Duality for Reidemeister Numbers

Richard Hill

Mathematisches Institut der
Georg August Universität
Bunsenstr. 3-5
37073 Göttingen

Germany

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn

Germany

MPI / 94-6

Duality for Reidemeister Numbers

Richard Hill
Mathematisches Institut der Georg August Universität,
Göttingen

Bonn, February 8, 1994

Abstract

Let $\phi : G \to G$ be an endomorphism of an abstract group. We prove when G has an Abelian subgroup of finite index that $R(\phi) = S(\phi)$, where $R(\phi)$ is the Reidemeister number of ϕ and $S(\phi)$ is the number of fixed points of the induced map $\hat{\phi}$ on the unitary dual $\hat{G}$ of G. We construct a functor $\mathbf{F}$ from groups with endomorphisms to groups with automorphisms and prove that $R(\mathbf{F}\phi) = R(\phi)$.

1 Introduction

Throughout this article G will be an abstract group and $\phi : G \to G$ an endomorphism (not necessarily injective or surjective). We shall refer to the unitary dual of G as $\hat{G}$. There is an induced map $\hat{\phi} : \hat{G} \to \hat{G}$. We shall study the Reidemeister number $R(\phi)$, and the number $S(\phi) = \#\mathrm{Fix}(\hat{\phi})$. These will be properly defined below. In [1] it was shown that when G is either finite or Abelian $R(\phi) = S(\phi)$. The classes of finite groups and of Abelian groups have a rather small intersection, and so the question arises, under what circumstances does the equality $R(\phi) = S(\phi)$ hold? This article is an attempt to answer this question. A group will be called almost Abelian if it has an Abelian subgroup of finite index. We prove here that if G is almost Abelian and finitely generated then $R(\phi) = S(\phi)$.

Reidemeister numbers arose first in topology as an estimate for the number of fixed points of a map $f : X \to X$ of a topological space X to itself. The treatment here is mainly group-theoretical, although I give an account of the geometric interpretations in §1.4.

I would like to thank A.L.Fel'shtyn, S.J.Patterson and M.Tadič for many many useful discussions. I would also loke to thank SFB 170 in Göttingen and the Max-Planck-Institut für Mathematik in Bonn for their hospitality.

1.1 ϕ-Conjugacy and Reidemeister Numbers.

Two elements $x, y \in G$ are said to be ϕ-conjugate iff there exists a $g \in G$ with

$$xg = \phi(g)y.$$

We shall write $\{x\}_\phi$ for the ϕ-conjugacy class of the element $x \in G$. The Reidemeister number $R(\phi)$ of ϕ is defined to be the number of ϕ-conjugacy classes in G. We shall also write $\mathcal{R}(\phi)$ for the set of ϕ-conjugacy classes of elements of G. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

1.2 Irreducible Representations and the number $S(\phi)$.

Let V be a Hilbert space. A unitary representation of G on V is a homomorphism $\rho : G \to \mathrm{U}(V)$ where $\mathrm{U}(V)$ is the group of unitary transformations of V. Two of these $\rho_1 : G \to \mathrm{U}(V_1)$ and $\rho_2 : G \to \mathrm{U}(V_2)$ are said to be equivalent if there is a Hilbert space isomorphism $V_1 \cong V_2$ which commutes with the G-actions. A representation $\rho : G \to \mathrm{U}(V)$ is said to be irreducible if there is no decomposition

$$V \cong V_1 \oplus V_2$$

in which V_1 and V_2 are non-zero, closed G-submodules of V.

One defines the unitary dual $\hat{G}$ of G to be the set of all equivalence classes of irreducible, unitary representations of G.

If $\rho : G \to \mathrm{U}(V)$ is a representation then $\rho \circ \phi : G \to \mathrm{U}(V)$ is also a representation, which we shall denote $\hat{\phi}(\rho)$. If ρ_1 and ρ_2 are equivalent then $\hat{\phi}(\rho_1)$ and $\hat{\phi}(\rho_2)$ are equivalent. Therefore the endomorphism ϕ induces a map $\hat{\phi} : \hat{G} \to \hat{G}$ from the unitary dual to itself.

Definition 1 *Define the number $S(\phi)$ to be the number of fixed points of the induced map $\hat{\phi} : \hat{G} \to \hat{G}$. We shall write $S(\phi)$ for the set of fixed points of $\hat{\phi}$. Thus $S(\phi)$ is the set of equivalence classes of irreducible representations $\rho : G \to U(V)$ such that there is a transformation $M \in U(V)$ satisfying*

$$\forall x \in G, \quad \rho(\phi(x)) = M \cdot \rho(x) \cdot M^{-1}. \tag{1}$$

Note that if ϕ is an inner automorphism $x \mapsto gxg^{-1}$ then we have for any representation ρ,

$$\rho(\phi(x)) = \rho(g) \cdot \rho(x) \cdot \rho(g)^{-1},$$

implying that the class of ρ is fixed by the induced map. Thus for an inner automorphism the induced map is trivial and $S(\phi)$ is the cardinality of $\hat{G}$.

If G is an Abelian group then all of its irreducible representations are one dimsional. If ρ_1 and ρ_2 are two 1-dimensional representations then their pointwise product $(\rho_1 \cdot \rho_2)(g) := \rho_1(g) \cdot \rho_2(g)$ is also a one-dimensional representation of G. This multiplication makes $\hat{G}$ into a group. There is a natural topology on $\hat{G}$ for which G can be identified with the set of continuous 1-dimensional representations of $\hat{G}$. In this identification $g \in G$ is identified with the representation $\rho \mapsto \rho(g)$. When G is Abelian the group $\hat{G}$ is called the Pontryagin dual of G.

The Pontryagin dual of a finite Abelian group G has the same cardinality as G. If $G \cong \mathbb{Z}^r$ is a free Abelian group then $\hat{G} \cong \mathbb{R}^r/\mathbb{Z}^r$ is a torus. The dual of a direct sum is that direct sum of the duals. This is all proved in [5]

1.3 Statements of Results

We shall prove the following

Theorem 1 *If G is a finitely generated almost Abelian group and ϕ an endomorphism of G then*

$$R(\phi) = S(\phi). \tag{2}$$

By specialising to the case when G is finite and ϕ is the identity map, we obtain the classical result equating the number of irreducible representations of a finite group with the number of conjugacy classes of the group.

The equation (1) was first conjectured in [1], where it was proved in the following cases:

1 If there is a natural number n such that $\phi^n(G)$ is Abelian.

2 If G is a finite group.

In §5 we shall describe a functor

$$\mathbf{F} : \begin{array}{ccc} \left[\begin{array}{c} \text{Groups with a chosen} \\ \text{endomorphism} \end{array} \right] & \longrightarrow & \left[\begin{array}{c} \text{Groups with a chosen} \\ \text{automorphism} \end{array} \right] \\ (G, \phi) & \longmapsto & (\mathbf{F}G, \mathbf{F}\phi) \end{array} \tag{3}$$

with the property that that $R(\phi) = R(\mathbf{F}\phi)$.

1.4 Geometric Interpretation of Reidemeister Numbers

In this paragraph let X be a topological space with fundamental group $\pi_1(X)$ and universal cover $\mathrm{pr} : \tilde{X} \to X$. If $f : X \to X$ is a self map of X, a lifting of f is a commuting square

$$\begin{array}{ccccc} & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{X} & \\ \mathrm{pr} & \downarrow & & \downarrow & \mathrm{pr}\ . \\ & X & \xrightarrow{f} & X & \end{array}$$

Two of these are said to be equivalent if there is a commuting cube:

$$\begin{tikzcd}
 & \tilde{X} \arrow[r, equal, "\sigma"] \arrow[dl, "\tilde{f}_1"'] & \tilde{X} \arrow[dl, "\tilde{f}_2"'] \arrow[d] \\
\tilde{X} \arrow[r, equal, "\sigma"] \arrow[d] & \tilde{X} \arrow[d] & X \arrow[dl, "f"'] \\
X \arrow[r, leftrightarrow, "\mathrm{id}"] & X &
\end{tikzcd}$$

The Reidemeister number $R(f)$ of the self-map $f : X \to X$ was defined by K.Reidemeister to be the number of equivalence classes of liftings of f. This was intended as an estimate on the number of fixed points of f in X. It is known, for example that when X is a compact polyhedron, there is a self-map $g : X \to X$ homotopic to f such that g has $\leq R(f)$ fixed points. The number $R(f)$ is a homotopy invariant of f.

The map f induces an endomorphism $\pi_1(f)$ of the fundamental group $\pi_1(X)$ which is defined upto composition with an inner automorphism. Using the fundamental group to parametrize the liftings $\tilde{f}$ of f, one finds that lifting classes correspond to $\pi_1(f)$-conjugacy classes in $\pi_1(X)$. One therefore has (see [2])

$$R(f) = R(\pi_1(f)),$$

the right hand side being the group-theoretical Reidemeister number defined in §1.1. The fact that $\pi_1(f)$ is only definied modulo inner automorphisms corresponds to the following which is easily proved:

Proposition 1 *Let $\phi : G \to G$ be any group endomorphism and let $g \in G$. Let ψ be the endomorphism given by $\psi(x) = g^{-1}\phi(x)g$. Then two elements $x, y \in G$ are ϕ-conjugate iff xg and yg are ψ-conjugate. In particular $R(\phi)$ depends only on ϕ modulo inner automorphisms of G.*

Note also that $\hat{\phi}$ (and therefore also $\mathcal{S}(\phi)$ and $S(\phi)$) depends only on ϕ modulo inner automorphisms.

Let T_f be the mapping torus of the map $f : X \to X$, ie. the quotient of the space $X \times [0,1]$ obtained by identifying the point $(x, 0)$ with $(f(x), 1)$ for every $x \in X$. There is a canonical projection $\tau : T_f \to \mathbb{R}/\mathbb{Z}$ given by $(x,t) \mapsto t$. This induces a map $\pi_1(\tau) : \pi_1(T_f) \to \mathbb{Z}$.

It turns out that the Reidemeister number $R(f)$ is equal to the number of homotopy classes of closed paths γ in T_f whose projections onto $\mathbb{R}/\mathbb{Z}$ are homotopic to the path

$$\begin{array}{cccc} \sigma : & [0,1] & \to & \mathbb{R}/\mathbb{Z} \\ & t & \mapsto & t. \end{array}$$

Corresponding to this there is a new group theoretical interpretation of $R(\pi_1(f))$ as the number of usual conjugacy classes of elements $\gamma \in \pi_1(T_f)$ satisfying $\pi_1(\tau)(\gamma) = 1$. Here the symbol 1 means the generator of the group $\mathbb{Z} = \pi_1(\mathbb{R}/\mathbb{Z})$. The functor $\mathbf{F}$ mentioned in §1.3 takes the group $\pi_1(X)$ to the kernel of $\pi_1(\tau) : \pi_1(T_f) \to \mathbb{Z}$ and the endomorphism $\pi_1(f)$ to the restriction to $\ker(\pi_1\tau)$ of the inner automorphism $\gamma \mapsto \tilde{\sigma}\gamma\tilde{\sigma}^{-1}$, where $\pi_1(\tau)(\tilde{\sigma}) = \sigma$.

In this context it is interesting to note the following:

Proposition 2 *Let ρ be an irreducible representation of $\pi_1(X)$. Then the class of ρ is fixed by $\hat{\phi}$ iff ρ is the restriction to $\pi_1(X)$ of an irreducible representation $\overline{\rho}$ of $\pi_1(T_f)$. Thus $S(\phi)$ is the number of irreducible representations of $\pi_1(X)$ which are restrictions of representations of $\pi_1(T_f)$.*

Proof. Let $\Gamma = \pi_1(T_f)$, $G = \pi_1(X)$ and $\phi = \pi_1(f)$. If G has a presentation

$$G =< \text{gen}|\text{reln} >$$

then it is known that Γ has the presentation

$$\Gamma =< \text{gen} \cup \{t\}|\text{reln} \cup \{t^{-1}gt\phi(g)^{-1} : g \in \text{gen}\} > .$$

Let ρ be a representation of G. If $\hat{\phi}(\rho)$ is equivalent to ρ then there is a matrix M with

$$\rho \circ \phi = M \cdot \rho \cdot M^{-1}. \tag{4}$$

Define

$$\tilde{\rho}(t) := M, \quad \tilde{\rho}(g) := \rho(g) \quad g \in G.$$

We then have

$$\tilde{\rho}(tgt^{-1}) = \tilde{\rho}(\phi(g)),$$

from which it follows that $\tilde{\rho}$ can be extended to a representation of Γ. Clearly ρ is the restriction of $\tilde{\rho}$ to G. Since ρ is irreducible it follows that $\tilde{\rho}$ is irreducible.

On the other hand if ρ is an irreducible representation of Γ whose restriction $\rho|_G$ to G is irreducible then (1) holds with $M = \rho(t)$, so $\hat{\phi}(\rho|_G)$ is equivalent to $\rho|_G$.

1.5 Some Old Results on Reidemeister Numbers

We now describe some known results on Reidemeister numbers.

Lemma 1 ([2]) *If G is a group and ϕ is an endomorphism of G then an element $x \in G$ is always ϕ-conjugate to its image $\phi(x)$.*

Proof. If $g = x^{-1}$ then one has immediately $gx = \phi(x)\phi(g)$. The existence of a g satisfying this equation implies that x and $\phi(x)$ are ϕ-conjugate.

1.5.1 Abelian Groups

In this paragraph let G be an Abelian group, whose group law we shall write additively. The unitary dual $\hat{G}$ of G is the Pontryagin dual of G. If ϕ is an endomorphism of G then x and y are ϕ-conjugate iff $x - y = \phi(g) - g$ for some $g \in G$. Therefore $R(\phi)$ is the number of cosets of the image of the endomorphism

$$\begin{array}{rcl} (\phi - 1) : G & \to & G \\ g & \mapsto & \phi(g) - g. \end{array}$$

We thus have

Lemma 2 ([2]) *If G is Abelian then $R(\phi) = \#\text{coker}(\phi - 1)$.*

Note that $\text{coker}(\phi - 1)$ is canonically isomorphic to the Pontryagin dual of $\ker(\hat{\phi} - 1)$, where

$$\begin{array}{rcl} (\hat{\phi} - 1) : \hat{G} & \to & \hat{G} \\ g & \mapsto & \hat{\phi}(g) - g. \end{array}$$

From this it follows that when $\text{coker}(\phi - 1)$ is a finite group, its order is equal to that of $\ker(\hat{\phi} - 1)$. On the other hand an element of $\ker(\hat{\phi} - 1)$ is the same thing as a fixed point of $\hat{\phi}$. The number of fixed points of $\hat{\phi}$ is $S(\phi)$. We therefore have

Lemma 3 ([1]) *If G is Abelian and $R(\phi)$ is finite then $R(\phi) = S(\phi)$*

1.5.2 Finite Groups

In this paragraph let G be a finite group. In [1] the following theorem was proved using a counting argument.

Theorem 2 ([1]) *Let ϕ be an endomorphism of a finite group G. Then ϕ maps (usual) congruency classes in G to congruency classes in G. The number of congruency classes in G which are mapped to themselves by ϕ is precisely the Reidemeister number $R(\phi)$.*

Now let V be the complex vector space of class functions on the group G. A class function is a function which takes the same value on every element of a (usual) congruency class. The map ϕ induces a map

$$\begin{aligned} \varphi : V &\to V \\ f &\mapsto f \circ \phi \end{aligned}$$

We shall calculate the trace of φ in two ways. The characteristic functions of the congruency classes in G form a basis of V, and are mapped to one another by φ (the map need not be a bijection). Therefore the trace of φ is the number of elements of this basis which are fixed by φ. By Theorem 3, this is equal to the Reidemeister number.

Another basis of V, which is also mapped to itself by φ is the set of traces of irreducible representations of G (see [4] chapter XVIII). From this it follows that the trace of φ is the number of irreducible representations ρ of G such that ρ has the same trace as $\hat{\phi}(\rho)$. However, representations of finite groups are charcterized upto equivalence by their traces. Therefore the trace of φ is equal to the number of fixed points of $\hat{\phi}$, ie. $S(\phi)$. We therefore have by Theorem 2

Theorem 3 ([1]) *Let ϕ be an endomorphism of a finite group G. Then $R(\phi) = S(\phi)$.*

2 Proof of Theorem 1

In this section we shall prove Theorem 1. It seems plausible that one could prove the same theorem for the so - called "tame" topological groups (see [3]). However we shall be interested mainly in discrete groups, and it is known that the discrete tame groups are almost Abelian.

We shall introduce the profinite completion $\overline{G}$ of G and the corresponding endomorphism $\overline{\phi} : \overline{G} \to \overline{G}$. This is a compact totally disconnected group in which G is densely embedded. The proof will then follow in three steps:

$$R(\phi) = R(\overline{\phi}), \quad S(\phi) = S(\overline{\phi}), \quad R(\overline{\phi}) = S(\overline{\phi}).$$

If one omits the requirement that G is almost Abelian then one can still show that $R(\phi) \geq R(\overline{\phi})$ and $S(\phi) \geq S(\overline{\phi})$. The third identity is a general fact for compact groups (Theorem 4).

2.1 Compact Groups

Here we shall prove the third of the above identities.

Let C be a compact topological group and ϕ a continuous endomorphism of C. We define the number $S^{\mathrm{top}}(\phi)$ to be the number of fixed points of $\hat{\phi}$ in the unitary dual of C, where we only consider continuous representations of C. The number $R(\phi)$ is defined as usual.

Theorem 4 *For a continuous endomorphism ϕ of a compact group C one has $R(\phi) = S^{\mathrm{top}}(\phi)$.*

The proof uses the Peter-Weyl Theorem:

Theorem 5 (Peter - Weyl) *If C is compact then there is the following decomposition of the space $L^1(C)$ as a $C \oplus C$-module.*

$$L^1(C) \cong \bigoplus_{\lambda \in \hat{C}} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_\lambda).$$

and Schur's Lemma:

Lemma 4 (Schur) *If V and W are two irreducible unitary representations then*

$$\mathrm{Hom}_{\mathbb{C}C}(V, W) \cong \begin{cases} 0 & V \not\cong W \\ \mathbb{C} & V \cong W. \end{cases}$$

Proof of Theorem 4. The ϕ-conjugacy classes, being orbits of a compact group, are compact. Since there are only finitely many of them, they are also open subsets of C and thus have positive Haar measure.

We embed C in $C \oplus C$ by the map $g \mapsto (g, \phi(g))$. This makes $L^2(C)$ a C-module with a twisted action. By the Peter-Weyl Theorem we have (as C-modules)

$$L^2(C) \cong \bigoplus_{\lambda \in \hat{C}} \mathrm{Hom}_{\mathbb{C}}(V_\lambda, V_{\hat{\phi}(\lambda)}).$$

We therefore have a corresponding decomposition of the space of C-invariant elements:

$$L^2(C)^C \cong \bigoplus_{\lambda \in \hat{C}} \mathrm{Hom}_{\mathbb{C}C}(V_\lambda, V_{\hat{\phi}(\lambda)}).$$

We have used the well known identity $\mathrm{Hom}_{\mathbb{C}}(V, W)^C = \mathrm{Hom}_{\mathbb{C}C}(V, W)$.

The left hand side consists of functions $f : C \to \mathbb{C}$ satisfying $f(gx\phi(g)^{-1}) = f(x)$ for all $x, g \in C$. These are just functions on the ϕ-conjugacy classes. The dimension of the left hand side is thus $R(\phi)$. On the other hand by Schur's Lemma the dimension of the right hand side is $S^{\mathrm{top}}(\phi)$.

2.2 The End of the Proof

Let G be an almost Abelian group with an Abelian subgroup A of finite index $[G : A]$. Let A^0 be the intersection of all subgroups of G of index $[G : A]$. Then A^0 is an Abelian normal subgroup of finite index in G and one has $\phi(A^0) \subset A^0$ for every endomorphism ϕ of G.

$$\begin{array}{ccccccccc} 1 & \to & A^0 & \to & G & \to & F & \to & 1 \\ & & \downarrow \phi|_{A^0} & & \downarrow \phi & & \downarrow & & \\ 1 & \to & A^0 & \to & G & \to & F & \to & 1 \end{array} .$$

Lemma 5 *If $R(\phi)$ is finite then so is $R(\phi|_{A^0})$.*

Proof. A ϕ-conjugacy class is an orbit of the group G. A $\phi|_{A^0}$-conjugacy class is an orbit of the group A^0. Since A^0 has finite index in G it follows that every ϕ-conjugacy class in A^0 can be the union of at most finitely many $\phi|_{A^0}$-conjugacy classes. This proves the lemma.

Let $\overline{G}$ be the profinite completion of G with respect to its normal subgroups of finite index. There is a canonical injection $G \to \overline{G}$ and the map ϕ can be extended to a continuous endomorphism $\bar{\phi}$ of $\overline{G}$.

There is therefore a canonical map

$$\mathcal{R}(\phi) \to \mathcal{R}(\bar{\phi}).$$

Since G is dense in $\overline{G}$, the image of a ϕ-conjugacy class $\{x\}_\phi$ is its closure in $\overline{G}$. From this it follows that the above map is surjective. We shall actually see that the map is bijective. This will then give us

$$R(\phi) = R(\bar{\phi}).$$

However $\bar{\phi}$ is an endomorphism of the compact group $\overline{G}$ so by Theorem 7

$$R(\bar{\phi}) = S^{\mathrm{top}}(\bar{\phi}).$$

It thus suffices to prove the following two lemmas:

Lemma 6 *If $R(\phi)$ is finite then $S^{\mathrm{top}}(\bar{\phi}) = S(\phi)$.*

Lemma 7 *If $R(\phi)$ is finite then the map $\mathcal{R}(\phi) \to \mathcal{R}(\bar{\phi})$ is injective.*

Proof of Lemma 6. By Mackey's Theorem (see [3]), every representation ρ of G is contained in a representation which is induced by a 1-dimensional representation χ of A. If ρ is fixed by $\hat{\phi}$ then for all $a \in A^0$ we have $\chi(a) = \chi(\phi(a))$. Let $A^1 = \{a \cdot \phi(a)^{-1} : a \in A^0\}$. By Lemma 5 $R(\phi|_{A^0})$ is finite and by Lemma 2 $R(\phi|_{A^0}) = [A^0 : A^1]$. Therefore A^1 has finite index in G. However we have shown that χ and therefore also ρ is constant on cosets of A^1. Therefore ρ has finite image, which implies that ρ is the restriction to G of a unique continuous irreducible representation $\bar{\rho}$ of $\overline{G}$. One verifies by continuity that $\hat{\bar{\phi}}(\bar{\rho}) = \bar{\rho}$.

Conversely if $\bar{\rho} \in \mathcal{S}(\bar{\phi})$ then the restriction of $\bar{\rho}$ to G is in $\mathcal{S}(\phi)$.

Proof of Lemma 7. We must show that the intersection with G of the closure of $\{x\}_\phi$ in $\overline{G}$ is equal to $\{x\}_\phi$. We do this by constructing a coset of a normal subgroup of finite index in G which is contained in $\{x\}_\phi$. For every $a \in A^0$ we have $x \sim_\phi xa$ if there is a $b \in A^0$ with $x^{-1}bx\phi(b)^{-1} = a$. It follows that $\{x\}_\phi$ contains a coset of the group $A_x^2 := \{x^{-1}bx\phi(b)^{-1} : b \in A^0\}$. It remains to show that A_x^2 has finite index in G.

Let $\psi(g) = x\phi(g)x^{-1}$. Then by Proposition 1 we have $R(\psi) = R(\phi)$. This implies $R(\psi) < \infty$ and therefore by Lemma 5 that $R(\psi|_{A_0}) < \infty$. However by Lemma 2 we have $R(\psi|_{A_0}) = [A^0 : A_x^2]$. This finishes the proof.

3 A Useful Lemma

The following lemma is useful for calculating Reidemeister numbers. It will also be used in the proof that $R(\phi) = R(\mathbf{F}\phi)$.

Lemma 8 *Let $\phi : G \to G$ be any endomorphism of any group G, and let H be a subgroup of G with the properties*

$$\phi(H) \subset H$$

$$\forall x \in G\ \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in H.$$

Then

$$R(\phi) = R(\phi\,|_H),$$

where $\phi\,|_H : H \to H$ is the restriction of ϕ to H. If all the numbers $R(\phi^n)$ are finite then

$$R_\phi(z) = R_{\phi|_H}(z).$$

From this follows immediately:

Corollary 1 *Let $H = \phi^n(G)$. Then $R(\phi) = R(\phi\,|_H)$.*

Proof of Lemma 8. Let $x \in G$. Then there is an n such that $\phi^n(x) \in H$. From Lemma 1 it is known that x is ϕ-conjugate to $\phi^n(x)$. This means that the ϕ-conjugacy class $\{x\}_\phi$ of x has non-empty intersection with H.

Now suppose that $x, y \in H$ are ϕ-conjugate, ie. there is a $g \in G$ such that

$$gx = y\phi(g).$$

We shall show that x and y are $\phi|_H$-conjugate, ie. we can find a $g \in H$ with the above property. First let n be large enough that $\phi^n(g) \in H$. Then applying ϕ^n to the above equation we obtain

$$\phi^n(g)\phi^n(x) = \phi^n(y)\phi^{n+1}(g).$$

This shows that $\phi^n(x)$ and $\phi^n(y)$ are $\phi|_H$-conjugate. On the other hand, one knows by Lemma 1 that x and $\phi^n(x)$ are $\phi|_H$-conjugate, and y and $\phi^n(y)$ are $\phi|_H$ conjugate, so x and y must be $\phi|_H$-conjugate.

We have shown that the intersection with H of a ϕ-conjugacy class in G is a $\phi|_H$-conjugacy class in H. We therefore have a map

$$\begin{array}{rccc} Rest: & \mathcal{R}(\phi) & \to & \mathcal{R}(\phi|_H) \\ & \{x\}_\phi & \mapsto & \{x\}_\phi \cap H \end{array}$$

This clearly has the two-sided inverse

$$\{x\}_{\phi|_H} \mapsto \{x\}_\phi.$$

Therefore *Rest* is a bijection and $R(\phi) = R(\phi|_H)$.

4 Reduction to the case of Automorphisms

In this § we begin with a group endomorphism $\phi : G \to G$ and we construct a group $\mathbf{F}G$ and an automorphism $\mathbf{F}\phi : \mathbf{F}G \to \mathbf{F}G$ with the property

$$R(\mathbf{F}\phi) = R(\phi).$$

Our reduction will be in two steps. We begin by reducing to the case of injective endomorphisms. After that we reduce from injective endomorphisms to automorphisms.

4.1 Reduction to Injective Endomorphisms

Let G be a group and $\phi : G \to G$ an endomorphism. We shall call an element $x \in G$ *nilpotent* if there is an $n \in \mathbb{N}$ such that $\phi^n(x) = \mathrm{id}$. Let N be the set of all nilpotent elements of G.

Proposition 3 *The set N is a normal subgroup of G. We have $\phi(N) \subset N$ and $\phi^{-1}(N) = N$. Thus ϕ induces an endomorphism $[\phi/N]$ of the quotient group G/N given by.*

$$[\phi/N](xN) := \phi(x)N.$$

The endomorphism $[\phi/N] : G/N \to G/N$ is injective, and we have

$$R(\phi) = R([\phi/N]), \qquad S(\phi) = S([\phi/N]).$$

Proof. (i) Let $x \in N$, $g \in G$. Then for some $n \in \mathbb{N}$ we have $\phi^n(x) = \mathrm{id}$. Therefore $\phi^n(gxg^{-1}) = \phi^n(gg^{-1}) = \mathrm{id}$. This shows that $gxg^{-1} \in N$ so N is a normal subgroup of G.

(ii) Let $x \in N$ and choose n such that $\phi^n(x) = \mathrm{id}$. Then $\phi^{n-1}(\phi(x)) = \mathrm{id}$ so $\phi(x) \in N$. Therefore $\phi(N) \subset N$

(iii) If $\phi(x) \in N$ then there is an n such that $\phi^n(\phi(x)) = \mathrm{id}$. Therefore $\phi^{n+1}(x) = \mathrm{id}$ so $x \in N$. This shows that $\phi^{-1}(N) \subset N$. The converse inclusion follows from (ii).

(iv) We shall now show that the map $x \mapsto xN$ induces a bijection

$$\mathcal{R}(\phi) \mapsto \mathcal{R}([\phi/N]).$$

Suppose $x, y \in G$ are ϕ-conjugate. Then there is a $g \in G$ with

$$gx = y\phi(g).$$

Projecting to the quotient group G/N we have

$$gNxN = yN\phi(g)N,$$

so

$$gNxN = yN[\phi/N](gN).$$

This means that xN and yN are $[\phi/N]$-conjugate in G/N. Conversely suppose that xN and yN are $[\phi/N]$-conjugate in G/N. Then there is a $gN \in G/N$ such that

$$gNxN = yN[\phi/N](gN).$$

In other words

$$gx\phi(g)^{-1}y^{-1} \in N.$$

From this it follows that there is an $n \in \mathbb{N}$ with

$$\phi^n(gx\phi(g)^{-1}y^{-1}) = \mathrm{id}.$$

Therefore

$$\phi^n(g)\phi^n(x) = \phi^n(y)\phi^n(\phi(g)).$$

This shows that $\phi^n(x)$ and $\phi^n(y)$ are ϕ-conjugate. However by Lemma 1 x and $\phi^n(x)$ are ϕ-conjugate, as are y and $\phi^n(y)$. Therefore x and y are ϕ-conjugate.

(v) We have shown that x and y are ϕ-conjugate iff xN and yN are $[\phi/N]$-conjugate. From this it follows that $x \mapsto xN$ induces a bijection from $\mathcal{R}(\phi)$ to $\mathcal{R}([\phi/N])$. Therefore $R(\phi) = R([\phi/N])$.

(vi) We shall now show that $S(\phi) = S([\phi/N])$. Let $\rho \in \mathcal{S}(\phi)$ and let M be a matrix for which

$$\rho \circ \phi = M \cdot \rho \cdot M^{-1}.$$

If $x \in N$ then there is an $n \in \mathbb{N}$ with $\phi^n(x) = \mathrm{id}$. Therefore

$$M^n \cdot \rho(x) \cdot M^{-n} = \rho(\phi^n(x)) = \mathrm{id},$$

which implies that $\rho(x) = \mathrm{id}$. Thus N is contained in the kernel of ρ and there is a representation $[\rho/N]$ of G/N given by

$$[\rho/N](gN) := \rho(g).$$

Since $[\rho/N]$ satisfies the identity

$$[\rho/N] \circ [\phi/N] = M \cdot [\rho/N] \cdot M^{-1},$$

we have $[\rho/N] \in \mathcal{S}([\phi/N])$.

(vii) Conversely if $\rho \in \mathcal{S}([\phi/N])$ then we may construct a $\overline{\rho} \in \mathcal{S}(\phi)$ by

$$\overline{\rho}(x) := \rho(xN).$$

It is clear that

$$\overline{[\rho/N]} = \rho \text{ and } \overline{\rho}/N = \rho$$

so it follows that $S(\phi) = S([\phi/N])$.

4.2 Reduction of Injective Endomorphisms to Automorphisms

Now suppose that $\phi : G \to G$ is any injective endomorphism of an almost Abelian group G. Consider the directed system

$$G_0 \xrightarrow{\phi} G_1 \xrightarrow{\phi} G_2 \xrightarrow{\phi} G_3 \xrightarrow{\phi} \cdots,$$

where each G_i is a copy of the group G. We may form the limit of this system

$$\overline{G} := \varinjlim G_i.$$

This is the union of the sets G_i in which we identify the element $x \in G_i$ with the element $\phi^n(x)$ in G_{i+n}. We now give $\overline{G}$ a group law. If $x, y \in \overline{G}$ then both x and y are represented by elements x_i, y_i in G_i for sufficiently large i. We define the product $xy \in \overline{G}$ to be the element of $\overline{G}$ represented by the element $x_i y_i$ of G_i. The group axioms are trivial to check.

By identifying G with G_0, we can think of G as being a subgroup of $\overline{G}$.

We now extend the map ϕ to an endomorphism of $\overline{G}$. For any element $\overline{x} \in \overline{G}$ there is a representative $x_i \in G_i$ of x in G_i for some i. We may define $\overline{\phi}(\overline{x})$ to be the element represented by $\phi(x_i)$ in G_i (NOT in G_{i+1}, otherwise $\overline{\phi}$ would be the identity map). This definition is independent of i.

Theorem 6 *In the notation introduced above, $\overline{\phi}$ is an automorphism of the group $\overline{G}$ and $R(\phi) = R(\overline{\phi})$.*

Proof. (i) Let $x \in \overline{G}$ be in the kernel of $\overline{\phi}$. The element x is represented by some $x_i \in G_i$. Since $\overline{\phi}(\overline{x}) = \mathrm{id}$ we know that $\phi(x_i) \in G_i$ is equivalent to $\mathrm{id} \in G_0$. From this it follows (since ϕ is injective) that $\phi^i(\mathrm{id}) = \phi(x_i)$ in G. Clearly this means that $x_i = \mathrm{id}$ in G_i. Therefore $x = \mathrm{id}$ in $\overline{G}$, so $\overline{\phi}$ is injective.

(ii) Let $x \in \overline{G}$ be represented by some $x_i \in G_i$. Let y be the element of $\overline{G}$ represented by x_i in G_{i+1}. Then $\overline{\phi}(y)$ is represented by $\phi(x_i)$ in G_{i+1}, which in turn is equivalent to $x_i \in G_i$. Therefore $\overline{\phi}(y) = x$, so $\overline{\phi}$ is surjective.

(iii) Let $x \in \overline{G}$ be represented by some $x_i \in G_i$. Let y be the element of $\overline{G}$ represented by x_i in G_0. Then $\overline{\phi}^i(x)$ is represented by $\phi^i(x_i) \in G_i$, which is equivalent to $x_i \in G_0$. Therefore for every element x of $\overline{G}$ there is an $i \in \mathbb{N}$ such that $\overline{\phi}^i(x) \in G$. In addition we have $\overline{\phi}(G) \subset G$. From this it follows by Lemma 8 that $R(\phi) = R(\overline{\phi})$.

References

[1] A.L.Felshtyn and R.Hill : Dynamical Zeta-Functions, Nielsen Theory and Reidemeister Torsion

[2] B. Jiang, Nielsen Fixed Point Theory, Contemp. Math. 14, Birkh"auser, 1983.

[3] Kirillov, Representation Theory, Springer Verlag.

[4] S.Lang, Algebra.

[5] W. Rudin : Fourier Analysis on Groups, Interscience tracts in pure and applied mathematics number 12, 1962.