Spherical algebraic knots increase with dimension

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Abstract: The set of knots which occur as the link of an isolated critical point of a complex hypersurface increases with the dimension.

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Let $f(z_0, z_1, \ldots, z_n)$ be a polynomial in $n+1$ complex variables having an isolated critical point at the origin, as considered in [Mi]. We make the additional assumption that $\Sigma^{2n-1} = S_\varepsilon^{2n+1} \cap f^{-1}(0)$ is a topological sphere for sufficiently small $\varepsilon > 0$, where S_ε^{2n+1} denotes the $(2n+1)$-sphere of radius ε centered at the origin of $\mathbb{C}^{n+1}$. (As will become clear in the proof, this is equivalent to the condition that every eigenvalue of the local monodromy has order divisible by at least two different primes.) Then we have a $(2n-1)$-knot $k = (S^{2n+1}, \Sigma^{2n-1})$, which we refer to as a <u>spherical algebraic</u> knot. As Milnor shows in [Mi], the complement K of Σ^{2n-1} in S^{2n+1} is fibred over the circle, and the fibre $\mathring{F}$ is $(n-1)$-connected. Of course, the submanifold $F = \Sigma \cup \mathring{F}$ is a Seifert surface of k. As explained in [D], an excellent introduction to this material, the Seifert surface gives rise to a linking pairing

$$H_n(F) \times H_n(F) \longrightarrow \mathbb{Z}$$

which by a choice of basis of the free abelian group $H_n(F)$ yields a unimodular integer matrix V, known as a Seifert matrix of k. Naturally, V is only determined up to unimodular congruence, corresponding to a change of basis. By a result of Levine [L], the converse is also true; that is, the congruence class of V determines the knot k. Note that we are relying here on the fact that V is unimodular, a property that corresponds to k being fibred. The assumption that Σ is a sphere is also easily expressed in terms of V: it corresponds to $V + (-1)^n V'$ being unimodular, since $V + (-1)^n V'$ represents the intersection pairing on $H_n(F)$. (For $n = 2$, we get only that Σ is a homology sphere.)

Let A_n denote the set of unimodular congruence classes of Seifert matrices V which arise in this way. Thus A_n is in one-one correspondence with the set of spherical algebraic $(2n-1)$-knots. By a well-known result (see [KN]), the polynomial $f(z_0, z_1, \ldots, z_n) + z_{n+1}^2 + z_{n+2}^2$ also has an isolated critical point at the origin and gives rise to a spherical algebraic knot having the same Seifert matrix as k . Thus we have an inclusion $A_n \subseteq A_{n+2}$. It was shown in [K] that this inclusion is strict for $n = 1$ and $n = 2$; the purpose of the present paper is to prove the following result.

<u>Theorem</u>. The inclusion $A_n \subset A_{n+4}$ is strict for all n .

Note that in the present paper we are using Levine's classification of simple knots in terms of Seifert matrices, instead of the classification in terms of homology modules and duality pairings used in [K].

As Durfee remarks in [D], the monodromy h is represented by the matrix $(-1)^{n+1} V^{-1} V'$; knot theorists are accustomed to the equivalent formulation of $H_n(F)$ as a $\mathbb{Z}[t, t^{-1}]$-module presented by the matrix $tV + (-1)^n V'$. Our argument relies on the monodromy theorem, that for some integer $d > 0$, $(h^d - \mathrm{id})^{n+1} = 0$, and on Malgrange's polynomial

$$f_n(z_1, \ldots, z_n) = (z_1 \ldots z_n)^2 + z_1^{2n+2} + \ldots + z_n^{2n+2} .$$

In [M], it is shown that the monodromy α of f_n satisfies $(\alpha^d - \mathrm{id})^{n-1} \neq 0$ for every $d > 0$. In fact, Malgrange shows that α has a Jordan block of size $n \times n$ corresponding to the eigenvalue -1.

Now let $d > 0$ be an integer such that $(\alpha^d - \mathrm{id})^n = 0$, and choose a prime p not dividing d. Set

$$g(z_0, z_1, \ldots, z_n) = z_0^p + f_n(z_1, \ldots, z_n) .$$

The monodromy of the polynomial z_0^p is known (see [KN], page 389). It has Jordan block decomposition $\mathrm{diag}(\zeta_p, \zeta_p^2, \ldots, \zeta_p^{p-1})$ where ζ_p is a primitive p^{th} root of unity. Denoting this monodromy by β, the polynomial g has monodromy $\alpha \otimes \beta$, by the Thom-Sebastiani Theorem [KN]. Thus the monodromy of g has an $n \times n$ Jordan block corresponding to the eigenvalue $-\zeta_p$.

Unfortunately we cannot stop here, for by results of Varchenko [V] the monodromy of f_n has 1 as an eigenvalue (compare [A'C; page 246] for the case $n = 3$). So let q be a prime not dividing pd, and set

$$\varphi(z_0, z_1, \ldots, z_{n+1}) = z_0^p + f_n(z_1, \ldots, z_n) + z_{n+1}^q .$$

If γ denotes the monodromy of the polynomial z_{n+1}^q, then the monodromy of φ is $\alpha \otimes \beta \otimes \gamma$ and has an $n \times n$ Jordan block corresponding to the eigenvalue $-\zeta_p \zeta_q$. The other eigenvalues of $\alpha \otimes \beta \otimes \gamma$ are all of the form $\xi \zeta_p^a \zeta_q^b$, where ξ is a d th root of unity and $1 \leqq a < p$, $1 \leq b < q$. Thus all the irreducible factors of $\Delta(t)$, the characteristic polynomial of $\alpha \otimes \beta \otimes \gamma$, are m th cyclotomic polynomials where pq divides m. Therefore $\Delta(1) = \pm 1$, and so the link Σ of the critical point of φ at the origin of $\mathbb{C}^{n+2}$ is a homology $(2n+1)$ - sphere [Mi, page 68]. And assuming $n \geqq 2$, Σ is in fact a $(2n+1)$ - sphere.

Thus we have produced a spherical algebraic $(2n+1)$ - knot k_{n+1}, and hence an element of A_{n+1} which has an $n \times n$ Jordan block in its monodromy. This element cannot therefore belong to A_{n-3}.

We end with two problems related to the proof above.

$(1)_n$ Does there exist a polynomial $f(z_0, \ldots, z_n)$ with an isolated singularity at the origin such that the local monodromy has only eigenvalues different from 1 and has a Jordan block of size $(n+1) \times (n+1)$?

$(2)_n$ Does there exist such a polynomial where the order of each eigenvalue of the monodromy is divisible by at least two different primes and such that the monodromy has a Jordan block of size at least $n \times n$?

Note that a positive (negative) answer to the first question would enable us to prove that the inclusion $A_{n-1} \subset A_{n+1}$ ($A_{n+1} \subset A_{n+3}$) is strict, and that a positive (negative) answer to the second question would imply that the inclusion $A_{n-2} \subset A_n (A_n \subset A_{n+2})$ is strict. Note also that for $n = 1$, Question 1 has been settled in the negative by Lê [Le]. (See also [A'C 2].) Finally, if $(1)_n$ has a positive answer then so does $(2)_{n+1}$, by adding a suitable power of z_{n+1}.

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