COUNTING POINTS IN MULTIPLICATIVE POLYHEDRA AND SPLIT TORIC VARIETIES

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MPI / 91-57

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Introduction

In this note, we consider two counting problems.

Problem A. Let $B \subset \mathbf{Z}^{r+1}_{\geq 0}$ be a finite set with the following properties:

i) $\sum_{i=0}^{r} b_i$ does not depend on the choice of $b = (b_o, ..., b_r) \in B$.
ii)For every $i \in [0, r]$, there exists $b \in B$ with $b_i = 0$.
iii) The set $\{b - b' | b, b' \in B\}$ spans r-dimensional sublattice in $\mathbf{Z}^{r+1}_{\geq 0}$.
For $x = (x_i) \in \mathbf{Z}^{r+1}_{>0}$, $b = (b_i)$, we put $x^b = x_0^{b_0} ... x_r^{b_r}$, $\quad x^B = \{x^b | b \in B\}$, and

$$h(x^B) = \frac{\max\{x^b | b \in B\}}{gcd\{x^b | b \in B\}} \tag{0.1}$$

We shall prove below that h is a counting function on $\mathbf{Z}^{r+1}_{>0,prim} = \{x | gcd(x_i) = 1\}$, that is, it tends to infinity on this set. We want to estimate

$$N(B; H) = card\{x \in \mathbf{Z}^{r+1}_{>0,prim} | h(x^B) \leq H\}, \tag{0.2}$$

or at least, to calculate

$$\beta(B) = \limsup_{H \to \infty} (\log N(B; H) / \log H). \tag{0.3}$$

We call the set (0.2) " a multiplicative polyhedron".

Problem B. Let $X = T_N emb(\Delta)$ be a split toric variety defined by a fan Δ in $N_{\mathbf{R}}$ over an algebraic number field k (we use notations of [O] and call X split if the Galois group $Gal(\bar{k}/k)$ acts trivially upon N, the lattice of the 1-parametric algebraic subgroups of the basic torus over $\bar{k}$).

Let X be proper. Consider a support function $\eta : |\Delta| \to \mathbf{R}$ ([O], p.66), that is a function taking integer values on $N \cap |\Delta|$ and linear on all $\sigma \in |\Delta|$. This function defines a divisor "at infinity" (i.e. outside the large orbit X_0)

$$D_\eta = - \sum_{\rho \in \Delta(1)} \eta(n(\rho)) V(\rho).$$

(Oda denotes support functions by h, but we use h for height). Here $\Delta(1) =$ the set of 1-dimensional components of Δ, $n(\rho)$ is the smallest (primitive) element of n in ρ, $V(\rho)$ is the closure of $orb(\rho)$, and generally ([O], p.10)

$$orb(\sigma) = \{u : \hat{N} \cap \sigma^{\perp} \to C \qquad \text{(group homomorphisms)}\}.$$

The sections of $\mathcal{O}(D_\eta)$ are spanned by characters (or "monomials") $e(m), m \in M_{\mathbf{R}}, M = \hat{N}$, of the following type ([O], pp.72,76)

$$H^0(X, \mathcal{O}(D_\eta)) = \oplus\{ke(m) | m \in M \cap \square_\eta\}$$

$$\square_\eta = \{m \in M_R | \forall n \in N_R, \langle m, n\rangle \geq \eta(n)\}.$$

Consider the height function, for $x \in X_0(k)$:

$$h_{\mathcal{O}(D_\eta)}(x) = \prod_{\nu \in M_k} \max_{m \in M \cap \square_\eta} \{|e(m)|_\nu(x)\}, \tag{0.4}$$

where ν runs over all places of k, and $|.|_\nu$ is the multiplier of the additive Haar measure on k_ν.

If $\mathcal{O}(D_\eta)$ is very ample, (0.4) is a Weil height and a counting function on $X_0(k)$, so that we can define

$$N_{X_0}(\mathcal{O}(D_\eta); H) = card\{x \in X_0(k) | h_{\mathcal{O}(D_\eta)}(x) \leq H\}, \tag{0.5}$$

$$\beta(\mathcal{O}(D_\eta)) = \limsup\,(\log N_{X_0}(\mathcal{O}(D_\eta); H)/\log H). \tag{0.6}$$

(Notice, however, that (0.4) can be a counting function even if $\mathcal{O}(D_\eta)$ is not very ample).

For $k = \mathbf{Q}$, one can reduce the split torus count to a multiplicative polyhedra count.

Namely, choose a basis in M, that is, a multiplicative basis $\{z_1, ... z_r\}$ in $\{e(m) | m \in M\}$. Identifying $M \cap \square_\eta \subset M$ with a subset $B^0 \subset \mathbf{Z}^r$, we can construct the respective $B \subset \mathbf{Z}^{r+1}_{\geq 0}$ as follows. First, translating B^0 by a vector in $\mathbf{Z}^r$, we do not change the isomorphism class of $\mathcal{O}(D_\eta)$. Thus, we may and will assume that $B^0 \subset \mathbf{Z}^r_{\geq 0}$, and that for every $i \in [1, r]$, there exists $b^0 \in B^0$ with $b^0_i = 0$. Second, we shall "homogenize" B^0 by putting $d = \max\{\sum_{i=1}^r b^0_i | b^0 = (b_i)^0 \in B^0\}$ and

$$B = \{(d - \sum_{i=1}^r b^0_i, b^0_1, ..., b^0_r) | b^0 \in B^0\} \subset \mathbf{Z}_{\geq 0} \tag{0.7}$$

If a point $x \in X_0(\mathbf{Q})$ is represented by the vector $(z_1(x), \ldots, z_r(x)) \in \mathbf{Q}^r$, and $z_i(x) = x_i/x_0$ for $(x_0, \ldots, x_r) \in \mathbf{Z}^r_{>0,prim}$, one can check that the D_η -height (0.4) of x coincides with (0.1) evaluated on $(x_0, ..., x_r)$. (The rest of the points of $X_0(\mathbf{Q})$ can be obtained from this subset by changing signs of $x_1, ... x_r$ which does not influence heights.)

The note is structured as follows. In §1 we recall an elementary summation formula which is probably well known and is easily deduced from Delange's Tauberian theorem. In §2 we deduce a lower and an upper bound for $\beta(B)$, (0.3), using linear programming, and state the conditions on B, ensuring the coincidence of these bounds. Finally in §3, we prove an upper bound for toric count, using an idea due to V.V. Batyrev, and discuss open questions.

§1 A summation formula

1.1. Notation. Let $b = (b_0, ... b_r) \in \mathbf{R}^{r+1}_{>0}$, $a = (a_0, ... a_r) \in \mathbf{R}^{r+1}$. Variable $x = (x_0, ... x_r)$ takes values in $\mathbf{Z}^{r+1}_{>0}$ or $\mathbf{Z}^{r+1}_{>0,prim}$, $x^a = x_o^{a_0} ... x_r^{a_r}$. Put

$$\sigma = \max_i \{\frac{1+a_i}{b_i}\}, \qquad m = card\{i | \frac{1+a_i}{b_i} = \sigma\} \tag{1.1}$$

Assume that $\sigma > 0$, and put

$$C = \prod_{\{i|\sigma b_i - a_i > 1\}} \zeta(\sigma b_i - as_i) \tag{1.2}$$

(C =1 if the index set is empty);

$$C' = C\zeta((\sum_i b_i)\sigma - \sum_i a_i)^{-1} \tag{1.3}$$

Finally, let

$$\psi(H) = \sum_{\substack{x \in Z^{r+1}_{>0} \\ x^b \le H}} x^a; \tag{1.4}$$

$$\psi'(H) = \sum_{\substack{x \in Z^{r+1}_{>0,prim} \\ x^b \le H}} x^a \tag{1.5}$$

1.2 PROPOSITION. *We have*

$$\psi(H) = \frac{C}{\Gamma(m)} \frac{H^\sigma}{\sigma} (\log H)^{m-1} (1 + o(1))$$

$$\psi'(H) = \frac{C'}{\Gamma(m)} \frac{H^\sigma}{\sigma} (\log H)^{m-1} (1 + o(1))$$

PROOF: We start with recalling Delange's theorem [Del]. Consider a monotone non-decreasing function $\psi(t)$, $t \ge 0$. Put $Z(s) = \int_1^\infty t^{-s}\psi(t)$, and assume that $Z(s)$ is analytic for $Re(s) \ge \sigma > 0$ except of a singularity at $s = \sigma$. Assume moreover that

$$Z(s) = \frac{g_0(s)}{(s-\sigma)^m} + g_1(s)$$

where g_0, g_1 are analytic for $Re(s) \ge \sigma$, with $C = g_0(\sigma) \ne 0$ and $m > 0$. Then

$$\psi(t) = \frac{C}{\Gamma(m)} \frac{t^\sigma}{\sigma} (\log t)^{m-1} (1 + o(1)) \tag{1.6}$$

(Delange proves this for $\sigma = 1$; the general case reduces to this one by the variable change $t \mapsto t^\sigma$).

Now, $\psi(H)$ in (1.4) is the summatory function of the Dirichlet series

$$A(s) = \prod_{i=0}^{r} \sum_{x_i=1}^{\infty} \frac{x_i^{a_i}}{x_i^{sb_i}} = \prod_{i=0}^{r} \zeta(sb_i - a_i),$$

and Delange's theorem is clearly applicable with σ and C from (1.1), (1.2).

Similarly $\psi'(H)$ in (1.5) is summatory function of

$$B(s) = \sum_{x \in \mathbf{Z}^{r+1}_{>0,prim}} \frac{x^a}{x^{bs}},$$

and

$$A(s) = (\sum_{d=1}^{\infty} \frac{d^{(\Sigma a_i)}}{d^{(\Sigma b_i)s}})B(s) = \zeta((\sum_i b_i)s - \sum a_i)B(s) \tag{1.7}$$

The rightmost pole of $\zeta((\sum_i b_i)s - \sum a_i)$ is

$$\sigma = \frac{\sum a_i + 1}{\sum b_i} \leq \max_i \frac{a_i + \frac{1}{r+1}}{b_i} < \max_i \frac{a_i + 1}{b_i} = \sigma$$

Hence the rightmost pole of $B(s)$ is still at σ , and only C should be replaced by C'.

§2 Points in multiplicative polyhedra.

2.1 Notation. In the situation of Problem A, consider the following polyhedron $P = P(B)$ in the $\mathbf{R}$ - space of the dual lattice $\hat{\mathbf{Z}}^{r+1} \otimes \mathbf{R}$:

$$P = \{\xi \in \mathbf{R}^{r+1} | \forall b \in B, \langle b, \xi \rangle \leq 1; \forall i \in [0, r], \xi_i \geq 0\}.$$

Put

$$\beta_-(B) = \beta_- = \max\{\sum_{i=0}^{r} \xi_i | \xi \in P\} \tag{2.1}$$

Let $\xi^0 \in P(b)$ be a point at which the maximal value β_- of $\sum_{i=0}^{r} \xi_i$ is achieved. Put

$$I^0 = I^0(\xi^0) = \{i \in [0, r] | \xi_i^0 \neq 0\}.$$

Consider the following list of conditions that may or may not be satisfied by B.

C1). There exists $\xi^0 \in P(B)$ with $\sum_{i=0}^{r} \xi^0 = \beta_-$ and a weighted average $\tilde{b} = \sum_{b \in B} \epsilon_b b$, $\epsilon_b \geq 0$, $\sum_{b \in B} \epsilon_b = 1$ such that:

a) $\epsilon_b \neq 0$ only for $b \in B$ with $\langle b, \xi^0 \rangle = 1$.

b) On $I^0(\xi^0)$, coordinates of $\sum_{b \in B} \epsilon_b b$ are all equal, and take their minimal value.

c)$\tilde{b}_i \neq 0$ for all $i \in [0, r]$

C2) B satisfies C1), and moreover,

$$\text{min pos}_{\lambda\in\hat{Z}^{r+1}_{\geq 0}}\{\langle\tilde{b},\lambda\rangle\} = \text{min pos}_{\lambda\in\hat{Z}^{r+1}_{\geq 0}}\{\langle\tilde{b},\lambda\rangle - \min_{b\in B}\langle b,\lambda\rangle\}$$

where "min pos" = "minimal positive value".

We will prove the following result.

2.2. THEOREM. *a) We always have*

$$\beta(B) \geq \beta_-(B) \tag{2.2}$$

b)If, in addition, B satisfies C2), then

$$\beta(B) = \beta_-(B) \tag{2.3}$$

PROOF: a) Choose a vector $\xi^0 = (\xi^0_i) \in \mathbf{R}^{r+1}_{\geq 0}$ with $\sum_i b_i\xi_i \leq 1$ for all $b \in B$. There are $> const.H^{\Sigma\xi^0_i}$ of vectors $x = (x_i) \in \mathbf{Z}^{r+1}_{>0,prim}$ with $x_i \leq H^{\xi_i}$. On the other hand, $x^b \leq H^{\Sigma b_i\xi^0_i} \leq H$ for every $b \in B$. This proves (2.2).

b) To get an upper bound for $N(B,H)$ we take first an arbitrary weight vector (ϵ_b), $b \in B, \epsilon_b \geq 0, \sum_b \epsilon_b = 1$. We have then from (0.1):

$$h(x^B) \leq H \Rightarrow \frac{x^{\sum_b \epsilon_b b}}{gcd_{b\in B}(x^b)} \leq H \tag{2.4}$$

Therefore, for $\tilde{b} = \sum \epsilon_b b$,

$$N(B;H) \leq card\{x \in \mathbf{Z}^{r+1}_{>0,prim} | \frac{x^{\tilde{b}}}{gcd(x^b)} \leq H\} \tag{2.5}$$

We want to undestand when the r.h.s. of (2.5) is bounded by $O(H^{\beta_+ + \epsilon})$ for every $\epsilon > 0$, that is, when the upper estimate of $\beta(B)$ that can be deduced from (2.5) coincides with $\beta_-(B)$. If we replace in the r.h.s. of (2.6) the $gcd(x^b)$ by 1 , we will only diminish our upper estimate of β. Hence we must first understand when this diminished upper bound coincides with β_-. Obviously, we must have $\tilde{\beta}_i \neq 0$ for all i (this is C1 c.).

Applying Proposition 1.2 to the case $a = 0, b = \tilde{b}$, we get:

$$card\{x \in \mathbf{Z}^{r+1}_{>0,prim} | x^{\tilde{b}} \leq H\} = (1+o(1))cH^{\beta_+}(\log H)^{m-1}, \tag{2.6}$$

$$\beta_+ := \min_i \tilde{b}_i^{-1}; \qquad m := card\{i | \tilde{b}_i = \beta_+^{-1}\}.$$

Let us now compare β_+ and β_-. Take a point $\xi^0 \in P(B)$ such that $\sum_{i=0}^r \xi^0_i = \beta_-$. Define the subset $B_0 \subset B$ by:

$$b \in B_0 \Leftrightarrow \langle b, \xi^0\rangle = 1; \qquad b \notin B_0 \Leftrightarrow \langle b, \xi^0\rangle < 1.$$

Define $I^0 = I^0(\xi)$ as in 2.1. We have

$$\sum_{b\in B}\langle\epsilon_b b,\xi^0\rangle = \langle\tilde{b},\xi^0\rangle \geq \min_i(\tilde{b}_i)\sum_j \xi^0_j = \beta_+^{-1}\beta_- \tag{2.7}$$

and the equality in (2.7) is achieved precisely when

$$\{i|\tilde{b}_i \text{takes its minimal value}\} \supset I_0 = \{i|\xi^0_i \neq 0\}.$$

This is C1 b).

On the other hand,

$$\sum_{b\in B}\langle\epsilon_L b,\xi^0\rangle = \sum_{b\in B_0}\epsilon_L + \sum_{b\notin B_0}\epsilon_B\langle b,\xi^0\rangle \leq 1, \tag{2.8}$$

and since $\sum_b \epsilon_b = 1$, the equality in (2.8) is achieved precisely when $\epsilon_b = 0$ for $b \notin B_0$. (This is C1 a.). Taken together, (2.7) and (2.8) show that $\beta_+^{-1}\beta_- \leq 1$ which of course was to be expected.

The net result so far is that C1 is necessary and sufficient for $\beta_- = \beta_+$.

Put now

$$\beta'_+ = \limsup \log card\{x \in \mathbf{Z}^{r+1}_{>0,prim}| \frac{x^{\tilde{b}}}{gcd(x^b)} \leq H\}.$$

Obviously, $\beta'_+ \geq \beta_+$, and we want to prove that C2 implies $\beta'_+ = \beta_+ (= \beta_-)$.

In the semigroup of products of fractional powers of primes in $\mathbf{R}_{>0}$ consider the equation

$$\frac{x^{\tilde{b}}}{gcd(x^b)} = n, x \in \mathbf{Z}^{r+1}_{>0,prim}.$$

Let $c(n)$ be the number of its solutions. Notice first that $c(mn) = c(n)c(m)$, if $(m,n) = 1$. To prove this, it suffices to show that if $ord_p(x_i) = 0$, then $p \not| x_i$ for all i. Otherwise, putting $\xi_i = ord_p(x_i)$ and comparing p-orders, we obtain:

$$\langle\tilde{b},\xi\rangle = \sum_i \tilde{b}_i\xi_i = \min_{b\in B}(\sum_i b_i\xi_i) = \min_{b\in B}(\langle b,\xi\rangle) > 0$$

with $\xi \neq 0$. But

$$\langle\tilde{b},\xi\rangle = \sum_{b\in B}\epsilon_b\langle b,\xi\rangle \geq (\sum_{b\in B}\epsilon_b)\min_{b\in B}\langle b,\xi\rangle,$$

and the equality is achieved precisely when $\langle b,\xi\rangle$ does not depend on b. This happens only if $\langle b-b',\xi\rangle = 0$ for all $b,b' \in B$, which implies that ξ is proportional to $(1,...,1)$ in view of the condition iii) in the statement of Problem A (Introduction). But since $gcd(x_i) = 1$, at least one of the coordinates of ξ vanishes so that $\xi = 0$, leading to a contradiction.

Let now $n = p^r, r \in \mathbf{Q}_{\geq 0}$. Then $c(p^r)$ depends only on r, and equals

$$c(p^r) = card\ \{\lambda \in \mathbf{Z}^{r+1}_{\geq 0} | \exists i, \lambda_i = 0; \langle \tilde{b}, \lambda \rangle - \min_{b \in B} \langle b, \lambda \rangle = r\} \tag{2.9}$$

(The condition $\exists i, \lambda_i = 0$ follows from $x \in \mathbf{Z}^{r+1}_{prim}$).

We want to estimate

$$\sum_{\substack{n \in \mathbf{R}_{>0} \\ n \leq H}} c(n)$$

by calculating the convergence abscisse of $D(s) = \sum_n c(n)/n^s$. From the multiplicativity of $c(n)$ it follows that

$$D(s) = \prod_p \sum_{r \in \mathbf{Q}_{\geq 0}} \frac{c(p^r)}{p^{rs}} \tag{2.10}$$

From (2.9) it is clear that $c(p^r)$ counts the number of lattice points in a compact domain linearly depending on r . Hence $c(p^r)$ is bounded by a polynomial in r. Therefore, the convergence abscisse of (2.10) is determined by the first positive value of r such that $c(p^r) \neq 0$, and equals to the inverse of this value. On the other hand,

$$\beta_+^{-1} = \min_i \tilde{b}_i = \min\{\langle \tilde{b}, \lambda \rangle | \lambda \in \mathbf{Z}^{r+1}_{\geq 0}, \exists i, \lambda_i = 0; \langle \tilde{b}, \lambda \rangle > 0\},$$

because $\min \tilde{b}_i$ is clearly achieved on a vector λ of the form (0...010...0)

In this way, we obtain the condition C2.

§3 Toric varieties

3.1. Finite heights. Recall that if L is an invertible sheaf on a projective variety V defined over an algebraic number field k , and h_L is a Weil height, then a representation of L in the form $\mathcal{O}(D)$, D a divisor, allows one to construct a decomposition $h_L(x) = h_{D,\infty}(x) h_{D,f}(x)$. For example, if L is very ample, $(s_0, ... s_n)$ is a basis of $\Gamma(L)$, and $s_0 = 0$ is an equation of D, we can put

$$h_{D,f}(x) = \prod_{\nu\, finite} max_i(|s_i/s_0|_\nu) \tag{3.1}$$

The following result is a version of an idea due to V.V.Batyrev, and a slight generalisation of Lemma 1.2 in [MaTschi]. Denote by ρ the rank of the group of units of k.

3.2. PROPOSITION. *Let $D_0, ... D_n \subset V$ be a family of pairwise distinct effective divisors; $U = V \setminus \cup_{i=0}^n D_i$; $L = \mathcal{O}(D_o + ... + D_n)$. Assume that for some $m > 0$, there exists a family of sections s_0, ... $s_N \in \Gamma(L^m)$ whose zeroes are supported by $D_0 \cup ... \cup D_n$, such that the map $\sigma : U \to \mathbf{P}^N$, $\sigma(x) = (s_0(x) :\ ...\ s_N(x))$ is well defined and has finite fibers. Then, for $H \to \infty$, among the points $x \in U(k)$ with $h_L(x) \leq H$, there can exist no more than $O((\log H)^{n\rho})$ points having the same family of finite heights $(h_{D_0,f}(x), ..h_{D_n,f}(x))$.*

SKETCH OF PROOF: Denote by E_i the divisor $s_i = 0$. Representing $\sigma(x)$ by "almost relatively prime" integer homogeneous coordinates in k , $(s_0(x), ... s_n(x)) \in A^{N+1}$, ($A$ is the ring of integers in k), we see that $h_{E_i,f}(x)$ coincides with $N_{k/\mathbf{Q}}(s_i(x))$ up to a finite valued factor. For fixed $h_{D_j,f}(x)$, ... , $j = 0$, ... , n, the finite heights $h_{E_i,f}$ are also fixed. The norms of the coordinates $s_i(x)$ being known, the remaining freedom of choice of $s_i(x)$ reduces to coordinate-wise multiplication by units. Looking now at $h_{D_i,\infty}(x)$ and using the Dirichlet theorem, one achieves the desired conclusion.

3.3. THEOREM. *With the assumptions of the Prop.3.2, one has for* $(b_0, \ldots, b_n) \in \mathbf{Z}^{n+1}_{>0}$:

$$N_U(\mathcal{O}(b_0 D_0 + \ldots + b_n D_n); H) = O(H^\sigma (\log H)^{\rho n + t}),$$

where

$$\sigma^{-1} = \min(b_i), \quad t = card\{i | b_i = b\} - 1.$$

In particular

$$\beta_U(L) \leq 1.$$

PROOF: We may and will assume that $h_{D_i,f}(x)$ takes integer values on U. For $M = \mathcal{O}(b_0 D_0 + \ldots + b_n D_n)$, we have

$$h_M(x) \geq c \prod_{i=0}^{n} h^{b_i}_{D_i,f}(x)$$

for some $c > 0$. Putting into one class all points with $h_M(x) \leq H$ and a fixed system of finite heights $(h_{D_i,f}(x))$ we see that there are $O(H^\sigma (\log H)^t)$ such classes (apply Prop.1.2 with $r = n, a = 0, b = (b_0, \ldots, b_n)$). It remains to apply Prop. 3.2.

3.4. Application to toric varieties. Let X be a complete split toric variety, X_0 the big orbit, $\{D_i\}$ the set of all irreducible divisors at infinity, $i \in \Delta(1)$. Then $-K_X = \sum_i D_i$. If $-K_X$ is very ample, or at least verifies the conditions of Prop. 3.2, we have

$$N_{X_0}(-K; H) = O(H (\log H)^{(\rho+1)n}),$$

$$\beta_{X_0}(-K) \leq 1.$$

Furthermore,

$$\alpha(b_0 D_0 + \ldots b_n D_n) \leq \min(b_i)^{-1},$$

where $\alpha(M)$, in notation of [BaMa], is defined by

$$\alpha(M) = \inf\{\gamma \in \mathbf{R} | \gamma[L] + K_V \in N^1_{eff}(V)\}.$$

3.5. Open questions. a) Generalize §2 to arbitrary number fields k.

b) In toric interpretation, elucidate the algebro-geometric meaning of conditions C1, C2 of §2. Is there an optimal choice of coordinates in N leading to the maximal value of $\beta_-(B)$?

Can one prove the [BaMa]-type equality $\beta_{X_0}(L) = \alpha(L)$ for toric varieties with non necessarily ample $-K$?

Acknowledgement. I would like to thank the Max-Planck-Institut für Mathematik in Bonn, where this note was written, for financial support and excellent working conditions.

References

[Del] H. Délange. Généralisation du théorème de Ikehara. Ann.Sci.ENS, 71 (1954), 213-242.

[O] T.Oda. Convex bodies and algebraic geometry. An Intoduction to the theory of toric varieties. Springer, 1985.

[MaTschi] Yu.I.Manin, Yu. Tschinkel. Points of bounded height on del Pezzo surfaces. Preprint, 1991.

[Ba] V.V.Batyrev. Rational points of bounded height on toric varieties. Preprint, 1991.

[BaMa] V.V.Batyrev, Yu.I.Manin. Sur le nombre des points rationnels de hauteur bornée des variétés algébriques. Math.Annalen, Bd.286 (1990), 27-43.