VARIATIONS OF EQUIMULTIPLICITY AND GRADED COHEN-MACAULAY RINGS

Manfred Herrmann

Sonderforschungsbereich 40
Theoretische Mathematik

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These lectures have their geometric roots in the resolution and classification of singularities of algebraic varieties. These two aspects are not independent ef each other: Any canonical process of desingularization leads to some kind of classification, and conversely one may expect that a "good" classification contains enough information to give a recipe for resolution. For plane curve singularities these two aspects coincide: Blowing up points for resolution gives a multiplicity sequence which determines the topological type of the singularity, and vice versa.

In generalizing the plane curve case to higher dimensions (and codimensions), the procedure of Zariski, Hironaka and Abhyankar was to blow up non-singular centers contained in the singular locus of the given variety. To decide about an improvement of a singularity under this process, a <u>numerical measure of the singularity</u> was needed, and the choice of the center of blowing up was related to this measure. Here is a hierarchy of numerical conditions on a <u>non-singular</u> center D :

i) all points of D have the same multiplicity;
ii) all points of D have the same Hilbert polynomial;
iii) all points of D have the same Hilbert function .

These conditions coincide for hypersurfaces, but they differ in general. For each condition there is an algebraic description : For i) by reduction of ideals ([H-O1]), for ii) and iii) by flatness conditions on the associated graded ring ([O-R], [Be]).

Besides blowing up <u>non-singular</u> centers, there are also approaches which amount to blowing up singular centers:

(1) When Zariski - following Jung - used generic projections of surfaces and embedded resolution of the discriminant locus, blowing up points on the discriminant induced blowing up of "thick points" on the original surface. Zariski's procedure also used normalization; whereas a "good" proof of desingularization should combine the following features : it should be canonical, it should not use normalization and globalizing should be no problem. The first canonical proof for surfaces not using normalization was given by Zariski in 1967.

(2) Another possibility to blow up a singular center is to blow up the intersection of the given (embedded) variety with a linear subvariety of a suitable embedding space. Then the center on the given variety will be singular in general.

An obvious problem is how to generalize the numerical conditions, their algebraic descriptions and the consequences to singular centers. In [H-O-1] , [H-O-2], [H-S-V], [O] , [O-R] , [Li] such a generalization was developped by using generalized Hilbert functions. In the first section we will give an account of this theory. Here we view in particular 3 types of numerical conditions as three possibilities to make precise the naive idea of equimultiplicity along a subvariety.

The important role of multiplicities and Hilbert functions is that they furnish some way of measuring and comparing singularities. Measuring singularities by multiplicities is much cruder than by Hilbert functions. One aspect of the difference is to be seen in the fact that lower dimensional components do not enter into multiplicities. Therefore, in order to get satisfactory results on multiplicities, one has to use a restricted class of rings for which certain dimension formulas hold. From this point of view Ratliff's extension theory of quasi-unmixed ("formally equidimensional") rings furnishes an appropriate frame for multiplicity theory. [Therefore already Chevalley 1943 assumed his local rings to be quasi-unmixed.] This opinion is supported by the nice behaviour of quasi-unmixed rings under blowing-up, and by the connection of these rings to multiplicities via reduction of ideals.

In the zero section we recall to some of Ratliff's results. We also include some consequences that don't seem to have been noticed explicitely before.

The idea of hyperplane sections leads to the investigation of Hilbert functions and multiplicities for ideals of the principal class. In section 1 we describe some results in this direction. One essential problem in this connection is to give a characterization of stability of multiplicities resp. Hilbert functions under blowing-up.

In section 3 and 4 we turn to the problem how to link the Cohen-Macaulay (CM for short) property of a local ring R to the CM properties of the various graded rings related to the blowing-up along an ideal $I \subset R$:

a) the blowing-up $Bl_I(R) = \mathrm{Proj} \oplus I^n$

b) the associated graded ring $gr_I(R)$ with respect to I

c) the Reesring $Re(I,R) = R[It,t^{-1}]$

d) the Reesring $Re^+(I,R) = R[It]$.

We will also treat the question how far it is necessary to assume R to be Cohen-Macaulay. Our main tools to give some answers to these questions come from multiplicity theory. On the other hand we also indicate some cohomological approach in section 4 .

§ 0 . Notations. Auxiliary results.

0.1. Quasi-unmixed rings ([Ra-0] - [Ra-3] , [Na])

Definition 0.1.1: A semilocal ring R is said to be quasi-unmixed, if each minimal prime ideal $\underline{P}$ in the completion $\hat{R}$ satisfies :

$$\dim(\hat{R}/\underline{P}) = \dim(\hat{R}) \ (= \dim R) .$$

These rings can be characterized by chain conditions. For that we recall to the following definitions.

Definition 0.1.2 : (i) An (integral) domain A satisfies the "altitude formula" if for each finitely generated (integral) domain B over A and for any prime ideal $\underline{P} \subset B$ the following relation is fulfilled:

$$\dim(B_{\underline{P}}) + \operatorname{trd}(B/P \big/ A/A\cap\underline{P}) = \dim(A_{A\cap\underline{P}}) + \operatorname{trd}(B/A) .$$

(ii) The ring R satisfies the altitude-formula if $R/\underline{P}$ satisfies the altitude formula for each minimal prime ideal $\underline{P}$.

Remark : Note that a noetherian ring R satisfies the altitude-formula if and only if R is universal catenarian.

Definition 0.1.3 : (i) A local ring R is said to be formal-catenarian, if $R/\underline{P}$ is quasi-unmixed for every minimal prime ideal $\underline{P} \subset R$.

(ii) A noetherian ring R is said to be locally-formal-catenarian, if every localization of R is formally catenarian.

Theorem 0.1.4 [Ra-1] : Let R be a noetherian ring. Then the following are equivalent:

(i) R is locally formal-catenarian

(ii) R is satisfies the altitude formula

(iii) R satisfies the chain condition [Na] for prime ideals.

If R in the theorem 0.1.4 is a domain then condition (i) may be replaced by R is locally quasi-unmixed , i.e. every localization is quasi-unmixed.

The next propositions describe the situation for a local ring, see [Ra-0].

Proposition 0.1.5 : A local ring R is quasi-unmixed if and only if R is formal-catenarian and equidimensional (i.e. $\dim R/\underline{P} = \dim R$ for all minimal primes).

<u>Proposition 0.1.6 :</u> A quasi-unmixed local ring R satisfies the following conditions :

(i) $R_{\underline{P}}$ is quasi-unmixed for all primes $\underline{P} \subset R$,

(ii) $\dim R = \dim R/a + ht(\underline{a})$ for all ideals $\underline{a} \subset R$,

(iii) the polynomialring R[X] is locally quasi-unmixed,

(iv) if R is a domain then any finitely generated R-algebra is also quasi-unmixed.

<u>Proposition 0.1.7 :</u> Let R be a quasi-unmixed local ring and $\underline{a}$ an ideal in R . Then $R/\underline{a}$ is quasi-unmixed if and only if all minimal primes of $\underline{a}$ have the same height.

<u>0.1.8.</u> From the geometric point of view it is interesting that quasi-unmixed local rings R can be characterized by the "equimultiple" ideals I of R , s. [Ra-3]. To describe this result we have to recall in section 0.2 and 0.3 the properties of the analytic spread of an ideal I and the two types of Reesrings Re(I,R) and $Re^{+}(I,R)$.

0.2 <u>Analytic spread.</u>

<u>Proposition 0.2.1 :</u> Let I and J be two ideals in a local ring (R,M) with $I \supset J$. Then the following statements are equivalent:

(i) $J \cdot I^n = I^{n+1}$ for some n

(ii) $\bigoplus_{n\geq 0} I^n$ is a finitely generated $\bigoplus_{n\geq 0} J^n$-module

(iii) $\bigoplus_{n\geq 0} I^n/MI^n$ is integral over the R/M-subalgebra (of $\oplus I^n/MI^n$) generated by J+MI/MI .

<u>Definition 0.2.2 :</u> If one the statements (i), (ii) or (iii) holds, J is called a <u>reduction</u> of I . A reduction J of I is said to be <u>minimal</u> if the properties $J' \subseteq J$ and J' is a reduction of I imply J' = J.

Consider the blowing up of the ideal I :
$\mathrm{Proj}(\bigoplus_{n\geq 0} I^n) \longrightarrow \mathrm{Spec}\, R$, let $Y = \mathrm{Proj} \bigoplus (I^n/MI^n)$ be the "closed" fibre $f^{-1}(M)$. Let $l(I) = \dim Y + 1 = \dim(\bigoplus_{i\geq 0} I^i/MI^i)$.

<u>Proposition 0.2.3 :</u> Let (R,M) be a local ring with infinite residue field and I an ideal in R . Then $l(I)$ is the least number of generators of any minimal reduction J of I .

<u>Definition 0.2.4 :</u> $l(I) = \dim(\bigoplus_{n\geq 0} I^i/MI^i)$ is called the <u>analytic spread</u> of I .

Note that reductions of ideals are closely related to multiplicity. So we know by Rees that two M-primary ideals $Q_1 \subset Q_2$ in a quasi-unmixed ring R have the same multiplicity if and only if Q_1 is a reduction of Q_2 . A corresponding statement is true for ideals $J \subset I$ of height <u>less</u> then $\dim(R)$ by Böger [Bö].

<u>Theorem 0.2.4*(Böger) :</u> Let R be quasi-unmixed and let $J \subset I$ be ideals in R such that

(i) J and I have the same radical

(ii) $ht(J) = l(J)$

(iii) $e(IR_P) = e(JR_P)$ for every minimal prime P of I .

Then J is a <u>reduction</u> of I .

See furthermore the remark to Teissier's "principal of specialization of integral dependence" in section 1.2.

<u>Proposition 0.2.5 :</u> Let (R,M) be a local ring and I an ideal in R . Then the following is true :

(i) $l(I) \leq \dim R$; equality holds if I is M-primary.

(ii) $l(I) \geq l(IR_{\underline{P}})$ for every prime P containing I

(iii) $l(I) \geq ht(P)$ for every minimal prime P of I ;
in particular $l(I) \geq ht(I)$.

(iv) $ht(I) \leq \dim R - \dim R/I \leq l(I)$.

Note that (ii) describes the upper semicontinuity of the fibre dimension of the blowing-up of the ideal I, [EGA IV], 13.1.3.

0.3 Auxiliary results on Reesrings.

0.3.1. Definition : Let (R,M) be a local ring and I a proper ideal. Let t be an indeterminate over R and let $u = t^{-1}$ (in the total quotient ring of R[t]) . Then we consider the following subrings of R[t,u], which are called Reesrings of R with respect to I, defined by

$$Re^+(I,R) = R[It] \cong \bigoplus_{n\geq 0} I^n$$

and

$$Re(I,R) = R[It,u] = Re^+(I,R)[u]$$

(The last one has been introduced by Rees).

0.3.1*. We have the obvious relations :

(0) $Re(I,R) = \bigoplus_{-\infty}^{+\infty} I^i t^i$ with $I^i = R$ if $i \leq 0$.

(1) $Re^+(I,R)/IRe^+(I,R) \cong Re(I,R)/uRe(I,R) \cong gr_I R$.

(2) $Re(I,R)/(It)Re(I,R) \cong R/I\,[\bar{u}]$, where the residue class $\bar{u}$ of u is algebraically independent over R/I ;

in particular it follows that

(2') $\dim(Re(I,R)/(It)Re(I,R) = \dim(R/I)+1$.

(3) $u^n \cdot Re(I,R) \cap R = I^n$ for all $n \geq 0$; and u is a non-zero-divisor in Re(I,R).

If $\mathcal{R} := Re(I,R)$, $\bar{\mathcal{R}}$ the normalization of $\mathcal{R}$, i.e. the integral closure of $\mathcal{R}$ in its total quotientring, and $\bar{I}$ the integral closure of the ideal I in R , then

(4) $u^n \cdot \bar{\mathcal{R}} \cap R = \overline{I^n}$.

<u>Lemma 0.3.2 :</u> Let (R,M) be a local ring, I a proper ideal of R and let N denote the unique homogeneous maximal ideal of $Re(I,R)$. Then we have :

(i) $\dim(Re(I,R)) = ht(N) = \dim R+1$

(ii) $\dim(gr_I(R)) = ht(N/uRe(I,R)) = \dim R$

(iii) $\dim(Re(I,R)/MRe(I,R) + uRe(I,R)) = ht(N/MRe(I,R) + uRe(I,R)$
$= l(I)$, the analytic spread of I .

Statement (i) has been proved by Ratliff, (ii) follows directly from (i), and (iii) is an easy consequence of the definition of analytic spread.

Later on we will ask for the existence of a homogeneous system of parameters for graded rings of the type $Re^+(I,R)$, $Re(I,R)$ and $gr_I R$. If A is any of these rings, then it has a unique homogeneous maximal ideal, let's say Q , and to have a homogeneous system of parameters we need to know that $\dim A = \dim A_Q$.

<u>Remark 0.3.3 :</u> For a graded ring, a homogeneous system of parameters <u>need not</u> exist in general. There is an important special case of graded rings A for which the existence of homogeneous systems of parameters is guaranted, namely if $A = \bigoplus_{n\geq 0} A_n$ is positively graded (noetherian) and A_0 is a field k . Since systems of parameters are invariant modulo nilpotent elements, the same is true if A_0 is an artinian local ring. In particular, if Q is an M-primary ideal in a local ring (R,M), then $gr_Q R$ <u>has</u> a homogeneous system of parameters. We shall indicate in § 2 that for a proper ideal I in a quasi-unmixed local ring (R,M) the associated graded rings $gr_I R$ and $Re(I,R)$ have a homogeneous system of parameters if and only if I is "equimultiple" in the sense that $ht(I) = l(I)$; see § 1 .

For $Re^+(I,R)$ one has the following answer (see § 2) : If $ht(I) > 0$ and if R/M is infinite, then $Re^+(I,R)$ has a homogeneous system of parameters if and only if $l(I) = 1$: If $ht(I) = l(I) = 1$, and $d = \dim R$, then $\dim R/I = d-1$. Hence there exists a system $b_1,\dots,b_{d-1}$ of parameters mod I; furthermore we have one parameter element a_1 = minimal reduction of I . It is easy to show that therefore

$$a_1 , a_1 t , b_1,\dots, b_{d-1}$$

is a homogeneous system of parameters of $Re^+(I,R) \cong R[It]$.)

In particular, if $\dim R \geq 2$, then $Re^+(M,R)$ cannot have a homogeneous system of parameters.

More exactly one can show that the number of homogeneous elements in $Re^+(Q,R)$, where Q is an M-primary ideal in R , is at most 2 . For $\dim R = 1$ we get the homogeneous system $\{a_1, a_1 t\}$ in $Re^+(M,R)$, see [H-O-G].

Now we want to explain one direction of the proof of Rafliff's theorem on the characterization of quasi-unmixed rings (see 0.1.8).

<u>Theorem 0.3.4 :</u> A local ring (R,M) is <u>quasi-unmixed</u> if and only if for every ideal I such that <u>$ht(I) = l(I)$</u> , the integral closure $\bar{I}$ has <u>no embedded prime divisors.</u>

<u>Sketch of the proof</u> of "$\Longrightarrow$" (after Ratliff) :

Let $\mathfrak{P}$ be any prime divisor of $\bar{I}$. For $x \in R$, $x \in \bar{I}$ iff $x' \in \overline{I+P/P}$ for all minimal primes P in R (where $x' = x \bmod P$). Hence there exists a minimal prime ideal Po of R <u>contained</u> in the <u>given</u> $\mathfrak{P}$, so that $\mathfrak{P}/P_o$ is a prime divisor of $\overline{I + P_o/P_o}$. By $\mathcal{R}^*$ we denote the Reesring $Re(I+P_o/P_o, R/P_o)$. Since $u\overline{\mathcal{R}^*} \cap R/P_o = \overline{I+P_o/P_o}$ by 0.3.1*,(4) we find a prime divisor P** of $u\cdot\overline{\mathcal{R}^*}$ such that

$$P^{**} \cap R/P_o = \mathfrak{P}/P_o .$$

We have $ht(P^{**}) = 1$ since $\overline{\mathcal{R}^*}$ = integral closure of a noetherian domain is a Krullring, s. [Na], 33.10 .

Now there is a big step to show that, since R and therefore R/P_o are quasi-unmixed (s. prop. 0.1.7), the Reesring $\mathcal{R}^*$ is locally quasi-unmixed. Hence $\mathcal{R}^*$ satisfies the altitude formula (see theorem 0.1.4 and prop. 0.1.5). Note that $ht(P^{**} \cap \mathcal{R}^*) = 1$, since $\mathcal{R}^*$ satisfies the chain-cond. for prime ideals. Denote $P^{**} \cap \mathcal{R}^*$ by P^*. Then the altitude formula tells us that for the situation $A = R/P_o$ and $B = \mathcal{R}^*$ we have :

$$(+) \quad \underset{\substack{\| \\ 1}}{ht(P^*)} + trd(\mathcal{R}^*/P^* / R/\mathfrak{P}) = ht(\mathfrak{P}/P_o) + \underset{\substack{\| \\ 1}}{trd(\mathcal{R}^*/R/P_o)} .$$

From this we get finally

$$(*) \; trd(\mathcal{R}^*/P^*/R/P) = \dim\left(Re(IR_{\mathfrak{P}}, R_{\mathfrak{P}})/\hat{P}Re(IR_{\mathfrak{P}}, R_{\mathfrak{P}})\right) \le l(IR \quad),$$

where $\hat{P}$ is an ideal in $\mathcal{R}$ corresponding to P^* under $\mathcal{R}^* \cong \mathcal{R}/P_o'$, where $\mathcal{R} = Re(I,R)$, $P_o' \cong \bigoplus_{i \in \mathbb{Z}} (P_o \cap I^i)t^i$

From (+) and (*) we get :

$$(3) \quad ht(I) \le ht(\mathfrak{P}) = ht(\mathfrak{P}/P_o) \le l(IR_{\mathfrak{P}}) \le l(I) .$$

By assumption, $ht(I) = l(I)$, hence (3) yields

$$ht(\mathfrak{P}) = ht(I) = ht(\bar{I}) ,$$

q.e.d.

Ratliff's proof of "$\Leftarrow$" is purely algebraic and rather complicated, see [Ra-3]. A simplification of Ratliff's proof from the algebraic point of view is contained in the forthcoming thesis of U. Grothe; see [Gr].

Remark 0.3.5 : We have indicated in two cases that the condition $ht(I) = l(I)$ is useful from the algebraic point of view. In the next section we describe geometric meanings of this condition. In particular, since the notion of quasi-unmixed rings gives the correct frame for the study of multiplicities it seems reasonable to clarify the link between Zariski's equimultiplicity $e(R) = e(R_P)$ and the condition $ht(P) = l(P)$ for a prime ideal $P \subset R$.

<u>§ 1 Equimultiple ideals</u>

1.1. Definitions, facts and examples.

For a local (noetherian) ring (R,M) and an ideal $I \subset R$ we recall the following two definitions that grew out of the <u>naive</u> idea of <u>equimultiplicity along a subvariety :</u>

<u>(1.1.1) Definition.</u> For a prime ideal $I = P$, R is said to be <u>equimultiple along P in the sense of Zariski</u>, if

$$e(R) = e(R_P) .$$

<u>(1.1.2) Definition.</u> R is said to be <u>normally flat along I</u> if <u>I^n/I^{n+1} is a free R/I - module</u> for all $n \geq 0$.

Let us first look at the case that $I = P$ is a <u>regular</u> prime, i.e. R/P is regular. If furthermore R is a hypersurface, i.e. $R = Q/fQ$ for some regular local ring Q , then equimultiplicity in the sense of Zariski and normal flatness coincide, whereas they differ considerably in general.

For regular primes P normal flatness is generally stronger then equimultiplicity.

<u>Example</u> s. [H-O-2]) : Let

$$R = k[[X,Y,Z,W]]/(Z^2-W^5,Y^2-XZ) \cong k[[x,y,z,w]] .$$

Note that $R \cong k[[u^2,ut^5,t^{10},t^4]]$.

Let $P = (y,z,w) \cong (ut^5,t^{10},t^4)$.

Then we have : $R/P \cong k[[x]]$ and $e(R) = e(R_P) = 4$.

But R is <u>not</u> normally flat along P , since P/P^2 has torsion: $x \cdot z \in P^2$, but $x \notin P$.

Note that here R is even a strict complete intersection, i.e. also the graded ring $gr_M R \cong k[X,Y,Z,W]/(Z^2, Y^2-XZ)$ is a complete intersection.

Let $f : Bl_I(R) \to Spec(R)$ be the blowing up of R along I, and let $f_o : f^{-1}(Spec\ R/I) \to Spec(R/I)$ denote the restriction of f to the exceptional divisor. For a <u>regular</u> curve $V(I) = V(P)$ on a surface Spec R (analytically irreducible), Zariski proved that $e(R) = e(R_p)$ if and only if f_o is a <u>finite</u> morphism. In [H-O-1] we have generalized this fact to any dimension, using Böger's theorem. We recall this result in the following Theorem, which also contains the well-known translation of normal flatness into a depth-condition (see e.g. [A-K]) :

<u>(1.1.3) Theorem.</u> Let P be a <u>regular</u> prime ideal in a local ring R and let f_o denote the restriction to the exceptional divisor of the blowing up morphism $f : Bl_P(R) \to Spec(R)$. Then the following hold :

a) Assume that R is quasi-unmixed. Then $e(R) = e(R_P)$ if and only if f_o is equidimensional (i.e. all fibres of f_o have the same dimension).

b) R is normally flat along P if and only if depth $P^n/P^{n+1} = \dim R/P$ for all $n \geq 0$.

A proof of a) is included in the proof of the following prop. 1.1.5.

Normal flatness implies (Zariski-)equimultiplicity in the case R/P is regular, but this conclusion needs not hold for <u>arbitrary</u> R/P .

(1.1.4) <u>Example.</u> Let k be a field and

$$R = k[[X,Y,Z]]/(Z^2-(X^2-Y^3)) = k[[x,y,z]] \quad ,$$

$$P = (z, x^2-y^3)R = zR \text{ ; i.e. } R_P \text{ regular.}$$

Then $e(R) = 2 \neq e(R_P) = 1$, but R is normally flat along P , since P is generated by a non-zerodivisor.

On the other hand (see [H-O-3]) normal flatness of R along I implies the equidimensionality of f_o for any I. [It should be remarked that the flatness of the morphism $\mathrm{Proj}(gr_I R) \longrightarrow \mathrm{Spec}(R/I)$ implies "projective normal flatness" (s. [O-R]), i.e. I^n/I^{n+1} is flat over R/I for $n >> 0$ and vice versa. And this last condition also implies the equidimensionality of f_o, provided that $\dim R = \dim(R/I) + ht(I)$ see [H-O-3].] Also the condition depth $I^n/I^{n+1} = \dim R/I$ for all n implies that f_o is equidimensional, provided that $\dim(R) = \dim(R/I) + ht(I)$. Now $ht(I) - 1$ is the smallest dimension of the fibres of f_o at the generic points of Spec R/I, while the dimension of the fibre of f_o at the closed point is, as we have pointed out, $l(I)-1$, where $l(I)$ denotes the analytic spread of I. Because of the semicontinuity of the dimensions of the fibres of f_o, we see that f_o is equidimensional if and only if $ht(I) = l(I)$. We will view this last equation as an equimultiplicity condition, and in fact it can be translated into an equality of certain multiplicities to be introduced below.

Choosing elements $\underline{x} = \{x_1,\ldots,x_r\} \subset R$ whose images in R/I form a system of parameters, and using the multiplicity symbol $e(\underline{x};-)$ of Northcott and Wright (see [No]), we define a numerical function by

$$H^{(o)}[\underline{x},I,R](n) = e(\underline{x};I^n/I^{n+1}) .$$

Then

$$H^{(o)}[\underline{x},I,R] = \sum_{P\in \mathrm{Assh}(R/I)} e(\underline{x};R/P)\cdot H^{(o)}[IR_P,R_P],$$

where $\mathrm{Assh}(R/I) = \{P \in \mathrm{Ass}(R/I) \mid \dim R/P = \dim R/I\}$, and where $H^{(o)}[IR_P,R_P] = H^{(o)}[\emptyset,IR_P,R_P]$ is the usual Hilbert function of the PR_P-primary ideal $I\cdot R_P$. From the above equation we se that the values of $H^{(o)}[\underline{x},I,R](n)$ for large n are given by a polynomial, of degree d and with highest coefficient a_d let's say. We define the multiplicity of I with respect to $\underline{x}$ by

$$e(\underline{x},I,R) = d!a_d .$$

(This multiplicity has been introduced indepently, and with different methods, by E.Dade [D] and R. Schmidt. Our approach is due to R. Schmidt.) If we have $\dim(R) = \dim(R/I) + ht(I)$, then *the "multiplicity-formula"*

(*) $$e(\underline{x},I,R) = \sum_{P \in Assh(R/I)} e(\underline{x},R/P)e(IR_P,R_P) \;;$$

and the following semicontinuity property hold :

$$e(\underline{x},I,R) \leq e(I+\underline{x}R,R) \quad \text{for any } \underline{x} .$$

We assume R/M to be infinite. Then this inequality can be seen by using step 1 in the proof of the following proposition 1.1.5. By this step we know that for $V := I+\underline{x}R$ there exists a sequence $x_1^*,\dots,x_r^*$ of superficial elements for V such that

(1) $V := I+\underline{x}R = I+\underline{x}^*R$

(2) $e(\underline{x},I,R) = e(x^*,I,R)$ and

(3) $e_o(V) = e_o(V/\underline{x}^*R)$.

Then by the second step in the proof of proposition 1.1.5 we have:

$$e(\underline{x},I,R) = e(\underline{x}^*,I,R) \leq e_o(I+\underline{x}^*R) = e_o(I+\underline{x}R) .$$

Now the interpretation of $ht(I) = l(I)$ as a <u>multiplicity condition</u> is given in the next proposition :

(1.1.5) <u>Proposition.</u> Let R be a local ring, I an ideal of R and $\underline{x} = \{x_1,\dots,x_r\}$ any elements whose images in R/I are a system of parameters. Then

$$ht(I) = l(I) \Rightarrow e(\underline{x},I,R) = e(I+\underline{x}R,R) ,$$

and both conditions are <u>equivalent</u> if R is <u>quasi-unmixed.</u>

<u>Proof:</u> We assume $ht(I) > o$ and $|R|M| = \infty$. [For $ht(I) = o$ s. [H-O-4]].

I) Let <u>$ht(I) = l(I)$</u> and let $x_1,\dots,x_r$ any system of parameters for R/I . We put $s := ht(I)$ and $r := \dim R/I$.

Let $z_1,\dots,z_s$ generate a minimal reduction B of I . Then B and I have the same minimal primes, and for each such prime P we have

$$(1) \qquad e_o(BR_P) = e_o(IR_P) .$$

Note that for the proof $e_o(q) := e(q,R)$, q any M-primary ideal in R , denotes the Samuelmultiplicity of a primary ideal q in R, whereas $e(y_1,\dots,y_t,S)$ denotes the multiplicity of a multiplicity-system $\underline{y}$ of the ring S in the sense of Northcott (s. [H-S-V], p. 100).

Furthermore $\underline{x}R+B$ is a reduction of $\underline{x}R+I$, hence

$$(2) \qquad e_o(\underline{x}R+B) = e_o(\underline{x}R+I) .$$

Since $\sqrt{\underline{x}R+\underline{z}R} = \sqrt{\underline{x}R+I}$ we get $\dim R/\underline{x}R+\underline{z}R = 0$; and we know by the assumption $ht(I) = l(I)$ that $r+s \leq d$, hence $r+s = d$. Therefore $x_1,\dots,x_r$, $z_1,\dots,z_s$ is a system of parameters for R . [Note that this argument shows in particular that $ht(I) = l(I)$ implies $\dim(R) = \dim (R/I) + ht(I)$.].

Now, for a system of parameters we have :

$$e_o(\underline{x}R+\underline{z}R) = e(x_1,\dots,x_r,z_1,\dots,z_s,R) = \sum_{P\in \mathrm{Min} B} e(\underline{x},R/P)\cdot e(\underline{z},R_P)$$

$$= \sum_{P\in \mathrm{Min} B} e(\underline{x},R/P)\cdot e_o((\underline{z})R_P)$$

Since $\mathrm{Min}\, B = \mathrm{Min}\, I$ we obtain from (1) and (2)

$$e(\underline{x},I,R) = e_o(\underline{x}R+B) = e_o(\underline{x}R+I) .$$

II) To prove the converse, assume that there is a s.o.p. $\underline{y} = (y_1,\dots,y_r)$ for R/I such that $e(\underline{y},I,R) = e_o(\underline{y}R+I)$ and $ht(I) > o$ as before.

We may assume that $r > o$ (for $r = o$ the statement of prop. 1.1.5 is trivial).

Step 1: Put $V = I+\underline{y}R$. Then there is a sequence of superficial elements $x_1,\ldots,x_r$ for V with the following properties (see [H-O-1]):

(1) $V = I + \underline{x}R$

(2) $e(\underline{x},I,R) = e(\underline{y},I,R)$

(3) $e_o(V) = e_o(V/\underline{x}R)$

(4) $x_1,\ldots,x_r$ is part of a system of parameters for R .

In the following we use the system $\underline{x}$ instead of $\underline{y}$.

Step 2: Since R/M is infinite, we can choose a system $\underline{z} = \{z_1,\ldots,z_s\}$ of elements in I , which is a system of parameters for $R/\underline{x}R$ such that

$$e_o(V/\underline{x}R) = e_o(\underline{z}R+\underline{x}R/\underline{x}R) \quad .$$

By definition, $s = d-r$, i.e. $x_1,\ldots,x_r,z_1,\ldots,z_s$ is a system of parameters in R , and $\dim R/\underline{z}R = \dim R/I$, hence $\mathrm{Assh}(R/I) \subsetneq \mathrm{Assh}(R/\underline{z}R)$.

Then we get the following relations:

$$e(\underline{x},I,R) \underset{\substack{\nwarrow \\ \text{associativity} \\ \text{law}}}{\le} e(\underline{x},\underline{z}R,R) \underset{\substack{\nwarrow \\ \underline{x,z} \text{ is s.o.p.}}}{=} e_o(\underline{x}R+\underline{z}R) \underset{\substack{\nwarrow \\ \underline{x} \text{ is part of s.o.p.}}}{\le} e_o(\underline{x}R+\underline{z}R/\underline{x}R) = e_o(V/\underline{x}R) = e_o(V)$$

Hence by assumption we get equality everywhere, so that

$$(*) \qquad e(\underline{x},I,R) = e(\underline{x},\underline{z}R,R) \ .$$

[Not that $s > o$, since we assume $ht(I) > o$] .

Since R is quasi-unmixed, the "dimension formulas" for I and $\underline{z}R$ hold. Therefore we may use the corresponding multiplicity-formulas for $e(\underline{x},I,R)$ and $e(\underline{x},\underline{z}R,R)$. Then, using (*), it comes out (see [H-O-1]) that

(**) $\mathrm{Assh}(R/I) = \mathrm{Assh}(R/\underline{z}R)$ and $e_o(IR_P) = e_o(\underline{z}R_P)$

for all $P \in \mathrm{Assh}(R/I)$.

<u>Step 3</u>: Since R is quasi-unmixed, hence $\dim R = \dim(R/\underline{z}R) + ht(\underline{z}R)$, we know that $\underline{z}R$ is an ideal of the principal class, i.e. $ht(\underline{z}R) = l(\underline{z}R)$. Therefore the integral closure $\overline{\underline{z}R}$ in R has no embedded primes, and all (minimal) primes of $\overline{\underline{z}R}$ have the same height (s. proof of thm. 0.3.4); hence all minimal primes of $\underline{z}R$ have the same height. So we get $Assh(R/I) = Assh(R/\underline{z}R) = Min(\underline{z}R)$, hence $Assh(R/I) = Min(I)$.
Using Bögers theorem 0.2.4* , we see that $z_1,\dots,z_s$ generate a reduction of I and consequently $ht(I) = l(I)$, q.e.d.

For a <u>general</u> prime ideal P, Zariski-equimultiplicity and the condition $ht(P) = l(P)$ are <u>totally unrelated</u>, as we can see from the following examples.

<u>(1.1.6) Examples.</u>
a) Let $R = k[[s^2,s^3,st,t]] \subset k[[s,t]]$, k any field, and take $P = (st,t)R$. Then R is a Buchsbaum ring of multiplicity 2 ,and R_P is regular, so $e(R) \neq e(R_P)$. But $ht(P) = l(P) = 1$ since $t \cdot P = P^2$. Note that R is not normally flat along P. This follows from lemma 3.19 p. 76 in [H-S-V], saying that P/P^2 is R/P-free, R/P is CM and R_P is regular imply R is Cohen-Macaulay.

b) Example (1.1.4) shows that $e(R) \neq e(R_P)$ and $ht(P) = l(P)$ is also possible for <u>R</u> a <u>Cohen-Macaulay ring</u>.

c) Let P be any prime ideal in a <u>regular</u> local ring R, so that $e(R) = e(R_P) = 1$. In [H-O-1] we have shown that $ht(P = l(P)$ if and only if P is generated by a regular sequence, i.e. P defines a complete intersection. For example, if $R = k[[x,y,z]]$ and $P = (y^2-xz;x^3-yz;z^2-x^2y)$, then $ht(P) \neq l(P)$ (see [Ha]).

These considerations, together with the fact that $ht(I) = l(I)$ implies a nice behaviour of multiplicities under blowing up (see section 1.2) lead us to the following

<u>(1.1.7)</u> <u>Definition</u>. Let R be a local ring. An ideal I of R will be called <u>equimultiple</u> if $ht(I) = l(I)$.

Similarly, for <u>R/I non-regular</u>, the condition of normal flatness along I should be replaced by the depth-condition given in Theorem (1.1.3), b):

<u>(1.1.8) Definition.</u> Let R be a local ring and I an ideal of R . R will be called <u>normally Cohen-Macauly along I</u> if

$$\underline{\text{depth } (I^n/I^{n+1}) = \dim R/I} \quad \text{for all } n \geq 0 .$$

This means that $gr_I(R)$ is a so-called <u>balanced big Cohen-Macaulay module</u> over R/I .

If R is normally Cohen-Macaulay along I , then the Hilbert function behaves well under blowing up (see [O]; actually it is sufficient for this purpose that depth $(I^n/I^{n+1}) = \dim R/I$ <u>for large n</u> , see [O-R]). Using the numerical characterization of normally Cohen-Macaulay ([H-S-V], Satz 3.13) it was shown in [H-O-3], [H-O-4] that normally Cohen-Macaulay and equimultiple (in the sence defined above) coincide for hypersurfaces under some additional assumptions, which are trivially satisfied for R/I regular.

<u>(1.1.9) Remarks:</u>
a) If R is normally flat along I and R/I is Cohen-Macaulay then, of course, R is normally Cohen-Macaulay along I . This need not be true in general, a trivial example being I = (0) in any local ring R which itself is <u>not</u> Cohen-Macaulay.

b) R is normally Cohen-Macaulay along any M-primary ideal I without I^n/I^{n+1} being R/I-flat. For example, take R = k[[x,y]] and $I = M^2 = (x^2,xy,y^2)$.

<u>(1.1.10) Remark:</u> Normal flatness is somewhat more difficile than Normal Cohen-Macaulayness in the following sense:

For any ideal $I \subset (R,M)$ and for a regular sequence $\underline{x} = \{x_1,\dots,x_r\}$ w.r.t. R/I ($\dim R/I =: r$) we have the statement

<u>x is superregular</u> (i.e. $\underline{x}$ is regular on $gr_I(R)$)

if and only if the canonical map

$$(*) \quad gr_I R \otimes R/(I,\underline{x})R\,[X_1,\dots,X_r] \xrightarrow{\varphi} gr_{(I,\underline{x})}(R)$$

is an isomorphism. (X_i are indeterminates)

In the same situation R is normally flat along I if and only if

(i) φ is an isomorphism and

(ii) $gr_{(I,\underline{x})}R$ is flat over $R/(I,\underline{x})$;

see [O-R].

1.2 Blowing up equimultiple centers

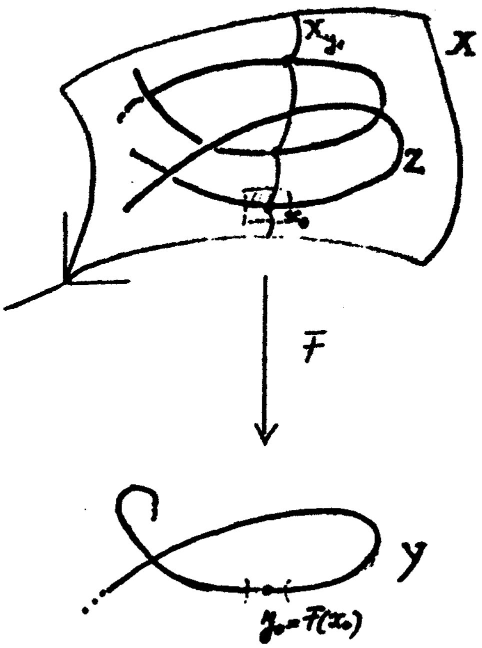
Our claim is that for an arbitrary prime ideal $P \subset (R,M)$, the "correct" equimultiplicity condition on P should be $ht(P) = l(P)$ instead of $e(R) = e(R_P)$. One reason for this is the nice behaviour of multiplicity under blowing up ideals P for which $ht(P) = l(P)$. This we will indicate in the following. Another reason for our viewpoint is that $ht(P) = l(P)$ implies equimultiplicity conditions for flat families of ideals parametrized by Spec R/P.
Let me indicate the geometric background:

Let $F : X \to Y$ be a morphism of complex analytic varieties, take $x_o \in X$ and $y_o = F(x_o)$. The corresponding local rings of germs of holomorphic functions are denoted by $S := O_{Y,y_o}$ and $R := O_{X,x_o}$. Let $\phi : S \to R$ be the ring homomorphism induced by F . We choose a coherent O_X-ideal $\mathfrak{I}$ such that for $I := \mathfrak{I}_{x_o} \subseteq R$:

(i) R/I is a finite S-module,

(ii) $\phi^{-1}(I) = o$.

Let Z be the subspace of X defined by $\mathfrak{I}$. Furthermore we assume that F is flat at x_o .



Perhaps after shrinking X and Y, we have $Z \cap F^{-1}(\{y_o\}) = \{x_o\}$, and F induces a proper surjective map $Z \to Y$ with finite fibres. The problem is to characterize algebraically the constance of the multiplicity $e_y(\mathfrak{J})$ of the fibres $X_y = F^{-1}(\{y\})$ in points of Z; where

$$e_y(\mathfrak{J}) := \sum_{x \in X_y \cap Z} e_o(\mathfrak{J}O_{X_y,x})$$

($e_o(..)$ = Samuel-multiplicity).
Then the following are equivalent:

(i) $e_y(\mathfrak{J})$ is constant in a neighborhood of y_o("geometric" equimultiplicty)

(ii) ht(I) = l(I) . ("static" equimultiplicity of $R = O_{X,x_o}$ along $\mathfrak{J}_{x_o} = I$.)

Note that if $\mathfrak{J}_o$ is a coherent ideal contained in $\mathfrak{J}$ such that $\mathfrak{J}_o O_{X_{yo},x_o}$ is a reduction of $\mathfrak{J}\, O_{X_{y_o},x_o}$, then $\mathfrak{J}_o$ is a reduction of $\mathfrak{J}$ in some neighborhood of x_o. Refering to Teissier we may call this fact "principle of specialization of reduction".

Let me finally mention some more technical reasons for using the equimultiplicity ht(I) = l(I) :

1) Normal flatness of R along I implies ht(I) = l(I) .
(it is enough to assume I^n/I^{n+1} is flat for infinitely many

values of n for this implication; s. [H-O-3], Thm. 1). Since ht(I) = l(I) implies that there are s = ht(I) elements in I giving the same radical as I, this implication contains the result - due to Grothendieck - that normal flatness along I makes Spec(R/I) a set-theoretic complete intersection (in a very weak sense).

2) The condition ht(I) = l(I) is an open condition on V(I). This can be deduced from Chevalley's semicontinuity theorem [EGA IV_3], 13-1-5 .

3) There exists a transivity property of equimultiplicity ht(I) = l(I) as in case of normal flatness, s. [H-O-3].

<u>1.2.1 Theorem [O]:</u> Let R_1 be a local ring of the blowing up of I in a <u>quasi-unmixed</u> local ring R which dominates R . Let $\underline{x} = (x_1,\dots,x_r)$ be any system of parameters with respect to I . and let r = dim R/I . Then the following holds:

If ht(I) = l(I) , then $e(R_1) \leq e(\underline{x},I,R)$.

To give a glimpse of the proof let me give the proof of the following corollary which is interesting for itself.

<u>1.2.2 Corollary:</u> Let R be quasi-unmixed and let P be a regular prime in R such that $e(R) = e(R_P)$. Then for a blowing up ring R_1 as above we get

$$e(R_1) \leq e(R) .$$

Note that for a 2-dimensional variety embedded in a 3-dimensional regular space this result is due to Zariski. After the La-Rabida conference on Algebraic Geometry in 1981 Teissier sketched a proof of the corollary in the complex-analytic case. Inspired by ideas of Dade, U. Orbanz gave an elegant proof of Thm. 1.2.1 in [O]. He also proved a similar result for the <u>behaviour of Hilbert functions</u> under blowing up an ideal I such that <u>R is normally Cohen-Macaulay along I</u> . Here one of the main ingredients is the use of generalized Hilbert-functions introduced in § 1.1; see [O], Theorem and Corollary, page 6.

Proof of the corollary:

1) We know that $e(R_1) \leq e(R_1/PR_1)$, where $PR_1 = t\cdot R_1$ for some non-zero-divisor t .
Let $R^* := gr_P(R) \supset M^* := M/P \oplus \sum_{i>0}^{\oplus} P^i/P^{i+1}$ and

$$P^* := \bigoplus_{i>0} P^i/P^{i+1} \triangleq gr_P(P,R) .$$

$R_1/t\, R_1$ is the local ring of a point in the exceptional locus of the blowing up, hence

$$R_1/t\cdot R_1 = R^*_{(Q^*)} \quad \text{for some homogeneous prime } Q^* \subset R^* .$$

Taking advantage of the good behaviour of blowing-up vis-à-vis completion we may assume R is complete . By [Ra-3] and [Gr] also R^* is quasi-unmixed, hence $R^*_{M^*}$ is quasi-unmixed (see § 0.1).

Furthermore R^* is finitely generated over the excellent ring R/P , hence we have the Lech-Nagata inequality $e(R^*_{Q^*}) \leq e(R^*_{M^*})$, since $Q^* \neq M^*$. So we know that

$$(1) \qquad e(R_1) \leq e(R^*_{(Q^*)}) = e(R^*_{Q^*}) \leq e(R^*_{M^*}) .$$

2) We always have $l(P) = l(P^*_{M^*})$ and $ht(P) = ht(P^*_{M^*})$ and $e(R^*_{P^*}) = e(R_P)$. By assumption $R^*/P^* \cong R/P$ is regular, hence by theorem 1.1.3 we obtain: $e(R^*_{M^*}) = e(R^*_{P^*})$. So (1) implies finally:

$$(2) \qquad e(R_1) \leq e(R_P) = e(R) \quad , \qquad \text{q.e.d.}$$

1.2.3 Remarks:

1) The same idea works for the proof of theorem 1.2.1. Then one uses prop. 1.1.5 instead of theorem 1.1.3 (for the regular case).

2) Quasi-unmixedness is always used in the previous prop. 1.1.5 or 1.1.3 for having the appropriate dimension-conditions. For blowing up R along centers for which R is normally Cohen-

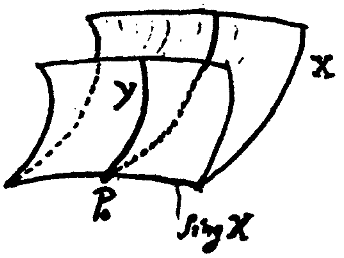
Macaulay, and looking for the behaviour of Hilbert-functions.
one also needs in Orbanz'proof that in our terminology

$$\dim (R^*/Q^*)-1 = \dim (R) - \dim (R_1) .$$

Question: Can one describe "stability-conditions" for the situations in theorem 1.2.1 and Cor. 1.2.2 ? Note that theorem 1.2.1 doesn't tell us anything about those points ξ' in the blowing up having the same multiplicity as ξ (ξ, ξ' are to be supposed to be corresponding points). One advantage of "permissibility" of the centers (i.e. normal flatness and regularity) in Hironaka's work seems to be that one can characterize stable points ξ' ("stable" with respect to Hilbert-functions).

1.3 Blowing up some singular centers.

Let me indicate the forthcoming situation by the following picture:



Y has a singular point $P_o \in \operatorname{Sing} X$, but $Y \not\subset \operatorname{Sing} X$; but $e_{P_o}(X) = e_{P_o}(Y)$

Let us say, we have "bounded multiplicities". Our claim is that under blowing up equimultiple ideals I with bounded multiplicities, the multiplicity doesn't become worse.

1.3.1. Proposition: Let (R,M) be a quasi-unmixed local ring. Let I be an equimultiple ideal, which is locally a complete intersection in R , satisfying $e(R) = e(R/I)$. Then for any local ring of the blowing up of I dominating R , we get:

$$e(R_1) \leq e(R).$$

Proof: By theorem 1.2.1, we have $e(R_1) \leq e(\underline{x},I,R)$ for all systems of parameters with respect to I . Now:

$$(1) \quad e(\underline{x},I,R) = \sum_{P \in \operatorname{Assh}(R/I)} e(\underline{x},R/P) \cdot e_o(I \cdot R_P)$$

(2) $e(\underline{x},R/I) = \sum_P l_{R_P}(R_P/I_P)\cdot e(\underline{x},R/P)$

But $l_{R_P}(R_P/I_P) \geq e_o(I\,R_P)$, since I_P is an ideal of the principal class.
Hence we get:

(3) $e(\underline{x},I,R) \leq e(\underline{x},R/I)$

Now choose $\underline{x}$ in such a way that $e(\underline{x},R/I) = e(R/I)$. Then (3) implies: $e(R_1) \leq e(R)$, q.e.d.

For a prime ideal $I = P$ we know that $ht(P) = l(P)$ implies $e(R) \leq e(R/P)\cdot e(R_P)$. So it is natural to consider ideals of "bounded multiplicity" in the sense that: $e(R) = e(R/P)\cdot e(R_P)$. Under the assumption that R_P is regular, this means that $e(R) = e(R/P)$, corresponding to the special fact that R_P is CM for $P \supseteq I$ and I is locally a complete intersection.

B. Singh has recently noticed that for any ideal $I \subset R$ we always have

(*) $H^1[R](n) \leq H^1[R/I] * H^o[I](n) = \sum_{i=o}^{n} H^1[R/I](i)\cdot H^o[I](n-i)$

where $H^o[I](n)$ is the Hilbert function $\dim_{R/M}(I^N/MI^n) = \mu(I^n)$ ($\mu(\mathfrak{a})$ is the least number of generators of an ideal $\mathfrak{a}$).

Note that for $I = P$ we don't get from Singh's inequality the inequality $e(R) \leq e(R/P)\cdot e(R_P)$ since the last one means to pass from Hilbert-functions above to certain Hilbert-polynomials, and for this transition we have to use the condition $ht(P) = l(P)$.

B. Singh also remarked that equality holds in (*) iff "$gr_M(R)$ is flat over $gr_I(R) \otimes_{R/I} R/M$."

Question 1: In which relation is normal flatness of R along I to this last condition?

Question 2: "How far" is $e(R) = e(R/P)\cdot e(R_P)$ from $ht(P) = l(P)$?

1.3*: Remarks to low multiplicities.

The proof of the next well known lemma contains in some sense a situation of equimultiple ideals $(\underline{x})R$ of "bounded multiplicity".

1.3*.1 Lemma: Let (R,M) be a local Cohen-Macaulay-ring and let $e(R) = 1$. Then R is regular.

Proof: We may assume R/M is infinite. Then we can construct a sequence of superficial elements $x_1,\dots,x_d$ of order 1 for M, which gives a minimal reduction of M.
Therefore we have

$$e(R) = e(R/\underline{x}R) = e(\underline{x},R) .$$

Hence $l(R/\underline{x}R) = e(R) = 1$, i.e. $M = (x_1,\dots,x_d)R$, q.e.d.

Another use of "bounded multiplicity" yields a statement of Ikeda:

1.3*.2 Lemma: If (R,M) is an equicharacteristic complete Cohen-Macaulay ring of multiplicity $e(R) = 2$, then R is a hypersurface, i.e. $R = Q/(f)$, where Q is regular and $f \neq 0$.

Proof: As in the proof of lemma 1.3*.1 we get $e(R) = e(R/\underline{x}R) = e(\underline{x},R) = l(R/\underline{x}R)$ where $\underline{x} = x_1,\dots,x_d$ is again a minimal reduction of M.
Therefore $l(R/\underline{x}R) = 2$, hence $l(M/\underline{x}R) = 1$. So $\underline{M/x}$ is principal, generated by an element $\bar{y} = y \bmod \underline{x}$ with $y \in M$.
Take the map $S := k[[X_1,\dots,X_d,Y]] \xrightarrow{\varphi} R$, sending X_i to x_i and Y to y. Then $ht(\ker\varphi) = 1$, and the ideal $\ker\varphi$ is unmixed, since R is CM. As S is factorial, $\ker\varphi = (f)$ is principal, q.e.d.

1.3*.3. Remark. Huneke has the weaker assumption "Serre-condition S_2" instead of "R is CM". Then he can show that „$e(R) \leq n$ and Serre-cond. S_n" imply „R is CM". Here the direct summand-theorem of Hochster and the syzygy-theorem of Griffith-Evans are used, which are only known to hold when R is equicharacteristic.
We will generalize lemma 1.3*.2 in some sense in section 1.4, where we use testideals to check the CM-property of R or of the Reesring $R[Mt]$.

1.4 Testideals for Cohen-Macaulay-singularities

Equimultiple ideals imply a nice behaviour of multiplicities under blowing up. Here we want to show that equimultiple prime ideals can also be used to check whether (R,M) or the arithmetical blowing up of R in the maximal ideal M is Cohen-Macaulay or not.

1.4.1 Lemma: Let R be a local ring. If the prime ideal P in R is equimultiple then $e(R) \leq e(R/P) \cdot e(R_P)$.

Proof: For any system $\underline{x} = \{x_1,\dots,x_l\}$ of parameters with respect to the equimultiple ideal P we have (s. prop. 1.1.5)

$$e(\underline{x},P,R) = e(P+\underline{x}R,R) \ ,$$

hence $e(P+\underline{x}R,R) = e(\underline{x},R/P) \cdot e(R_P)$.

Now choose a special system $\underline{x}$ of parameters such that $e(\underline{x},R/P) = e(R/P)$. Since $e(R) \leq e(P+\underline{x}R,R)$, this implies the claim.

Testideals.

1.4.2. Definition: R is said to be a hypersurface, if $R = Q/(f)$, where Q is a regular local ring and $f \neq 0$.

1.4.3. Propostion: Let (R,M) be an excellent local ring of dim $R \geq 2$, containing a field. If there exists an equimultiple prime ideal $P \subset R$ such that $e(R/P) \leq 2$, if R_P is regular and if R satisfies S_2 then R is a hypersurface.

Proof: Since R is excellent, we may assume that R is complete. Lemma 1.4.1 tells us that $e(R) \leq 2$. Then, by the result of Ikeda (s. 1.3*.2) we know that S_2 and $e(R) \leq 2$ imply that R is a hypersurface, q.e.d.

1.4.4 Remark: Prop. 1.4.3 shows, that a non-hypersurface singularity (R,M), dim $R \geq 2$ (R containing a field) cannot be CM if there exists an equimultiple „testideal" P with R_P regular which has a point of multiplicity 2 in this singularity.

<u>1.4.5. Example 1:</u> Let $R = k[[s^2,s^3,st,t]]$, k any field, and take $P = (st,t)R$. Then $e(R/P) = 2$, $ht(P) = l(P) = 1$, R_P regular, and R is certainly not a hypersurface. And R is indeed not Cohen-Macaulay (depth $R = 1 < \dim R = 2$)

<u>Example 2:</u>

Let
$R = k[[x^2,xy,y^2,xz,yz,z]]$
$\cup$
$P = (xz,yz,z)$; then
we have: R is CM
$e(R/P) = 2$ (and R/P CM)
$ht(P) = l(P) = 1$

But R is not a hypersurface; hence R_P cannot be <u>regular</u>, and indeed $PR_P = (xz,yz,z)\,R_P$ cannot be generated by 1 element only.

<u>1.4.6. Proposition: Let R be a Cohen-Macaulay ring of dim $R \geq 3$. If there exists an equimultiple ideal $P \subset R$ such that $e(R/P) \leq 3$ and if R_P is regular, then the Reesring $R[Mt] \cong \bigoplus_{n \geq 0} M^n$ is Cohen-Macaulay.</u>

Note that here we are controlling the CM-property of $R[Mt]$ by using testideals $P \neq M$. The idea is that R/P might be "simpler" (e.g. a hypersurface) than R .

<u>Proof:</u> The existence of the testideal P implies $e(R) \leq 3$, hence $e(R) \leq \dim R$.
Furthermore $e(R) \leq 3$ and R CM imply [Sa] that $gr_M(R)$ is CM. Therefore the Reesring $R[Mt]$ is CM.
A slight generalization of Prop. 1.4.6 is the following proposition.

<u>1.4.7 Proposition:</u> <u>Let (R,M) be a local ring and let $gr_M(R)$ be Cohen-Macaulay. If there exists an equimultiple ideal $P \subset R$ such that $e(R/P) \leq \dim R$ and R_P is regular, then the Rees-ring $R[Mt]$ is CM.</u>

1.4.8 Remarks: (1) If (R,M) is a hypersurface-singularity then we know by [H-O-G] that $R[Mt]$ is CM if and only if $e(R) \leq \dim R$.

(2) In concrete cases it might be easier to compute $e(R/P)$ than $e(R)$. For instance, R/P might be an hypersurface, but R itself is not. So in the case of example 1 in 1.4.5 it is immediately clear that $e(R/P) = 2$ and for $e(R)$ we get in this example $e(R) \leq 2$. Since R is a non-regular complete equidimensional ring containing a field, it cannot have multiplicity 1, hence $e(R) = 2$. [Of course this is also clear by considering the semigroup of exponents.]

1.4.9 Lemma (Orbanz) Let (R,M) be quasi-unmixed, $|R/M| = \infty$ and let $\underline{x} = \{x_1,\ldots,x_r\}$ be a part of a system of parameters in M. Then the following are equivalent:

a) $e(R) = e(R/\underline{x}R)$

b) $\underline{x}$ is part of a minimal reduction of M, and R_P is CM for all $P \in \mathrm{Assh}(R/\underline{x}R)$.

For $r=d=\dim R$ we have: $e(R) = e(R/x_1,\ldots,x_d)$ iff $\underline{x}$ is a minimal reduction of M and R is Cohen-Macaulay. —
Ikeda's result can immediately be used for the following proposition.

1.4.10 Proposition: Let (R,M,k) be an equicharacteristic, quasi-unmixed complete ring. If there exists a system $x_1,\ldots,x_d$ of paramters in R such that $e(R/\underline{x}R) = 2$, then R is a hypersurface (in particular a CM-ring).

[Note: generally if $R \neq$ CM, then $e(R) \neq e(R/\underline{x}R)$, by Orbanz' lemma 1.4.9.]

Proof: $R/\underline{x}R$ is a ring of dimension zero, hence is Cohen-Macaulay. Therefore we have

$$2 = e(R/\underline{x}R) = e(\emptyset;\ R/\underline{x}R) = l(R/\underline{x}R)\ .$$

This implies $l(M/\underline{x}) = 1$, hence $M/\underline{x} = (\bar{y})$ with $\bar{y} = y \bmod(\underline{x})R$. Take the map $S := k[[X_1,\ldots,X_d,Y]] \xrightarrow{\mathcal{L}} R$ by sending $X_i \to x_i$ and $Y \to y$.

Then $\ker \varphi =: I$ has $ht(I) = 1$.
Since $R \cong S/I$ (and S) is <u>quasi-unmixed</u>, I is (height-) unmixed (see [Ra]). As S is factorial, $I = (f)$ is principal.

<u>Remark:</u> $e(R) \neq 2$ is possible in our case.

<u>1.4.11 Corollary (to 1.4.10):</u> R equichar., q.u., complete ring. Assume, there exists a system of parameters, say $x_1,\dots,x_d$, such that $l(R/\underline{x}R) = 2$. Then $R[Mt]$ is CM iff $e(R) \le \dim R =: d$.

The next step is to indicate that equimultiple ideals guarantee a kind of "transitivity" of the CM-property for Reesrings.

<u>1.4.12 Proposition:</u> Let (R,M) be a local Cohen-Macaulay-ring and let $J \subset I$ be ideals in R such that $l(I/J+MI) = ht(I) - ht(J)$ and $ht(J) = l(J)$.
Then we have the implication:

$$R\overset{+}{e}(J,R) \text{ is CM} \Longrightarrow R\overset{+}{e}(I,R) \text{ is CM} .$$

This shows that generally the CM-property of $R\overset{+}{e}(J,R)$ for a smaller ideal $J \subset I$ is "stronger" than the CM-property of $R\overset{+}{e}(I,R)$.

<u>Example:</u> $R = k[[x^2,xy,y^2,xz,yz,z]]$; consider the ideals:
$J = (x^2)R \subset I = (x^2,y^2,z)R$.
$R\overset{+}{e}(J,R)$ is clearly CM, hence $R\overset{+}{e}(I,R)$ is CM.

In a forthcoming preprint (with U. Orbanz and S. Ikeda we shall prove this proposition as well as some "converse" of propostion 1.4.12.

<u>Supplement:</u> I should remark that our methods are directed to <u>singular</u> rings (R,M). And most of the results can only be proved for <u>equimultiple</u> ideals I in R .

R.C. Cowsik asked me the following question: Start with a <u>regular</u> local ring R and consider a non-regular prime ideal $P \subset R$. How does the blowing up of R along P looks like? The interesting case concerns prime ideals P which are <u>not</u> equimultiple.

In our example 1.1.6 c) the prime ideal P in $R = k[[x,y,z]]$ was generated by $f_1 = y^2-xz$, $f_2 = x^3-y\cdot z$, $f_3 = z^2-x^2\cdot y$. This means that P can be generated by the maximal minors of the 2x3 matrix $\begin{vmatrix} y, & z, & x^2 \\ x, & y, & z \end{vmatrix}$

Therefore P is generated by a d-sequence [Hu-3]. This implies [Hu-2]

$$R\overset{+}{e}(P,R) \cong Sym\,(P,R)\ ,$$

where $Sym(P,R)$ is the symmetric algebra w.r.t. P. It is defined by all linear forms of $R[X_1,X_2,X_3]$ vanishing at f_1,f_2,f_3 (i.e. by the syzygies of P).

Hence we get:

$$R\overset{+}{e}(P,R) = R[X_1,X_2,X_3] \Big/ \left(yX_3+zX_2+x^2X_1;xX_3+yX_2+zX_1\right),$$

i.e. in particular that $R\overset{+}{e}(P,R)$ is CM. (Note that in this case $gr_P R$ is Gorenstein, [Hu-2]). Here $R\overset{+}{e}(P,R)$ is also <u>normal</u>.

Cowsik observed that for the monomial curve, defined by the equations:

$$f_1 = x^4y^3 - z^5 = o$$
$$f_2 = x^7 - yz^3 = o$$
$$f_3 = x^3z^2 - y^4 = o$$

(i.e. the generic point is (t^{14}, t^{23}, t^{25})), $R\overset{+}{e}(P,R)$ is <u>not</u> normal.

So the question arises: which are the necessary and sufficient conditions on a prime ideal P in a regular local ring R for $R\overset{+}{e}(R,P)$ being normal. For monomial curves multiplicity $e(R/P) = 3$ is sufficient.

G. Valla has indicated (Bonn, December 1983) that for monomial curves $R\overset{+}{e}(R,P)$ is always normal if $e(R/P) \leq 5$. An essential observation is that one only has to check if the ideal MS (where $S := Sym\,(P,R)$) is a regular prime or not, since every other height one prime ideal in S is regular.

Note that generally R is not normally Cohen-Macaulay along P for monomial curves. More general we know that for $R = k[[X_{ij}]]$, X_{ij} indeterminates, $1 \leq i \leq n$ and $1 \leq j \leq n+1$, and the ideal $I := I_n(X)$, generated by the n-minors of (X_{ij}), the Reesring $R\overset{+}{e}(I,R)$ is CM (since I can be generated by a d-sequence). But I/I^2 is not CM for $n \geq 2$, hence R is not CM along I; see also [I-H.] prop. 1.5.

§ 2. Auxiliary computations in graded rings.

2.1 Homogeneous systems of paramters in graded rings.

2.2 Remarks on systems of parameters and generalized multiplicities of $R\dot{e}(I,R)$

In this section we will roughly indicate the proofs of some results. The details will appear in a forthcoming paper of Herrmann-Orbanz-Grothe. According to the title of these lectures the <u>main</u> technique used in § 2 and § 3, comes from multiplicity-theory.

2.1 Homogeneous system of parameters in graded rings.

We have already mentioned in 0.3.2 and 0.3.3 that we will define a homogeneous system of parameters in a graded ring A as follows, where A is always one of the <u>graded</u> rings $R\dot{e}(I,R)$, $Re(I,R)$ and $gr_I R$. Those rings have a unique homogeneous maximal ideal, say N, such that $\dim A = \dim A_N$.

<u>Definition:</u> Homogeneous elements $a_1,\ldots,a_d$, where $d = \dim A = \dim A_N$, form a <u>homogeneous system of parameters</u> if $\sqrt{(a_1,\ldots,a_d)} = N$.

<u>Remark:</u> Recall that Nagata gave the following general definition for a system of parameters in any noetherian ring A , s. [Na], § 24, p. 77 : A set of d elements $a_1,\ldots,a_r$ is called a system of parameters in R if $\dim (R/(a_1,\ldots,a_r)R) = 0$ and $\max ht(P) = r$, where P runs over the minimal prime divisors of $(a_1,\ldots,a_r)R$. (Note that by the altitude theorem of Krull we always have $\max ht(P) \leq r$.) Hence a homogeneous system of parameters in all the cases mentioned above is a system of parameters in the sense of Nagata.

<u>2.1.1. Proposition:</u> Let (R,M) be a quasi-unmixed local ring and I an ideal of R . Then the following conditions are equivalent:

(i) I is *equimultiple* .

(ii) $gr_I R$ has a *homogeneous* system of parameters.

(iii) $Re(I,R)$ has a *homogeneous* system of parameters.

<u>Proof:</u> Note that in quasi-unmixed rings condition (i) is equivalent to

(i') $\qquad \dim R = \dim R/I + l(I)$.

Now we can prove the equivalence of (i'), (ii) and (iii) without any assumption on R:

(i') =>(ii): Let $\bar{b}_1,\ldots,\bar{b}_s \in R/I$ a system of parameters.

Then $A = gr_I(R) / (\bar{b}_1,\ldots,\bar{b}_s)\, gr_I(R)$ is of dimension $l(I)$ and has a homogeneous system of parameters $\hat{a}_1,\ldots,\hat{a}_\ell$ $(\ell := l(I))$, since A_o is an artinian local ring.
Let $\bar{a}_1,\ldots,\bar{a}_\ell \in gr_I(R)$ be any <u>homogeneous</u> inverse images of $\hat{a}_1,\ldots,\hat{a}_\ell$ respectively. Since

$\dim gr_I(R)/(\bar{b}_1,\ldots,\bar{b}_s;\bar{a}_1,\ldots,\bar{a}_\ell)gr_I(R) = o$ and $l+s = \dim R$

by <u>assumption (i')</u> , we conclude that $\{\bar{b}_1,\ldots,\bar{b}_s,\bar{a}_1,\ldots,\bar{a}_\ell\}$ is a <u>homogeneous</u> system of parameters of $gr_I(R)$.
[Note that the ideal $S = (\bar{b}_1,\ldots,\bar{b}_s,\bar{a}_1,\ldots,\bar{a}_\ell)gr_I(R)$ is contained in the unique homogeneous maximal ideal N of $gr_I(R)$ and that rad S = N.]

(ii) => (iii): is trivial since $Re(I,R)/(u) \cong gr_I R$.

(iii) =>(i'): Assume that

$$\{q_1u^{r_1},\ldots,q_ku^{r_k};\ b_1,\ldots,b_m;\ c_1t^{s_1},\ldots,c_nt^{s_n}\}$$

is a homogeneous system of parameters of $Re(I,R)$, where $q_1,\ldots,q_k \in R$, $b_1,\ldots,b_m \in M$;. $c_j \in I^{s_j}$, $j = 1,\ldots,n$ and $r_i > 0$, $s_j > 0$ and $k+m+n = \dim(R) + 1$. Then we obtain the following inequalities for k,m,n :

(1) $\dim Re(I,R)/MRe(I,R) + uRe(I,R) = l(I) \leq n$

(2) $\dim Re(I,R)/(It)Re(I,R) = \dim R/I + 1 \leq k+m$.

(1) and (2) imply that $l(I) + \dim R/I \leq \dim R$. On the other hand we have: $ht(I) \leq \dim(R) - \dim R/I \leq l(I)$, hence (i') follows, q.e.d.

__2.1.2. Remark:__

An analogous possibility to prove (i') $\Rightarrow$ (iii) $\Rightarrow$ (ii) is the following ($Re := Re(I,R)$): Since IRe uRe, the ideal $(b_1,\dots,b_s,u)Re$ has the same radical as $(M,u)Re$. Hence dividing out $(\underline{b},u)$ gives - up to the radical - the fibre of the blowing up of R along I in the closed point M. The dimension of the fibre is $l(I) = (d+1) - (s+1) = d-s$ by (i').

Since $Re/(M,u)Re$ is an algebra over a field, we find $l(I) = d-s$ homogeneous parameters in this ring (take the __initial forms__ $in_I a_i$ of any minimal reduction $a_1,\dots,a_l$ of I in R, which are by definition a homogeneous system of parameters of $gr_I(R) \otimes R/M$). Hence

$$u, b_1,\dots,b_s,\ a_1 t^1,\dots,a_l t^1$$

is a homogeneous system of parameters in $Re = Re(I,R)$, i.e. $n = l(I)$ and $k = 1$.

__2.1.3. Corollary:__ Assume that $a_1,\dots,a_l$ generate a minimal reduction of I, where $l = l(I)$. Assume also that $b_1,\dots,b_s$ is a system of parameters mod I. If $l+s = \dim R$, then $\{in_I a_1,\dots, in_I a_l;\ in_I b_1,\dots, in_I b_s\}$ is a homogeneous system of parameters for $gr_I R$.

It is easy to see that contrary to the rings $gr_I R$ and $Re(I,R)$, the rings $Re^+(I,R)$ have "almost never" a homogeneous system of parameters.

__2.1.4 Lemma:__ Let (R,M) be a local ring and let I be an ideal of R such that $ht(I) > o$. Let M^* be the irrelevant homogeneous ideal of $Re^+(I,R)$, and let $\{h_1,\dots,h_r\} \in M^*$ be a part of a system of parameters of $Re^+(I,R)$. Then the number of homogeneous element among $\{h_1,\dots,h_r\}$ is __at most__ __$\dim R - l(I) + 2$__. If $Re^+(I,R)$ has a homogeneous system of parameters, then $l(I)$ must be 1.

__2.1.5 Remark:__ If $\dim R \geq 2$ (therefore $ht(M) = l(M) = 2$), $Re^+(M,R)$ does __not__ have a homogeneous system of parameters.

<u>2.1.6 Example:</u> (see 1.1.6., c)): $R = k[[x,y,z]]$, $P = (y^2-xy;\ x^3-yz;\ z^2-x^2y)$. Since $ht(P) \neq l(P) > 1$ we know that there is no homogeneous system of parameters for any of the rings $gr_I R$, $Re(I,R)$ and $R\dot{e}(I,R)$.

2.2 Remarks on systems of parameters and generalized multiplicities for $R\dot{e}(I,R)$.

<u>Program:</u>

For R Cohen-Macaulay we can compute the length of $R\dot{e}(I,R)$ mod a special system of parameters (first defined by G. Valla). Then we are able to relate generalized Hilbert functions and generalized multiplicities of $R\dot{e}(I,R)$ to those of the ground ring. These computations and results are our main technical tools for a characterization of the Cohen-Macaulay property of $R\dot{e}(I,R)$ for <u>equimultiple</u> ideals I, without using local cohomology. Since these computations are very technical I will only sketch the ideas. For details - in particular for the proofs- see the forthcoming paper [H-O-G].

Throughout this section we fix a quasi-unmixed local ring (R,M) and a proper ideal I of R such that $ht(I) > o$. $R\dot{e}(I,R) = R[It]$ will be denoted by R^*.

We are looking for a special system of parameters. The idea due to G. Valla is as follows: If $a_1,\ldots,a_s$ generate a minimal reduction of an <u>equimultiple</u> ideal I, and if $\{x_1,\ldots,x_r\}$ is a system of parameters mod I, then a system of parameters of $R\dot{e}(I,R)$ will be given by $\{a_1, a_1t-a_2,\ldots,a_{s-1}t-a_s; a_st, x_1,\ldots,x_r\}$. The reason is that $\{a_1,a_1t-a_2,\ldots,a_st\}$ is a <u>reduction of $V = I\cdot R\dot{e}(I,R)+(It)R\dot{e}(I,R)$</u> , and that the elements $x_1,\ldots,x_r$ form a system of parameters for $R\dot{e}(I,R)/V \cong R/I$. To indicate this, let us fix some notations:

For any ideal J in R , we define the ideal J^* of $R\dot{e}(I,R)$ by

$$J^* = \bigoplus_{n\geq o} (J\cap I^n)t^n .$$

Recall that in the proof of theorem 0.3.4 we used this type of ideal in $Re(I+P_o/P_o,R/P_o) \cong Re(I,R)/ \bigoplus_{-\infty}^{+\infty}(P_o \cap I^i)t^i$

$$\cong Re(I,R)/P_o \cdot R[t,t^{-1}] \cap Re(I,R)$$

Putting $R^* = R\overset{+}{e}(I,R)$ we get the corresponding relation:

$$R^*/J^* \cong R\overset{+}{e}(I+J/J,\ R/J)\ .$$

For any sequence $\underline{a} = (a_1,\ldots,a_s)$ of elements in I we consider "associated" sequences $\underline{a}^* := (a_1,a_1t-a_2;\ldots,a_st)$ and $\underline{at} := (a_1t,\ldots,a_st)$.

Then one can easily check that the following statement holds.

<u>2.2.1 Lemma:</u> For any sequence $\underline{a} = (a_1,\ldots,a_s)$ of elements of I and any ideal J of R , we have

$$(\underline{a}R^* + \underline{at}R^* + JR^*)(\underline{a}R+J)^* = (\underline{a}^*R^*+JR^*)\cdot(\underline{a}R+J)^*$$

Lemma 2.2.1 yields the following proposition:

<u>2.2.2 Proposition:</u>

a) If $ht(\underline{a}R) > o$, then $\underline{a}^*R^* + JR^*$ is a reduction of $\underline{a}R^* + \underline{at}R^* + JR^*$.

b) In particular, if $\underline{a}R$ is a reduction of I , then $\underline{a}^*R^*$ is a reduction of $I^* = IR^*+(It)R^*$.

<u>Sketch of proof:</u>

a) It is enough to prove the assertion in R^*/P , where P is any minimal prime ideal of R^* .

Since $o < ht(\underline{a}R) \le ht(\underline{a}R+J) \le ht(\underline{a}R+J)^*$, the image of $(\underline{a}R+J)^*$ in R^*/P is nonzero for all P . Therefore, if $\mathcal{V}$ is any (discrete) valuation ring containing R^*/P, then $(\underline{a}^*,J)R^*$ and $(\underline{a},\underline{at},J)R^*$ generate the same ideal in $\mathcal{V}$, since we may cancel the principal ideal $(\underline{a}R+J)^*\mathcal{V} \neq o$. Therefore they have the same integral closure, which proves the assertion.

b) follows from a).

<u>2.2.3 Remark.</u> In the case $J=o$ and $s=2$ proposition 2.2.2 tells us that $(a_1,a_1t-a_2,a_2t)R^*$ is a reduction of $(a_1,a_2,a_1t,a_2t)R^*$, in particular a_1t is integral over (a_1,a_1t-a_2,a_2t). And indeed we find at once one of the <u>equations of integral dependence:</u>

$$(a_1t)^2 - (a_1t-a_2)(a_1t) - a_1a_2t = o .$$

In this case it is easy to see that $IR^* + (\underline{a}R)^* + ItR^* = I^* \subseteq \underline{a}^*R^*:a_2$.
If $\underline{a} = (a_1,a_2)$ is a regular sequence then we have even equality:

$$IR^* + (a_1,a_2)^* + It(R^* = \underline{a}^*R^* : a_2$$

These relations "contain" the equation of integral dependence, mentioned above.
More general one can show the following lemma.

<u>2.2.4 Lemma:</u> Let $s \geq 2$, let $\underline{a} = (a_1,\ldots,a_s)$ by any sequence of elements of I and let J be any ideal of R. Then

$$IR^* + (\underline{a}R+J)^* + (ItR^*)^{s-1} \subseteq (\underline{a}^*R^* + JR^*) : \alpha_2 .$$

If $\underline{a}$ is a regular sequence mod J, then equality holds.

The first statement in a consequence of lemma 2.2.1. The second statement is obtained by "comparing coefficients", for this one can use the regularity of $\underline{a}$ mod J.

As a corollary of proposition 2.2.2 we get the result [s. also [H-I], prop. 1.5]:

<u>2.2.5 Corollary:</u> Let $ht(I) = l(I) =: s>o$ and let $a_1,\ldots,a_s$ generate a minimal reduction of I. Choose any system $x_1,\ldots,x_r$ of parameters mod I. Then

$$\{a_1,a_1t-a_2,\ldots,a_{s-1}t-a_s,\ a_st,\ x_1,\ldots,x_r\}$$

is a system of parameters of $R^+_e(I,R) = R^*$. Furthermore $I^*R^*_{M^*}$ is also <u>equimultiple.</u>

Now we are able to relate the length $l_{R^*}(R^*/\underline{a}^*R^*+JR^*)$ to the corresponding length $l_R(R/\underline{a}R+J)$ for the ground ring R .

<u>2.2.6 Proposition:</u> Let J be an ideal of R such that R/J is <u>Cohen-Macaulay</u>. Let $a_1,\ldots,a_s \in I$ be a system of parameters <u>mod J</u>. Assume that $\underline{a}I^n = I^{n+1}$. Then

$$l_{R^*}(R^*/\underline{a}^*R^*+JR^*) = s\cdot l(R/\underline{a}R+J) + \sum_{i=s}^{n} l_R(I^i+\underline{a}R+J/\underline{a}R+J) + \\ + \sum_{i=o}^{n} l_R((I^i \cap (\underline{a}R+J)/\underline{a}I^{i-1}+JI^i) .$$

<u>Remarks</u> to the line of the proof:

<u>1. Step:</u> By proposition 2.2.2 the radical of $\underline{a}^*R^* + JR^*$ coincides with the radical of $\mathfrak{b}R^* + \underline{at}\,R^*$, where $\mathfrak{b} = \underline{aR} + J$. Since $\underline{a}t$ is a reduction of It , this radical must be M^* ,

Since $R^*/M^* \cong R/M$, we get

$$l_o := l_{R^*}(R^*/\underline{a}^*R^* + JR^*) = l_R(R^*/\underline{a}^*R^* + JR^*) ,$$

hence

$$l_o = l_R(R^*/(\underline{a}R+J)^*) + l_R((\underline{a}R+J)^*/JR^* + \underline{a}R^* + \underline{at}R^*) + \\ + l_R(JR^*+\underline{a}R^*+\underline{at}R^*/\underline{a}^*R^* + JR^*) .$$

<u>2. Step:</u> By induction on s , one can check that

$$l_R(\underline{a}R^*+J\cdot R^* + \underline{at}R^*/\underline{a}^*R^* + JR^*) = \sum_{i=o}^{s-1} l_R(R/\underline{a}R+J+I^i),$$ following

from the fact that:

$$(\underline{a}R^*+JR^* + \underline{at}R^*) \cap (\underline{a}^*R^* + JR^* + (a_1R)^*) \subseteq \underline{a}^*R^* + JR^* + (a_1t)R^*$$

Therefore we obtain:

$$l_o = \sum_{i=o}^{n} l_R(I^i/I^i \cap (\underline{a}R+J)$$

$$+ \sum_{i=o}^{n} l_R(I^i \cap (\underline{a}R+J)/\underline{a}I^{i-1}+JI^i)$$

$$+ \sum_{i=o}^{s-1} l_R(R/\underline{a}R+J+I^i) .$$

(Note that the first two sums are finite, since $\underline{a}I^n = I^{m+1}$ for $m \geq n$).
This proves the claim.

Generalized Hilbert functions and multiplicities in $R^{*}_{\dagger}(I,R)$.

Let $x_1,\ldots,x_r \in M$ be a s.o.p. mod I . Then the images of x_i are a s.o.p. of $R^*/I^* \cong R/I \cong (R^*_{M^*}/I^*R^*_{M^*}$.

Therefore we may consider the Hilbert functions $H^{(i)}[\underline{x},I^*R^*_{M^*},R^*_{M^*}]$.

Since R^*/I^* is local, there is no ambiguity in writing these functions simply as $H^{(i)}[\underline{x},I^*,R^*]$ and the corresponding multiplicities as $e(\underline{x},I^*,R^*)$, instead of $e(\underline{x},I^*R^*_{M^*}.R^*_{M^*})$.

Now it is easy to check that if in particular
$\underline{\dim R = \dim R/I + ht(I)}$, then

$$e(\underline{x},I^*,R^*) = ht(I)\cdot e(\underline{x},I,R) .$$

This implies the following proposition.

2.2.7 Proposition: Let I be an equimultiple ideal of R and let $\underline{a} = (a_1,\ldots,a_s)$ generate a minimal reduction of I , where $s = ht(I)$. Let $x_1,\ldots,x_r$ be a system of parameters mod I , Then

$$e(\underline{x}R^*+\underline{a}^*R^*,R^*) = ht(I)\cdot e(\underline{x}R+\underline{a}R,R) .$$

For the proof apply proposition 2.2.2 and corollary 2.2.5.

§ 3 Graded Cohen-Macaulay-rings

Following our general viewpoint we will study the Cohen-Macaulay property for equimultiple ideals only.
First we recall to some well known facts:

3.1 Definition : A noetherian ring A will be called Cohen-Macaulay, if A_P is CM for all $P \in$ Spec A.

3.2 If A is any graded noetherian ring with unique homogeneous maximal ideal N, then A is CM if and only if A_N is CM.

If $y_1,\dots,y_m$ are homogeneous elements of A, then $(y_1,\dots,y_m)$ is a regular sequence in A if and only if it is so in A_N.

3.3 If R is a local ring and I a proper ideal of R, then $Re(I,R)$ is CM if and only if $gr_I R$ is CM.

Using results of [Ro-2] and prop. 2.1.1. one can prove the following result (the details of the proof will be given in [H-O-G].

<u>3.4 Proposition:</u> Let (R,M) be a local <u>Cohen-Macaulay</u> ring and I an <u>equimultiple</u> ideal of R. Let $\{x_1,\dots,x_r\}$ be a system of parameters mod I. If $\underline{a} = (a_1,\dots,a_s)$, $s = ht(I)$, generates a minimal reduction of I, then the following conditions are equivalent:

(i) $gr_I R$ is Cohen-Macaulay

(ii) $(\underline{x}R+\underline{a}R)\cap I^i = \underline{x}I^i + \underline{a}I^{i-1}$ for all $i \geq 1$.

Now we can prove the main theorem. The idea of the following proof is due to U. Grothe.

<u>3.5 Theorem :</u> Let R be a local ring and I an <u>equimultiple</u> ideal of R of height $> o$. Let $\underline{a} = (a_1,\dots,a_s)$ generate a minimal reduction of I, $s = ht(I) = l(I)$. Then the following conditions are equivalent:

(i) $R\overset{+}{e}(I,R)$ is Cohen-Macaulay and R is Cohen-Macaulay

(ii) $gr_I R$ is Cohen-Macaulay and $I^S \subset \underline{a}R$.

If $gr_I R$ is Cohen-Macaulay (and $ht(I) = l(I)$), then $I^S \subset \underline{a}R$ is equivalent to $\underline{a}\cdot I^{s-1} = I^s$.

<u>Proof :</u> We may assume <u>R to be CM</u> from the beginning, since this is a consequence of $gr_I R$ being CM.

Let $\underline{x} = \{x_1,\dots,x_r\}$ be a system of parameters mod I and let $\underline{a}^* = (a_1, a_1t-a_2,\dots,a_st)$ as in 2.2. Then, by corollary 2.2.5,

$\{x_1,\dots,x_r,a_1,a_1t-a_2,\dots,a_st\}$ is a system of parameters of $R^* = R\dot{e}(I,R)$, generating an ideal primary to M^*. Hence we have the following implications:

$$R^* \ CM \overset{3.2}{\Longleftrightarrow} R^*_{M^*}\text{- } CM$$

$$\Longleftrightarrow e(\underline{x}R^*+\underline{a}^*R^*,R^*) = l_{R^*}(R^*/\underline{x}R^*+\underline{a}^*R^*) \ .$$

By proposition 2.2.7, we have:

$$e(\underline{x}R^*+\underline{a}^*R^*,R^*) = s\cdot e(\underline{x}R+\underline{a}R,R) = s\cdot l_R(R/\underline{a}R+\underline{x}R) \ ,$$

hence proposition 2.2.6 implies that

$$R^* \ CM \Longleftrightarrow (1) \ \sum_{i=s}^{n} l_R \ (I^i+\underline{a}R+\underline{x}R/\underline{a}R+\underline{x}R) = o \ \text{ and}$$

$$(2) \ \sum_{i=o}^{n} l_R(I^i \cap (\underline{a}R+\underline{x}R)/\underline{a}I^{i-1}+\underline{x}I^i) = o \ ,$$

where n is any integer such that $I^{n+1} = \underline{a}I^n$.

Therefore we get finally the following implications:

$$R^* \ CM \Longleftrightarrow \begin{cases} I^s \subset \underline{a}R+\underline{x}R & \text{and} \\ (\underline{a}R+\underline{x}R) \cap I^i = \underline{x}I^i+\underline{a}I^{i-1} \ , & i \geq 1 \end{cases}$$

$$\Longleftrightarrow \begin{cases} I^s \subset (\underline{a}R+\underline{x}R) \cap I^s & \text{and} \\ (\underline{a}R+\underline{x}R) \cap I^i = \underline{x}I^i+\underline{a}I^{i-1} \ , & i \geq 1 \ . \end{cases}$$

a) If $I^s \subset \underline{a}I^{s-1}$ and $(\underline{a}R+\underline{x}R) \cap I^s = \underline{x}I^s + \underline{a}I^{s-1}$, then $I^s \subset (\underline{a}R+\underline{x}R) \cap I^s$.

b) If $I^s \subset (\underline{x}R+\underline{a}R) \cap I^s \Longrightarrow I^s \subset \underline{x}I^s+\underline{a}I^{s-1}$, hence $I^s \subset \underline{a}I^{s-1}$.

Therefore:

$$R^* \ CM \Longleftrightarrow \begin{cases} I^s \subset \underline{a}I^{s-1} & \text{and} \\ (\underline{a}R+\underline{x}R) \cap I^i = \underline{x}I^i+\underline{a}I^{i-1} \ , & i \geq 1 \ . \end{cases}$$

$$\Longleftrightarrow \begin{cases} I^s \subset \underline{a}I^{s-1} & \textit{and} \\ gr_I(R) \ \text{is} \ CM \ , \end{cases}$$

q.e.d.

<u>3.6 Remarks :</u> By an easy application of cohomological methods Huneke [Hu-2] has shown that $R\overset{+}{e}(I,R)$ and R Cohen-Macaulay imply $gr_I(R)$ Cohen-Macaulay without any assumption on I . This we will mention in section 4.1. But note that the harder part in the proof of theorem 3.5 is to show (ii) ⟹ (i) .

A result similar to theorem 3.5 has been announced by P. Schenzel, but we don't know of a written proof. In this announcement the condition $I^s \subset \underline{a}R$ is replaced by vanishing conditions on the local cohomology of $gr_I R$ (without assuming $ht(I) = l(I)$).

<u>3.7 Corollary:</u> Let (R,M) be a local Cohen-Macaulay ring with infinite residue field. Let I be an equimultiple ideal of height I > o . If $R\overset{+}{e}(I,R)$ is Cohen-Macaulay, then $R\overset{+}{e}(I^t,R)$ is Cohen-Macaulay for all $t \geq 1$.

Another consequence of theorem 3.5 is the following statement for <u>height 1</u> ideals:

<u>3.8 Proposition:</u> Let R be a local ring with infinite residue field and let I be an <u>equimultiple</u> ideal of R of <u>height 1</u> . Then the following conditions are equivalent:

(i) $R\overset{+}{e}(I,R)$ is Cohen-Macaulay

(ii) I is principal and $gr_I R$ is Cohen-Macaulay.

<u>Proof:</u> (ii) ⟹ (i) : follows from theorem 3.5.
(i) ⟹ (ii) : Let x be any element of I and let (a) be a minimal reduction of I . Then the elements a,at of $R\overset{+}{e}(I,R)$ are part of a homogeneous system of parameters, hence they form a regular sequence. Furthermore we have the relation

$$x(at) = a(xt) \equiv o \mod (a)\cdot R\overset{+}{e}(I,R) .$$

Therefore $x \in (a)\, R\overset{+}{e}(I,R)$, hence (as an element of degree o) $x \in (a)\cdot R$, i.e. I is generated by the non-zero divisor a .

Now $R/aR = R\overset{+}{e}(I,R)/aR\overset{+}{e}(I,R) + (at)R\overset{+}{e}(I,R)$ is CM , so R is CM, hence $gr_I R$ is CM, q.e.d.

From this proposition the question arises of how far the CM-property of $R\overset{+}{e}(I,R)$ (where I is any equimultiple ideal) implies that R itself is CM . The following examples show that without any additional assumptions this implication is not true.

<u>3.9.1 Example.</u> (s. Ikeda [I]): Let k be a field and let $X_1, X_2, X_3, Y_1, Y_2, Y_3$ indeterminates over k . Let

$$A = k[[X_1,X_2,X_3,Y_1,Y_2,Y_3]]/(X_1Y_1+X_2Y_2+X_3Y_3),\ (Y_1,Y_2,Y_3)^2) ,$$

and let N denote the maximal ideal of A .

Then $\dim A = 3$, depth $A = 2$, $e(A) = 2$ and $R\overset{+}{e}(N,A)$ is CM .

From this example one can easily get an example with <u>any</u> embedding dimension of R/P :

$R := A[[T_1,\ldots,T_n]]$, Ti indeterminates; $P := NR$.
Since R is faithfully flat over A we get $R\overset{+}{e}(P,R)$ is CM and $ht(P) = l(P) = 3$, and R is not CM .

<u>3.9.2 Example</u> (the same example was recently found by Goto-Shimoda - oral communication by N.Suzuki):

$A = k[[s^2,s^3,st,t]]$. Consider the following system a_1,a_2 of parameters with $a_1 = s^2$, $a_2 = t$. Then $(a_2^2:a_1^n) = (a_2^2:a_1) \subseteq (a_2)R$. Therefore by a result of U. Grothe $R\overset{+}{e}((a_1,a_2)A,A)$ is CM , but A is not.

As in example 3.9.1 we can pass to $R = A[[T_1,\ldots,Tn]]$ and $I := (a_1,a_2)R$ to get an example with big embedding-dimension of R/I , and $ht(I) = l(I) = 2$.

The following proposition states a sufficient condition for the implication $R\overset{+}{e}(I,R)$ is CM $\Longrightarrow$ R is CM.

<u>3.10 Proposition:</u> Let (R,M) be a complete equicharacteristic local ring such that R_Q is CM für all $Q \in \operatorname{Spec}R \setminus M$. Let I be an equimultiple ideal in R of height $I > o$ with the follo-

wing properties:

(i) $R\overset{+}{e}(I,R)$ is Cohen-Macaulay

(ii) $e(R) + e(R/I) \leq \text{emdim}\,(R/I) + 2$.

Then R is CM .

<u>3.11 Corollary:</u> (R,M) as in 3.10. Let $I=P_o$ be an equimultiple prime ideal (height $P_o > o$) with the following properties:

(i) $R\overset{+}{e}(P_o,R)$ is Cohen-Macaulay

(ii) $2e(R/P_o) \leq \text{emdim}\,(R/P_o) + 2$, and R_{P_o} is regular.

Then R is CM .

<u>3.12.1 Example:</u> In example 3.9.1 condition (ii) in 3.10 for $P = N \subset A$ is not fulfilled, and indeed A is not CM . If we take $R = A[[T_1,\dots,T_n]]$, n big enough, $P = NR$ then (i) and (ii) in 3.10 are fulfilled, but $R_P \cong A[[T_1,\dots,T_n]]_{N[[Ti]]}$ is <u>not</u> CM . Again R is not CM .

<u>3.12.2 Example (s. example 1.1.6 a)) :</u> $R = k[[s^2,s^3,st,t]]$ is a non-CM-Buchsbaum-ring with $e(R) = 2$. Let $P_o = (st,t)$. Then R_{P_o} is regular and $ht(P_o) = 1(P_o) = 1$. Now we have $e(R) = e(R/P_o)=2$, $\text{emdim}\,(R/P_o)) = 2$, hence (ii) in Cor. 3.11. is fulfilled. Since R is <u>not</u> CM , $R\overset{+}{e}(P_o,R)$ cannot be CM .

<u>3.12.3 Example:</u> In example 3.9.2 A and $I = (s^2,t)$ satisfy all assumptions of prop. 3.10 <u>except (ii)</u> , since (ii) implies in case of a M-primary ideal I that $e(A) \leq 1$. Again A is not CM.

<u>Question 1 :</u> Let (R,M) be a complete equicharacteristic local ring such that R_Q is CM for all $Q \in \text{Spec}R \setminus M$. Let I be an <u>equimultiple</u> ideal of height $I > o$ such that $R\overset{+}{e}(I,R)$ is CM . Are the following conditions equivalent:

(1) R is CM

(2) $e(R) + e(R/I) \leq \text{emdim}\,(R/I)+2$?

<u>Question 2 :</u> For <u>Cohen-Macaulay</u> rings (R,M) and <u>Cohen-Macauly</u> ideals $I \subset R$ (i.e. R/I is CM) we have [Sa], p. 80

$$\nu(I) \leq e(R/I)^{t-1} \cdot e(R) + t - 1 ,$$

where t = height I . This implies:

(1) $\nu(I) \leq e(R)$ for $t = 1$.

For height 2 ideals one knows a better bound due to Rees:

(2) $\nu(I) \leq e(R) + e(R/I)$ for $t = 2$.

So, giving a bound for $e(R)$ or $e(R)+e(R/I)$ means in these cases restricting the number of elements in a minimal base of I.

Is there a "natural" condition on $e(R)$ and $e(R/I)$ which implies $\nu(I_P) \leq ht(P)$ for $P \supset I$? (See also section 4)

Proof of proposition 3.10 : 1. Step: Let $a_1,\ldots,a_s$ be a minimal reduction of I and $x_1,\ldots,x_r$ a system of parameters with respect to I . Then by corollary 2.2.5 and the assumption (i)

$$\{a_1, a_1t-a_2, \ldots, a_{s-1}t-a_s, a_st, x_1, \ldots, x_r\}$$

is an $R^*_{M^*}$-sequence. [$R^* = R\overset{+}{e}(I,R)$, M^* the unique homogeneous maximal ideal of R^*]. Using the fact that a_1 is also a non-zero-divisor on R^* one can show that

$$(a_1R^* : a_1t) = IR^* .$$

Therefore we get the exact sequence

$$o \to gr_I(R)(-1) \to R^*/a_1R^* \longrightarrow R^*/(a_1,a_1t)R^* \longrightarrow o .$$

The case $r = o$ is trivial. Se we may use induction on $r = \dim R/I$. If $r > o$, then a_1,x_1 is a $R^*_{M^*}$-sequence, hence (by the exact sequence) x_1 is a non-zero-divisor on $gr_I(R)$. Therefore $x_1R \cap I^n = x_1I^n$ for $n \geq o$. This implies that

$$R\overset{+}{e}(I+x_1R/x_1R,\ R/x_1R) \cong R^*/x_1R^* \text{ is CM} .$$

By induction hypothesis we have

$$\text{depth } R/x_1R \geq \dim (R/I+x_1R) +1 = \dim R/I ,$$

i.e. $$\text{depth } R \geq \dim R/I + 1 .$$

Furthermore, it is easy to see that R/I is <u>Cohen-Macaulay</u>. (One can even show, that <u>depth $I^n/I^{n+1} = \dim R/I$</u> for all $n \geq o$; see [I-H], proof of prop. 1.5.).

<u>2. step:</u> We set $n_o = \dim R/I + 1$. Since depth $R \geq n_o$ by step 1 and since R_Q is CM for all primes $Q \neq M$, R satisfies <u>Serre's condition</u> S_{n_o} .

Since R/I is CM we have the Abhyankar-inequality:

$$e(R/I) \geq \text{emdim}\,(R/I) - \dim(R/I) + 1 .$$

Hence by condition (ii) of proposition 3.10 we get

$$e(R) \leq n_o .$$

Since R is a complete equicharacteristic ring we get by [Hu-1] that R is CM , q.e.d.

Note that the conditions (i) and (ii) of prop.3.10 imply (s. theorem 3.5) in particular $I^s = \underline{a}I^{s-1}$, where $s = ht(I) = l(I)$. Furthermore (by [Ro-2] and prop. 2.1.1 they imply R is normally CM along I . The following proposition shows that this implications doesn't depend on the special assumptions on R in prop. 3.10 .

<u>3.13 Proposition:</u> Let (R,M) be a local ring and I an <u>equimultiple</u> ideal in R of $s = ht(I) > o$. Let $(a_1,\ldots,a_s)$ be a minimal reduction of I . If $R\mathring{e}(I,R)$ is <u>Cohen-Macaulay</u> then

(i) $(a_1,\ldots,a_s)I^{s-1} = I$

(ii) depth $R \geq \dim(R/I) + 1$

(iii) R is normally Cohen-Macaulay along I .

For the details see [I-H].

The method used in step 2 of the proof of prop. 3.10 can also be applied to get statements about the CM-property of monoidal transforms R_1 of R with center I .

We will say "(A,M) is <u>locally CM</u>" if A_P is CM for all $P \neq M$.

<u>3.14 Proposition:</u> Let (R,M) be a quasi-unmixed excellent ring containing a field k. Let I be a Cohen-Macaulay-ideal of the principal class satisfying the following properties:

(i) R is normally Cohen-Macaulay along I for $n >> o$

(ii) $e(R/I) \leq \dim(R/I) + 1$.

If R_1 is locally Cohen-Macaulay then it is Cohen-Macaulay.

For the proof one can use the fact that (i) implies depth $R_1 \geq \dim(R/I) + 1$. This follows from [O-R], Cor. 1.7, p.8.

Furthermore we get in particular the following proposition.

<u>3.15 Proposition:</u> Let (R,M) be an unmixed complete local ring containing a field. Let P be a prime ideal in R satisfying the following properties:

(i) R_P is regular and R/P is Cohen-Macaulay

(ii) $f : \mathrm{Proj}\,(gr_P R) \longrightarrow \mathrm{Spec}\,(R/P)$ is flat

(iii) $e(R/P) = 2$.

Then the completion of any monoidal transform R_1 of R with center P, satisfying S_2, is a <u>hypersurface</u>.

<u>Proof:</u> The flatness of f implies [O-R] projective normal flatness of R along P (i.e. P^n/P^{n+1}, is flat over R/P for $n >> o$). Since R/P is CM, we get [O]:

$$H^{(s+1)}[R_1] \leq e(R/P)\, H^{(r+1)}[R_P] \quad ,$$

where $r = \dim R/P$ and $s = \dim R - \dim R_1$, hence by (i), (iii): $e(R_1) \leq 2$. Then if R_1 satisfies S_2, $\hat{R}_1$ is a hypersurface.

§ 4 Cohomological approach

We make several remarks of how to use homological methods for the questionings of these lectures. In 3.6 we remarked that $R\overset{+}{e}(I,R)$ and R Cohen-Macaulay imply $gr_I R$ Cohen-Macaulay without any assumption on I. The proof of this statement (which is even true for any commutative ring and any ideal I of height > 0) is due to Huneke.

4.1 Proposition: Let R be a local Cohen-Macaulay-ring, I an ideal of height at least 1. If $R\overset{+}{e}(I,R)$ is Cohen-Macaulay then $gr_I(R)$ is Cohen-Macaulay.

Proof: We set:

$$R^* = R\overset{+}{e}(I,R), \qquad M^* = (M, It),$$

$$T = gr_I R, \qquad Q = M/I \oplus \sum_{n\geq 1} I^n/I^{n+1} .$$

Then there are exact sequences:

(1) $0 \longrightarrow It \longrightarrow R^*_{M^*} \longrightarrow R \longrightarrow 0$

(2) $0 \longrightarrow I \longrightarrow R^*_{M^*} \longrightarrow T_Q \longrightarrow 0$.

Note that $d := \dim R = \operatorname{depth} R$ and $\operatorname{depth} R^*_{M^*} = d+1$ by assumption. Furthermore we set $K = R^*/M^* = R/M$.

Applying $\operatorname{Hom}_{R^*_{M^*}}(K,-)$ we get the exact sequence

$$\ldots \longrightarrow \underset{\overset{\parallel}{0}}{\operatorname{Ext}^{d-1}(K,R)} \longrightarrow \operatorname{Ext}^d(K,It) \longrightarrow \underset{\overset{\parallel}{0}}{\operatorname{Ext}^d(K,R^*_{M^*})} \longrightarrow \ldots ,$$

hence $\operatorname{Ext}^{\nu}(K,It) = o$ for $\nu < d+1$.

Since $It \cong I$ as R^*-modules, we also have $\operatorname{Ext}^{\nu}(K,I) = o$ for $\nu < d+1$.

Now apply $\mathrm{Hom}_{R^*_{M^*}}(K,-)$ to the sequence (2):

$$\ldots \longrightarrow \underset{\substack{\| \\ 0}}{\mathrm{Ext}^{d-1}(K,R^*_{M^*})} \longrightarrow \mathrm{Ext}^{d-1}(K,T_Q) \longrightarrow \underset{\substack{\| \\ 0}}{\mathrm{Ext}^{d}(K,I)} \longrightarrow \ldots ,$$

i.e. depth $T_Q = d$, hence $gr_I R$ is CM.

In [I-H] we have characterized the Cohen-Macaulay-structure of $R\overset{+}{e}(I,R)$ by a vanishing-theorem of local cohomology of $gr_I R$ with respect to $M^* = M \oplus \sum_{n\geq 1} I^n$; see [I-H], prop. 2.1 and 1.5. There we assumed that R is normally Cohen-Macaulay along I. Recently S. Ikeda has observed that the proof of prop. 2.1 in [I-H] also works if we replace this strong condition on I by equimultiplicity, i.e. ht(I) = l(I). By using local duality one can even omit equimultiplicity as S. Ikeda and N.V. Trung have pointed out; see theorem 4.5 and remark 4.7.

<u>4.2 Résumé of local cohomology:</u>

If Q is a homogeneous ideal of a Noetherian ring S , which here we assume to be a finitely generated (nonnegatively) graded algebra over a Noetherian ring S_o, and if $\mathrm{rad}(Q) = \mathrm{rad}((f_o,\ldots,f_n)S)$, where f_i are forms of S, then the local cohomology modules $H^i_Q(S)$ can be expressed as direct limits of Koszul cohomology

$$H^i_Q(S) = \varinjlim_t H^i(K^\cdot(\underline{f}^t;S)) \ , \ f^t = f_1^t ,\ldots, f_t^t$$

acquiring a $\mathbb{Z}$-grading (which is <u>independent</u> of the choice of the f_i).

The corresponding formula holds for a graded S-Module G which may have finitely many negative pieces: $H^i_Q(G) = \varinjlim_t H^i(K^\cdot(\underline{f}^t;G))$.

Recall that if f_o has degree d_o , the right complex $K^\cdot(f_o,G)$ of graded modules and maps of degree 0 is

$$K^{o}(f_o;G) = G$$
$$K^{1}(f_o;G) = G(d_o) \ , \qquad G(d_o)_n = G_{d_o+n}$$
$$K^{i}(f_o;G) = o \qquad \text{for} \quad i \neq o,1 \ ,$$

where the map $K^{o}(f_o;G) \longrightarrow K^{1}(f_o;G)$ is induced by multiplication by f_o . The complexes $K^{\bullet}(\underline{f}^{t};G)$ form a direct limit system. For $f = f_o$, this system is given by the diagram:

$$\begin{array}{lccccccc}
K^{\bullet}(f_o^{t+t'},G): & 0 & \longrightarrow & G & \xrightarrow{f_1^{\,t+t'}} & G((t+t')d_o)) & \longrightarrow & 0 \\
\uparrow & & & \| & & \uparrow f_o^{t'} & & \| \\
K^{\bullet}(f_o^{t},G): & 0 & \longrightarrow & G & \longrightarrow & G(t\cdot d_o) & \longrightarrow & 0 ;
\end{array}$$

the graded limit complex may be identified [EGA I, No4] , 1.6.1 with

$$0 \longrightarrow G \longrightarrow G_{f_o} \longrightarrow 0 \ ;$$

where $\frac{g}{f_o^{\,k}} \in (G_{f_o})_n$ if $\deg g = k\cdot d_o+n$;

this is what one denotes by $K^{\bullet}(f_o^{\infty},G)$.

For $\underline{f} = f_o,\dots,f_n$ and $\underline{f}^{t} = f_o^{t},\dots,f_n^{t}$ we define

$$K^{\bullet}(\underline{f},G) = \Big(\bigotimes_{i=o}^{n} K^{\bullet}(f_i;R)\Big) \otimes_R G \quad ,$$

where G is concentrated in degree 0 and $\otimes_R$ is graded. Then the limit $\varinjlim K^{\bullet}(\underline{f}^{t},E)$ may be identified with

$$\bigotimes_{i=o}^{n} K^{\bullet}(f_i^{\infty};R) \otimes_R G =: K^{\bullet}(\underline{f}^{\infty},G) \ .$$

If we forget the grading of S and if we consider any ideal I and any elements $f_o,\dots,f_n$ such that rad I = rad$(f_o,\dots,f_n)$, then the cohomology modules $H_I^{\bullet}(G)$ are usually defined in this <u>affine</u> case as $\varinjlim_t \mathrm{Ext}_{\mathcal{S}}^{\bullet}(S/I^t, G) \cong H^{\bullet}(K^{\bullet}(\underline{f}^{\infty};G))$.

Therefore, for the graded ring S and for an <u>homogeneous</u> ideal $I \subset \mathcal{S}$ we may first choose <u>forms</u> $f_o,\dots,f_n$ such that rad (I) = rad $(f_o,\dots,f_n)$; then we can also choose a part of a system of parameters, say $g_1,\dots,g_t$ $(t = l(I))$ which is a reduction of I, so that

$$\mathrm{rad}(I) = \mathrm{rad}(f_o,\dots,f_n) = \mathrm{rad}(g_1,\dots,g_t) \ ,$$

and we get

$$H_I^{\bullet}(G) \cong H^{\bullet}(K^{\bullet}(\underline{f}^{\infty},G)) \cong H^{\bullet}(K^{\bullet}(\underline{g}^{\infty},G)) \ , \text{ which is therefore}$$

graded.

In particular, take a M-primary ideal I in the local ring R of dimension d and let $a_1,\dots,a_d$ be a minimal reduction of I . Set $S = R^* = R\ddagger(I,R)$, $M^* = M \oplus \sum I^n$, $G = \mathrm{gr}_I R$ and a_i^* = initial form of a_i in I/I^2. Note that $\underline{a}R^* + \underline{at}R^* = (a_1,\dots,a_d,a_1t,\dots,a_dt)$ is a reduction of M^* and $a_1, a_2-a_1t,\dots,a_d-a_{d-1}t, a_dt$ is a system of parameters of R^*. The images of these inhomogeneous elements in $G = R^*/IR^*$ are the initialforms $a_1^*,\dots,a_d^*$ $(a_i \mapsto 0; a_i-a_{i-1}t \mapsto a_i^*$ since $a_it \notin I^2t)$, and $\mathrm{rad}(a_1^*,\dots,a_d^*)$ = maximal <u>homogeneous</u> ideal of G . Therefore we get

$$(\#) \qquad \bigoplus_{i=1}^{d} G_{a_1^*\dots\check{a}_i^*\dots a_d^*} \xrightarrow{\ \varphi\ } G_{a_1^*\dots a_d^*} \longrightarrow H^d_{M^*}(G) \longrightarrow 0$$

$$G_{a_1^*\dots a_d^*} \cong R^*_{\underline{f}}/IR^*_{\underline{f}} \ , \quad \underline{f} = \{a_1, a_2-a_1t,\dots,a_dt\},$$

$$\text{and } \ \mathrm{rad}((\underline{f})) = M^* \ ;$$

where $\varphi(g_1,\dots,g_d) = \sum_{i=1}^{d} (-1)^i g_i$ for $g_i \in G_{a_1^*\dots\check{a}_i^*\dots a_d^*}$.

As in 4.1 we will use the following exact sequences (now setting R^*_+ for $(It)R^*$ and $R^*_+(1)$ for IR^*):

(1) $\quad 0 \longrightarrow R^*_+ \longrightarrow R^* \longrightarrow R \longrightarrow 0$

(2) $\quad 0 \longrightarrow R^*_+(1) \longrightarrow R^* \longrightarrow G \longrightarrow 0$;

they imply the exact sequences:

$$(1') \longrightarrow H^i_{M^*}(R^*) \longrightarrow H^i_M(R) \longrightarrow H^{i+1}_{M^*}(R^*_+) \longrightarrow H^{i+1}_{M^*}(R^*) \longrightarrow$$

$$(2') \longrightarrow H^i_{M^*}(R^*) \longrightarrow H^i_{M^*}(G) \longrightarrow H^{i+1}_{M^*}(R^*_+)(1) \longrightarrow H^{i+1}_{M^*}(R^*) \longrightarrow$$

[Note that in (1') we get $H^i_M(R)$ as second term, for the images of $a_1, a_2-a_1t, \ldots, a_d-a_{d-1}t, a_dt$ (which also determine the cohomology $H^i_{M^*}(R^*/R^*_+)$) are $a_1, a_2, \ldots, a_d$, hence:

$$\bullet\ R_{a_1 \ldots \check{a}_i \ldots a_d} \xrightarrow{\varphi} R_{a_1 \ldots a_d} \longrightarrow H^i_{M^*}(R^*/R^*_+) \longrightarrow 0.]$$

These sequences (1') and (2') are the main tools for the cohomological approach.

A version of the following theorem 4.3 was first proved in [I-H] prop. 2.1.

__4.3 Theorem:__ Let (R,M) be a local ring such that $l(H^i_M(R)) < \infty$ for $i < d = \dim R$. Let I be an ideal in R. Then the following conditions are equivalent:

(i) $R^* = R\dot{e}(I,R)$ is Cohen-Macaulay and I is equimultiple.

(ii) a) $H^i_{M^*}(G)_n = \begin{cases} H^i_M(R) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$ and $i < d$

b) $H^d_{M^*}(G)_n = 0$ for $n \geq 0$.

c) R nCM I

__Sketch of the proof:__ (i) => (ii) : Since R^* is CM we obtain from (1') and (2') for $i < d$:

$$H^i_M(R) = H^{i+1}_{M^*}(R^*_+) \qquad \text{and}$$

$$H^i_{M^*}(G) = H^{i+1}_{M^*}(R^*_+)(1) ,$$

hence (ii) a). [Note that $[H^{\bullet}_{M^*}(E)]_y \cong H^{\bullet}(K^{\bullet}_y(\underline{f}^{\infty}, E))$;, and $H^i_M(R)$ is concentrated in degree o, hence $H^{i+1}_{M^*}(R^*_+) = 0$ for $\nu \neq o$. Therefore $H^i_{M^*}(G)_n = H^{i+1}_{M^*}(R^*_+)_{n+1} = 0$ for all $n \neq -1$.]

To prove <u>(ii) b) and c)</u> we recall that by corollary 2.2.5, page 36 $\{a_1, a_1t-a_2, \ldots, a_st, x_1, \ldots, x_r\}$ is a system of parameters of R^*, where $a_1, \ldots, a_s$ is a minimal reduction of I and $x_1, \ldots, x_r$ is a system of parameters mod I. (Here we use that fact that $ht(I) = l(I) = s$). Therefore by assumption, $\{a_1, a_1t-a_2, \ldots, a_st\}$ is a $R^*_{M^*}$-sequence.

Furthermore R is <u>normally CM along I</u> as we have indicated in the 1. step of the proof of proposition 3.10. Therefore we find elements $b_1, \ldots, b_r$ ($r = \dim R/I$), forming a G-sequence $\underline{b}$. Now take

$$\overline{G} = gr_{I+\underline{b}R/\underline{b}R}(R/\underline{b}R) = G/\underline{b}G.$$

Then one can show that

$$(3) \qquad H^s_{M^*}(\overline{G})_n = o \quad \text{for } n \geq o,$$

using the previous relation (#) (page 50) for $\overline{G}$ and the $M/\underline{b}R$ - primary ideal $I+bR/\underline{b}R$.

From the exact sequence

$$0 \longrightarrow G \xrightarrow{b_1} G/b_1G \longrightarrow 0$$

we obtain

$$(4) \quad \ldots \longrightarrow H^{i-1}_{M^*}(G) \longrightarrow H^{i-1}_{M^*}(G/b_1G) \longrightarrow H^i_{M^*}(G) \longrightarrow H^i_{M^*}(G) \longrightarrow$$

hence by (ii) a): $\qquad H^i_{M^*}(G/b_1G)_n = o \quad$ for $n \neq -1$, $i < d-1$.

For $i = d-1$ we get the equivalence:

$$H^d_{M^*}(G)_n = 0 \text{ for } n \geq o \Longleftrightarrow H^{d-1}_{M^*}(G/b_1G)_n = o \quad \text{for } n \geq o$$

Repeating this argument we get

$H^d_{M*}(G)_n = o$ for $n \geq o \iff H^{d-r}_{M*}(\bar{G})_n = o$ for $n \geq o$.

Note that $d-r = s$. Now, using the previous sequence (#) in 4.2 and the fact that $I^s = (\underline{a})I^s$ (which in easy to be proved), we can show indeed that $H^s_{M*}(\bar{G})_n = o$ for $n \geq o$.

To (ii) $\Longrightarrow$ (i): Now we know by assumption that R is normally CM along I , hence - as before - we find a G-sequence $b_1,\ldots,b_r$ such that

$$(b_1,\ldots,b_r) \cap I^n = (b_1,\ldots,b_r)I^n \quad \text{for} \quad n \geq o ,$$

hence

$$R\ddagger(I+\underline{b}R/\underline{b}R, R/\underline{b}R) = \bigoplus_{n\geq o} I^n/\underline{b}R \cap I^n = \bigoplus_{n\geq o} I^n/bI^n = R\ddagger(I,R)/\underline{b}R\ddagger(I,R).$$

(see proof of 3.10).

Since b_1 is a non-zero-divisor on $R\ddagger(I,R)$ we can conclude that $\underline{b}$ is a homogeneous $R\ddagger(I,R)$-sequence by induction on s. Therefore we have only to prove that

$$\overline{R^*} := R\ddagger(I+\underline{b}R/\underline{b}R,\ R/\underline{b}R) \quad \text{is Cohen-Macaulay.}$$

From (4) and (ii) a) we obtain for $i < s$ by induction on r :

(5) $\quad H^i_{M*}(\bar{G})_n = o \quad$ for $n \neq -1$ and

$\quad l(H^i_{M*}(\bar{G})) < \infty$.

(5) implies by an essential result of Goto that

$$l(H^i_{M*}(\overline{R^*})) < \infty \quad \text{for} \quad i \lneqq s .$$

As in the first part of the proof we have again

(6) $\quad H^s_{M*}(\bar{G})_n = O \quad$ for $n \geq o$.

From the analogous exact sequences (1') and (2') in 4.2, now applied on $\overline{R^*}$, we get for $i < s$:

$$H^i_{M^*}(\overline{R^*_+})_\nu \cong H^i_{M^*}(\overline{R^*})_\nu \quad , \quad \text{for } \nu \neq o$$

$$H^i_{M^*}(\overline{R^*_+})_{\nu+1} \cong H^i_{M^*}(\overline{R^*})_\nu \quad , \quad \text{for } \nu \neq -1 \; .$$

But

(7) $H^i_{M^*}(\overline{R^*})_\nu = o$ for $\nu >> o$ or $\nu << o$, $i \leq s$,

since $l(H^i_{M^*}(\overline{R^*})) < \infty$. Therefore by diagram-chasing we have

$$H^i_{M^*}(\overline{R^*}) = o \quad \text{for } i < s \; .$$

So it <u>remains to prove</u> that $H^s_{M^*}(\overline{R^*}) = o$ (note that $\dim \overline{R^*} = \dim(R/\underline{b}R) + 1 = s+1$). But this follows from (5),(6) and (7), q.e.d.

4.4 Blowing up conductor ideals.

In a recent paper [Shi] M. Shinagawa investigated the Cohen-Macauly-property of the blowing up of (R,M) along the conductor. His ideas and theorem 4.3 can be used to get some idea of the gap between R is Cohen-Macaulay and $R^+_e(I,R)$ is Cohen-Macaulay for special ideals I; compare 3.9 - 3.12.

<u>4.41 Proposition:</u> Let (R,M) be reduced with a finite normalization $\bar{R}$ oder R . Let I be an unmixed ideal of height 1 with the following properties:

(i) $\quad R/I \longrightarrow \bar{R}/I\bar{R}$ is flat

(ii) $\quad IR = I\bar{R}$ is principal in $\bar{R}$.

Then we get the two statements:

(1) $\mathrm{Proj}(R^+_e(I,R))$ is Cohen-Macaulay iff R is Cohen-Macaulay.

(2) $R^+_e(I,R)$ is Cohen-Macaulay iff

a) $H^i_{M^*}(G)_n = o$ for all n and $i < d = \dim R$

b) $H^d_{M^*}(G)_n = o$ for all $n \geq o$.

c) R is Cohen-Macaulay.

<u>Proof:</u> (following the ideas in [Shi]).
By the assumptions the blowing-up morphism

$$\Pi : \operatorname{Proj}(R\overset{+}{e}(I,R)) \longrightarrow \operatorname{Spec} R$$

is finite, since $ht(I) = l(I) = 1$.

Since I is the conductor we have $\bigoplus_{n>o} I^n = \bigoplus_{n>o} (I\bar{R})^n$, hence (by [EGA II], 2.4.7)

$$\operatorname{Spec} \bar{R} \cong \operatorname{Proj}(R\overset{+}{e}(I\bar{R},\bar{R})) = \operatorname{Proj}(Re^{+}(I,R))$$

Now I^n/I^{n+1} is free over $\bar{R}/I\bar{R}$ for all n , since $I\bar{R}$ is principal, generated by a regular element. Since $\bar{R}/I\bar{R}$ is free over R/I by condition (i), we get that R is normally flat along I . Therefore by [Shi], thm. 3, R is CM iff $\operatorname{Proj}(R\overset{+}{e}(R,I))$ is CM , proving (1).

(2) follows from thm. 4.3 and statement (1).

<u>4.4.2 Remark:</u> Conditions (i) and (ii) have been used to get normal flatness of R along I . This doesn't mean, that $I \subset R$ is principal.

<u>Example:</u> $R = k[[X,Y,Z]]/(X^3-Y^2) \cong k[[t^2,t^3,z]]$,

k = infinite field

$I = (t^2,t^3)\cdot R$, $ht(I) = 1 = l(I)$

Then : $\bar{R} = k[[t,z]]$ and $I\bar{R} = (t^2)k[[t,z]]$. Here $I \subset R$ is not principal, hence $R\overset{+}{e}(I,R)$ is <u>not</u> CM by theorem 3.5. But of course R is normally flat along I, and $\operatorname{Proj}(R\overset{+}{e}(R,I))$ is CM.

Note that R is a hypersurface with $e := e(R) = 2 = \dim R =: d$; clearly $gr_M R$ is CM . But for hypersurfaces we have always $M^e = (\underline{x})M^{e-1}$ for some minimal reduction of M; see [H-O-G], 5.1. Therefore we get in our case: $M^d = \underline{x}M^{d-1}$, hence $R\overset{+}{e}(M,R)$ <u>is</u> CM.

As we have already mentioned that S. Ikeda and N.V. Trung can prove the equivalence of (i) and (ii), a), b) in theorem 4.3 without any additional condition on R and I using local duality. We will sketch this proof but we assume for the sake of simplicity that R is quasi-unmixed. Furthermore we add to (ii) a), b) the condition "Proj R* is CM" ; see theorem 4.5.

<u>4.5 Theorem:</u> Let (R,M) be a quasi-unmixed (noetherian) ring and I an ideal of R with $ht(I) > o$. Then the following conditions are equivalent:

(1) $R^* = R\overset{+}{e}(I,R)$ is Cohen-Macaulay

(2) a) Proj R^* is Cohen-Macaulay

b) $H^i_{M^*}(G)_n = \begin{cases} H^i_M(R) & \text{for } n = -1 \\ 0 & \text{for } n \neq -1 \end{cases}$ and $i < d = \dim R$

c) $H^d_{M^*}(G)_n = o$ for $n \geq o$.

<u>Idea of the proof</u>,(after S. Ikeda in a letter from March /83 to M. Herrmann): (1) => (2), a) is clear and (1) => (2), b) we have shown in the proof of theorem 4.3;

<u>(1) =>(2),c)</u> can be seen as follows (see again proof of thm.4.3):

Take the exact sequences (1) and (2) of 4.2; since R^* is CM they imply the exact sequences:

(1') $0 \longrightarrow H^d_M(R) \longrightarrow H^{d+1}_{M^*}(R^*_+) \longrightarrow H^{d+1}_{M^*}(R^*) \longrightarrow 0$.

$H^d_M(R)$ is concentrated in degree o; therefore we get:

(∓) $H^{d+1}_{M^*}(R^*_+)_i \cong H^{d+1}_{M^*}(R^*)_i$ for $i > o$.

Since $H^{d+1}_{M^*}(R^*_+)$ is artinian we have $H^{d+1}_{M^*}(R^*_+)_n = o$ for $n >> o$, hence $H^{d+1}_{M^*}(R^*)_n = o$ for $n >> o$.

The second sequence (2) implies

(2') $0 \longrightarrow H^d_{M^*}(G) \longrightarrow H^{d+1}_{M^*}(R^*_+)(+1) \xrightarrow{\delta} H^{d+1}_{M^*}(R^*) \longrightarrow 0$.

The map δ implies:

$$H^{d+1}_{M^*}(R^*_+)(+1)_n = H^{d+1}_{M^*}(R^*_+)_{n+1} \longrightarrow\!\!\!\!\rightarrow H^{d+1}_{M^*}(R^*)_n \overset{(7)}{=} H^{d+1}_{M^*}(R^*_+)_n \quad ,$$

in particular: $H^{d+1}_{M^*}(R^*_+)_1 \longrightarrow\!\!\!\!\rightarrow H^{d+1}_{M^*}(R^*)_o$.

Therefore $H^{d+1}_{M^*}(R^*_+)(+1)_{n-1} = H^{d+1}_{M^*}(R^*_+)_n = 0$ for $n \geq 1$, hence

by (2') we have: $H^d_{M^*}(G)_m = 0$ for $m \geq 0$.

For <u>(2) => (1)</u>: I) We want to use duality-theory of graded rings over a given local ring. Recall that the canonical module of a noetherian graded ring $S = \bigoplus_{n\geq} S_n$ with $S_o = (A,N)$ a complete local ring is defined by:

$$K_S = \underline{\mathrm{Hom}}_S\,(H^{d_s}_{M_S}(S),\, E_S) \, ,$$

where $d_s = \dim S$, M_S the maximal homogeneous ideal of S, and E_S is the injective hull of S/M_S in the category of graded S - modules.

Then the "Poineare - Serre - duality" - theorem implies for <u>Cohen-Macaulay</u>-rings S and for any <u>finitely generated S</u> - module V :

(1) $\underline{\mathrm{Hom}}_S(H^i_{M_S}(V),\, E_S) = \mathrm{Ext}_S^{d_s-i}(V,K_S)$,

where $o \leq i \leq d_s$.

(2) S is Gorenstein iff $K_S = S(n)$ for some $n \in \mathbb{Z}$.

II) For the proof of (2) => (1) we may assume that the given ring (R,M) is complete. Then there exists a (complete) local Gorenstein ring B such that $R = B/b$ and $\dim R = \dim B$. Obviously R^* is an homomorphic image of the graded <u>Gorenstein</u> ring $S := B[X_1,\ldots,X_m]$, where X_i are indeterminates of degree 1 , $m = \nu(I)$ and $\dim S = d+m$. Setting $V = G = gr_I R$, we get by the duality theorem:

$$(*) \quad \underline{Hom}_S(H^i_{M*}(G), E_S) = Ext_S^{d+m-i}(G, K_S) .$$

[Note that $H^i_{M_S}(G) \cong H^i_{M*}(G) = H^i_{M_G}(G)$, where M_G is the maximal homogeneous ideal of G ; see 4.2].

By (*) the conditions (2) b) and (2) c) of theorem 4.5 are equivalent to:

$$(\bar{2})\ b) \quad Ext_S^{d+m-i}(G, K_S)_\mu = \begin{cases} Ext_B^{d-i}(R,B) & \text{for } \mu = 1 \\ 0 & \text{for } \mu \neq 1 \end{cases} \quad \text{for } i < d$$

$$(\bar{2})\ c) \quad Ext_S^m(G, K_S)_\mu = 0 \quad \text{for } \mu \leq 0 .$$

III) <u>Claim:</u> $l(H^i_{M*}(R^*)) < \infty$ for $0 \leq i \leq d$.

<u>Proof of the claim:</u> Since R is quasi-unmixed, R^*_{M*} is quasi-unmixed. To prove the claim it is enough to show that for any <u>graded</u> prime $P^* \neq M^*$ of R^* we have R^*_{P*} is Cohen-Macaulay. If P^* is relevant, R^*_{P*} is CM by condition 2) a). Hence we may assume $P^* \supset \bigoplus_{n \geq 0} I^n$, i.e. $P^* \cap R =: P \neq M$. Again by 2) a) we may assume <u>$P \supset I$.</u>

Assume first that P is a minimal prime of I . Then we have $ht(I \cdot R_P) = l(I \cdot R_P)$. Furthermore the conditions $(\bar{2})$ b) and $(\bar{2})$ c) are satisfied by R_P and $G(IR_P)$ since S_P is Gorenstein.

(Note that $K_S = S(n)$ for $n = -\nu(I)$ and use the Gorenstein-property for computing the Ext's).

But these conditions are equivalent to (2) a) and (2) b) for R_P , $G(IR_P)$ by duality.Therefore (since $ht(IR_P) = l(IR_P)$) by theorem 4.3, R^*_{P*} is CM .

Now by induction on $\dim R/I$, we may assume that R^*_P is CM for all $P \nsupseteq M$ [by dividing out parameter-elements $x_1,\ldots,x_{r-1}$ with respect to R/I] . This prove the claim.

IV) Now we can continue in analogy to the proof of theorem 4.3 to finish the proof of theorem 4.5.

<u>4.6 Remarks to the proof of theorem 4.5:</u>

1) The assumption "R is quasi-unmixed" was used in step III to conclude that $R^*_{M^*}$ is quasi-unmixed. What one really needs is some dimension-condition in $R^*_{M^*}$. Therefore Ikeda and Trung can avoid this assumption on R .

2) "Proj R^* is CM" is used in the proof (2) $\Longrightarrow$ (1) to deal with the case $P \not\supset I$. Ikeda observed that this case can also be settled without the assumption Proj R^* is CM.

<u>4.7 Remark.</u> There exists another homological approach to the question of how to get $R^+(I,R)$ or $gr_I R$ Cohen-Macaulay if R is Cohen-Macaulay by Herzog, Simis and Vasconcelos [H-Si-Va]:

<u>4.8 Theorem:</u> Let R be a local Cohen-Macaulay ring. Let I be an ideal with the properties:

(i) I is generically a complete intersection.

(ii) The homology modules of the Koszul complex on a system of generators of I are Cohen-Macaulay

(iii) $\nu(I_P) \leq ht(P)$ for every prime $P \supset I$.

Then $R^+(I,R)$ is Cohen-Macaulay.

The condition (iii) is used to show that $R^+(I,R)$ is isomorphic to the symmetric algebra with respect to I, which is "easier" to work with. The hard problem is of course how to check the very strong condition (ii).[Avramov and Herzog have considered certain special cases where (ii) is fulfilled.]

REFERENCES

[A-K] A. Altman - St. Kleiman, Introduction to Grothendieck duality theory, Lecture Notes in Math. 146 (1970)

[Be] B.M. Bennett, On the characteristic functions of a local ring, Ann. of Math. 91 (1970), 25 - 87

[Bö] E. Böger, Eine Verallgemeinerung eines Multiplizitätensatzes von D. Rees, J. Algebra 12 (1969), 207 - 215

[D] E.C. Dade, Multiplicity and monoidal transformations, Thesis, Princeton Univ. 1960

[EGA] A Grothendieck - J. Dieudonné, Eléments de géométrie algébrique, Publ. Math. I.H.E.S. 1960 - 67

[HO-1] M. Herrmann - U. Orbanz, Faserdimensionen von Aufblasungen lokaler Ringe und Äquimultiplizität, J. Math. Kyoto Univ. 20(1980), 651 - 659

[HO-2] --- Normale Flachheit und Äquimultiplizität für vollständige Durchschnitte, J. Algebra 70 (1981), 437 - 451

[HO-3] --- On equimultiplicity, Math.Proc. Camb. Phil. Soc. (1982), 91, 207 - 213

[HO-4] --- Between equimultiplicty and normal flatness; Proceedings of the La Rabida conference on Singularities, Springer Lecture notes 961 (1981)

[H-O-G] M. Herrmann, U. Grothe, U. Orbanz. Graded Cohen-Macaulay rings associated to equimultiple ideals. Preprint 1983

[Hu-1] C. Huneke, A remark concerning multiplicities, Proc. Amer. Math. Soc. 85 (1982). 331 - 332

[Hu-2] --- On the associated graded ring of an ideal, Illinois J. of Math. 26 (1982), 121 - 137

[Hu-3] --- The theory of d-sequences and powers of ideals, Adv. Math. 46 (1982), 249-270

[HSV] M. Herrmann, R. Schmidt, W. Vogel. Theorie der normalen Flachheit. Teubner Leipzig 1977

[H-I] M. Herrmann, S. Ikeda, Remarks on lifting of Cohen-Macaulay property. To appear in Nagoya Math. J.Vol.92

[I] S. Ikeda, The Cohen-Macaulayness of the Rees algebras of local rings, Nagoya Math. J. 89 (1983), 47 - 63

[G-S-1] S. Goto, Y. Shimoda, On the Rees algebras of Cohen-Macaulay local rings, Commutative algebra (analytic methods), Lect. Notes in Pure and Appl. Math. 68 (1982), 201 - 231

[G-S-2] --- On Rees algebras over Buchsbaum rings, J. Math. Kyoto Univ. 20 (1980), 691 - 708

[Li] J. Lipman, Equimultiplicity, reduction and blowing up. Commutative algebra (analytic methods), Lect. Notes in Pure and Appl. Math. 68 (1982)

[M] H. Matsumura, Commutative algebra, New York 1970

[Na] Nagata, Local rings. New York 1962

[N-R] D.G. Northcott, D. Rees, Reductions of ideals in local rings. Proc. Camb Phil. Soc. 50(1954), 145 - 158

[O] U. Orbanz, Multiplicities and Hilbert functions under blowing up, Manuscr. math. 36 (1981), 179 - 186

[O-R] U. Orbanz - L. Robbiano, Projective normal flatness and Hilbert functions, to appear in Trans. Amer. Math. Soc. 1984

[Ra-O] L.J. Ratliff, On quasi-unmixed semi-local rings and the altitude formula, Am.J. Math. 87(1965), 278 - 284

[Ra-1] --- On quasi-unmixed local domains, the altitude formula and the chain condition for prime ideals (I), Am. J.Math. 91 (1969), 508 - 528

[Ra-2] --- On quasi-unmixed local domains, the altitude formula and the chain condition for prime ideals (II), Am. J. Math. 92 (1970), 99 - 144

[Ra-3] --- Locally quasi-unmixed noetherian rings and ideals of the principal class, Pacific J. Math. 52 (1974), 185 - 205

[Ro-1] L. Robbiano, Remarks on blowing-up divisorial ideals, Manuscr. math. 30 (1980)

[Ro-2] --- On normal flatness and some related topics. In proceedings of the Trento-conference, Lecture notes in pure and appl. math. 84, Marcel Dekker 1983, 235 - 251

[S] J.D. Sally, Numbers of generators of ideals in local rings, Marcel Dekker 1978

[Sh] M. Shinagawa, Some results on the normalization and normal flatness, Hiroshima Math. J. 12 (1982), 209 - 224

[T] B. Teissier, Résolution simultanée et cycles évanescent, Lect. Notes in Math. 777(1980), 82 - 146

[V-V] P. Valabrega - G. Valla, Form rings and regular sequences, Nagoya Math. J. 72 (1978), 93 - 101

[V] G. VAlla, Certain graded algebras are always Cohen-Macaulay, J. Algebra 42 (1976), 537 - 548

M. Herrmann
Math. Inst. der
Universität zu Köln
SFB Theoret. Math. Bonn