^\zeta$ be a nonempty wall of type $(\tilde{c}_1,c_2)$ and let $l_\zeta(\tilde{c}_1,c_2)$ be the integer $c_2+(\zeta^2-\tilde{c}_1^2)/4$. We define $E_\zeta(\tilde{c}_1,c_2)$ to be the set of isomorphism classes of 2-vector bundles $E$ given by nontrivial extensions

$$0\to\mathcal{O}_X(F)\to E\to\mathcal{O}_X(\tilde{c}_1-F)\otimes I_Y\to 0,$$

where $F$ is a divisor such that $\zeta$ is the numerical equivalence class of $(2F-\tilde{c}_1)$, and where $Y \subset X$ is a locally complete intersection of codimension 2 in $X$ such that $\deg(Y) = l_\zeta(\tilde{c}_1,c_2)$.

**Remark 4** The definitions (i)-(iv) depend only on the numerical type $(c_1, c_2)$, where $c_1$ is the numerical equivalence class of $\tilde{c}_1$. We fix the numerical Chern classes $c_1 = (\alpha, \beta) \in \mathrm{Num}(X)$, $c_2 \in \mathbb{Z}$ and the integers $d$, $r$ such that the conditions $2d > \alpha$, $l(c_1, c_2, d, r) \geq 0$ are satisfied. We denote by $\zeta = (d - d')C_0 + (r - s)f_0$ and we have that the condition $l(c_1, c_2, d, r) \geq 0$ is equivalent to the condition $-(4c_2 - c_1^2) \leq \zeta^2$, and that $\zeta + c_1$ is the numerical equivalence class of $2F$ for $F$ a divisor on $X$. If we suppose, moreover, that $\zeta^2 < 0$ and there exists an ample line bundle $L$ over $X$ such that $c_1(L).\zeta = 0$, then the element $\zeta$ represents a nonempty wall of (numerical) type $(c_1, c_2)$ and we have $E_\zeta(\tilde{c}_1, c_2) \subset M(c_1, c_2, d, r)$ (see [A-B1]).

In the next result we shall investigate the stability of vector bundles in the moduli space $M(c_1, c_2, d, r)$. For $F$ a torsion-free sheaf on $X$ and $H$ an ample line bundle, we use the notation $\mu_H(F) := c_1(F).H/rank(F)$.

**Theorem 5** *Let $E \in M(c_1, c_2, d, r)$. Then, there exists an ample line bundle $H$ such that $E$ is $H$-stable if and only if $2r - \beta < \min\{0, e(2d - \alpha)/2\}$ and the extension (1) of $E$ is non-splitting.*

*Proof:* Let us suppose that $E \in M(c_1, c_2, d, r)$ is $H$-stable for some ample line bundle $H$ on $X$. From Theorem 1 we know that $E$ is given by an extension

$$0 \to N_2 \to E \to N_1 \otimes I_Y \to 0, \tag{2}$$

where

$$N_2 = \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2, \; N_1 = \mathcal{O}_X(d'C_0 + sf_0) \otimes \pi^* L_1,$$

with $L_1$, $L_2 \in \mathrm{Pic}_0(C)$. As a stable vector bundle $E$ is non- splitting. Let $\zeta = (2d - \alpha)C_0 + (2r - \beta)f_0 \in \mathrm{Num}(X)$. From the definition of the invariant $d$ we have $2d \geq \alpha$, i.e. $\zeta.f \geq 0$. Let us suppose that $2r - \beta \geq 0$. Then, $(2d - \alpha)C_0 + (2r - \beta)f_0$ is an effective divisor and, therefore, $H.\zeta \geq 0$ for any ample line bundle $H$ on $X$. It follows from the exact sequence (2) that

$$\mu_H(N_2) \geq \mu_H(E),$$

i.e. $N_2$ is destabilising, contradiction.

Now, let us suppose that $2r - \beta \geq e(2d - \alpha)/2$. If $2d = \alpha$ we get the above case $2r - \beta \geq 0$. Assume $2d > \alpha$. We shall prove that $H.\zeta > 0$, which gives as above a contradiction. A simple computation gives

$$2r - \beta \geq e(2d - \alpha)/2 \iff \zeta^2 \geq 0.$$

If $H.\zeta = 0$, by the index theorem we get $\zeta^2 \leq 0$. It follows $\zeta^2 = 0$, i.e. $\zeta$ is numerically trivial. But $\zeta.f = 2d - \alpha > 0$, contradiction. If $H.\zeta < 0$, let $D = (H.f)\zeta - (H.\zeta)f$. Since $H.D = 0$, $D^2 \leq 0$ by the index theorem. But

$$D^2 \leq 0 \iff (H.f)^2\zeta^2 - 2(H.f)(H.\zeta)(\zeta.f) \leq 0,$$

and we get a contradiction, since $(H.f)(H.\zeta)(\zeta.f) < 0$. It follows

$$2r - \beta < \min\{0,\ e(2d - \alpha)/2\}. \tag{3}$$

Conversely, suppose that $E$ is given by a non-splitting extension (1) and the inequality (3) is satisfied.

*Case 1.* $2d > \alpha$.

We show firstly that $\zeta$ defines a nonempty wall of type $(c_1, c_2)$. From the extension (1) we get

$$\zeta^2/4 - (c_1^2/4 - c_2) = l(c_1, c_2, d, r) \geq 0.$$

Since $2r - \beta < e(2d - \alpha)/2$, we get $\zeta^2 < 0$, so

$$-(4c_2 - c_1^2) \leq \zeta^2 < 0.$$

Therefore $\zeta$ is a normalized numerical equivalence class of type $(c_1, c_2)$ and defines a wall $W^\zeta$ of type $(c_1, c_2)$. We show that $W^\zeta$ is nonempty, i.e. there exists $a \in \mathbb{Q}$, $a > \max\{e,\ e/2\}$ such that the polarization $D = C_0 + af_0$ satisfies $D.\zeta = 0$. But

$$D.\zeta = 0 \iff a = e - (2r - \beta)/(2d - \alpha).$$

From $2r - \beta < 0$ we get $a > e$ and from $2r - \beta < e(2d - \alpha)/2$ we get $a > e/2$, i.e. $W^\zeta$ is nonempty. Now, take the chamber $\mathcal{C}$ below the nonempty wall $W^\zeta$ such that $W^\zeta \cap \text{Closure}(\mathcal{C}) \neq \emptyset$. Then, by the Theorem 1.2.3, Chap.II in [Q3], every non-splitting 2-vector bundle $E$ of the extension (1) is $H$-stable for any ample line bundle $H \in \mathcal{C}$.

*Case 2.* $2d = \alpha$.

In this case the inequality from hypothesis is equivalent to the inequality $2r - \beta < 0$. Let $E_1 = E(-dC_0)$. Then, $E_1 \in M((0, \beta), l, 0, r)$, where $l := l(c_1, c_2, d, r) \geq 0$ and $c_1(E_1) = \bar{c}_1 = (0, \beta)$, $c_2(E_1) = \bar{c}_2 = l$. Since, for an ample line bundle $H$, $E$ is $H$-stable if and only if $E_1$ is $H$-stable, it sufficies

to show that $E_1$ is $H$-stable for some ample line bundle $H$. One may remark that $l \geq 0$ is equivalent to $c_1^2 - 4c_2 \leq 0$.
*Subcase (a)* $c_1^2 - 4c_2 < 0$
We shall prove that $E_1$ is $H$-stable for any ample line bundle $H \in \mathcal{C}_{f_0}$, where $\mathcal{C}_{f_0}$ is the chamber of type $(c_1, c_2)$ such that the $[f_0]$-axis in $\mathrm{Num}(X) \otimes \mathbb{R}$ is part of the boundary of $\mathcal{C}_{f_0}$. Clearly, by the definition, the chambers of type $(c_1, c_2)$ coincide with the chambers of type $(\bar{c}_1, \bar{c}_2)$. The 2-vector bundle $E_1$ is given by an extension

$$0 \to \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E_1 \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0,$$

and $E_1|_f \cong \mathcal{O}_f \oplus \mathcal{O}_f$, for a general fibre $f$ of $X$.

Let $\mathcal{O}_X(D) \subset E_1$ be a subsheaf of rank 1 with the quotient torsion-free and let $\eta$ be the numerical equivalence class $2D - \bar{c}_1$. We shall show that, for any ample line bundle $H \in \mathcal{C}_{f_0}$, we have

$$\mu_H(\mathcal{O}_X(D)) < \mu_H(E_1).$$

For a general fibre $f$ of $X$ we get

$$\deg(\mathcal{O}_X(D)|_f) \leq 0,$$

i.e. $D.f \leq 0$. Since $\bar{c}_1.f = 0$ we get $\eta.f \leq 0$ and we have two subcases:
*(a1)* $\eta.f = 0$. It follows

$$\mathcal{O}_X(D) = \mathcal{O}_X(qf_0) \otimes \pi^* L, \ L \in \mathrm{Pic}_0(C).$$

Since $\mathcal{O}_X(D) \subset E_1$, by the definition of $r = r_E$, we get $q \leq r$, hence

$$\mu_H(\mathcal{O}_X(D)) \leq \mu_H(\mathcal{O}_X(rf_0) \otimes \pi^* L_2).$$

But $2r < \beta$ implies

$$\mu_H(\mathcal{O}_X(rf_0) \otimes \pi^* L_2) < \mu_H(E_1),$$

for any $H \in \mathcal{C}_{f_0}$ and we are done.
*(a2)* $\eta.f < 0$. We show that $H.\eta < 0$ for any $H \in \mathcal{C}_{f_0}$ ( the inequality $H.\eta < 0$ is equivalent to the inequality $\mu_H(\mathcal{O}_X(D)) < \mu_H(E_1)$). If $H.\eta > 0$, by the index theorem applied to the divisor $(H.\eta)f - (H.f)\eta$, which is orthogonal on $H$, we get

$$(H.f)^2\eta^2 - 2(H.\eta)(f.\eta)(H.f) \leq 0.$$

Counting the signs it follows $\eta^2 < 0$. Since $-(4c_2 - c_1^2) \leq \eta^2$ (from the extension corresponding to the inclusion $\mathcal{O}_X(D) \subset E_1$), we get that $\eta$ is a numerical equivalence class of type $(c_1, c_2)$. For any $H \in \mathcal{C}_{f_0}$, $H$ and $f_0$ are not separated by any wall, hence $\text{sign}(f.\eta) = \text{sign}(H.\eta)$, contradiction. If $H.\eta = 0$ then, by the index theorem, it follows $\eta^2 \leq 0$. If $\eta^2 < 0$, it follows that $\eta$ is a numerical class of type $(c_1, c_2)$, since $-(4c_2 - c_1^2) \leq \eta^2$. Then $H \in W^\eta$, contradicting the inclusion $H \in \mathcal{C}_{f_0}$. If $\eta^2 = 0$, by the index theorem, we get $\eta$ numerically trivial. Then $2D \equiv \bar{c}_1 = (0, \beta)$, hence $D.f = 0$. But $\mathcal{O}_X(D) \subset E_1$ and, by the definition of $r = r_E$, we get $\beta/2 \leq r$, contradiction.
*Subcase (b)* $c_1^2 - 4c_2 = 0 \Leftrightarrow l = 0$
Then the canonical extension of $E_1$ becomes

$$0 \to \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E_1 \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \to 0.$$

In this case we shall prove that $E_1$ is $H$-stable for any ample line bundle $H$. Indeed, let $a \in \mathbb{Q}$ , $a > \max\{e, e/2\}$ and $\mathcal{O}_X(F) \subset E_1$ an invertible sheaf with torsion-free quotient $\det(E_1) \otimes \mathcal{O}_X(-F) \otimes I_Z$, where $Z$ is a zero-dimensional subscheme of $X$ and $F \equiv -mC_0 + nf_0$ with $m \geq 0$. We have
$F.(\bar{c}_1 - F) + \deg(Z) = c_2(E_1) = 0 \Rightarrow F.(\bar{c}_1 - F) \leq 0$ $(\Leftrightarrow me + 2n - \beta \leq 0)$. But now, if $m > 0$, since $a > e/2$, so $me - ma + n < me/2 + n \leq \beta/2$ leading us to $F.(C_0 + af_0) < \beta/2 = c_1(E_1).(C_0 + af_0)/2$, i.e. $E_1$ is $H$-stable. If $m = 0$, then $n \leq r$ (by the definition of $r = r_E$), hence $n < \beta/2$ and, again, $E_1$ is $H$-stable.

We obtain, in the particular case of ruled surfaces, the Bogomolov inequality, which was proved in this case by Takemoto; see Theorem 3.7 in [T1].

**Corollary 6** *Let $X$ be a ruled surface and let $E$ be an algebraic 2-vector bundle over $X$, with Chern classes $c_1(E) = c_1 \in \text{Num}(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$. If $E$ is $H$-stable for some ample line bundle $H$, then $\Delta(E) = (c_2 - c_1^2/4)/2 \geq 0$.*

*Proof.* With the notations of the previous theorem, we obtain from the first implication that $\zeta^2 \leq 0$ for a stable vector bundle $E$. Since $-(4c_2 - c_1^2) \leq \zeta^2$, hence $\Delta(E) \geq 0$.

## 2 Non-emptiness of moduli spaces $\mathcal{M}_H(c_1, c_2)$

Let $\pi : X \to C$ be a ruled surface and let $c_1 = (\alpha, \beta) \in \mathrm{Num}(X)$, $c_2 \in \mathbb{Z}$ be fixed numerical Chern classes. Let $H$ be an ample line bundle over $X$. We investigate the question of existence of $H$-stable 2-vector bundles $E$ with $c_1(E) = c_1$ and $c_2(E) = c_2$, i.e. the question when $\mathcal{M}_H(c_1, c_2) \neq \emptyset$. If $\tilde{c}_1 \in \mathrm{Pic}(X)$ and $c_1$ is the numerical equivalence class of $\tilde{c}_1$ then, clearly, $\mathcal{M}_H(c_1, c_2) \neq \emptyset$ if and only if $\mathcal{M}_H(\tilde{c}_1, c_2) \neq \emptyset$.

By the Bogomolov inequality, if $4c_2 - c_1^2 < 0$ then, $\mathcal{M}_H(c_1, c_2) = \emptyset$ for any polarization $H$ on $X$. The next case, $4c_2 - c_1^2 = 0$ (projectively flat bundles), which follows by the proof of Theorem 5, has been studied by Takemoto; see [T1], Theorem 3.7:

**Corollary 7** *Let $H$ be an ample line bundle over a ruled surface $X$. An algebraic 2-vector bundle $E$ over $X$ with $\Delta(E) = 0$ is $H$-stable if and only if there is a stable 2-vector bundle $F$ over the curve $C$ and a line bundle $L$ over $X$ such that $E = \pi^*(F) \otimes L$.*

**Remark** Thus, in the case $4c_2 - c_1^2 = 0$, the non-emptiness of the moduli spaces $\mathcal{M}_H(c_1, c_2)$ is reduced to the case of moduli spaces of stable bundles over curves.

From now on, we shall assume $4c_2 - c_1^2 > 0$. As we have seen in Definition 3, there exist in this case walls and chambers of type $(c_1, c_2)$ in the ample cone $\mathbf{C}_X$. Let $H \equiv aC_0 + bf_0$ be an ample divisor over the ruled surface $X$. Recall that $a > 0$ and $b > ae$ if $e \geq 0$ and, $a > 0$ and $b > ae/2$ if $e < 0$ (see [Ha], p. 382). Therefore, in the case of a ruled surface, the ample cone has a simple description. Moreover, from the conditions in Definition 3 (ii)

$$-(4c_2 - c_1^2) \leq \zeta^2 < 0,$$

we get that there exist always a finite number of walls and chambers. Recall that we denoted by $\mathcal{C}_{f_0}$ the chamber of type $(c_1, c_2)$ such that the $[f_0]$-axis in $\mathrm{Num}(X) \otimes \mathbb{R}$ is part of the boundary of $\mathcal{C}_{f_0}$.

Firstly, suppose that the ample line bundle $H$ belongs to some chamber. It is well-known that if $H_1$ and $H_2$ lie in the same chamber of type $(c_1, c_2)$ then, $\mathcal{M}_{H_1}(c_1, c_2)$ and $\mathcal{M}_{H_2}(c_1, c_2)$ can be naturally identified (see, for example [F], [Q2]). In [A-B2], as a consequence of the Theorem 2, we obtained the following result:

**Corollary 8** *Let $X$ be a ruled surface. Assume that $X$ is not $\mathbb{P}^1 \times \mathbb{P}^1$ . Let $\mathcal{C}$ be any chamber of type $(c_1, c_2)$ different from $\mathcal{C}_{f_0}$. Then the moduli space $\mathcal{M}_{\mathcal{C}}(c_1, c_2) \neq \emptyset$.*

**Remark** In fact, in the case $X = \mathbb{P}^1 \times \mathbb{P}^1$, $C_0$ defines the other axis $[C_0]$ in Num$(X)$ which lies on the boundary of $\mathbf{C}_X$ and , by the last remark in [A-B2], if $\mathcal{C}$ is a chamber different from $\mathcal{C}_{f_0}$ lying below a non-empty wall $W$ defined by a normalized class $\zeta \equiv uC_0 + vf_0$ of type $(c_1, c_2)$ such that either $l_\zeta(c_1, c_2) > 0$ or $v < -1$ then $\mathcal{M}_{\mathcal{C}}(c_1, c_2) \neq \emptyset$. It is easy to see that if $l_\zeta(c_1, c_2) = 0$ and $v = -1$ then $\mathcal{C} = \mathcal{C}_{C_0}$, where we denoted by $\mathcal{C}_{C_0}$ the chamber that has the $[C_0]$-axis on its boundary.

Indeed, let $H \equiv C_0 + af_0$ , $a \in \mathbb{Q}$ , $a > 0$ be a class lying on a nonempty wall $W^\zeta$, where $\zeta \equiv uC_0 + vf_0$ is a normalized class of type $(c_1, c_2)$. Then $a = -v/u$ and we have to prove that $\zeta_0.H \geq 0$ for $\zeta_0 = u_0C_0 - f_0$, i.e.

$$(((4c_2 - c_1^2)/2)C_0 - f_0).(C_0 + af_0) \geq 0.$$

Since $v^2 \geq 1$, then $-2u/v \leq -2uv \leq 4c_2 - c_1^2$ so $-v(4c_2 - c_1^2)/2u - 1 \geq 0$.

Let us consider now the case $H \in \mathcal{C}_{f_0}$:

**Corollary 9** *Let $X$ be a ruled surface. Then $\mathcal{M}_{\mathcal{C}_{f_0}}(c_1, c_2)$ is nonempty if and only if $\alpha$ is even and the intersection $[\beta/2 - (g + c_2 - c_1^2/4)/2,\ \beta/2) \cap \mathbb{Z}$ is nonempty.*

*Proof*: Firstly, we remark that $2d = \alpha$ if the vector bundle $E$ is $\mathcal{C}_{f_0}$-stable. Indeed, the bundle $E$ is given by an extension (2) and, if we suppose $2d > \alpha$, then from the proof of Theorem 5 it follows that $\zeta = (2d - \alpha)C_0 + (2r - \beta)f_0$ is a normalized numerical class of type $(c_1, c_2)$ defining a nonempty wall $W^\zeta$. Since $\mathcal{C}_{f_0}$ is above $W^\zeta$, it follows that $H.\zeta > 0$ for any $H \in \mathcal{C}_{f_0}$, which is equivalent to the fact that the subsheaf $N_2$ of the extension (2) is a destabilising subsheaf of $E$, contradiction (compare also with [T1] theorem 3.7).

Secondly, in the case $2d = \alpha$, if $2r - \beta < 0$ then $E$ is non-splitting. Indeed, if $E$ would be splitting, then $E$ would be given by an extension (2) with $Y = \emptyset$. Since $N_1 \subset E$, by the definition of $r = r_E$, we get $s \leq r$, contradiction with $2r < \beta$.

By Theorem 5 it follows that $\mathcal{M}_{C_{f_0}}(c_1, c_2)$ is nonempty if and only if $\alpha$ is even ($\alpha = 2d$) and there exists an integer $r$ with $2r < \beta$ such that $M(c_1, c_2, d, r) \neq \emptyset$. By Theorem 2 we know that $M(c_1, c_2, d, r) \neq \emptyset$ if and only if $l = l(c_1, c_2, d, r) \geq 0$ and $\beta - 2r \leq g + l$. Thus, $\mathcal{M}_{C_{f_0}}(c_1, c_2)$ is nonempty if and only if $\alpha = 2d$ , $c_1^2 - 4c_2 < 0$ and there exists an integer $r$ such that the following conditions hold:

$$l(c_1, c_2, d, r) \geq 0,\ 0 < \beta - 2r \leq g + l(c_1, c_2, d, r),$$

which are equivalent to the conditions of the corollary.

**Corollary 10** *If $X = \mathbb{P}^1 \times \mathbb{P}^1$ then, with the notations from the above remark $\mathcal{M}_{C_{C_0}}(c_1, c_2) \neq \emptyset$ if and only if $\beta$ is even and the intersection $[\alpha/2 - (g + c_2 - c_1^2/4)/2,\ \alpha/2) \cap \mathbb{Z}$ is nonempty.*

Now, suppose that the ample line bundle $H$ lies on some nonempty wall. In principle, by using the formulae of Qin in [Q1], [Q2], [Q3] and the previous corollaries one should get the non-emptiness of the stable moduli spaces for polarizations lying on walls. We were able to obtain only the following particular result:

**Corollary 11** *Let $X$ be a ruled surface different from $\mathbb{P}^1 \times \mathbb{P}^1$ with nonnegative invariant $e$ and assume $g \leq e + 1$. Let $H \equiv aC_0 + bf_0$ be an ample line bundle lying on some nonempty wall $W$ of type $(c_1, c_2)$ and denote $b/a = k$. Assume either $\zeta.f_0 \geq 2$ for all normalized numerical equivalence classes $\zeta$ which represent the wall $W$ or $4c_2 - c_1^2 > 2k - e$. Then $\mathcal{M}_H(c_1, c_2)$ is nonempty.*

*Proof*: We shall use some results of Qin. Let $\tilde{c}_1 \in \mathrm{Pic}(X)$ such that $c_1$ is the numerical equivalence class of $\tilde{c}_1$. By Proposition 1.3.1, Chap.II in [Q3] we get

$$\mathcal{M}_H(\tilde{c}_1, c_2) = \mathcal{M}_C(\tilde{c}_1, c_2) - \bigsqcup_{\zeta} E_\zeta(\tilde{c}_1, c_2)\ ,$$

where $\zeta$ runs over all normalized numerical equivalence classes which define the wall $W$ and the chamber $C$ lies below the wall $W$ such that $W \cap$

Closure($\mathcal{C}$) $\neq \emptyset$. By Corollary 8 we have $\mathcal{M}_C(\tilde{c}_1, c_2) \neq \emptyset$. Then, by a well-known result on deformation theory of vector bundles (see, for example [B2], p. 144), we get

$$\dim \mathcal{M}_C(\tilde{c}_1, c_2) \geq 4c_2 - c_1^2 + 3g - 3,$$

where $4c_2 - c_1^2 + 3g - 3$ is the "expected dimension". We shall prove that the dimensions of all sets $E_\zeta(\tilde{c}_1, c_2)$ are strictly smaller than the expected dimension.

Following Qin, let us denote the dimension of $E_\zeta(\tilde{c}_1, c_2)$ by $D_\zeta(\tilde{c}_1, c_2)$ and put

$$d_\zeta(\tilde{c}_1, c_2) := D_\zeta(\tilde{c}_1, c_2) - (4c_2 - c_1^2 + 3g - 3).$$

By Proposition 1.7 in [Q1] we get

$$d_\zeta(\tilde{c}_1, c_2) = \zeta^2/4 - (4c_2 - c_1^2)/4 + \zeta.K_X/2 + 1 - g.$$

Let $\zeta = uC_0 + vf_0$ be a normalized numerical equivalence class which represents the wall $W$. From $H.\zeta = 0$, $a > 0$ and $k > e$ ($H$ ample) we get the condition $v = u(e - k) < 0$. By computation, we obtain:

$$d_\zeta(\tilde{c}_1, c_2) = (u-2)(2v - eu)/4 + (u-1)(g-1) - (4c_2 - c_1^2)/4.$$

Let us suppose that $u = \zeta.f_0 \geq 2$ for all $\zeta$. By Definition 3 we have

$$-(4c_2 - c_1^2) \leq \zeta^2 < 0,$$

hence

$$d_\zeta(\tilde{c}_1, c_2) \leq (u-1)(2v - 2 + 2g - eu)/2 \leq (u-1)(2v + e(2-u))/2 < 0.$$

Now, suppose there exist normalized numerical equivalence classes $\zeta$ with $u = \zeta.f_0 = 1$ ($u > 0$) and that $4c_2 - c_1^2 > 2k - e$. For these classes we get

$$d_\zeta(\tilde{c}_1, c_2) = (e - 2v)/4 - (4c_2 - c_1^2)/4.$$

But $v = e - k$, hence

$$d_\zeta(\tilde{c}_1, c_2) = ((2k - e) - (4c_2 - c_1^2))/4 < 0.$$

It follows $\mathcal{M}_H(c_1, c_2)$ nonempty.

**Remark** If $\zeta_0 = C_0 + v_0 f_0$ is a normalized class of type $(c_1, c_2)$ defining a nonempty wall $W$ and $4c_2 - c_1^2 = e - 2v_0$, then $W$ is a part of the boundary of $\mathcal{C}_{f_0}$.

Indeed, we have to prove that there are no walls between $W$ and the $[f_0]$-axis.
Let $\zeta = uC_0 + vf_0$ be a normalized class of type $(c_1, c_2)$ defining a nonempty wall $W^\zeta$. Then $4c_2 - c_1^2 \geq \zeta^2 > 0$, $u > 0$ and $v < \min\{0, ue/2\}$.
Let $H = C_0 + af_0 \in W^\zeta$, where $a \in \mathbb{Q}$, $a = (ue - v)/u$. We want to prove that $H.\zeta_0 \leq 0$.
But $H.\zeta_0 = (ue - 2v + u(c_1^2 - 4c_2))/2u$. Now $u^2 \geq 1 \Rightarrow (u^2 e - 2uv)/u^2 \leq u^2 e - 2uv \leq 4c_2 - c_1^2$, which implies $H.\zeta_0 \leq 0$.

**Remark** By using some proofs as in the previous corollaries one may obtain results about the non-emptyness of $\mathcal{M}_H(c_1, c_2)$ for $X = \mathbb{P}^1 \times \mathbb{P}^1$ and $H$ lying on walls.

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