COORDINATE-FREE CLASSIC GEOMETRIES
I. PROJECTIVE CASE

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Abstract. This is the first of a series of papers dedicated to a coordinate-free approach to several classic geometries such as hyperbolic (real, complex, quaternionic), elliptic (spherical, Fubini-Study), and lorentzian ones. These geometries carry a certain simple structure that is in some sense stronger than the riemannian one. Their basic geometrical objects have linear nature. Such objects provide natural compactifications of commonly studied geometries. The usual riemannian concepts are easily derivable from the strong structure and thus gain their coordinate-free form. Many examples show how this view can help when dealing with explicit classic geometries and illustrate fruitful features of the approach. In this paper, only projective aspects of classic geometries are studied.

The methods were first tested in [AGG] and [AGu] where they were successfully applied to constructing complex hyperbolic manifolds and to solving problems in complex hyperbolic geometry.

1. Classic geometries: introduction, definition, examples, and motivation

1.1. Introduction. This series of papers constitutes an attempt to systematically develop a coordinate-free view on several classic geometries. The approach originates from [AGG] where, in order to simplify formulae, we expressed several complex hyperbolic geometry concepts in a more invariant and convenient form.

As it turns out, the riemannian structure in many classic geometries (hyperbolic, spherical, Fubini-Study, etc.) is just a shadow of a simpler one. For example, the tangent vectors to the grassmannian $\mathrm{Gr}_{\mathbb{K}}(k,V)$ at a nondegenerate point p, where V is a vector space with a hermitian form, are linear maps $p\to p^\perp$. A more adequate object should be simply a linear map $V\to V$, a *footless* tangent vector: being composed with the two projectors related to p, i.e., being *observed* from p, it becomes a usual tangent vector. The *product* $t_1^*t_2$, where $t_1,t_2:V\to V$ and t_1^* stands for the adjoint to t_1, is the structure that provides the hermitian (riemannian) metric at p given by $\langle t_1,t_2\rangle:=\mathrm{tr}(t_1^*t_2)$ for t_1,t_2 observed from the same point p. The $(2,1)$-symmetrization of the triple product $tt_2^*t_1$ provides the curvature tensor $R(t_2,t_1)t$ for t,t_1,t_2 observed from the same point (see Subsection 4.4). Taking more observers in the previous examples, we obtain more geometric characteristics. Distance, for instance, appears when one observer sees the mote in the other observer's eye, i.e., when the projectors related to their points are composed.

The basic objects in a classic geometry are linear in nature. This makes grassmannians (and $\mathbb{C}$-grassmannians; see Subsection 1.7 and the forthcoming papers) a place where these objects naturally vary. So, grassmannians should be studied even if one is interested only in geometries embedded into projective spaces. Regarding a classic geometry as a homogeneous space related to the corresponding unitary (or orthogonal) group is **deficient** : it does not allow to go outside the absolute which would be useful for the following reasons. The absolute (formed by degenerate points) divides $\mathrm{Gr}_{\mathbb{K}}(k,V)$ into riemannian and pseudo-riemannian pieces. Only one of them is traditionally considered as a classic

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geometry. The grassmannian can be therefore seen as its compactification. The points in each piece are in fact basic geometrical objects (living in the traditional piece) whose type is related to the compactification. Thus, each piece is equipped with its natural (pseudo-)riemannian geometry. Such geometries fit each other: geometrical objects (geodesics, totally geodesic subspaces, equidistant loci, etc.) pass through the absolute, leaving one piece and entering another. Moreover, this global picture sheds light on the geometry of the absolute. In particular, the general structure described above (the one that provides the hermitian metric at nondegenerate points) is inherited by the absolute. In the case of real hyperbolic space, for instance, this explains the interrelation between the conformal structure on the absolute and the metric structure on the ball.

In classic geometries, the geometrical concepts and objects can be introduced and handled synthetically. This suggests the above modification of the usual riemannian tools and leads to simple linear and hermitian algebra.

The initial steps of this coordinate-free approach can be found in literature. The following is a (very likely incomplete) list of references to some contributions concerning classic geometries: [Kle] (the concept of a projective model); [Arn], [BeP] (coordinate-free description of the metric in a particular case); [ChG], [Hsi1], [Hsi2], (linear approach to elementary geometric objects such as geodesics, totally geodesic spaces, bisectors, etc.); [Gir] (linear description of equidistant hypersurfaces in the complex hyperbolic plane); [Thu], [San], [HSa], [Gol] (some use of linear and hermitian tools in real or complex hyperbolic geometry); [ChK] (lorentzian projective compactification of real hyperbolic space).

1.2. Definition. Let $\mathbb{K}$ denote one of the following fields: $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), or $\mathbb{H}$ (quaternions). A *classic geometry* is a right $\mathbb{K}$-vector space V equipped with a *hermitian form* $\langle -,-\rangle$. By definition (see, for instance, [Lan]), the form is hermitian if it takes values in $\mathbb{K}$, is biadditive, and satisfies the identities $\langle kv_1,v_2\rangle=\overline{k}\langle v_1,v_2\rangle$, $\langle v_1,v_2k\rangle=\langle v_1,v_2\rangle k$, and $\langle v_1,v_2\rangle=\overline{\langle v_2,v_1\rangle}$ for all $v_1,v_2\in V$ and $k\in\mathbb{K}$ ■

Behind this definition there is indeed more geometry than it might appear at the first glance. The tangent space to a point p in the projective space $\mathbb{P}_{\mathbb{K}}V$ has a well-known description as the $\mathbb{R}$-vector space ($\mathbb{C}$-vector space if $\mathbb{K}=\mathbb{C}$)

$$\mathrm{T}_p\,\mathbb{P}_{\mathbb{K}}V=\mathrm{Lin}_{\mathbb{K}}(p,V/p) \tag{1.3}$$

of all $\mathbb{K}$-linear transformations from p to V/p. Here and in what follows, we frequently do not distinguish the notation of a point in $\mathbb{P}_{\mathbb{K}}V$, of a chosen representative of it in V, and of the corresponding one-dimensional subspace when a concept or expression does not depend on interpretation. For instance, the subspace $p^\perp$ of V is well defined for any $p\in\mathbb{P}_{\mathbb{K}}V$.

If p is not isotropic, that is, if $\langle p,p\rangle\neq 0$, then we can naturally identify V/p with $p^\perp$. In this case, we interpret the tangent space as $\mathrm{T}_p\,\mathbb{P}_{\mathbb{K}}V=\mathrm{Lin}_{\mathbb{K}}(p,p^\perp)$. It inherits the $\mathbb{R}$-bilinear form

$$(t_1,t_2):=\pm\frac{\mathrm{tr}_{\mathbb{R}}(t_1^*t_2)}{\dim_{\mathbb{R}}\mathbb{K}}, \tag{1.4}$$

where $t_1,t_2:p\to p^\perp$ are tangent vectors, $t_1^*:p^\perp\to p$ stands for the adjoint[1] of t_1 in the sense of the hermitian form, and $\mathrm{tr}_{\mathbb{R}}(t_1^*t_2)$ denotes the trace of the $\mathbb{R}$-linear map $t_1^*t_2:p\to p$. We will refer to this form as the *metric* of a classic geometry. In the case of $\mathbb{K}=\mathbb{C}$, we have the *hermitian metric*

$$\langle t_1,t_2\rangle:=\pm\mathrm{tr}_{\mathbb{C}}(t_1^*t_2). \tag{1.5}$$

[1]In terms of the identification $\mathrm{Lin}_{\mathbb{K}}(p,p^\perp)=p^\perp\langle p,-\rangle$, we have $\big(v\langle p,-\rangle\big)^*=p\langle v,-\rangle$ and $\mathrm{tr}_{\mathbb{R}}\big(v\langle p,-\rangle\big)=\dim_{\mathbb{R}}\mathbb{K}\cdot\mathrm{Re}\langle p,v\rangle$, where $v\in p^\perp$. This treatment is useful while performing explicit calculations.

It is easy to see that $\operatorname{Re}\langle t_1, t_2\rangle = (t_1, t_2)$. Obviously, the (hermitian) metric depends smoothly on a nonisotropic p. If the hermitian form in V is nondegenerate, then the metric is nondegenerate. We warn the reader that the case $\mathbb{K} = \mathbb{H}$ contains some peculiarities. The tangent space $\mathrm{T}_p\,\mathbb{P}_{\mathbb{H}}V$ is **not** an $\mathbb{H}$-vector space and it makes no sense to speak of a hermitian metric on it.

The *signature* of a point divides $\mathbb{P}_{\mathbb{K}}V$ into three parts: *positive points, null points,* and *negative points,* defined respectively as

$$\mathrm{E}\,V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p, p\rangle > 0\}, \qquad \mathrm{S}\,V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p, p\rangle = 0\}, \qquad \mathrm{B}\,V := \{p \in \mathbb{P}_{\mathbb{K}}V \mid \langle p, p\rangle < 0\}.$$

1.6. Examples. We take

(1) $\mathbb{K} = \mathbb{C}$, $\dim_{\mathbb{C}} V = 2$, the form of signature $++$, and the sign $+$ in the definition of the hermitian metric. We obtain the usual 2-dimensional sphere of constant curvature.

(2) $\mathbb{K} = \mathbb{C}$, $\dim_{\mathbb{C}} V = 2$, the form of signature $+-$, and the sign $-$ in the definition of the hermitian metric. Let $p \in \mathbb{P}_{\mathbb{C}}V$ be nonisotropic. From the orthogonal decomposition $V = p \oplus p^\perp$ it follows that the hermitian metric in $\mathrm{T}_p\,\mathbb{P}_{\mathbb{C}}V$ is positive definite. We get two hyperbolic Poincaré discs $\mathrm{B}\,V$ and $\mathrm{E}\,V$.

(3) $\mathbb{K} = \mathbb{R}$, $\dim_{\mathbb{R}} V = 3$, the form of signature $++-$, and the sign $-$. We arrive at the hyperbolic Beltrami-Klein disc $\mathrm{B}\,V$.

(4) $\mathbb{K} = \mathbb{C}$, $\dim_{\mathbb{C}} V = 3$, the form of signature $++-$, and the sign $-$. The open 4-ball $\mathrm{B}\,V$ is the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$.

(5) $\mathbb{K} = \mathbb{H}$, $\dim_{\mathbb{H}} V = 2$, the form of signature $++$, and the sign $+$. We obtain the usual 4-sphere of constant curvature. There is no $\mathbb{H}$-action on the tangent space $\mathrm{T}_p\,\mathbb{P}_{\mathbb{H}}V$. However, fixing a geodesic in $\mathbb{P}_{\mathbb{H}}V$ leads to a curious action of $\mathbb{S}^3 \subset \mathbb{H}$ on the tangent bundle $\mathrm{T}\,\mathbb{P}_{\mathbb{H}}V$ (see Example 3.7). The same is applicable to Example 1.6 (6) that follows.

(6) $\mathbb{K} = \mathbb{H}$, $\dim_{\mathbb{H}} V = 2$, the form of signature $+-$, and the sign $-$. The open 4-ball $\mathrm{B}\,V$ is the real hyperbolic space $\mathbb{H}^4_{\mathbb{R}}$ (Example 3.7 shows a geometrical role of the 'additional' quaternionic structure).

In a similar way, we can describe many other geometries: elliptic geometries such as spherical and Fubini-Study ones, hyperbolic geometries including those of constant sectional or constant holomorphic curvature, some lorentzian geometries such as de Sitter and anti de Sitter spaces, etc. ∎

The most elementary geometrical objects are the 'linear' ones, i.e., those given by the projectivization $\mathbb{P}_{\mathbb{K}}W$ of an $\mathbb{R}$-vector subspace $W \subset V$. For instance, we can isometrically embed Examples (1) and (2) as projective lines in Example (4) by taking for W an appropriate 2-dimensional $\mathbb{C}$-vector subspace in V. (The negative part of a projective line of signature $+-$ is commonly known as a *complex geodesic* in $\mathbb{H}^2_{\mathbb{C}}$.) Let us take a look at some less immediate

1.7. Examples. (1) We take $\dim_{\mathbb{R}} W = 2$. Suppose that the hermitian form, being restricted to W, is real and does not vanish. It is easy to see that $W\mathbb{K} \simeq W \otimes_{\mathbb{R}} \mathbb{K}$. The circle

$$\mathrm{G}\,W := \mathbb{P}_{\mathbb{K}}W = \mathbb{P}_{\mathbb{R}}W \simeq \mathbb{S}^1$$

is said to be a *geodesic.* The projective line $\mathbb{P}_{\mathbb{K}}(W\mathbb{K})$ is the *projective line of* the geodesic. By Corollary 5.5, the introduced circle, out of its isotropic points, is indeed a geodesic with respect to the metric and every geodesic of the metric arises in this way.

(2) Let $\dim_{\mathbb{R}} W = 2$ in Example 1.6 (2). When W is not a $\mathbb{C}$-vector space (otherwise, $\mathbb{P}_{\mathbb{C}}W$ is simply a point in $\mathbb{P}_{\mathbb{C}}V$), the real part of the hermitian form over W can be nondegenerate indefinite, definite, nonnull degenerate, or null. The circle $\mathbb{P}_{\mathbb{C}}W$ is respectively said to be a *hypercycle, metric circle, horocycle,* or the *absolute.* Inside of either of the Poincaré discs $\mathrm{E}\,V$ and $\mathrm{B}\,V$, we get the usual hypercycles, metric circles, and horocycles.

(3) We can isometrically embed (here the normalizing factor in (1.4) plays its role) Example 1.6 (3) in Example 1.6 (4) by taking for W a 3-dimensional $\mathbb{R}$-vector subspace such that the hermitian form, being restricted to W, is real and nondegenerate. We obtain the $\mathbb{R}$-*plane* $\mathbb{P}_{\mathbb{C}}W=\mathbb{P}_{\mathbb{R}}W\simeq\mathbb{P}_{\mathbb{R}}^2$, a maximal lagrangian submanifold. The $\mathbb{R}$-planes play an important role in complex hyperbolic geometry (see, for instance, [Gol] and [AGG]).

(4) In Example 1.6 (4), let $S\subset V$ be an $\mathbb{R}$-vector subspace, $\dim_{\mathbb{R}}S=2$. Suppose that the hermitian form is real and nondegenerate over S. It is easy to see that $S^\perp$ is a one-dimensional $\mathbb{C}$-vector space. Taking $W=S+S^\perp$, we arrive at the *bisector* $B:=\mathbb{P}_{\mathbb{C}}W$. The geodesic $\mathrm{G}\,S$, the projective line $\mathbb{P}_{\mathbb{C}}(S\mathbb{C})$, and the point $\mathbb{P}_{\mathbb{C}}S^\perp$ are respectively the *real spine,* the *complex spine,* and the *focus* of the bisector. This description of a bisector immediately provides (see [AGG]) the well-known slice and meridional decompositions of a bisector (see [Gir], [Mos], and [Gol]). If the hermitian form is indefinite over S, then $\mathbb{P}_{\mathbb{C}}W\cap\mathrm{B}\,V$ is a usual bisector (= a hypersurface equidistant from two points) in $\mathbb{H}_{\mathbb{C}}^2$. Every bisector in $\mathbb{H}_{\mathbb{C}}^2$ is describable in this manner ■

We would like to illustrate the thesis that the basic linear objects form themselves spaces naturally endowed with a classic geometry structure:

In Example 1.6 (3), the real projective plane $\mathbb{P}_{\mathbb{R}}V$ consists of the usual Beltrami-Klein disc $\mathrm{B}\,V$ equipped with its riemannian metric and of the Möbius band $\mathrm{E}\,V$ endowed with a lorentzian metric. The hermitian form establishes a duality between points and projective lines (= geodesics) in $\mathbb{P}_{\mathbb{R}}V$: the point $p\in\mathbb{P}_{\mathbb{R}}V$ corresponds to the geodesic $\mathbb{P}_{\mathbb{R}}p^\perp$. In view of this duality, the classic lorentzian geometry of the Möbius band $\mathrm{E}\,V$ is nothing but the geometry of geodesics in the Beltrami-Klein disc $\mathrm{B}\,V$ and *vice versa.* For the same reason, the classic pseudo-riemannian geometry of $\mathrm{E}\,V$ in Example 1.6 (4) is the geometry of the complex geodesics in $\mathbb{H}_{\mathbb{C}}^2$.

In Example 1.7 (2), we will indistinctly refer to hypercycles, metric circles, horocycles, and the absolute as *circles.* A point W in the grassmannian $\mathrm{Gr}_{\mathbb{R}}(2,V)$ of 2-dimensional $\mathbb{R}$-vector subspaces of V determines in $\mathbb{P}_{\mathbb{C}}V$ a circle if W is not a $\mathbb{C}$-vector space and a point, otherwise. Clearly, $W,W'\in\mathrm{Gr}_{\mathbb{R}}(2,V)$ provide the same circle if and only if $W=W'c$ for some $c\in\mathbb{C}^*$. The $\mathbb{C}$-*grassmannian* $\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(2,V)$ is the quotient of $\mathrm{Gr}_{\mathbb{R}}(2,V)$ by this action.

The singular locus of $\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(2,V)$ is formed by the complex subspaces of V and, therefore, coincides with $\mathbb{P}_{\mathbb{C}}V$. It is easy to see that $\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(2,V)$ is topologically $\mathbb{P}_{\mathbb{R}}^3$ without an open 3-ball. It has $\mathbb{P}_{\mathbb{C}}V$ as its boundary. The absolute, a 2-sphere with a single double point, is formed by the horocycles and divides $\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(2,V)$ into two parts.

How can we equip the $\mathbb{C}$-grassmannian $\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(r,V)$ of r-dimensional $\mathbb{R}$-vector subspaces of V with a classic geometry structure? Let $W\in\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(r,V)$ be a *nondegenerate* point, that is, the real form $(-,-):=\operatorname{Re}\langle-,-\rangle$ is nondegenerate over W. A tangent vector in $\mathrm{T}_W\,\mathrm{Gr}_{\mathbb{C}|\mathbb{R}}(r,V)$ is an $\mathbb{R}$-linear transformation $t:W\to W^\perp$ such that $\mathrm{tr}_{\mathbb{R}}(\pi_W it)=0$, where the orthogonal $W^\perp$ is taken with respect to $(-,-)$ and π_W is the orthogonal projection onto W. The metric is given by $(t_1,t_2):=\mathrm{tr}_{\mathbb{R}}(t_1^*t_2)$, where $t_1^*:W^\perp\to W$ is the adjoint in the sense of $(-,-)$.

In this article, we study only projective classic geometries and describe in a coordinate-free way several features of such geometries. In particular, we obtain explicit expressions for the parallel displacement along geodesics in terms of the hermitian form (Corollaries 5.7 and 5.9). Applying these expressions to the case of complex hyperbolic geometry, we get a geometrical interpretation of the angle between cotranchal bisectors in $\mathbb{H}_{\mathbb{C}}^2$ (Examples 5.10 and 5.12). Other explicit formulae involving geodesics (Subsections 3.2 and 3.4), projective cones (Example 3.6), bisectors (Examples 3.6 and 5.13), the Levi-Civita connection (Proposition 4.3), the tensor of curvature (Subsection 4.4), and sectional curvatures (Subsection 4.5) are also provided.

In [AGoG] and [AGr], we treat grassmannians in the same spirit.

2. Preliminaries

Let $p \in \mathbb{P}_{\mathbb{K}}V$ be a nonisotropic point. We introduce the following notation of orthogonal decomposition

$$V = p \oplus p^{\perp}, \qquad v = v^p + {}^p v = \pi'[p]v + \pi[p]v,$$

where

$$v^p := \pi'[p]v := p\,\frac{\langle p, v\rangle}{\langle p, p\rangle} \in p\mathbb{K} \qquad \text{and} \qquad {}^p v := \pi[p]v := v - p\,\frac{\langle p, v\rangle}{\langle p, p\rangle} \in p^{\perp}$$

do not depend on the choice of a representative of p. Depending on circumstances, we choose the most convenient variant of notation.

The hermitian form over a 2-dimensional $\mathbb{K}$-vector subspace of V can be null, definite, nondegenerate indefinite, or nonnull degenerate. The corresponding projective line will be respectively called *null, spherical, hyperbolic,* or *euclidean.* We need a very rudimental form of Sylvester's criterion applicable to the case $\mathbb{K} = \mathbb{H}$.

2.1. Lemma. *Let W be a 2-dimensional $\mathbb{K}$-vector space equipped with a nonnull hermitian form. The hermitian form is respectively definite, nondegenerate indefinite, or degenerate if and only if $\mathrm{D}(p,q) > 0$, $\mathrm{D}(p,q) < 0$, or $\mathrm{D}(p,q) = 0$, where $\mathrm{D}(p,q) := \langle p,p\rangle\langle q,q\rangle - \langle p,q\rangle\langle q,p\rangle$ and p,q are any two $\mathbb{K}$-linearly independent vectors in W. (Obviously, $\mathrm{D}(p,q) = 0$ if p,q are $\mathbb{K}$-linearly dependent.)*

Proof. If one of p,q is nonisotropic (say, p) the result follows from ${}^p q \neq 0$, $\langle p, {}^p q\rangle = 0$, and

$$\langle p,p\rangle\langle {}^p q, {}^p q\rangle = \langle p,p\rangle\langle q, {}^p q\rangle = \langle p,p\rangle\Big(\langle q,q\rangle - \frac{\langle q,p\rangle\langle p,q\rangle}{\langle p,p\rangle}\Big) = \mathrm{D}(p,q).$$

If both p,q are isotropic, we take a nonisotropic $u \in W$. We can assume that $u = pk + q$ for some $k \in \mathbb{K}^*$. Clearly, ${}^u q \neq 0$, $\langle u, {}^u q\rangle = 0$, and $\mathrm{D}(u,q) = \langle u,u\rangle\langle {}^u q, {}^u q\rangle$. It remains to observe that

$$\mathrm{D}(u,q) = \langle pk+q, pk+q\rangle\langle q,q\rangle - \langle pk+q, q\rangle\langle q, pk+q\rangle = -\overline{k}\langle p,q\rangle\langle q,p\rangle k = |k|^2\,\mathrm{D}(p,q) \ \blacksquare$$

2.2. Remark. (1) Let L be a projective line. For every nonisotropic $p \in \mathrm{L}$ there exists a unique $q \in \mathrm{L}$ *orthogonal* to p, that is, such that $\langle p,q\rangle = 0$.

(2) Isotropic points in a hyperbolic projective line form an $(n-1)$-sphere, where $n = \dim_{\mathbb{R}}\mathbb{K}$. An euclidean projective line contains a single isotropic point $\blacksquare$

A linear transformation in (1.3) can be regarded as a tangent vector in usual differential terms: Let f be a $\mathbb{K}$-valued smooth function defined in a neighbourhood of $p \in \mathbb{P}_{\mathbb{K}}V$ and let $\hat f$ denote its lift to the corresponding neighbourhood of $p\mathbb{K} \setminus \{0\}$ in V. Clearly, $\hat f(vk) = \hat f(v)$ for all $k \in \mathbb{K}^*$. Every $\varphi \in \mathrm{Lin}_{\mathbb{K}}(p, V)$ defines a tangent vector $t_\varphi \in \mathrm{T}_p\,\mathbb{P}_{\mathbb{K}}V$ given by

$$\textbf{(2.3)} \qquad t_\varphi f := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \hat f\big((1+\varepsilon\varphi)p\big),$$

where $\varepsilon \in \mathbb{R}$. Notice that t_φ vanishes if and only if $\varphi p \in p\mathbb{K}$. Also, altering φ by adding pk to φp, where $k \in \mathbb{K}$, does not change the vector $t_\varphi \in \mathrm{Lin}(p, V/p)$.

Let $W \subset V$ be an $\mathbb{R}$-vector subspace. We call a point $p \in W$ *projectively smooth in W* if

$$\dim_{\mathbb{R}}(p\mathbb{K} \cap W) = \min_{0 \neq w \in W} \dim_{\mathbb{R}}(w\mathbb{K} \cap W).$$

It is not difficult to see that the projectively smooth points in W provide an open smooth region in $\mathbb{P}_{\mathbb{K}}W$. Moreover, we have the following

2.4. Lemma [AGG, Lemma 5.2.1]**.** *Let $W \subset V$ be an $\mathbb{R}$-vector subspace, let $p \in W$ be a projectively smooth point in W, and let $\varphi \in \operatorname{Lin}_{\mathbb{K}}(p, V)$. Then $t_\varphi \in \mathrm{T}_p \mathbb{P}_{\mathbb{K}} W$ if and only if $\varphi p \in W + p\mathbb{K}$* ■

The tangent vector to a smooth path is expressible in terms of (1.4) :

2.5. Lemma [AGG, Lemma 5.1.1]**.** *Let $c : [a, b] \to \mathbb{P}_{\mathbb{K}} V$ be a smooth curve and let $c_0 : [a, b] \to V$ be a smooth lift of c to V. If $c(t_0)$ is not isotropic, then the tangent vector $\dot c(t_0) : c_0(t_0) \to c_0(t_0)^\perp$ is given by $\dot c(t_0) : c_0(t_0) \mapsto {}^{c(t_0)}\dot c_0(t_0)$* ■

3. Geodesics

Let us remind the definition in Example 1.7 (1). We take a 2-dimensional $\mathbb{R}$-vector subspace $W \subset V$ such that the hermitian form, being restricted to W, is real and does not vanish. It is immediate that $W\mathbb{K} \simeq W \otimes_{\mathbb{R}} \mathbb{K}$. Hence, $\mathbb{P}_{\mathbb{K}} W = \mathbb{P}_{\mathbb{R}} W$. The circle $\mathrm{G}\, W := \mathbb{P}_{\mathbb{K}} W$ is, by definition, a geodesic. (Corollary 5.5 relates this concept to the common one.) The geodesic $\mathrm{G}\, W$ spans its projective line $\mathbb{P}_{\mathbb{K}}(W\mathbb{K})$. A geodesic is called *spherical, hyperbolic,* or *euclidean* depending on the nature of its projective line.

3.1. Lemma. (1) *Let $g_1, g_2 \in \mathbb{P}_{\mathbb{K}} V$ be distinct and nonorthogonal. Then there exists a unique geodesic containing g_1 and g_2.*

(2) *Let $p \in \mathbb{P}_{\mathbb{K}} V$ be nonisotropic and let $0 \neq t \in \mathrm{T}_p \mathbb{P}_{\mathbb{K}} V$, $t : p \to p^\perp$. Then there exists a unique geodesic having t as its tangent vector at p. It is given by the subspace $W = p\mathbb{R} + tp\mathbb{R}$.*

Proof. (1) Clearly, $g_1, g_2 \in \mathrm{G}\, W$ for $W = g_1\mathbb{R} + g_2\langle g_2, g_1\rangle\mathbb{R}$. If $g_1, g_2 \in \mathrm{G}\, W'$, then $W' = g_2k_2\mathbb{R} + g_1k_1\mathbb{R}$ for some $k_1, k_2 \in \mathbb{K}$ such that $\overline{k}_2\langle g_2, g_1\rangle k_1 \in \mathbb{R}^*$. Hence, $W' = g_2k_2\overline{k}_2\langle g_2, g_1\rangle k_1\mathbb{R} + g_1k_1\mathbb{R} = Wk_1$, that is, $\mathrm{G}\, W' = \mathrm{G}\, W$.

(2) The geodesic $\mathrm{G}\, W$, where $W = p\mathbb{R} + tp\mathbb{R}$, does not depend on the choice of $p \in p\mathbb{K}$. By Lemma 2.4, t is a tangent vector to $\mathrm{G}\, W$ at p. Let $\mathrm{G}\, W'$ be a geodesic with tangent vector t. We can choose W' so that $p \in W'$. By Lemma 2.4, $tp \in W' + p\mathbb{K}$. So, $tp \in p^\perp$ implies $tp \in W'$. In other words, $W' = p\mathbb{R} + tp\mathbb{R}$ ■

We denote by $\mathrm{G}\wr g_1, g_2\wr$ the geodesic that contains given distinct nonorthogonal $g_1, g_2 \in \mathbb{P}_{\mathbb{K}} V$.

Take distinct **orthogonal** $g_1, g_2 \in \mathbb{P}_{\mathbb{K}} V$. Assume that the projective line L spanned by g_1, g_2 is nonnull. One of g_1, g_2 is nonisotropic — say, g_1. Every geodesic in L passing through g_1 has the form $\mathrm{G}\, W$ with $W = q\mathbb{R} + g_1\mathbb{R}$, $g_1 \neq q \in \mathrm{L}$, and $\langle q, g_1\rangle \in \mathbb{R}^*$. So, ${}^{g_1}q \in \mathrm{G}\, W$. By Remark 2.2 (1), g_2 is the only point in L orthogonal to g_1. Hence, ${}^{g_1}q = g_2$ in $\mathbb{P}_{\mathbb{K}} V$. In other words, every geodesic in L that passes through g_1 also passes through g_2. In particular, every geodesic in an euclidean projective line passes through the isotropic point (see Remark 2.2 (2)). In this case, in the affine chart $\mathbb{K}$ of nonisotropic points of L, the geodesics correspond to the straight lines. This justifies the term 'euclidean.' Since the metric is actually null over euclidean lines, perhaps a more appropriate term would be *affine line.*

3.2. Length of non-euclidean geodesics. Take a spherical projective line L, take a point $g_1 \in \mathrm{L}$, and choose the sign + in the definition (1.4) of the metric. Let $g_1' \in \mathrm{L}$ denote the point orthogonal to g_1. Fixing representatives $g_1, g_1' \in V$ such that $\langle g_1, g_1\rangle = \langle g_1', g_1'\rangle = 1$, we parameterize a lift $c_0(t) := g_1 \cos t + g_1' \sin t$ to V of a segment of geodesic $c = c(t)$ joining g_1 and $g_2 := c(a)$, where $t \in [0, a]$ and $a \in [0, \pi/2]$. Since $\big\langle \dot c_0(t), c_0(t)\big\rangle = 0$ and $\big\langle c_0(t), c_0(t)\big\rangle = 1$, it follows from Lemma 2.5 that $\big(\dot c(t), \dot c(t)\big) = 1$. Hence, $\ell\, c = \displaystyle\int_0^a \sqrt{\big(\dot c(t), \dot c(t)\big)} = a$. Noticing that $\operatorname{ta}(g_1, g_2) = \cos^2 a$, where

$$\textbf{(3.3)} \qquad \operatorname{ta}(g_1, g_2) := \frac{\langle g_1, g_2\rangle\langle g_2, g_1\rangle}{\langle g_1, g_1\rangle\langle g_2, g_2\rangle},$$

we obtain

$$\ell\, c = \arccos \sqrt{\mathrm{ta}(g_1, g_2)}.$$

It follows immediately from Lemma 2.1 that, being L spherical, $0 \le \mathrm{ta}(g_1, g_2) \le 1$. The first equality occurs exactly when g_1, g_2 are orthogonal and the second, exactly when $g_1 = g_2$.

If L is a hyperbolic projective line, similar arguments involving cosh, sinh, and the sign $-$ for the metric show that the length of a segment of geodesic c that contains no isotropic points and joins $g_1, g_2 \in \mathrm{L}$ is given by

$$\ell\, c = \operatorname{arccosh} \sqrt{\mathrm{ta}(g_1, g_2)}.$$

In both cases, the distance is a monotonic function of the *tance* $\mathrm{ta}(g_1, g_2)$ (see also [AGG]) ■

3.4. Equations of a geodesic. Let the geodesic $\mathrm{G}\wr g_1, g_2\wr$ be non-euclidean and let L denote its projective line. We will show that $x \in \mathrm{L}$ belongs to $\mathrm{G}\wr g_1, g_2\wr$ if and only if

$$b(x, g_1, g_2) := \langle x, g_1\rangle\langle g_1, g_2\rangle\langle g_2, x\rangle - \langle x, g_2\rangle\langle g_2, g_1\rangle\langle g_1, x\rangle = 0.$$

The proof is straightforward. The above equation does not depend on the choice of representatives $x, g_1, g_2 \in V$. If $x \in \mathrm{G}\wr g_1, g_2\wr$, then $b(x, g_1, g_2) = 0$ since the hermitian form is real over W and we can assume $x, g_1, g_2 \in W$. Suppose that $b(x, g_1, g_2) = 0$ for some $x \in \mathrm{L}$. We can take $g_1, g_2 \in W$ and $x = g_1 k + g_2$ for some $k \in \mathbb{K}$. The condition $b(x, g_1, g_2) = 0$ is equivalent to $\big(\langle g_1, g_2\rangle\langle g_2, g_1\rangle - \langle g_1, g_1\rangle\langle g_2, g_2\rangle\big)(k - \overline{k}) = 0$. Since L is not euclidean, we conclude from Lemma 2.1 that $k \in \mathbb{R}$, that is, $x \in W$.

Let $g \in \mathrm{G}\wr g_1, g_2\wr$ and let $\varphi \in \mathrm{Lin}_{\mathbb{K}}(g, V)$ be such that $t_\varphi \in \mathrm{T}_g \mathrm{L}$. We will show that $t_\varphi \in \mathrm{T}_g \mathrm{G}\wr g_1, g_2\wr$ if and only if

$$t(\varphi g, g, g_1, g_2) := \langle \varphi g, g_1\rangle\langle g_1, g_2\rangle\langle g_2, g\rangle + \langle g, g_1\rangle\langle g_1, g_2\rangle\langle g_2, \varphi g\rangle -$$
$$-\langle \varphi g, g_2\rangle\langle g_2, g_1\rangle\langle g_1, g\rangle - \langle g, g_2\rangle\langle g_2, g_1\rangle\langle g_1, \varphi g\rangle = 0.$$

It follows from $b(g, g_1, g_2) = 0$ that

(3.5) $$t(\varphi g + gk, g, g_1, g_2) = t(\varphi g, g, g_1, g_2) + \overline{k} \cdot b(g, g_1, g_2) + b(g, g_1, g_2) \cdot k = t(\varphi g, g, g_1, g_2)$$

for every $k \in \mathbb{K}$. Also, the equation $t(\varphi g, g, g_1, g_2) = 0$ does not depend on the choice of representatives for g, g_1, g_2. We take $g, g_1, g_2 \in W$. If $t_\varphi \in \mathrm{T}_g \mathrm{G}\wr g_1, g_2\wr$, then $\varphi g \in W + g\mathbb{K}$ by Lemma 2.4. Due to (3.5), we can assume that $\varphi g \in W$. Hence, $t(\varphi g, g, g_1, g_2) = 0$. Conversely, suppose that $t(\varphi g, g, g_1, g_2\big) = 0$. We can take $g = g_1 r + g_2$ for some $r \in \mathbb{R}$ (interchanging g_1 and g_2 if necessary). Since $t_\varphi \in \mathrm{T}_g \mathrm{L}$, it follows from Lemma 2.4 that $\varphi(g) = g_1 k_1 + g_2 k_2$ for some $k_1, k_2 \in \mathbb{K}$. Due to (3.5), we can assume that $\varphi g = g_1 k$. Now, the condition $t(\varphi g, g, g_1, g_2) = 0$ means that $\big(\langle g_1, g_2\rangle\langle g_2, g_1\rangle - \langle g_1, g_1\rangle\langle g_2, g_2\rangle\big)(k - \overline{k}) = 0$. By Lemma 2.1, $k \in \mathbb{R}$, that is, $\varphi g \in W$ ■

3.6. Example: equations of the cone over a geodesic. We take $\dim_{\mathbb{K}} V = 3$ and a nondegenerate hermitian form $\langle -, -\rangle$. The hermitian form establishes a correspondence between points and projective lines in $\mathbb{P}_{\mathbb{K}} V$: the point $p \in \mathbb{P}_{\mathbb{K}} V$ corresponds to the projective line $\mathbb{P}_{\mathbb{K}} p^\perp$. We call p the *polar point* to $\mathbb{P}_{\mathbb{K}} p^\perp$.

Let $\mathrm{G}\, S$ be a non-euclidean geodesic. Clearly, $S^\perp$ is a $\mathbb{K}$-vector space and $p := \mathbb{P}_{\mathbb{K}} S^\perp$ is the (non-isotropic, by Lemma 2.1) polar point to the projective line of $\mathrm{G}\, S$. Therefore, $C := \mathbb{P}_{\mathbb{K}}(S + S^\perp)$ is the projective cone over $\mathrm{G}\, S$ with vertex p. All elements in $S + S^\perp$, except those in $S^\perp$, are projectively smooth.

A point $x \in \mathbb{P}_{\mathbb{K}} V$ that is distinct from p belongs to C if and only if ${}^p x \in \mathrm{G}\, S$. Hence, $x \in C$ means that $b({}^p x, g_1, g_2) = 0$ (see Subsection 3.4), where $g_1, g_2 \in \mathrm{G}\, S$ are distinct nonorthogonal points. This implies that C is given by the equation

$$b(x, g_1, g_2) = 0.$$

Let $c \in C$ be different from p and let $\varphi \in \operatorname{Lin}_{\mathbb{K}}(c, V)$. Define a linear map $\psi \in \operatorname{Lin}_{\mathbb{K}}(g, V)$ by putting $g := {}^p c$ and $\psi g := {}^p \varphi c$. Fix a representative $c \in S + S^\perp$. Clearly, $g \in S$. If $t_\varphi \in \mathrm{T}_c C$, then $\varphi c \in S + S^\perp + c\mathbb{K}$ by Lemma 2.4. This implies that $\psi g \in S + g\mathbb{K}$, that is, $t_\psi \in \mathrm{T}_g \mathrm{G}\, S$. Conversely, if $t_\psi \in \mathrm{T}_g \mathrm{G}\, S$, then $\psi g \in S + g\mathbb{K} \subset S + S^\perp + c\mathbb{K}$. Hence, $\varphi c \in S + S^\perp + c\mathbb{K}$. In other words, $t_\varphi \in \mathrm{T}_c \mathbb{P}_{\mathbb{K}} V$ is tangent to C if and only if $t({}^p \varphi c, g, g_1, g_2) = 0$, where g_1, g_2 are distinct nonorthogonal points in $\mathrm{G}\, S$. This is equivalent to

$$t(\varphi c, c, g_1, g_2) = 0.$$

In the case of $\mathbb{K} = \mathbb{C}$, the projective cone C is nothing but the bisector with the real spine $\mathrm{G}\, S$ (see Example 1.7 (4) and the references therein). From the equation for the tangent space to a point of a bisector, one derives the expression

$$n(q, g_1, g_2) = \Big(g_1 \frac{\langle g_2, q\rangle}{\langle g_2, g_1\rangle} - g_2 \frac{\langle g_1, q\rangle}{\langle g_1, g_2\rangle} \Big) i \langle q, -\rangle$$

for the normal vector $n(q, g_1, g_2)$ at q to the bisector whose real spine is $\mathrm{G} \wr g_1, g_2 \wr$ (see [AGG, Proposition 5.2.7]). This last expression permits to calculate, in terms of the hermitian form, the oriented angle between two bisectors with a common slice (see [AGG, Lemma 5.3.1] and Example 5.12) ■

3.7. Example: actions on tangent bundle given by the choice of a geodesic. We consider the case of $\mathbb{K} = \mathbb{H}$. The tangent space to a point in $\mathbb{P}_{\mathbb{H}} V$ is not an $\mathbb{H}$-vector space. In order to define an action of the sphere $\mathbb{S}^3 \subset \mathbb{H}$ over the tangent bundle $\mathrm{T}\, \mathbb{P}_{\mathbb{H}} V$, we assume that V is an $(\mathbb{H}, \mathbb{H})$-bimodule.

Let $p \in \mathbb{P}_{\mathbb{H}} V$ and let $\varphi \in \operatorname{Lin}_{\mathbb{H}}(p, V)$. Given $k \in \mathbb{S}^3 \subset \mathbb{H}$, we define the linear map $k\varphi \in \operatorname{Lin}_{\mathbb{H}}(kp, V)$ by putting $(k\varphi)(kp) := k(\varphi p)$. In this way, we arrive at the left action $(p, t_\varphi) \mapsto (kp, t_{k\varphi})$ of $\mathbb{S}^3$ over the tangent bundle $\mathrm{T}\, \mathbb{P}_{\mathbb{H}} V$ (notice that changing φp by $\varphi p + pk'$ results in the same $t_{k\varphi}$). It is easy to verify that $t_{k\varphi}$ is also the image of t_φ under the differential $d(k\cdot)_p$, where $k\cdot : \mathbb{P}_{\mathbb{H}} V \to \mathbb{P}_{\mathbb{H}} V$ is induced by $k\cdot : v \mapsto kv$.

Suppose that the $(\mathbb{H}, \mathbb{H})$-bimodule structure is *compatible* with the hermitian form, that is, $\langle v_1, kv_2\rangle = \langle \overline{k} v_1, v_2\rangle$ for all $v_1, v_2 \in V$ and $k \in \mathbb{H}$. Then, for a nonisotropic p and for $t : p \to p^\perp$, we have $d(k\cdot)_p\, t = kt : kp \to (kp)^\perp$. Hence, out of isotropic points, $k\cdot$ is an isometry.

It is well known that every $(\mathbb{H}, \mathbb{H})$-bimodule has the form $V = W \otimes_{\mathbb{R}} \mathbb{H}$, where $W = \{v \in V \mid kv = vk \text{ for every } k \in \mathbb{H}\}$ is the centre of the bimodule. The bimodule structure is compatible with $\langle -, -\rangle$ if and only if the form restricted to W is real. In other words, the choice of a bimodule structure compatible with the hermitian form is equivalent to the choice of a linear geometrical object $\mathbb{P}_{\mathbb{K}} W$ corresponding to a maximal real subspace W in V.

In the particular case of $\dim_{\mathbb{H}} V = 2$, we get an action of $\mathbb{S}^3$ over $\mathbb{P}_{\mathbb{H}} V$ by isometries that is determined by the choice of an arbitrary geodesic G. This geodesic is the fixed-point set of the action. The orbit of every other point is a 2-sphere. Thus, we obtain some foliation of $\mathbb{P}_{\mathbb{H}} V \setminus \mathrm{G}$ by 2-spheres. The actions over $\mathbb{P}_{\mathbb{H}} V$ for hyperbolic (Example 1.6 (6)) and elliptic (Example 1.6 (5)) geometries produce topologically distinct foliations ■

In Section 5, we show that the geodesics introduced in Example 1.7 (1) are indeed geodesics with respect to the metric, out of their isotropic points. Thus, for the classic geometries, we can forget about the variational characterization of geodesics and deal only with the 'linear' one, which is much easier.

4. Levi-Civita Connection

From now on, we assume the hermitian form $\langle -,-\rangle$ to be nondegenerate. In particular, $\mathrm{B}\,V$ and $\mathrm{E}\,V$ are endowed with pseudo-riemannian metrics.

Also, until the end of the article, we use the following conventions. Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic. Extending by zero, we consider any tangent vector $t:p\to p^\perp$ as a linear map $t\in\mathrm{Lin}_{\mathbb{K}}(V,V)$. So, $\mathrm{T}_p\,\mathbb{P}_{\mathbb{K}}V=\mathrm{Lin}_{\mathbb{K}}(p,p^\perp)\subset\mathrm{Lin}_{\mathbb{K}}(V,V)$. (Obviously, $t=t\pi'[p]$, $t=\pi[p]t$, $t\pi[p]=\pi'[p]t=0$, and $st=0$ for all tangent vectors $s,t\in\mathrm{Lin}_{\mathbb{K}}(V,V)$ at p.) Conversely, given an arbitrary linear map $t\in\mathrm{Lin}_{\mathbb{K}}(V,V)$, we define the tangent vector $t_p:=\pi[p]t\pi'[p]$ at p.

Let $U\subset V$ be a *saturated* open set (i.e., $U\mathbb{K}^*\subset U$) without isotropic points. A *lifted field* over U is a smooth map $X:U\to\mathrm{Lin}_{\mathbb{K}}(V,V)$ such that $X(p)_p=X(p)$ and $X(pk)=X(p)$ for all $p\in U$ and $k\in\mathbb{K}^*$. In other words, X correctly defines a smooth tangent field over $\mathbb{P}_{\mathbb{K}}U$. For example, every $t\in\mathrm{Lin}_{\mathbb{K}}(V,V)$ provides the *standard* lifted field T related to t : it is given by the rule $T(p)=t_p$ and is defined for all nonisotropic p.

For $t\in\mathrm{Lin}_{\mathbb{K}}(V,V)$, we put

$$\nabla_tX(p):=\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}X\big((1+\varepsilon t)p\big)\Big)_p.$$

Since $\pi[pk]=\pi[p]$ and $\pi'[pk]=\pi'[p]$ for all $p\in U$ and $k\in\mathbb{K}^*$, the field $p\mapsto\nabla_{Y(p)}X$ is lifted for arbitrary lifted fields X and Y over U. Obviously, ∇ enjoys the properties of an affine connection.

4.1. Lemma. *Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic and let t be a tangent vector at p. Then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi'[p+tp\varepsilon]=-\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+tp\varepsilon]=t+t^*.$$

Proof. By definition, $\pi'[p+tp\varepsilon]=(p+tp\varepsilon)\dfrac{\langle p+tp\varepsilon,-\rangle}{\langle p,p\rangle+\varepsilon^2\langle tp,tp\rangle}$. Derivating,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(p+tp\varepsilon)\frac{\langle p+tp\varepsilon,-\rangle}{\langle p,p\rangle+\varepsilon^2\langle tp,tp\rangle}=p\frac{\langle tp,-\rangle}{\langle p,p\rangle}+tp\frac{\langle p,-\rangle}{\langle p,p\rangle}.$$

The second term equals $t\pi'[p]=t$. Put $\varphi:=p\dfrac{\langle tp,-\rangle}{\langle p,p\rangle}$. Then

$$\langle tx,y\rangle=\langle tx^p,y\rangle=\Big\langle tp\frac{\langle p,x\rangle}{\langle p,p\rangle},y\Big\rangle=\frac{\langle x,p\rangle}{\langle p,p\rangle}\langle tp,y\rangle=\Big\langle x,p\frac{\langle tp,y\rangle}{\langle p,p\rangle}\Big\rangle=\langle x,\varphi y\rangle$$

for every $x,y\in V$. Hence, $t^*=\varphi$ ■

4.2. Lemma. *Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic. Let s and t be tangent vectors at p. Then*

$$\nabla_TS(x)=\big(s\pi[x]t-t\pi'[x]s\big)_x$$

for every nonisotropic $x\in\mathbb{P}_{\mathbb{K}}V$, where S and T are the standard fields related to s and t. In particular, $\nabla_TS(p)=0$.

Proof. By Lemma 4.1,

$$\nabla_TS(x)=\nabla_{t_x}S(x)=\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}S(x+t_xx\varepsilon)\Big)_x=\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[x+t_xx\varepsilon]s\pi'[x+t_xx\varepsilon]\Big)_x=$$

$$=\Big(-\big(t_x+(t_x)^*\big)s\pi'[x]+\pi[x]s\big(t_x+(t_x)^*\big)\Big)_x=\big(s\pi[x]t-t\pi'[x]s\big)_x$$

since $\pi[x](t_x)^*=(t_x)^*\pi'[x]=0$ ■

4.3. Proposition. ∇ *is the Levi-Civita connection for the* (*hermitian*) *metric on every component of* $\mathbb{P}_{\mathbb{K}}V\setminus\mathrm{S}\,V$.

Proof. Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic. Let S and T be lifted local fields with $S(p):=s$ and $T(p):=t$.

In order to show that $\big(\nabla_ST-\nabla_TS-[S,T]\big)(p)=0$, we can assume that S and T are the standard fields related to s and t. It follows from Lemma 4.2 that $\nabla_ST(p)=\nabla_TS(p)=0$. The proof of $[S,T](p)=0$ follows [AGG, Lemma 5.5.2] : Let f be an analytic function and let $\hat f$ denote its lift to V. By definition, $T(x)f=\dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f(x+{}^xtx\varepsilon)$. Therefore,

$$S(p)(Tf)=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f\big(p+sp\delta+{}^{p+sp\delta}t(p+sp\delta)\varepsilon\big)\Big)=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f(p+sp\delta+{}^{p+sp\delta}tp\varepsilon)\Big)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\bigg(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f\Big(p+sp\delta+tp\varepsilon-(p+sp\delta)\frac{k_0\varepsilon\delta}{1+\delta^2\langle sp,sp\rangle/\langle p,p\rangle}\Big)\bigg),$$

where $k_0:=\langle sp,tp\rangle/\langle p,p\rangle$. Since $\hat f(pk)=\hat f(p)$ for every $k\in\mathbb{K}^*$, it follows that

$$\hat f\big(p(1-k_0\varepsilon\delta)+sp\delta(1-k_0\varepsilon\delta)+tp\varepsilon\big)=\hat f\Big(p+sp\delta+tp\frac{\varepsilon}{1-k_0\varepsilon\delta}\Big).$$

Being f analytic,

$$S(p)(Tf)=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f\big(p+sp\delta+tp\varepsilon-(p+sp\delta)k_0\varepsilon\delta\big)\Big)=$$

$$=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f(p+sp\delta+tp\frac{\varepsilon}{1-k_0\varepsilon\delta})\Big)=\frac{d}{d\delta}\Big|_{\delta=0}\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\hat f(p+sp\delta+tp\varepsilon)\Big).$$

Hence, $S(p)(Tf)=T(p)(Sf)$, that is, $[S,T](p)=0$.

In order to verify that $v(S,T)(p)=\big(\nabla_vS(p),T(p)\big)+\big(S(p),\nabla_vT(p)\big)$ for a tangent vector v at p, we put $\varphi_1:=\dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}S(p+vp\varepsilon)$ and $\varphi_2:=\dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0}T(p+vp\varepsilon)$. So,

$$\pm\big(\nabla_vS(p),T(p)\big)\dim_{\mathbb{R}}\mathbb{K}=\pm\big(\pi[p]\varphi_1\pi'[p],T(p)\big)\dim_{\mathbb{R}}\mathbb{K}=\mathrm{tr}_{\mathbb{R}}\Big(\big(\pi[p]\varphi_1\pi'[p]\big)^*T(p)\Big)=\mathrm{tr}_{\mathbb{R}}\big(\varphi_1^*T(p)\big),$$

$\pm\big(S(p),\nabla_vT(p)\big)\dim_{\mathbb{R}}\mathbb{K}=\mathrm{tr}_{\mathbb{R}}\,\big(S^*(p)\varphi_2\big)$, and

$$\pm v(S,T)(p)\dim_{\mathbb{R}}\mathbb{K}=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\mathrm{tr}_{\mathbb{R}}\big(S^*(p+vp\varepsilon)T(p+vp\varepsilon)\big)=\mathrm{tr}_{\mathbb{R}}\big(\varphi_1^*T(p)\big)+\mathrm{tr}_{\mathbb{R}}\big(S^*(p)\varphi_2\big).$$

Similar arguments work for the hermitian case ■

4.4. Curvature tensor. Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic and let T_1,T_2,S be local lifted fields with $T_i(p)=t_i$ and $S(p)=s$. We wish to express the curvature tensor $R(T_1,T_2)S(p):=\big(\nabla_{T_2}\nabla_{T_1}S-\nabla_{T_1}\nabla_{T_2}S+\nabla_{[T_1,T_2]}S\big)(p)$ in terms of the hermitian form. We can assume that T_i and S are the standard fields related to t_i and s. By Lemma 4.2,

$$\nabla_{T_1}\nabla_{T_2}S(p)=\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+t_1p\varepsilon]\big(s\pi[p+t_1p\varepsilon]t_2-t_2\pi'[p+t_1p\varepsilon]s\big)\pi'[p+t_1p\varepsilon]\Big)_p.$$

By Lemma 4.1,

$$\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+t_1p\varepsilon]s\pi[p+t_1p\varepsilon]t_2\pi'[p+t_1p\varepsilon]\Big)_p=$$

$$=\big(-(t_1+t_1^*)s\pi[p]t_2\pi'[p]-\pi[p]s(t_1+t_1^*)t_2\pi'[p]+\pi[p]s\pi[p]t_2(t_1+t_1^*)\big)_p=-st_1^*t_2$$

and $\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[p+t_1p\varepsilon]t_2\pi'[p+t_1p\varepsilon]s\pi'[p+t_1p\varepsilon]\Big)_p=t_2t_1^*s$. In other words, $\nabla_{T_1}\nabla_{T_2}S(p)=-st_1^*t_2-t_2t_1^*s$. By symmetry, $\nabla_{T_2}\nabla_{T_1}T(p)=-st_2^*t_1-t_1t_2^*s$. Since $[T_1,T_2](p)=0$ (see the proof of Proposition 4.3), we arrive at

$$R(t_1,t_2)s=st_1^*t_2+t_2t_1^*s-st_2^*t_1-t_1t_2^*s\ \blacksquare$$

4.5. Sectional curvature. Constant curvature classic geometries. Let $p\in\mathbb{P}_{\mathbb{K}}V$ be non-isotropic. Let $W\subset\mathrm{T}_p\,\mathbb{P}_{\mathbb{K}}V$ be a 2-dimensional $\mathbb{R}$-vector subspace such that the metric, being restricted to W, is nondegenerate. The sectional curvature of W is given by

$$SW:=S(t_1,t_2):=\frac{\big(R(t_1,t_2)t_1,t_2\big)}{(t_1,t_1)(t_2,t_2)-(t_1,t_2)^2}$$

for $\mathbb{R}$-linearly independent $t_1,t_2\in W$. We can assume that $t_j=v_j\langle p,-\rangle$, where $v_j\in p^\perp$ and $\langle v_j,v_j\rangle=\sigma_j\in\{-1,0,+1\}$ for $j=1,2$. Denote $k:=\langle v_1,v_2\rangle$, $k\in\mathbb{K}$. In this way, using the same sign $\pm$ as in (1.4), we obtain

$$\pm(t_1t_1^*t_2,t_2)\dim_{\mathbb{R}}\mathbb{K}=\mathrm{tr}_{\mathbb{R}}(t_2^*t_1t_1^*t_2)=\dim_{\mathbb{R}}\mathbb{K}\cdot\langle p,p\rangle^2\langle v_1,v_2\rangle\langle v_2,v_1\rangle.$$

In this way,

$$\big(R(t_1,t_2)t_1,t_2\big)=\pm\langle p,p\rangle^2\big(|k|^2+\sigma_1\sigma_2-2\,\mathrm{Re}(k^2)\big),\quad(t_j,t_j)=\pm\langle p,p\rangle\sigma_j,\quad(t_1,t_2)=\pm\langle p,p\rangle\,\mathrm{Re}\,k.$$

Hence,

$$SW=\pm\frac{|k|^2+\sigma_1\sigma_2-2\,\mathrm{Re}(k^2)}{\sigma_1\sigma_2-(\mathrm{Re}\,k)^2}=\pm\Big(1+\frac{3|k-\overline{k}|^2}{4\big(\sigma_1\sigma_2-(\mathrm{Re}\,k)^2\big)}\Big),$$

where the last equality follows from the identity $|k|^2-2\,\mathrm{Re}(k^2)=\frac34|k-\overline{k}|^2-(\mathrm{Re}\,k)^2$. By Lemma 2.1, $\sigma_1\sigma_2\neq(\mathrm{Re}\,k)^2$ since $(-,-)$ is nondegenerate over W.

Obviously, $SW=\pm1$ if $\mathbb{K}=\mathbb{R}$. If $\mathbb{K}\neq\mathbb{R}$ and if v_1,v_2 are $\mathbb{K}$-linearly dependent, then $\sigma_1\sigma_2=|k|^2$ by Lemma 2.1. In this case, $|k|=\sigma_1\sigma_2=1$, and it follows from the identity $|k|^2=|k-\overline{k}|^2/4+(\mathrm{Re}\,k)^2$ that $SW=\pm4$. Since $v_1,v_2\in p^\perp$ are always $\mathbb{K}$-linearly dependent if $\dim_{\mathbb{K}}V=2$, we arrive at the

4.5.1. Remark. In every component of $\mathbb{P}_{\mathbb{R}}^n$, $\mathbb{P}_{\mathbb{C}}^1$, and $\mathbb{P}_{\mathbb{H}}^1$, the sectional curvature is constant $\blacksquare$

All the remaining possible values for SW can be extracted from the above formula. They are displayed in the following table, where $W=t_1\mathbb{R}+t_2\mathbb{R}$, $t_j=v_j\langle p,-\rangle$, and $v_1,v_2\in p^\perp$ are $\mathbb{K}$-linearly independent. The sign $\pm$ is the same as in (1.4).

Form over $v_1\mathbb{K}+v_2\mathbb{K}$, $\mathbb{K}\neq\mathbb{R}$	Metric over W	Sectional curvature
Indefinite	Indefinite	$\pm\,(-\infty,1]$
Definite	Definite	$\pm\,[1,4)$
Degenerate	Definite	$\pm\,4$
Indefinite	Definite	$\pm\,(4,\infty)$

$\blacksquare$

5. Parallel displacement along geodesics

Let $p\in\mathbb P_{\mathbb K}V$ be nonisotropic, let t be a tangent vector at p, and let T be the standard field related to t. The smooth lifted field

$$\operatorname{Tn}(t)(-):=\frac{T(-)}{\operatorname{ta}(p,-)}$$

is defined out of $\mathbb P_{\mathbb K}p^\perp\cup\mathrm S\,V$.

5.1. Lemma. *Let* $\mathrm G$ *be a geodesic and let* t *be a nonnull tangent vector to* $\mathrm G$ *at a nonisotropic* $p\in\mathrm G$. *Then the field* $\operatorname{Tn}(t)$ *is nonnull and tangent to* $\mathrm G$ *wherever defined.*

Proof. Let $g\in\mathrm G$ be nonisotropic and nonorthogonal to p. Clearly, $\varphi:=\operatorname{Tn}(t)(g)\neq0$ since $\pi[g]t\pi'[g]=0$ would imply $g\in p^\perp$. By Lemma 3.1 (2), $\mathrm G=\mathrm G\,W$ with $W=p\mathbb R+tp\mathbb R$. We can assume that $g\in W$. Hence, $\varphi g\in W$ and $\operatorname{Tn}(t)(g)$ is tangent to $\mathrm G$ at g by Lemma 2.4 $\blacksquare$

5.2. Lemma. *Let* $p,q\in\mathbb P_{\mathbb K}V$ *be distinct nonorthogonal with* p *nonisotropic. Denote* $\mathrm G[p,q]$ *the oriented segment*[2] *of the geodesic* $\mathrm G\wr p,q\wr$ *that does not contain the point orthogonal to* p. *Let* $\varphi:V\to V$ *be given by* $\varphi=q\langle p,q\rangle^{-1}\langle p,-\rangle$. *Then* φ_p *is tangent to the* oriented *segment* $\mathrm G[p,q]$ *at* p.

Proof. The tangent vector φ_p does not depend on the choice of representatives $p,q\in V$. We can assume that $\langle p,p\rangle=\sigma$ and $\langle p,q\rangle=\sigma a$, where $\sigma\in\{-1,+1\}$ and $a>0$. Clearly, $\varphi_p:p\mapsto{}^pq(1/a)$. The curve $c_0(t):=p(1-t)+qt$, $t\in[0,1]$, parameterizes a lift of $\mathrm G[p,q]$. Indeed, $\big\langle p,p(1-t)+qt\big\rangle=0$ means that $(1-a)t=1$, which is impossible. By Lemma 2.5, the linear map $\dot c(0):p\mapsto{}^pq$ is tangent to $\mathrm G[p,q]$ at p $\blacksquare$

5.3. Lemma. *Let* $p\in\mathbb P_{\mathbb K}V$ *be nonisotropic, let* t *be a tangent vector at* p, *and let* T *be the standard field related to* t. *Then, for every nonisotropic* x,

$$T(x)\big(\operatorname{ta}(p,-)\big)=-2\operatorname{ta}(p,x)\operatorname{Re}\frac{\langle tx,x\rangle}{\langle x,x\rangle}.$$

Proof is straightforward:

$$T(x)\big(\operatorname{ta}(p,-)\big)=\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\frac{\langle p,x+{}^xtx\varepsilon\rangle\langle x+{}^xtx\varepsilon,p\rangle}{\langle p,p\rangle\big(\langle x,x\rangle+\varepsilon^2\langle{}^xtx,{}^xtx\rangle\big)}=\frac{\langle p,{}^xtx\rangle\langle x,p\rangle+\langle p,x\rangle\langle{}^xtx,p\rangle}{\langle p,p\rangle\langle x,x\rangle}=$$

$$=-\frac{\langle p,x\rangle\langle x,tx\rangle\langle x,p\rangle+\langle p,x\rangle\langle tx,x\rangle\langle x,p\rangle}{\langle p,p\rangle\langle x,x\rangle^2}=-2\operatorname{ta}(p,x)\operatorname{Re}\frac{\langle tx,x\rangle}{\langle x,x\rangle}\ \blacksquare$$

5.4. Theorem. *Let* $\mathrm G$ *be a geodesic, let* t *be a nonnull tangent vector to* $\mathrm G$ *at a nonisotropic* $p\in\mathrm G$, *and let* $h\in\mathrm T_p\,\mathrm L$, *where* $\mathrm L$ *stands for the projective line of* $\mathrm G$. *Then, for every nonisotropic* $g\in\mathrm G$ *not orthogonal to* p,

$$\nabla_{\operatorname{Tn}(t)(g)}\operatorname{Tn}(h)=0.$$

Proof. Denote by H and T the standard fields related to h and t, respectively. It suffices to show that $\Big(\nabla_{T(g)}\dfrac{H(-)}{\operatorname{ta}(p,-)}\Big)g=0$. By Lemma 3.1 (2), $\mathrm G=\mathrm G\,W$ with $W=p\mathbb R+tp\mathbb R$. We can take $g\in W$. By Lemmas 4.2 and 5.3,

$$\Big(\nabla_{T(g)}\frac{H(-)}{\operatorname{ta}(p,-)}\Big)g=T(g)\Big(\frac{1}{\operatorname{ta}(p,-)}\Big)H(g)g+\frac{1}{\operatorname{ta}(p,g)}\big(\nabla_{T(g)}H\big)g=$$

[2]In the particular case of a spherical $\mathrm G\wr p,q\wr$, the segment $\mathrm G[p,q]$ is the shortest one from p to q.

$$=\frac{1}{\operatorname{ta}(p,g)}\pi[g]\Big(2\frac{\langle tg,g\rangle}{\langle g,g\rangle}hg+h\pi[g]tg-t\pi'[g]hg\Big).$$

It follows from Lemma 2.4 that $hp=tpk$ for some $k\in\mathbb{K}$ since both h and t are tangent to L at p. From $hp^\perp=tp^\perp=0$, we conclude that $hg=tgk$. Finally, from $\pi[g]=1-\pi'[g]$, $htg=0$, $\langle tg,g\rangle\in\mathbb{R}$, and $hg=tgk$, we obtain $h\pi[g]tg=-h\pi'[g]tg=-hg\dfrac{\langle g,tg\rangle}{\langle g,g\rangle}=-\dfrac{\langle tg,g\rangle}{\langle g,g\rangle}hg$ and $t\pi'[g]hg=t\pi'[g]tgk=tg\dfrac{\langle g,tg\rangle}{\langle g,g\rangle}k=\dfrac{\langle tg,g\rangle}{\langle g,g\rangle}hg$ ■

Theorem 5.4, Lemma 5.1, and Lemma 3.1 (2) have the following

5.5. Corollary. *Out of isotropic points, a geodesic in the sense of* Example 1.7 (1) *is a geodesic of the Levi-Civita connection* ∇. *Every geodesic of this connection appears in this way* ■

Let $p\in\mathbb{P}_{\mathbb{K}}V$ be nonisotropic, let t be a tangent vector at p, and let T be the standard field related to t. The smooth lifted field

$$\operatorname{Ct}(t)(-):=\frac{T(-)}{\sqrt{\operatorname{ta}(p,-)}}$$

is defined at every nonisotropic point in $\mathbb{P}_{\mathbb{K}}V\setminus\mathbb{P}_{\mathbb{K}}p^\perp$ that belongs to the component of $\mathbb{P}_{\mathbb{K}}V\setminus\mathrm{S}\,V$ containing p.

5.6. Theorem. *Let* G *be a geodesic, let* t *be a nonnull tangent vector to* G *at a nonisotropic* $p\in\mathrm{G}$, *and let* $v\in(\mathrm{T}_p\mathrm{L})^\perp$, *where* L *stands for the projective line of* G. *Then*

$$\nabla_{\operatorname{Tn}(t)(g)}\operatorname{Ct}(v)=0$$

for every nonisotropic $g\in\mathrm{G}\setminus\mathbb{P}_{\mathbb{K}}p^\perp$ *that belongs the component of* $\mathbb{P}_{\mathbb{K}}V\setminus\mathrm{S}\,V$ *containing* p.

Proof. Denote by U and T the standard fields related to v and t, respectively. It suffices to show that $\Big(\nabla_{T(g)}\dfrac{U(-)}{\sqrt{\operatorname{ta}(p,-)}}\Big)g=0$. By Lemma 3.1 (2), $\mathrm{G}=\mathrm{G}\,W$ with $W=p\mathbb{R}+tp\mathbb{R}$. We can take $g\in W$. By Lemmas 4.2 and 5.3,

$$\Big(\nabla_{T(g)}\frac{U(-)}{\sqrt{\operatorname{ta}(p,-)}}\Big)g=T(g)\Big(\frac{1}{\sqrt{\operatorname{ta}(p,-)}}\Big)U(g)g+\frac{1}{\sqrt{\operatorname{ta}(p,g)}}\big(\nabla_{T(g)}U\big)g=$$

$$=\frac{1}{\sqrt{\operatorname{ta}(p,g)}}\pi[g]\Big(\frac{\langle tg,g\rangle}{\langle g,g\rangle}vg+v\pi[g]tg-t\pi'[g]vg\Big).$$

By Lemma 2.4, $tpk\langle p,-\rangle\in\mathrm{T}_p\mathrm{L}$ for all $k\in\mathbb{K}$. Taking $v\in(\mathrm{T}_p\mathrm{L})^\perp$ in the form $v=w\langle p,-\rangle$ with $w\in p^\perp$, we obtain $\langle p,p\rangle\operatorname{Re}\langle w,tpk\rangle=0$. This implies that $w\in(p\mathbb{K}+tp\mathbb{K})^\perp$, $vg\in(p\mathbb{K}+tp\mathbb{K})^\perp$, and $\pi'[g]vg=0$. Finally, as in the proof of Theorem 5.4, $v\pi[g]tg=-vg\dfrac{\langle g,tg\rangle}{\langle g,g\rangle}=-\dfrac{\langle tg,g\rangle}{\langle g,g\rangle}vg$ ■

Let L be a non-euclidean projective line and let $\mathrm{L}\ni p$ be nonisotropic. It easily follows from the identification $\mathrm{T}_p\mathbb{P}_{\mathbb{K}}V=p^\perp\langle p,-\rangle$ that $\mathrm{T}_p\mathbb{P}_{\mathbb{K}}V=\mathrm{T}_p\mathrm{L}\oplus(\mathrm{T}_p\mathrm{L})^\perp$. Hence, every tangent vector $t\in\mathrm{T}_p\mathbb{P}_{\mathbb{K}}V$ decomposes as $t=h+v$, where $h\in\mathrm{T}_p\mathrm{L}$ and $v\in(\mathrm{T}_p\mathrm{L})^\perp$. This decomposition is called *horizontal-vertical.* Under the assumption that L is spanned by p and q, the horizontal-vertical decomposition is $t=\pi'[w]t+\pi[w]t$, where $w:={}^pq$.

5.7. Corollary. *Let* L *be a non-euclidean projective line spanned by distinct, nonisotropic, and nonorthogonal points* $p,q\in\mathbb{P}_{\mathbb{K}}V$ *of the same signature. Let* $t=h+v$ *be the horizontal-vertical* *decomposition of* $t\in\mathrm T_p\,\mathbb P_{\mathbb K}V$ *with respect to* L. *Then the parallel displacement of* t *from* p *to* q *along* $\mathrm G[p,q]$ *is given by* $\mathrm{Tn}\,\big(h)(q)+\mathrm{Ct}(v)(q)$ ■

The above corollary expresses the parallel displacement along geodesics in a component of $\mathbb P_{\mathbb K}V$. However, in particular cases, some parallel displacement can be performed even if the nonisotropic and nonorthogonal points p,q lie in **different** components of $\mathbb P_{\mathbb K}V$ (we just 'bat an eye' while passing through $\mathrm S\,V$) : For a horizontal vector h, $\mathrm{Tn}(h)(q)$ gives a parallel displacement of h along $\mathrm G[p,q]$. When $\mathbb K=\mathbb C$, for a vertical vector v, $\mathrm{Ct}(v)(q)$ gives a parallel displacement of v along $\mathrm G[p,q]$ (we fix the sign of $\sqrt{\mathrm{ta}(p,q)}\in\mathbb Ri$).

It remains to study the parallel displacement along euclidean geodesics. Let $p\in\mathbb P_{\mathbb K}V$ be nonisotropic, let s be a tangent vector at p, and let S be the standard field related to s. The smooth vector field

$$\mathrm{Eu}(s)(x):=\frac12\big(\pi[p]\pi'[x]s\big)_x+S(x)$$

is defined out of isotropic points. Clearly, $\mathrm{Eu}(s)(p)=S(p)=s$.

5.8. Theorem. *Let* G *be an euclidean geodesic, let* t *be a nonnull tangent vector to* G *at a nonisotropic* $p\in\mathrm G$, *and let* $s\in\mathrm T_p\,\mathbb P_{\mathbb K}V$. *Then, for every nonisotropic* $g\in\mathrm G$,

$$\nabla_{\mathrm{Tn}(t)(g)}\,\mathrm{Eu}(s)=0.$$

Proof. It suffices to show that $\big(\nabla_{T(g)}\,\mathrm{Eu}(s)\big)g=0$, where T is the standard field related to t. By Lemma 3.1 (2), $\mathrm G=\mathrm G\,W$ with $W=p\mathbb R+tp\mathbb R$. We can take $g\in W$. Notice that, being orthogonal to p, each one of tp, tg, and ${}^pg$ represents the only isotropic point $u\in\mathrm G$. Clearly, $\langle u,\mathrm G\rangle=0$. It follows that $\pi[g]t=\pi[p]t=t$. Hence, $s\pi[g]t=st=0$. Also, $\pi'[g]\pi[p]\pi'[g]=0$. Now, using $\pi[g](t_g)^*=(t_g)^*g=0$, we obtain

$$2\big(\nabla_{T(g)}\,\mathrm{Eu}(s)\big)g=\pi[g]\Big(\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\pi[g+t_gg\varepsilon]\pi[p]\pi'[g+t_gg\varepsilon]s\pi'[g+t_gg\varepsilon]\Big)g+2\pi[g]s\pi[g]tg-2\pi[g]t\pi'[g]sg=$$

$$=-\pi[g]\big(t_g+(t_g)^*\big)\pi[p]\pi'[g]sg+\pi[g]\pi[p]\big(t_g+(t_g)^*\big)sg+\pi[g]\pi[p]\pi'[g]s\big(t_g+(t_g)^*\big)g-2\pi[g]t\pi'[g]sg=$$
$$=\pi[g]\pi[p]\big(t_g+(t_g)^*\big)sg-2\pi[g]t\pi'[g]sg$$

by Lemmas 4.1 and 4.2. Since $(\varphi\psi)^*=\psi^*\varphi^*$, $\langle g,t^*g\rangle=\langle tg,g\rangle=0$, $\pi[g]\pi[p]g={}^pg$, and the projections are self-adjoint, we obtain

$$\pi[g]\pi[p](t_g)^*sg=\pi[g]\pi[p]\pi'[g]t^*\pi[g]sg=\pi[g]\pi[p]\pi'[g]\Big(t^*sg-t^*g\frac{\langle g,sg\rangle}{\langle g,g\rangle}\Big)=$$

$$=\pi[g]\pi[p]\Big(g\frac{\langle g,t^*sg\rangle}{\langle g,g\rangle}-g\frac{\langle g,t^*g\rangle\langle g,sg\rangle}{\langle g,g\rangle^2}\Big)={}^pg\frac{\langle tg,sg\rangle}{\langle g,g\rangle}.$$

It follows from $\pi[p]t=\pi[g]t=t$ and $sg\in p^\perp$ that $\pi[g]\pi[p]t_gsg=\pi[g]t\pi'[g]sg=tg\dfrac{\langle g,sg\rangle}{\langle g,g\rangle}=tg\dfrac{\langle {}^pg,sg\rangle}{\langle g,g\rangle}$. It remains to observe that ${}^pg$ and tg are $\mathbb R$-proportional ■

5.9. Corollary. *Let* $p,q\in\mathbb P_{\mathbb K}V$ *be distinct and nonisotropic points that span an euclidean projective line and let* $t\in\mathrm T_p\,\mathbb P_{\mathbb K}V$. *Then the parallel displacement of* t *from* p *to* q *along* $\mathrm G[p,q]$ *is given by* $\mathrm{Eu}(t)(q)$ ■

The following three examples concern complex hyperbolic geometry. For basic background on the subject, see [Gol] or Section 5 in [AGG]. As in Example 1.6 (4), we take $\mathbb K=\mathbb C$, $\dim_{\mathbb C}V=3$, the form of signature $++-$ and the sign $-$ in the definition (1.5) of the hermitian metric. Thus, $\mathrm{B}\,V$ is the complex hyperbolic plane $\mathbb{H}^2_{\mathbb{C}}$.

5.10. Example: area formula. Let $p_1, p_2, p_3 \in \mathrm{B}\,V \cup \mathrm{S}\,V$ be points in a complex geodesic L. With the use of vertical parallel displacement, we will show that the oriented area of the plane triangle $\triangle(p_1, p_2, p_3)$ is given by

$$\text{(5.11)} \qquad \operatorname{Area} \triangle(p_1, p_2, p_3) = -\frac{1}{2} \arg\big(-\langle p_1, p_2\rangle\langle p_2, p_3\rangle\langle p_3, p_1\rangle\big),$$

where arg varies in $[-\pi, \pi]$.

First, we take $p_j \notin \mathrm{S}\,V$, $j = 1,2,3$. We have $\mathrm{L} = \mathbb{P}_{\mathbb{C}} p^{\perp}$, where $p \in \mathrm{E}\,V$ is the polar point to L (see the beginning of Example 3.6). By Lemma 2.4, $(\mathrm{T}_q \mathrm{L})^{\perp} = p\mathbb{C}\langle q, -\rangle$ for every $q \in \mathrm{L} \setminus \mathrm{S}\,V$. Let $v := pc\langle p_1, -\rangle \in (\mathrm{T}_{p_1} \mathrm{L})^{\perp}$, $c \in \mathbb{C}^*$. Making the parallel displacement of v along the segment of geodesic $\mathrm{G}[p_1, p_2]$, then along $\mathrm{G}[p_2, p_3]$, and finally along $\mathrm{G}[p_3, p_1]$, we end up with some $v' \in (\mathrm{T}_{p_1} \mathrm{L})^{\perp}$. By Corollary 5.7,

$$v' = \frac{\pi[p_1]\pi[p_3]\pi[p_2]v\pi'[p_2]\pi'[p_3]\pi'[p_1]}{\sqrt{\operatorname{ta}(p_1, p_2)\operatorname{ta}(p_2, p_3)\operatorname{ta}(p_3, p_1)}} = \frac{pc\langle p_1, p_2\rangle\langle p_2, p_3\rangle\langle p_3, p_1\rangle\langle p_1, -\rangle}{\langle p_2, p_2\rangle\langle p_3, p_3\rangle\langle p_1, p_1\rangle\sqrt{\operatorname{ta}(p_1, p_2)\operatorname{ta}(p_2, p_3)\operatorname{ta}(p_3, p_1)}}$$

because $p \in p_j^{\perp}$. Clearly, $(\mathrm{T}_{p_1} \mathrm{L})^{\perp}$ is a one-dimensional $\mathbb{C}$-vector space. The oriented angle $\angle(v, v')$ from v to v', taken in $[-\pi, \pi]$, is an additive measure of a triangle. Hence, it is proportional to the oriented area of $\triangle(p_1, p_2, p_3)$. In terms of the hermitian metric (1.5),

$$\angle(v, v') = \arg\langle v, v'\rangle = \arg\big(-\langle p_1, p_2\rangle\langle p_2, p_3\rangle\langle p_3, p_1\rangle\big)$$

due to $p \in \mathrm{E}\,V$ and $p_2, p_3 \in \mathrm{B}\,V$. The formula is extendable to isotropic points. Considering a suitable ideal triangle, we find the factor of proportionality $-1/2$ in (5.11).

The obtained formula (without orientation taken into account) can be found in [Gol]. Using the horizontal parallel displacement instead of the vertical one, we would arrive at the well-known area formula in terms of the angles. Curiously, the formula (5.11) seems to appear more naturally in the context of complex hyperbolic geometry. A similar formula holds for a plane spherical triangle ■

5.12. Example: some geometry behind the angle between bisectors. Let B_1 and B_2 be bisectors in $\mathbb{H}^2_{\mathbb{C}}$ with hyperbolic real spines G_1 and G_2. Assume that these bisectors share a common slice S whose polar point is $p \in \mathrm{E}\,V$. Let $v_j \in \mathrm{S}\,V \cap \mathrm{G}_j$ denote some vertex of B_j, $j = 1, 2$. Then the point $q_j := {}^p v_j$ is the intersection point of the real spine of B_j with the slice S. Denote by $\mathrm{G}[q_j, v_j) \subset \mathrm{G}_j$ the oriented segment of the real spine that starts with q_j and ends with v_j. Let $B[q_j, v_j) \subset B_j$ denote the corresponding oriented segment of bisector: $B[q_j, v_j)$ is oriented with respect to the orientation of $\mathrm{G}[q_j, v_j)$ and to the natural orientation of its slices. Define

$$u := 1 - \frac{\langle v_2, v_1\rangle\langle p, p\rangle}{\langle v_2, p\rangle\langle p, v_1\rangle}.$$

In other words, $u = 1 - \dfrac{1}{\eta(v_1, v_2, p)}$, where $\eta(v_1, v_2, p)$ is *Goldman's invariant* [Gol].

Let $q \in S$. We choose representatives $p, v_1, v_2 \in V$ such that $\langle p, p\rangle = \langle p, v_j\rangle = 1$. Thus,

$$q_j = v_j - p, \quad \langle q_j, v_j\rangle = -1, \quad \langle q_j, q\rangle = \langle v_j, q\rangle, \quad \langle q_j, q_j\rangle = -1,$$

$${}^{q_j} v_j = p, \quad \langle q_2, q_1\rangle = \langle v_2, v_1\rangle - 1 = -u, \quad \operatorname{ta}(q_1, q_2) = |u|^2.$$

In particular, $u \neq 0$. According to [AGG, Proposition 5.2.7 and Lemma 5.2.9],

$$n(q,q_j,v_j)=\Big(q_j\frac{\langle v_j,q\rangle}{\langle v_j,q_j\rangle}-v_j\frac{\langle q_j,q\rangle}{\langle q_j,v_j\rangle}\Big)i\,\langle q,-\rangle=p\langle v_j,q\rangle i\,\langle q,-\rangle$$

is a normal vector to the oriented segment $B[q_j,v_j)$ at q. Both normal vectors in question belong to the $\mathbb{C}$-vector space $(\mathrm{T}_q\,S)^\perp$ and, therefore, the oriented angle $\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)$ from $B[q_1,v_1)$ to $B[q_2,v_2)$ at q can be calculated as

$$\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)=\arg\big\langle n(q,q_1,v_1),n(q,q_2,v_2)\big\rangle=\arg\big(-\langle q,q\rangle\langle q,v_1\rangle\langle v_2,q\rangle\big)=$$

$$=\arg\big(\langle q,v_1\rangle\langle v_2,q\rangle\big)=\arg\big(\langle q,q_1\rangle\langle q_2,q\rangle\big)=\arg\big(-u\langle q,q_1\rangle\langle q_1,q_2\rangle\langle q_2,q\rangle\big)$$

since $-u\langle q_1,q_2\rangle=|u|^2$. In other words, using the previous example,

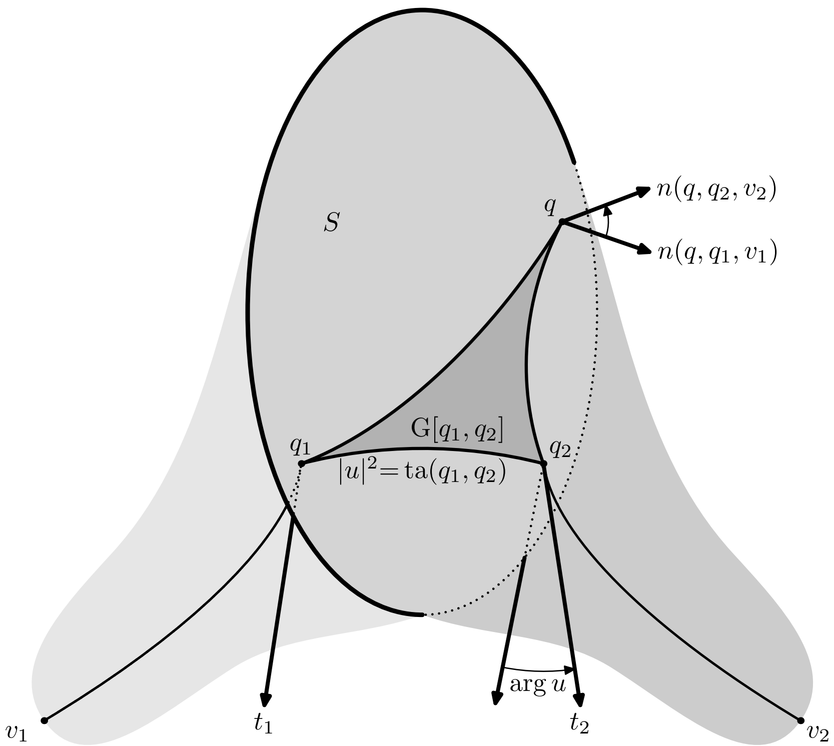
$$\angle\big(q,B[q_1,v_1),B[q_2,v_2)\big)\equiv\arg u-2\,\mathrm{Area}\,\Delta(q,q_1,q_2)\mod 2\pi.$$

We can see that the angle in question is composed of two parts. The *constant angle* $\arg u$ is independent of $q\in S$ (in [Hsi3], this angle is called prespinal). The *nonconstant angle* $-2\,\mathrm{Area}(q,q_1,q_2)$ depends only on the mutual position of q,q_1,q_2 in S. Let us show that the constant angle is the angle from the real spine $\mathrm{G}[q_1,v_1)$ to the real spine $\mathrm{G}[q_2,v_2)$ measured with the help of parallel displacement along the segment of geodesic $\mathrm{G}[q_1,q_2]$.

By Lemma 5.2, $t_j:={}^{q_j}v_j\langle q_j,v_j\rangle^{-1}\langle q_j,-\rangle=-p\langle q_j,-\rangle$ is tangent to $\mathrm{G}[q_j,v_j)$ at q_j. By Corollary 5.7, the parallel displacement of t_1 along $\mathrm{G}[q_1,q_2]$ is given by

$$\mathrm{Ct}(t_1)(q_2)=\frac{\pi[q_2]t_1\pi'[q_2]}{\sqrt{\mathrm{ta}(q_1,q_2)}}=-\frac{{}^{q_2}p\langle q_1,q_2\rangle\langle q_2,-\rangle}{|u|\langle q_2,q_2\rangle}=-\frac{\overline{u}}{|u|}p\langle q_2,-\rangle=\frac{\overline{u}}{|u|}t_2.$$

This implies the result, illustrated by the following picture:



It easily follows from Sylvester's criterion that u completely characterizes the configuration of $B[q_1, v_1)$ and $B[q_2, v_2)$ and that every $u \in \mathbb{C}$ with $|u| \geq 1$ is possible. The geometric meaning of u is clear now: $|u|^2$ is the tance between the complex spines of the bisectors and $\arg u$ is the angle between their real spines, in the above sense ■

5.13. Example: meridional and parallel displacements. Let B be a bisector in $\mathbb{P}_{\mathbb{C}}V$ as introduced in Example 1.7 (4), let G and L be the real and complex spines of B, and let $p_1, p_2 \in \mathrm{G}$ be distinct, nonisotropic, and nonorthogonal points. Denote by S_j the slice of B that contains p_j, $j = 1, 2$. Take $q_1 \in S_1$ different from the focus f of B. The slice S_j is spanned by p_j and f. By Lemma 2.4, the complex spine and the slices are orthogonal.

The vector $v := {}^{p_1}q_1\langle p_1, q_1\rangle^{-1}\langle p_1, -\rangle$ is tangent to $\mathrm{G}[p_1, q_1] \subset \mathrm{S}_1$ at p_1 by Lemma 5.2 and is thus orthogonal to the complex spine of B. Let $\mathrm{Ct}(v)(p_2)$ denote the parallel displacement of v from p_1 to p_2 along $\mathrm{G}[p_1, p_2]$ given by Corollary 5.7 and by the considerations right after it. Then there exists a unique $q_2 \in S_2$ such that

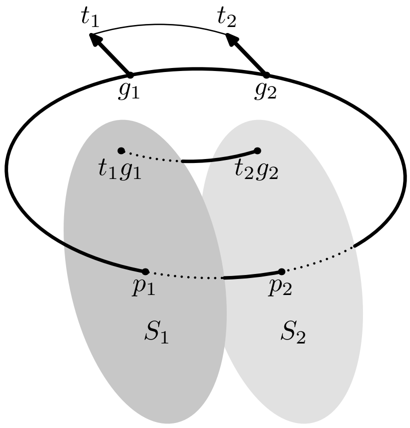
$$ {}^{p_2}q_2\langle p_2, q_2\rangle^{-1}\langle p_2, -\rangle = \mathrm{Ct}(v)(p_2). $$

(This can be seen by considering q_2 in the form $q_2 = p_2 + fc$, $c \in \mathbb{C}$.) We call q_2 the *meridional displacement* of q_1 from p_1 to p_2 along $\mathrm{G}[p_1, p_2]$. In explicit terms,

$$ q_2 = p_2\langle p_1, q_1\rangle\sqrt{\mathrm{ta}(p_1, p_2)} + {}^{p_1}q_1\langle p_1, p_2\rangle. $$

The meridional displacement identifies almost all slices of the bisector (the only exceptions are the slices tangent to $\mathrm{S}\,V$, if they exist). Such identification, called the *slice identification,* is an important tool for constructing and characterizing complex hyperbolic manifolds in [AGG] and [AGu].

The meridional and parallel displacements are related as follows. As is easy to see, every slice S of B has the form $S = \mathbb{P}_{\mathbb{C}}g^{\perp}$, where $g \in \mathrm{G}$ is the polar point to S. If g is nonisotropic, we associate to every nonnull tangent vector $t \in \mathrm{T}_g\,\mathbb{P}_{\mathbb{C}}V$ the point $tg \in S$. Denote by $g_j \in \mathrm{G}$ the polar points to S_j. The parallel displacement along $\mathrm{G}[g_1, g_2]$ produces the meridional displacement of the associated points:



Indeed, g_1, g_2 are nonorthogonal and nonisotropic. Let t_1 be a tangent vector at g_1. By Corollary 5.7, the parallel displacement of t_1 from g_1 to g_2 along $\mathrm{G}[g_1, g_2]$ is given by

$$ t_2 := \mathrm{Tn}(h)(g_2) + \mathrm{Ct}(v)(g_2) = \Big(\frac{h}{\mathrm{ta}(g_1, g_2)} + \frac{v}{\sqrt{\mathrm{ta}(g_1, g_2)}}\Big)_{g_2}, $$

where $t_1 = h + v$ is the horizontal-vertical decomposition of t_1 with respect to L, that is, $h \in \mathrm{T}_{g_1}\mathrm{L}$ and $v \in (\mathrm{T}_{g_1}\mathrm{L})^{\perp}$. We can assume that $h \neq 0$ (otherwise, the focus f is the point associated to both t_1

and t_2). It is easy to see that $\operatorname{ta}(g_1,g_2)=\operatorname{ta}(p_1,p_2)$. Since $\pi'[g_1]g_2$ and g_1 are $\mathbb{C}^*$-proportional, the point in S_2 associated to t_2 has the form

$$t_2g_2=\frac{\pi[g_2]hg_2}{\operatorname{ta}(g_1,g_2)}+\frac{\pi[g_2]vg_2}{\sqrt{\operatorname{ta}(g_1,g_2)}}\simeq\frac{\pi[g_2]hg_1\langle p_1,p_1\rangle\langle p_2,p_2\rangle}{\langle p_2,p_1\rangle}\sqrt{\operatorname{ta}(p_1,p_2)}+vg_1\langle p_1,p_2\rangle,$$

where $\simeq$ means $\mathbb{C}^*$-proportionality. By Lemma 2.4, $hg_1\in(p_1\mathbb{C}+g_1\mathbb{C})\cap g_1^\perp=p_1\mathbb{C}$ because $h\in\mathrm{T}_{g_1}\mathrm{L}$. Also, $vg_1\in f\mathbb{C}$. From $t_1=h+v$ and from the orthogonal decomposition $p_2\mathbb{C}+g_2\mathbb{C}$, it follows now that ${}^{p_1}t_1g_1=vg_1$ and $\pi[g_2]hg_1=\pi'[p_2]hg_1=p_2\dfrac{\langle p_2,hg_1\rangle}{\langle p_2,p_2\rangle}$. It remains to observe that $hg_1\in p_1\mathbb{C}$ implies that $\langle p_2,hg_1\rangle=\langle p_2^{p_1},hg_1\rangle=\dfrac{\langle p_1,hg_1\rangle\langle p_2,p_1\rangle}{\langle p_1,p_1\rangle}$ ■

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