

Notes on "Mother Body" Problem in Geographics

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Abstract

The simple definition of the mother body notion is proposed for the two-dimensional heavy body. In generic position, the local structure of mother body near singular points of continuation of the potential is investigated. The finite algorithm of constructing mother body in the sense of the introduced definition is also given.

Introduction

The problem of constructing families of graviequivalent bodies is a classical geophysical problem strongly connected with the inverse problem of geoprospecting. This problem was investigated by a lot of mathematicians and geophysicists (see P.S. Novikov [1], L. N. Sretenskii [2], D. Zidarov [3], [4], V. N. Strakhov and V.G.Filatov [5], A. V. Tsirulskii [6], O. I. Kounchev [7, 8], B. Gustafsson [9], B. Gustafsson and M. Sakai [10], and others).

In short words, the problem is as follows. Suppose that the external gravitational field generated by some body (below we use the term “heavy body”) is known. One

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wants to find the shape of this body and the mass distribution in it. Such a problem arises, for example, in the geoprospecting if one measures the gravitational field on the Earth surface and wants to find out what mass distribution (inside the Earth) can generate such a field. Clearly, the solution of this problem occurs not to be unique. Actually, from the Newton's times it is known that a sphere filled uniformly by masses generates the same gravitational field as the point mass of the same magnitude placed in the center of the sphere. So, there exist different heavy bodies producing one and the same gravitational field. Such heavy bodies are called *graviequivalent* ones, and one can consider the family of graviequivalent bodies generating one and the same gravitational field. Using the Poincaré sweeping method one can show that each family of graviequivalent bodies contains an infinite number of elements. Clearly, this is a good idea to find a minimal (in some sense) heavy body in each graviequivalent family, similar as a point mass is a minimal body in the family of concentric balls. Such an attempt was first done in 1968 by D. Zidarov [3], who named this minimal element "*mother body*" of the family in question. This notion was in a great extent heuristic. Later on, some authors ([7], [8], [9], [11], [12]) have done the attempts to give a more or less rigorous definition of the mother body notion. The most acceptable description of mother body notion is given by B. Gustafsson (see [9]).

It seems, however, that there exists a more simple and geometrically clear definition of the mother body. In this paper we present such a definition for the two-dimensional case. We give also a finite algorithm of constructing a mother body in the sense of the given definition.

To conclude this Introduction, we remark that the problem of constructing a mother body is not always solvable, and the solution is not always unique. For example, the unit disc D in the two-dimensional plane $\mathbf{R}_{(x,y)}^2$ filled with the mass with the density $f(x, y) = \exp(x)$ does not determine any mother body at all. This is a consequence of the fact that the continuation of the field generated by such a body inside the unit disc has an essential singularity at the origin. (Connection between the notion of mother body and the singularity set of continuation of the corresponding gravitational field will be explained later.) The non-uniqueness of the construction of mother body can be illustrated by the well-known Zidarov's example (see below). The algorithm of constructing a mother body for any heavy body, proposed in the paper, allows one, in particular, to find out whether the searched mother body exists or not. Besides, we give the investigation of the local structure of the mother body, at least in the generic position.

Acknowledgements. We are grateful to Harold Shapiro, who attracted our attention to this problem and to Björn Gustafsson who had acquainted us with the preprint of his interesting paper [9].

1 Mother body and continuation of the gravitational field

Consider the heavy body concentrated in the domain $\Omega \subset \mathbf{R}_{(x,y)}^2$ and having the mass density $f(x, y) \geq 0$. Suppose that the function $f(x, y)$ can be continued up to an entire function in the complex plane $\mathbf{C}_{(x,y)}^2$, and the domain Ω has an algebraic boundary, that is,

$$\Gamma = \partial\Omega = \{(x, y) | P(x, y) = 0\} \quad (1)$$

with some polynomial $P(x, y)$ with real coefficients.

The definition of the mother body used in this paper is as follows:

Definition 1 *A mother body for the given heavy body Ω is the body concentrated on the finite system of curve segments or/and points with positive integrable mass density, such that the support of this body does not bound a two-dimensional subdomain of Ω .*

Suppose that M is a mother body (in the sense of the above definition) corresponding to Ω . Then M consists of a finite set of segments of curves and, possibly, points. It is clear that the set of curves included into M forms a planar graph which is a tree, and, therefore, has "hanged" vertexes (vertexes of degree 1).

Later on, since the body M generates the same field as Ω does, one can see that the potential $V(x, y)$ generated by Ω can be continued into the complement \overline{M} of the set M in $\mathbf{R}_{(x,y)}^2$ as a harmonic function. As we shall see below, the harmonic continuation of the external field of the heavy body inside the domain occupied initially by masses is a ramifying function. So, the mother body M can be considered as the set of cuts selecting a univalued branch of (ramifying) continuation \tilde{V} of the potential V . (During this analysis we neglect the point components of the mother body M since they are simply univalued singularities of the continuation. These singularities must be at most of the logarithmic type.)

Further, one can notice that any hanged vertex of the set M must be a singular point of the continuation \tilde{V} . Actually, if some hanged vertex $(x_0, y_0) \in M$ is a regular point of the continuation \tilde{V} , then the line mass density on the curve originating from (x_0, y_0) vanishes identically (we recall that the mass density on the curve equals the jump of the normal derivative $\partial\tilde{V}/\partial n$ of the potential \tilde{V}).

Finally, each cut included into the mother body M has to satisfy the following three conditions:

1. This cut must be *admissible* in the sense that the limit values of the potential from both sides of the cut must coincide with each other. This follows from the absence of gravitational dipoles.

2. This cut must be *positive* in the sense that the sum of normal derivatives of the potential on the two sides of the cut (in the directions of the corresponding inner normals) must be a positive function. This is clear since this sum equals the line mass density which must be positive from the physical reasons.

3. All the cuts must be *contained in the domain* Ω .

Of course, as it was mentioned above, for the mother body to exist, one should require in addition that all singularities of continuation \tilde{V} of the potential V have not more than logarithmic growth. We shall suppose in the sequel that this latter requirement is fulfilled for all problems in question.

Thus, we can give another definition of the notion of a mother body which is equivalent to Definition 1 above.

Definition 2 For given heavy body Ω (defined as a pair (Ω, f) , where f is a positive function in Ω) satisfying the above requirements, the union of the set of singularities of continuation \tilde{V} of the potential V and a system of cuts subject to conditions 1 – 3 above will be called a *mother body* for Ω .

With this definition in hand, we shall describe the algorithm of constructing a mother body for any given body Ω (see Section 3 below). To do this, we need more detailed information about the continuation \tilde{V} of the potential V generated by the body Ω . In the rest part of this section and in the next one we shall obtain the required information.

We recall that the potential V generated by the heavy body Ω is defined as a solution to the Laplace equation

$$\Delta V = -f$$

and can be computed with the help of the formula

$$V(x, y) = -\frac{1}{2\pi} \int_{\Omega} \ln \sqrt{(x - x_0)^2 + (y - y_0)^2} f(x_0, y_0) dx_0 \wedge dy_0 \quad (2)$$

(we suppose that the domain Ω is oriented with the positive orientation of the space $\mathbf{R}_{(x,y)}^2$).

To describe the required continuation in more detail, it is useful to complexify the problem, that is, to consider the function $V(x, y)$ in the complex space $\mathbf{C}_{(x,y)}^2$. The variables (x_0, y_0) can also supposed to be complex ones; in this case integral (2) must be considered as an integral over a chain in the complex space $\mathbf{C}_{(x_0,y_0)}^2$ determined by the real domain Ω . After such a complexification, we perform the variable change

$$z = x + iy, \quad \zeta = x - iy \quad (3)$$

in integral (2). As a result we obtain¹

$$V(z, \zeta) = \frac{1}{8\pi i} \int_{\Omega} [\ln(z - z_0) + \ln(\zeta - \zeta_0)] f(z_0, \zeta_0) dz_0 \wedge d\zeta_0, \quad (4)$$

where the branches of the logarithms are chosen in such a way that formulas (2) and (4) coincide with each other. We remark that the variables (z, ζ) are characteristic variables for the Laplace operator in $\mathbf{C}_{(x,y)}^2$.

Clearly, integral (4) hardly can be computed for an arbitrary function $f(z_0, \zeta_0)$. However, all required information can be obtained from singular parts of the derivatives $\partial V/\partial z$ and $\partial V/\partial \zeta$. These singular parts can be computed in terms of the so-called Schwarz function (see [13], [14], [15]). We recall that the *Schwarz function* $S(z)$ is defined as a solution of the equation

$$P(z, \zeta) = 0 \quad (5)$$

with respect to ζ , where (5) is a complexified equation of the boundary Γ of the domain Ω in the characteristic variables (z, ζ) (see equation (1) above). We need also a *conjugate Schwarz function* $\tilde{S}(\zeta)$ which is defined as a solution to (5) with respect to ζ . Since the polynomial P has real coefficients in the variables (x, y) , one can easily verify that the functions $S(z)$ and $\tilde{S}(\zeta)$ are connected with each other with the help of the relation

$$\tilde{S}(\zeta) = \overline{S(\bar{\zeta})}.$$

Differentiating integral (4) with respect to z , we obtain

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i} \int_{\Omega} \frac{f(z_0, \zeta_0) dz_0 \wedge d\zeta_0}{z - z_0} = \frac{1}{8\pi i} \int_{\Omega} d \left\{ \frac{F_1(z_0, \zeta_0) dz_0}{z - z_0} \right\}, \quad (6)$$

where $F_1(z_0, \zeta_0)$ is any function satisfying the following relation:

$$\frac{\partial F_1(z_0, \zeta_0)}{\partial \zeta_0} = f(z_0, \zeta_0).$$

Using the Stokes formula, we arrive at the relation

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i} \int_{\Gamma} \frac{F_1(z_0, \zeta_0) dz_0}{z - z_0}, \quad (7)$$

¹From now on we do not distinguish the function $V(x, y)$ and its continuation $\tilde{V}(x, y)$, as well as functions of (x, y) and functions of (z, ζ) .

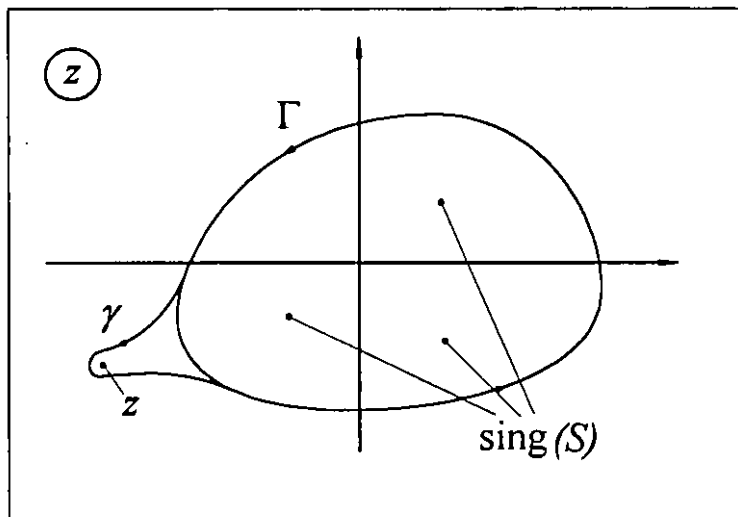


Figure 1: Changing integration contour.

where Γ is considered as an one-dimensional homology class in the complex-analytic surface $\Gamma_{\mathbb{C}}$ defined by (5). Deriving ζ from (5) and substituting it into integral (7), we have

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i} \int_{\Gamma} \frac{F_1(z_0, S(z_0)) dz_0}{z - z_0}, \quad (8)$$

where Γ is again the contour in the complex plane \mathbb{C}_{x_0} coinciding with the boundary of the domain Ω (here we identify the plane \mathbb{C}_{x_0} with the real plane $\mathbb{R}_{(x,y)}^2$ with the help of the relation $z = x + iy$).

We emphasize that the latter relation was obtained for values of z lying in the complement of the domain Ω in the complex plane \mathbb{C}_{x_0} (or, what is the same, in the real plane $\mathbb{R}_{(x,y)}^2$). To obtain the continuation of this function inside Ω , we change this contour to the contour γ encircling both the domain Ω and the point z but containing no singular points of the function $S(z_0)$ lying outside Ω (see Figure 1). The residue theorem shows that

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i} \int_{\gamma} \frac{F_1(z_0, S(z_0)) dz_0}{z - z_0} + \frac{1}{4} F_1(z, S(z)) = \frac{1}{4} F_1(z, S(z)) + \Phi_1(z),$$

where $\Phi_1(z)$ is holomorphic inside the domain Ω . The latter expression describes all singularities of continuation of the function $\partial V/\partial z$ inside the domain Ω .

Similar, one can derive the expression of the derivative $\partial V/\partial \zeta$:

$$\frac{\partial V}{\partial \zeta}(z, \zeta) = \frac{1}{4} F_2(\tilde{S}(\zeta), \zeta) + \Phi_2(\zeta)$$

with $\Phi_2(\zeta)$ regular in Ω , where $F_2(z, \zeta)$ is any function satisfying the relation

$$\frac{\partial F_2}{\partial z}(z, \zeta) = f(z, \zeta). \quad (9)$$

In what follows it will be convenient for us to fix the choice of the function $F_2(z, \zeta)$ in the following way:

$$F_2(z, \zeta) = \overline{F_1(\bar{\zeta}, \bar{z})}$$

(the reader can verify that this function satisfies (9) for any function f which is real for real values of (x, y)). Now, up to functions which are regular inside Ω , one has

$$V(z, \bar{z}) = \frac{1}{4} \int F_1(z, S(z)) dz + \frac{1}{4} \int F_2(\tilde{S}(\zeta), \zeta) d\zeta \Big|_{\zeta=\bar{z}} = \frac{1}{4} \operatorname{Re} \int F_1(z, S(z)) dz.$$

As one shall see from the above considerations, the function $\partial V/\partial z$ plays the crucial role in the investigation of singularities of continuation of the potential V inside the domain Ω . This function is known in the Russian geophysical literature as complex vector of gravitational field (see, e. g. [6]). One can see also that the location of singularities of continuation of the potential is uniquely determined by the Schwarz function $S(z)$, that is, by the equation of the boundary of the domain Ω . On the contrary, we shall see below, the geometry of admissible cuts strongly depends on the mass density $f(x, y)$.

2 Investigation of admissible cuts and the local structure of mother bodies

Let us turn our mind to the investigation of admissible cuts. As it was already told in the previous section, if we consider a cut as a part of the system of cuts which determines a univalued branch of the function V , then this cut is admissible if and only if the limit values of this branch from both sides of the cut coincide with each other. It is easy to see that in this case the admissible cut is none more than the set of zeros of the variation of the potential V along some element of the fundamental group

$$\pi_1(\Omega \setminus \operatorname{sing}(S), z^0)$$

of the domain $\Omega \setminus \operatorname{sing}(S)$ with a base point z^0 lying on the curve Γ (see Figure 2, where the loop determining the above mentioned element of the fundamental group is denoted by l). This leads us to the following definition:

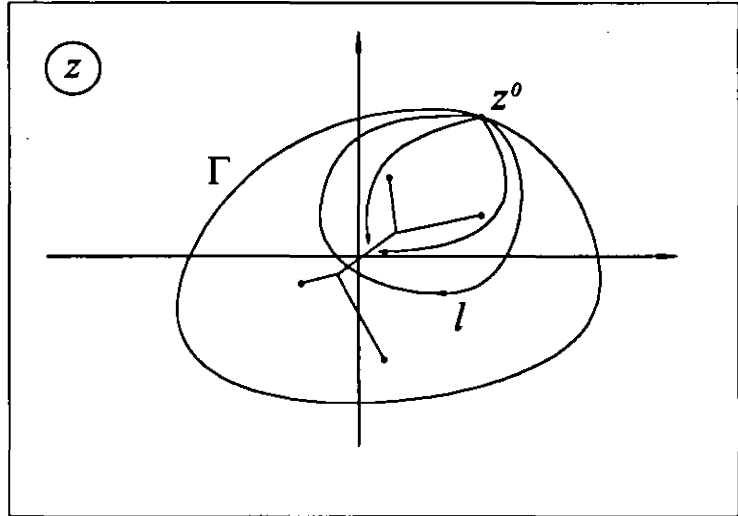


Figure 2: Definition of the element of fundamental group.

Definition 3 A cut c is called *admissible* with respect to the element

$$l \in \pi_1(\Omega \setminus \text{sing}(S), z^0)$$

if and only if $\text{var}_l V|_c = 0$.

Remark 1 All the considerations in this section will be carried out on the real plane $\mathbb{R}_{(z, \bar{z})}^2$. We shall use here the characteristic coordinates (z, ζ) , so that the equation of the real space is $\zeta = \bar{z}$.

Using the results obtained in the previous section, we can describe the set of admissible cuts as the set of integral curves of a certain vector field in the plane \mathbb{C}_z . To do this, we represent any vector tangent to the real space as a complex number c . Then, in terms of the variables (z, ζ) the derivative D_c of the function $V(z, \zeta)$ along the vector c is given by

$$D_c V(z, \bar{z}) = c \frac{\partial V}{\partial z}(z, \bar{z}) + \bar{c} \frac{\partial V}{\partial \zeta}(z, \bar{z}) \equiv \frac{1}{4} \left[c F_1(z, S(z)) + \bar{c} F_2(\tilde{S}(\bar{z}), \bar{z}) \right]$$

modulo regular in the domain Ω functions. Taking into account the connection between the functions F_1 and F_2 , we rewrite the latter formula in the following way:

$$D_c V(z, \bar{z}) \equiv \frac{1}{2} \text{Re} [c F_1(z, S(z))]. \quad (10)$$

Let l be, as above, some element of the fundamental group $\pi_1(\Omega \setminus \text{sing}(S), z_0)$. Then, taking the variation of the right- and left-hand sides of the latter relation we arrive at the equality

$$D_c \text{var}_l V(z, \bar{z}) = \frac{1}{2} \text{Re} [c \text{var}_l F_1(z, S(z))]$$

(regular terms vanish under the action of the operator var_l). Equating to zero the right-hand part of the last relation, we obtain the equation of the vector field c :

$$\text{Re} [c \text{var}_l F_1(z, S(z))] = 0. \quad (11)$$

Integral curves of this vector field are exactly the lines on which the function $V(z, \bar{z})$ is a constant. So, any admissible cut is an integral curve of the above constructed vector field c . We remark also that singular points of the vector field c are singular points of the Schwarz function $S(z)$ and points for which the function $F_1(z, S(z))$ vanishes.

Now we must derive the “initial conditions” for integral curves of the vector field c , so that they are admissible cuts (that is, zero levels of the function $F_1(z, S(z))$). Since admissible cuts can have singular points of the Schwarz function $S(z)$ as their origins, to do this it is necessary to investigate local structure of admissible cuts in a neighborhoods of these singular points. We shall carry out such an investigation in the generic position. In other words, we suppose that the singularity z_0 in question is brought from the finite regular characteristic point of the manifold $\Gamma_{\mathbb{C}}$ and that the tangency between this singular point and the corresponding characteristic ray is quadratic. Under such requirements the function $S(z)$ has at the point z_0 the singularity of the square root type:

$$S(z) = \sqrt{z - z_0} S_1(z) + S_2(z), \quad (12)$$

where $S_1(z)$ and $S_2(z)$ are regular in a neighborhood of the point z_0 .

Denote by $\zeta_0 = S(z_0) = S_2(z_0)$ the value of the Schwarz function at the singular point z_0 and expand the function $F_1(z, \zeta)$ into the Taylor series at point (z_0, ζ_0) :

$$F_1(z, \zeta) = \sum_{j \geq 0, k \geq 0} b_{jk} (z - z_0)^j (\zeta - \zeta_0)^k. \quad (13)$$

Substituting (12) into (13) and expanding the functions $S_1(z)$ and $S_2(z)$ into the Taylor series at the point z_0 , we obtain up to terms regular near z_0 :

$$F_1(z, S(z)) \equiv \sqrt{z - z_0} \sum_{j=0}^{\infty} c_j (z - z_0)^j$$

up to functions regular near z_0 . Using the relation $\partial F_1(z, \zeta) / \partial \zeta = f(z, \zeta)$, one can easily verify that

$$c_0 = f(z_0, \zeta_0) S_1(z_0).$$

We remark that (z_0, ζ_0) is the characteristic point of the surface $\Gamma_{\mathbf{C}}$ generating the singularity z_0 of the Schwarz function $S(z)$, so that the singularity of the function $\partial V/\partial z$ (as well as $\partial V/\partial \zeta$) is determined by values of the right-hand part f at characteristic points of $\Gamma_{\mathbf{C}}$. As we shall see below, the number c_0 determines the behavior of admissible cuts in a neighborhood of the point z_0 . Hence, we see, that the set of admissible cuts depends on the mass density $f(x, y)$ (unlike the set of singularities of the potential $V(x, y)$ which is determined only by the geometry of the domain Ω).

Now we have

$$V(z, \bar{z}) = \frac{1}{2} \operatorname{Re} \int F_1(z, S(z)) dz \equiv \frac{1}{2} \operatorname{Re} \left[\frac{2}{3} c_0 (z - z_0)^{3/2} (1 + \psi(z)) \right],$$

where $\psi(z)$ is a regular near z_0 function vanishing at this point. Taking the variation of both sides of the latter relation, we arrive at the formula

$$\operatorname{var}_l V(z, \bar{z}) = \frac{1}{2} \operatorname{var}_l \operatorname{Re} \left[\frac{2}{3} c_0 (z - z_0)^{3/2} (1 + \psi(z)) \right].$$

Let us introduce the polar coordinates in a neighborhood of the point z_0 :

$$z = z_0 + \rho e^{i\varphi}.$$

Then the equation of admissible cuts near this point reads

$$\operatorname{var}_l V(z, \bar{z}) = \frac{2R}{3} \rho^{3/2} \operatorname{Re} \left[e^{(\frac{3i\varphi}{2} + i\theta)} (1 + \psi(z_0 + \rho e^{i\varphi})) \right] = 0,$$

where R and θ are determined by the relation $c_0 = R e^{i\theta}$, or

$$\cos \left(\frac{3\varphi}{2} + \theta \right) \operatorname{Re} (1 + \psi(z_0 + \rho e^{i\varphi})) - \sin \left(\frac{3\varphi}{2} + \theta \right) \operatorname{Im} (1 + \psi(z_0 + \rho e^{i\varphi})) = 0. \quad (14)$$

Let us consider this equation as an equation with respect to φ for small values of ρ . Then, since the function $\psi(z_0 + \rho e^{i\varphi})$ is of order $O(\rho)$, the principal term of this equation gives

$$\cos \left(\frac{3\varphi}{2} + \theta \right) = 0, \text{ or } \varphi = \varphi_k = \frac{\pi}{3} - \frac{2\theta}{3} + \frac{2\pi k}{3}, \quad k = 0, 1, 2. \quad (15)$$

Since at each point φ_k the derivative with respect to φ of the left-hand side of equation (14) does not vanish for $\rho = 0$, this equation has the unique smooth solution $\varphi = \varphi_k(\rho)$ near $\rho = 0$ such that $\varphi_k(0) = \varphi_k$. Each of these solutions determine an admissible (with respect to a small loop l encircling the point z_0) cut near the point z_0 . So, the initial condition for an admissible cut near singular points is

$$\lim_{\rho \rightarrow 0} \varphi(\rho) = \varphi_k,$$

where φ_k are given by formula (15). The above considerations can be summarized in the form of the theorem.

Theorem 1 *Each admissible cut is an integral curve of the vector field c determined by relation (11). Moreover, under the generic position requirement, in a neighborhood of each point of ramification of the Schwarz function there exist three directions of admissible cuts given by relation (15). These cuts are positive or negative depending of the sign of the derivative with respect to φ of the left-hand side of relation (14) at the point ($\varphi = \varphi_k, \rho = 0$).*

3 Algorithm of constructing a mother bodies

At this point, we have the sufficient information to formulate the algorithm of constructing a mother body for a given heavy body (Ω, f) . We remind that the following conditions must be valid:

1. The boundary Γ of the domain Ω must be an algebraic surface.
2. The function $f(x, y)$ must admit the continuation up to an entire function in the complex space $\mathbb{C}_{(x,y)}^2$.
3. The singularities of the continuation of the potential inside the domain Ω are not more than of logarithmic type.

The last requirement can be verified with the help of the formulas for singular parts of the continuation of the potential obtained in Section 1.

The algorithm in question goes through the following four steps:

1. Determination of the set of singularities of the Schwarz function of the domain Ω lying inside this domain (that is, of the set of singularities of the continuation of the potential $V(x, y)$ into the domain initially occupied by the mass distribution). This step requires only solving algebraic equations describing the boundary in terms of the characteristic variables (z, ζ) .

2. Constructing the set of cuts which are admissible for each simple loop with base point z^0 surrounding one or more points of ramification of the Schwarz function lying inside Ω . To carry out this step one has to compute integral curves of the above constructed vector field c . The initial conditions for this integral curves are defined as follows:

For a loop l encircling a single point of singularity, (e. g., for the loops l_1 or l_2 on Figure 3) the initial conditions are given by the local structure of the set of admissible cuts in a neighborhood of this point.

For a loop l encircling two points of singularity, (see loop l_3 on Figure 3) the initial condition for the set of admissible cuts is given by the intersection of admissible cuts

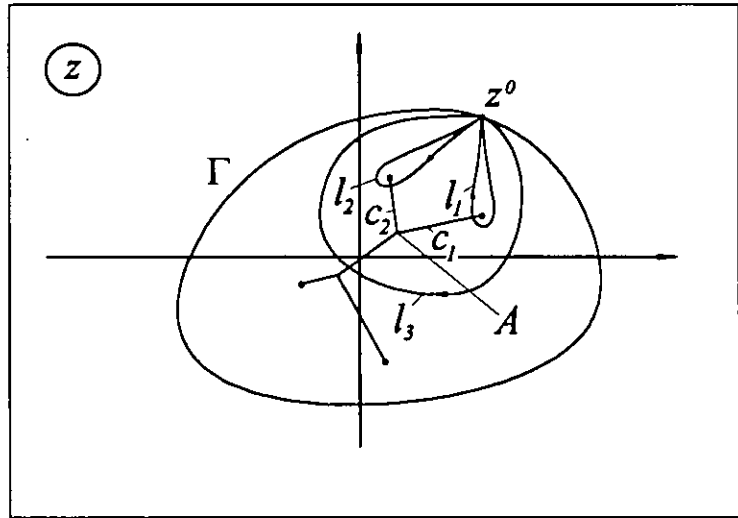


Figure 3: Non-elementary loop.

for loops l_1 and l_2 corresponding to each of these points (on Figure 3 one of admissible cut for the loop l_3 comes through the point A of intersection of the admissible cuts c_1 and c_2 corresponding to the loops l_1 and l_2).

For admissible cuts corresponding to a loop l encircling n points of singularity, initial points can be found as an intersections of admissible cuts corresponding to a loop l_1 encircling the first $n - 1$ of these points and of that corresponding to the loop l_2 corresponding to the last of them.

It is clear that the described process will be completed in the finite number of steps. Certainly, if there exist points of singularity of the vector field c different from points of singularities of the Schwarz function (that is, points of singularity of c determined by zeros of the function $F_1(z, S(z))$; see relation (11)), then one have to investigate the local structure of the vector field c near such point. This can be done with the help of the definition (11) of the field c .

3. Constructing the set of finite admissible cuts corresponding to simple loops with base point z_0 surrounding one or more points of ramification of the Schwarz function lying inside Ω . This step can be fulfilled simply by deleting those edges from the graph constructed on the previous step which intersect the complement of the domain Ω in the plane $\mathbf{R}_{(x,y)}^2$.

4. Constructing a mother body for (Ω, f) . This can be done by examining of each maximal tree of the above constructed graph (or of each maximal forest, if this graph is a non-connected one). If the corresponding maximal tree involves only positive cuts,

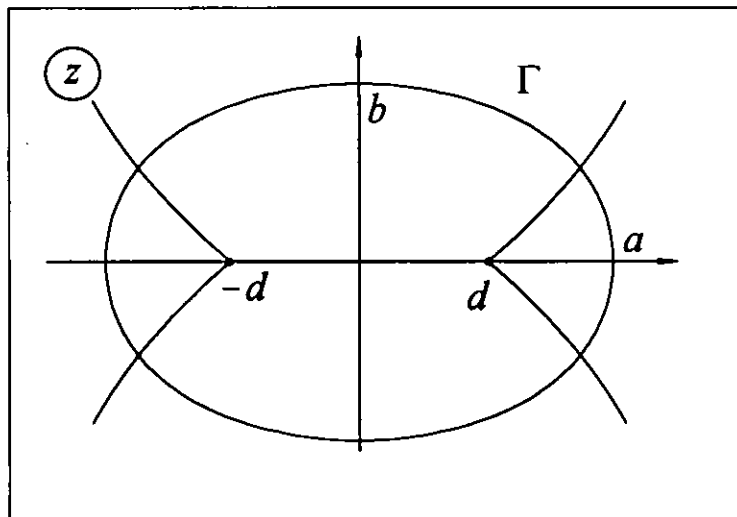


Figure 4: Admissible cuts for an ellipse.

then it determines a mother body for (Ω, f) .

In the next section we shall illustrate the work of the described algorithm on different examples.

4 Examples

To be short, in all the examples below we suppose that the mass density of the heavy body in question equals 1 identically. Thus, all mother bodies below are uniquely determined by the geometry of the domain Ω .

4.1 An ellipse

Let

$$\Omega = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

be an ellipse with the half-axes a and b , $a > b$ (see Figure 4). Then it is easy to compute that the Schwarz function is given by

$$S(z) = \frac{a^2 + b^2}{d^2} + \frac{2ab}{d^2} \sqrt{z^2 - d^2},$$

where $d = \sqrt{a^2 - b^2}$ is the half of the interfocal distance. In this simple case the complex field vector $\partial V / \partial z$ can be computed in the explicit way with the help of

formula (8):

$$\frac{\partial V}{\partial z}(z, \zeta) = \frac{1}{8\pi i} \int_{\Gamma} \frac{S(z_0) dz_0}{z - z_0} = \frac{ab}{2d^2} (\sqrt{z^2 - d^2} - z).$$

Similar,

$$\frac{\partial V}{\partial \zeta}(z, \zeta) = \frac{ab}{2d^2} (\sqrt{\zeta^2 - d^2} - \zeta).$$

This allows us to compute the potential V up to the constant term:

$$V(z, \zeta) = \frac{ab}{d^2} \operatorname{Re} \left\{ z\sqrt{z^2 - d^2} - d^2 \ln \left(z + \sqrt{z^2 - d^2} \right) - z^2 \right\} + \text{const.}$$

Since the function

$$\operatorname{var}_l \frac{\partial V}{\partial z}(z, \zeta) = \frac{ab}{d^2} \sqrt{z^2 - d^2}$$

does not vanish at any point except for the singular points $\pm d$ of the Schwarz function, the vector field c determining the set of admissible cuts has singularities only at these points, too. Let us determine the directions of admissible cuts originating, say, from the point $z = d$. Using formula (15) above, we obtain

$$\varphi_k = \frac{\pi}{3} + \frac{2\pi k}{3}, \quad k = 0, 1, 2,$$

since $c_0 = 2ab/d^2$ and, hence, $\theta = 0$. Similar, the directions of admissible cuts at $z = -d$ are

$$\varphi_k = -\frac{\pi}{3} + \frac{2\pi k}{3}, \quad k = 0, 1, 2.$$

From the symmetry reasons, it is clear that one of the admissible cuts is an interfocal segment; the rest four cuts come to infinity in the plane $\mathbf{R}_{(x,y)}^2$. Hence, the picture of the admissible cuts is such as it is drawn on Figure 4. So, the finite graph mentioned in the point 3 of the above formulated algorithm contains only the interfocal segment, and the only thing rest is to verify the positivity of this cut. The simple calculations lead us to the following expression of the mass density for the mother body consisting of the interfocal segment:

$$\sigma = \frac{ab}{d^2} \sqrt{d^2 - x^2}.$$

Since this expression is a positive one, *the interfocal segment is the only mother body of the ellipse uniformly filled by masses.*

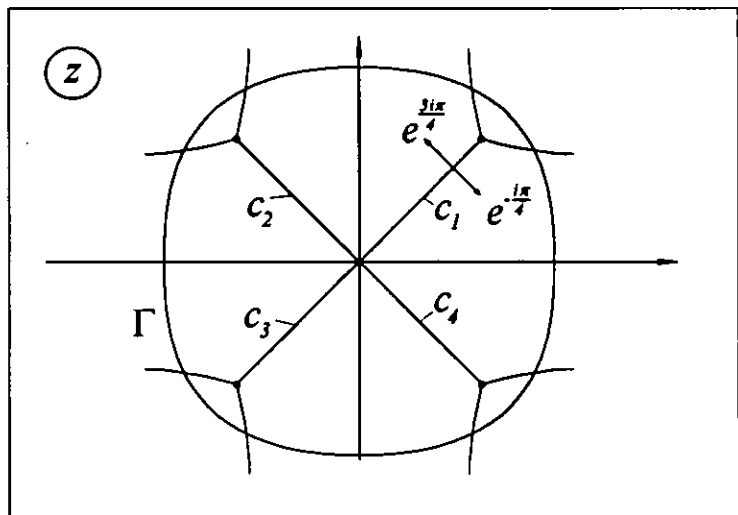


Figure 5: Admissible cuts for $x^4 + y^4 = 1$.

4.2 A curve of the fourth order

Here we consider the heavy body $(\Omega, 1)$ with the domain Ω given by

$$x^4 + y^4 \leq 1$$

(see Figure 5). The corresponding Schwarz function is

$$S(z) = \sqrt{-3z^2 + 2\sqrt{2}\sqrt{z^4 + 1}}. \quad (16)$$

This function have singularities at eight points

$$z = e^{i(\pi/4 + \pi k/2)}, \quad k = 0, 1, 2, 3,$$

$$z = 2\sqrt{2}e^{i\pi k/2}, \quad k = 0, 1, 2, 3.$$

The last four of these points lye in the complement of the domain Ω and, hence, are not of interest for us. The four points of singularity lying inside Ω are marked on Figure 5.

Similar to the previous example, the vector field c determining the set of admissible cuts does not vanish at any point except for the points of singularity of the Schwarz function. Let us investigate the local structure of admissible cuts in a neighborhood of each point of singularity in question. For the point $z = e^{i\pi/4}$ one has, due to equation (15), the expression for the directions of the admissible cuts:

$$\varphi_k = \frac{7\pi}{12} + \frac{2\pi k}{3}, \quad k = 0, 1, 2,$$

since the argument θ of the number c_0 for this case equals $-3\pi/8$ (we leave to the reader the verification of the last equality). Thus, the picture of admissible cuts is such as it is drawn on Figure 5. The fact that the straight line segment connecting points $e^{i\pi/4}$ and $e^{5i\pi/4}$, as well as the segment connecting $e^{-i\pi/4}$ and $e^{3i\pi/4}$, is an admissible cut follows from the symmetry reasons. The rest of the cuts are not contained in the domain Ω and, hence, are of no interest for us. So, the graph mentioned in the point 3 of the above algorithm, consists of the two above mentioned straight line segments. This graph is a tree, and the only thing rest is to verify the positivity of all the cuts involved in this graph.

The verification of this last assertion is a little bit more complicated task than it was in the previous example, and we shall point out the main steps of this verification. Clearly, from the symmetry reasons it is sufficient to carry out this verification for one of the four singular points of $S(z)$, lying inside Ω , say, for the point $z = e^{i\pi/4}$.

The expression for the sum of normal derivatives of the potential V from both sides of the cut c_1 (see Figure 5) equals

$$D_{e^{-i\pi/4}} [V^-] + D_{e^{3i\pi/4}} [V^+],$$

where V^+ and V^- are values of the potential V from the upper and lower sides of the cut c_1 , respectively. Here, as above, for any given complex number c we denote the derivative in the direction of the vector $\vec{c} \in \mathbf{R}^2$. Due to formula (10), this expression can be rewritten as

$$\frac{1}{2} \operatorname{Re} [e^{-i\pi/4} S(z)]^- + \frac{1}{2} \operatorname{Re} [e^{3i\pi/4} S(z)]^+. \quad (17)$$

Let us first compute the first summand of the latter expression up to derivatives of functions which are regular near the point $e^{i\pi/4}$ (such terms clearly do not contribute in the final expression). We have

$$\frac{1}{2} \operatorname{Re} [e^{-i\pi/4} S(z)]^- = \frac{1}{2} \operatorname{Re} \left[e^{-i\pi/4} \sqrt{-3z^2 + 2\sqrt{2}\sqrt{z^4 + 1}} \right]^-. \quad (18)$$

To determine the right branches of the square roots involved into the last expression, we perform the analytic continuation of the function (16) along the path l_1 shown on Figure 6. At the origin A of this path we have $\zeta = S(z)$, $\zeta = z = 1$, and, hence, both square roots must have positive real values. Now one can verify that the expression under the square root on the right in (16) is changed along the path \tilde{l}_1 shown on Figure 7. This path consists of the straight line segment $[1, 2\sqrt{2}]$ of the real axis and the segment of the ellipse

$$\frac{x^2}{8} + \frac{y^2}{9} = 1$$

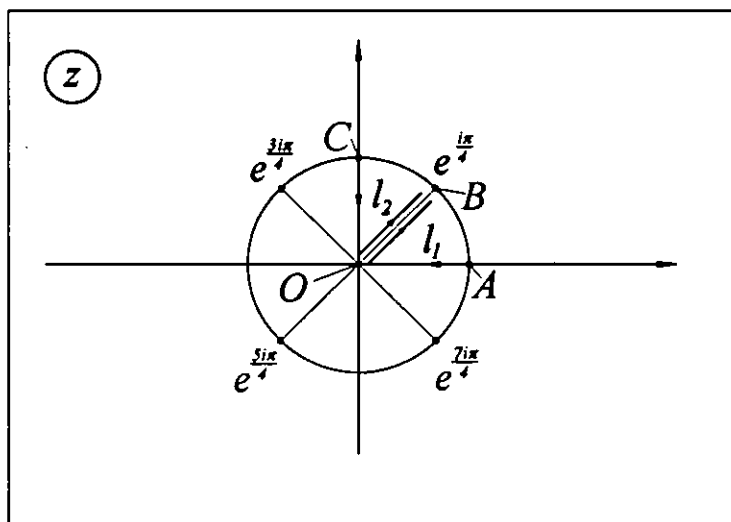


Figure 6: Paths of analytic continuation.

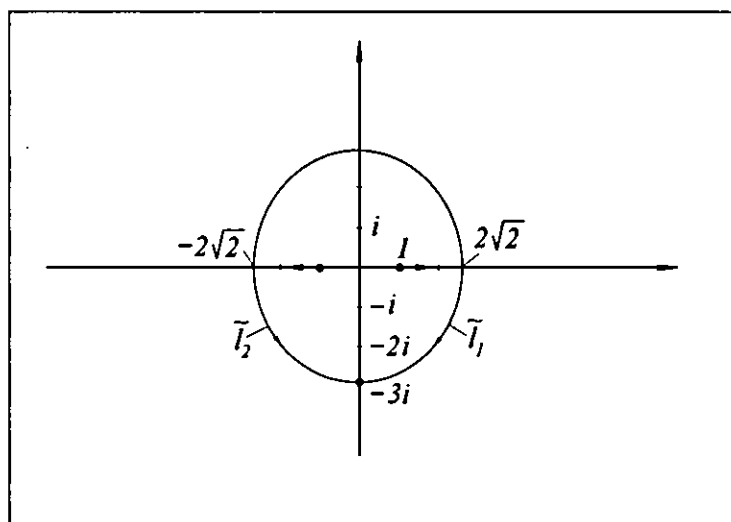


Figure 7: Expression under the square root sign.

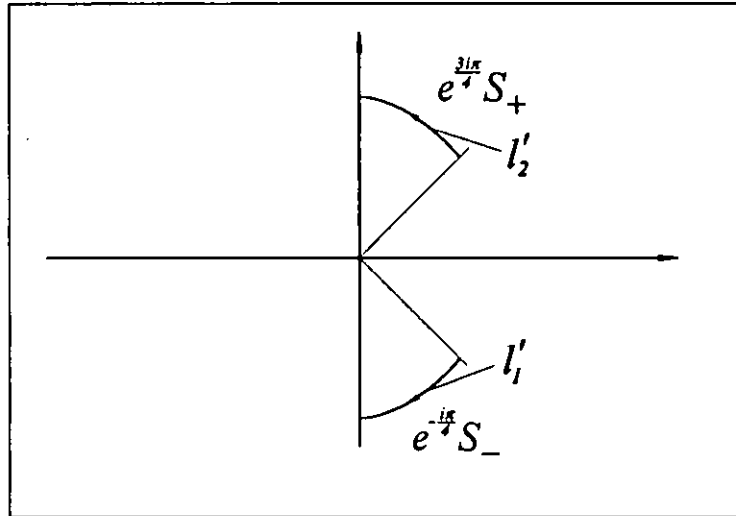


Figure 8: Jump of the normal derivative.

lying in the fourth quarter of the plane $\mathbf{R}_{(x,y)}^2$. Extracting the square root of this expression and multiplying the result by $e^{-i\pi/4}$ one can see that the expression under the Re sign on the right of (18) is changed along the path l'_1 drawn on Figure 8 when z is changed along the cut c_1 . Hence, we see that the first summand in the expression (17) has the positive sign.

Similar, one can see that the expression under the square root on the right in (16) is changed along the path \tilde{l}_2 shown on Figure 7 when the point z is changed along the path l_2 (see Figure 6). Hence, the expression under the Re sign in (17) changes along the path l'_2 (Figure 8), and the first summand in (17) is also positive.

So, *the union of the four cuts c_j , $j = 1, 2, 3, 4$ form the (unique) mother body of $(\Omega, 1)$.*

4.3 One more curve of the fourth order (Cassini oval)

Here we consider the heavy body $(\Omega, 1)$, where the domain Ω is given by the equation

$$(x^2 + y^2)^2 - 2b^2(x^2 - y^2) = a^4 - b^4,$$

where a and b are positive constants (the so-called Cassini oval). This curve consists of two closed curves for $a < b$, and of one closed curve for $a > b$. We shall investigate this last case.

This example is interesting from the following point of view. If we compute the

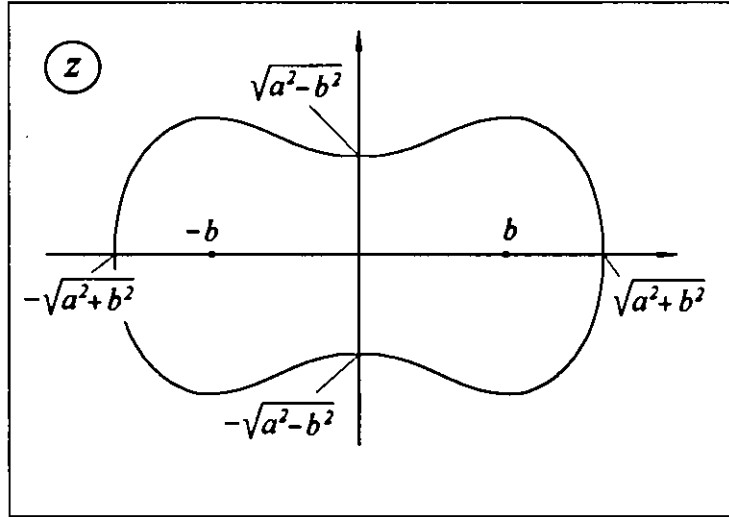


Figure 9: Cassini oval.

Schwarz function of the Cassini oval, we shall obtain the expression

$$S(z) = \frac{\sqrt{b^2 z^2 + a^4 - b^4}}{\sqrt{z^2 - b^2}}$$

which has inside the domain Ω two singular points $z = \pm b$ with singularities of the type of the inverse square root (the domain with points of singularity is shown on Figure 9). This happens since these singularities are generated not by regular characteristic points of the surface $\Gamma_{\mathbb{C}}$ but by singular points of this surface.

Due to this fact we cannot apply directly the result of the investigation of the local structure of the set of admissible cuts obtained in Section 2. However, the computations similar to those in the mentioned section show that each of the two points of singularity possesses exactly two directions of admissible cuts emanated from these points, namely $\varphi = 0$ and $\varphi = \pi$. The symmetry reasons show immediately that the only admissible cuts for the heavy body in question are three segments $[-\infty, -b]$, $[-b, b]$, and $[b, +\infty]$ of the real axis. Since only one of these segments is contains in the domain Ω , the only possibility of constructing a mother body is to consider the segment $[-b, b]$. Computation of the mass distribution which has to be considered on this segment to generate the same gravitational field as the initial body leads us to the expression

$$\sigma = \frac{\sqrt{b^2 x^2 + a^4 - b^4}}{8\sqrt{b^2 - x^2}}.$$

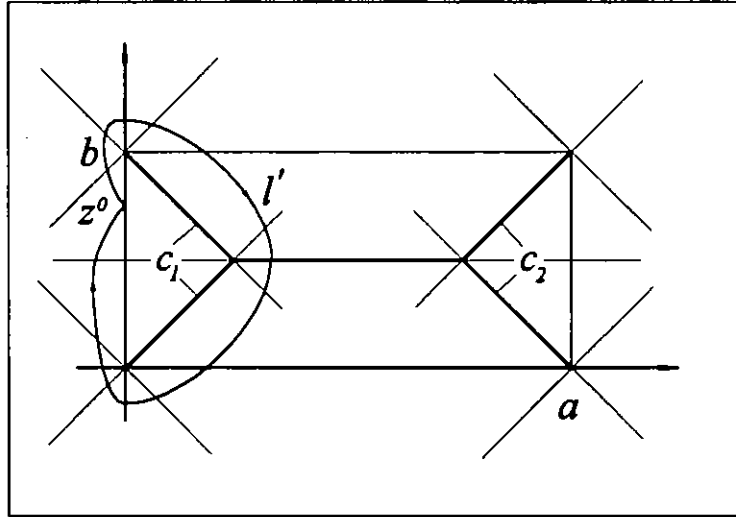


Figure 10: Admissible cuts for a rectangle.

Since this expression is positive and integrable function on the segment $[-b, b]$, *this segment is the only mother body for the Cassini oval.*

4.4 A rectangle

This last example shows that even in the case when we cannot use the Schwarz function for the investigation of singularities, the above described algorithm of constructing a mother body is applicable. Consider the heavy body $(\Omega, 1)$, where the domain Ω is a rectangle

$$\begin{cases} 0 \leq x \leq a, \\ 0 \leq y \leq b \end{cases}$$

in the plane $\mathbf{R}_{(x,y)}^2$. Here a and b are positive constants and we suppose, to be definite, that $b < a$ (see Figure 10).

Using directly relation (6) for the complex field vector, one has

$$\begin{aligned} \frac{\partial V}{\partial z} \equiv & \frac{1}{8\pi i} \{-2z \ln(-z) - 2(a-z) \ln(a-z) \\ & + 2(a+ib-z) \ln(a+ib-z) - 2(ib-z) \ln(ib-z)\} \end{aligned}$$

and, consequently,

$$\begin{aligned} \int \frac{\partial V}{\partial z} dz \equiv & \frac{1}{8\pi i} \{-z^2 \ln(-z) + (a-z)^2 \ln(a-z) \\ & - (a+ib-z)^2 \ln(a+ib-z) + (ib-z)^2 \ln(ib-z)\} \end{aligned}$$

up to regular terms (we omit rather long but simple computations). Hence, we have

$$\text{Var}_l \left[\int \frac{\partial V}{\partial z} dz \right] = -\frac{1}{4}z^2,$$

where the variation is taken over the loop l encircling the point $z = 0$. Consequently,

$$\text{Var}_l [V] = 2\text{Re} \text{Var}_l \left[\int \frac{\partial V}{\partial z} dz \right] = -\frac{1}{2}(x^2 - y^2).$$

So, the point $z = 0$ determines four directions of admissible cuts emanated from this point. Moreover, it is clear that the lines $x = \pm y$ are admissible cuts.

Similar, investigation of all other vertexes of the rectangle leads us to six more admissible cuts which are also straight lines emanated from these vertexes under the angles $\pm\pi/4$ (see Figure 10). The constructed set of admissible cuts determines two open rectangles c_1 and c_2 with hanged vertexes at those of the rectangle. However, these two cuts do not determine a univalued branch of the potential V . One has to include into consideration the admissible cuts corresponding to the loop l' encircling the two left (or the two right, which leads a posterior to the same result) vertexes of the rectangle. The computation of the variation along this loop gives

$$\text{Var}_{l'} [V] = b \left(y - \frac{b}{2} \right)$$

and, hence, the admissible cut corresponding to this loop is the straight line $y = b/2$. Adding the segment of this straight line connecting the two angular points of the open rectangles c_1 and c_2 , we arrive at the system of cuts (shown on Figure 10 by thick lines) which determines a univalued branch of the potential V . We leave to the reader the verification of the fact that this system consists of positive cuts, so that *the mother body of the rectangle occupied by the uniform mass distribution has the form shown on the above mentioned figure.*

To conclude the paper, we present Zidarov's example of the heavy body for which the corresponding mother body is not uniquely defined. Consider the heavy body supported in the square in the plane with deleted quarter in the left upper corner (see Figure 11) with constant mass density. Then, one can construct a corresponding mother body in the following way. First, one can divide the considered body into two rectangles, and, second, construct the mother body for each rectangle as it was described above. Then the union of the two constructed mother body will be the mother body corresponding to the body in question. However, one can divide the above described domain into two rectangles by two different ways, and, therefore, to construct two different mother bodies (they are drawn on Figures 11 a) and b), respectively).

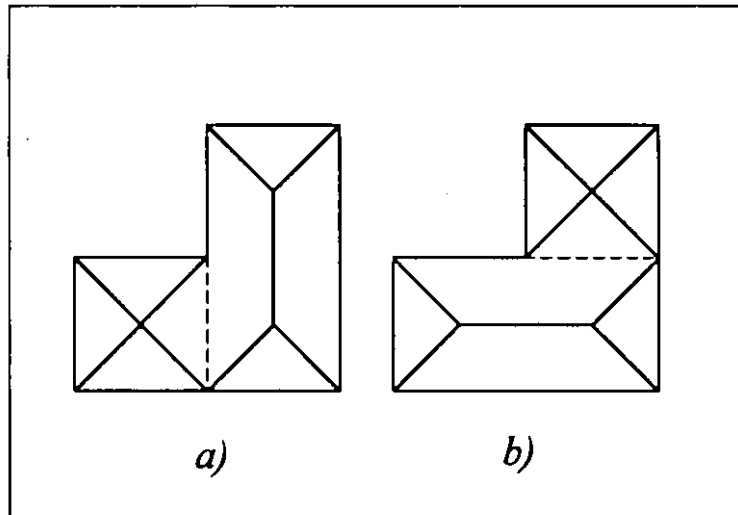


Figure 11: Non-uniqueness of a mother body.

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