DYNAMICS, LAPLACE TRANSFORM AND SPECTRAL GEOMETRY*

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ABSTRACT. We consider a vector field X on a closed manifold which admits a Lyapunov one form. We assume X has Morse type zeros, satisfies the Morse–Smale transversality condition and has non-degenerate closed trajectories only. For a closed one form η, considered as flat connection on the trivial line bundle, the differential of the Morse complex formally associated to X and η is given by infinite series. We introduce the exponential growth condition and show that it guarantees that these series converge absolutely for a non-trivial set of η. Moreover the exponential growth condition guarantees that we have an integration homomorphism from the deRham complex to the Morse complex. We show that the integration induces an isomorphism in cohomology for generic η. Moreover, we define a complex valued Ray–Singer kind of torsion of the integration homomorphism, and compute it in terms of zeta functions of closed trajectories of X. Finally, we show that the set of vector fields satisfying the exponential growth condition is C^0–dense.

CONTENTS

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*This paper is a new version of work which has circulated as preprint [6] under the name "Laplace transform, dynamics and spectral geometry." The mathematical perspective of this version is however very different.

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1. Introduction

Let M be a closed smooth manifold. We consider a vector field X which admits a Lyapunov form, see Definition 3. We assume X has Morse type zeros, satisfies Morse–Smale transversality and has non-degenerate closed trajectories only. These assumptions imply that the number of instantons as well as the number of closed trajectories in a fixed homotopy class are finite. Moreover, we assume that X satisfies the exponential growth condition, a condition on the growth of the volume of the unstable manifolds of X, see Definition 4 below. Using a theorem of Pajitnov we show that the set of vector fields with these properties is C^0–dense, see Theorem 2.

Let $\eta \in \Omega^1(M;\mathbb{C})$ be a closed one form, and consider it as a flat connection on the trivial bundle $M \times \mathbb{C} \to M$. Using the zeros and instantons of X one might try to associate a More type complex to X and η. Since the number of instantons between zeros of X is in general infinite, the differential in such a complex is given by infinite series. The exponential growth condition guarantees that this series converges absolutely for a non-trivial set of closed one forms η. For these η we thus have a Morse complex $C^*_\eta(X;\mathbb{C})$, see section 1.5, which, as a 'function' of η, can be considered as the 'Laplace transform' of the Novikov complex. The exponential growth condition also guarantees that we have an integration homomorphism $\mathrm{Int}_\eta : \Omega^*_\eta(M;\mathbb{C}) \to C^*_\eta(X;\mathbb{X})$, where $\Omega^*_\eta(M;\mathbb{C})$ denotes the deRham complex associated with the flat connection η. It turns out that this integration homomorphism induces an isomorphism in cohomology, for generic η. These results are the contents of Theorem 1 and Proposition 12.

For those η for which Int_η induces an isomorphism in cohomology we define the (relative) torsion of Int_η with the help of zeta regularized determinants of Laplacians in the spirit of Ray–Singer. Our torsion however is based on non-positive Laplacians, is complex valued, and depends holomorphically on η. While the definition requires the choice of a Riemannian metric on M we add an appropriate correction term which causes our torsion to be independent of this choice, see Proposition 14. Combining results of Hutchings–Lee, Pajitnov and Bismut–Zhang we show that the torsion of Int_η coincides with the 'Laplace transform' of the counting function for closed trajectories of X, see Theorem 3. Implicitly, the set of closed one forms η for which the Laplace transform of the counting function for closed trajectories converges absolutely is non-trivial, providing an (exponential) estimate on the growth of the number of closed trajectories in each homology class, as the class varies in $H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))$. Moreover, the torsion of Int_η provides an analytic continuation of this Laplace transform, considered as a function on the space of closed one forms, beyond the set of η for which it is naturally defined.

The rest of the paper is organized as follows. The remaining part of section 1 contains a thorough explanation of the main results including all necessary definitions. The proofs are postponed to sections 2 through 5 and two appendices.

1.1. Morse–Smale vector fields.

Let X be a smooth vector field on a smooth manifold M of dimension n. A point $x \in M$ is called a *rest point* or a *zero* if $X(x) = 0$. The collection of these points will be denoted by $\mathcal{X} := \{x \in M \mid X(x) = 0\}$.

Recall that a rest point $x \in \mathcal{X}$ is said to be of *Morse type* if there exist coordinates $(x_1, \dots, x_n)$ centered at x so that

$$X = \sum_{i \leq q} x_i \frac{\partial}{\partial x_i} - \sum_{i > q} x_i \frac{\partial}{\partial x_i}. \tag{1}$$

The integer q is called the *Morse index* of x and denoted by $\operatorname{ind}(x)$. A rest point of Morse type is non-degenerate and its Hopf index is $(-1)^{n-q}$. The Morse index is independent of the chosen coordinates $(x_1, \dots, x_n)$. Denote by $\mathcal{X}_q$ the set of rest points of Morse index q. Clearly, $\mathcal{X} = \bigsqcup_q \mathcal{X}_q$.

Convention. Unless explicitly mentioned all vector fields in this paper are assumed to have all rest points of Morse type, hence isolated.

For $x \in \mathcal{X}$, the *stable* resp. *unstable set* is defined by

$$D_x^{\pm} := \{y \mid \lim_{t \to \pm\infty} \Psi_t(y) = x\}$$

where $\Psi_t : M \to M$ denotes the flow of X at time t. The stable and unstable sets are images of injective smooth immersions $i_x^{\pm} : W_x^{\pm} \to M$. The manifold W_x^- resp. W_x^+ is diffeomorphic to $\mathbb{R}^{\operatorname{ind}(x)}$ resp. $\mathbb{R}^{n-\operatorname{ind}(x)}$.

Definition 1 (Morse–Smale property, MS)**.** The vector field X is said to satisfy the *Morse–Smale property*, MS for short, if the maps i_x^- and i_y^+ are transversal, for all $x, y \in \mathcal{X}$.

If the vector field X satisfies MS, and $x \neq y \in \mathcal{X}$, then the set $D_x^- \cap D_y^+$, is the image of an injective immersion of a smooth manifold $\mathcal{M}(x,y)$ of dimension $\operatorname{ind}(x) - \operatorname{ind}(y)$. Moreover, $\mathcal{M}(x,y)$ is equipped with a free and proper $\mathbb{R}$–action. The quotient is a smooth manifold $\mathcal{T}(x,y)$ of dimension $\operatorname{ind}(x) - \operatorname{ind}(y) - 1$, called the manifold of *trajectories* from x to y. Recall that a collection $\mathcal{O} = \{\mathcal{O}_x\}_{x \in \mathcal{X}}$ of orientations of the unstable manifolds, $\mathcal{O}_x$ being an orientation of W_x^-, provides (coherent) orientations on $\mathcal{M}(x,y)$ and $\mathcal{T}(x,y)$. If $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$ then $\mathcal{T}(x,y)$ is zero dimensional and its elements are isolated trajectories called *instantons.* The orientations $\mathcal{O}$ provide a sign $\epsilon^{\mathcal{O}}(\sigma) \in \{\pm 1\}$ for every instantons $\sigma \in \mathcal{T}(x,y)$.

1.2. **Closed trajectories.** Recall that a *parameterized closed trajectory* is a pair (θ, T) consisting of a non-constant smooth curve $\theta : \mathbb{R} \to M$ and a real number T such that $\theta'(t) = X(\theta(t))$ and $\theta(t+T) = \theta(t)$ hold for all $t \in \mathbb{R}$. A *closed trajectory* is an equivalence class σ of parameterized closed trajectories, where two parametrized closed trajectories (θ_1, T_1) and (θ_2, T_2) are equivalent if there exists $a \in \mathbb{R}$ such that $T_1 = T_2$ and $\theta_1(t) = \theta_2(t+a)$, for all $t \in \mathbb{R}$. Recall that the *period* $p(\sigma)$ of a closed trajectory σ is the largest integer p such that for some (and hence every) representative (θ, T) of σ the map $\theta : \mathbb{R}/T\mathbb{Z} = S^1 \to M$ factors through a map $S^1 \to S^1$ of degree p. Also note that every closed trajectory gives rise to a homotopy class in $[S^1, M]$.

Suppose (θ, T) is a parametrized closed trajectory and $t_0 \in \mathbb{R}$. Then the differential of the flow $T_{\theta(t_0)}\Psi_T : T_{\theta(t_0)}M \to T_{\theta(t_0)}M$ fixes $X(\theta(t_0))$ and hence descends to a linear isomorphism $A_{\theta(t_0)}$ on the normal space to the trajectory $T_{\theta(t_0)}M/\langle X(\theta(t_0))\rangle$, called the *return map.* Note that the conjugacy class of $A_{\theta(t_0)}$ only depends on the closed trajectory represented by (θ, T). Recall that a closed trajectory is called *non-degenerate* if 1 is not an eigen value of the return map. Every non-degenerate closed trajectory σ has a sign $\epsilon(\sigma) \in \{\pm 1\}$ defined by $\epsilon(\sigma) := \operatorname{sign}\det(\mathrm{id} - A_{\theta(t_0)})$ where $t_0 \in \mathbb{R}$ and (θ, T) is any representative of σ.

Definition 2 (Non-degenerate closed trajectories, NCT)**.** A vector field is said to satisfies the *non-degenerate closed trajectories property*, NCT for short, if all of its closed trajectories are non-degenerate.

1.3. **Lyapunov forms.** The existence of a Lyapunov form for a vector field has several important implications: it implies finiteness properties for the number of instantons and closed trajectories, see Propositions 4 and 5 below; and it permits to complete the unstable manifolds to manifolds with corners, see Theorem 4 in section 4.1.

Definition 3 (Lyapunov property, L)**.** A closed one form $\omega \in \Omega^1(M;\mathbb{R})$ for which $\omega(X) < 0$ on $M \setminus \mathcal{X}$ is called *Lyapunov form* for X. We say a vector field satisfies the *Lyapunov property*, L for short, if it admits Lyapunov forms. A cohomology class in $H^1(M;\mathbb{R})$ is called *Lyapunov cohomology class for* X if it can be represented by a Lyapunov form for X.

The Kupka–Smale theorem [11, 22, 20] immediately implies

Proposition 1. *Suppose X satisfies L, and let $r \geq 1$. Then, in every C^r–neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies L, MS and NCT.*

In appendix A we will prove

Proposition 2. *Every Lyapunov cohomology class for X can be represented by a closed one form ω, so that there exists a Riemannian metric g with $\omega = -g(X, \cdot)$. Moreover, one can choose ω and g to have standard form in a neighborhood of $\mathcal{X}$, i.e. locally around every zero of X, with respect to the coordinates $(x_1, \dots, x_n)$ in which X has the form (1), we have $\omega = -\sum_{i \leq q} x_i dx^i + \sum_{i>q} x_i dx^i$ and $g = \sum_i (dx^i)^2$.*

For the structure of the set of Lyapunov cohomology classes we obviously have

Proposition 3. *The set of Lyapunov cohomology classes for X constitutes an open convex cone in $H^1(M;\mathbb{R})$. Consequently we have: If X satisfies L, then it admits a Lyapunov class contained in the image of $H^1(M;\mathbb{Z}) \to H^1(M;\mathbb{R})$. If X satisfies L, then it admits a Lyapunov class ξ such that $\xi : H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z})) \to \mathbb{R}$ is injective. If $0 \in H^1(M;\mathbb{R})$ is a Lyapunov class for X then every cohomology class in $H^1(M;\mathbb{R})$ is Lyapunov for X.*

The importance of Lyapunov forms stems from the following two results. Both propositions are a consequence of the fact that the energy of an integral curve γ of X satisfies $E_g(\gamma) = -\omega(\gamma)$ where g and ω are as in Proposition 2.

Proposition 4 (Novikov [16])**.** *Suppose X satisfies MS, let ω be a Lyapunov form for X, let $x, y \in \mathcal{X}$ with $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$, and let $K \in \mathbb{R}$. Then the number of instantons σ from x to y which satisfy $-\omega(\sigma) \leq K$ is finite.*

Proposition 5 (Fried [7], Hutchings–Lee [9])**.** *Suppose X satisfies MS and NCT, let ω be Lyapunov for X, and let $K \in \mathbb{R}$. Then the number of closed trajectories σ which satisfy $-\omega(\sigma) \leq K$ is finite.*

1.4. **Counting functions and their Laplace transform.** Let us introduce the notation $\mathcal{Z}^1(M;\mathbb{C}) := \{\eta \in \Omega^1(M;\mathbb{C}) \mid d\eta = 0\}$. Similarly, we will write $\mathcal{Z}^1(M;\mathbb{R})$ for the set of real valued closed one forms. For a homotopy class γ of paths joining two (rest) points in M and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ we will write $\eta(\gamma) := \int_\gamma \eta$.

For a vector field X which satisfies L and MS, and two zeros $x, y \in \mathcal{X}$ with $\operatorname{ind}(x) - \operatorname{ind}(y) = 1$, we define the *counting function of instantons from x to y* by

$$\mathbb{I}_{x,y} = \mathbb{I}_{x,y}^{X,\mathcal{O}} : \mathcal{P}_{x,y} \to \mathbb{Z}, \qquad \mathbb{I}_{x,y}(\gamma) := \sum_{\sigma\in\gamma} \epsilon^{\mathcal{O}}(\sigma).$$

Here $\mathcal{P}_{x,y}$ denotes the space of homotopy classes of paths from x to y, and the sum is over all instantons σ in the homotopy class $\gamma \in \mathcal{P}_{x,y}$. Note that these sums are finite in view of Proposition 4. For notational simplicity we set $\mathbb{I}_{x,y} := 0$ whenever $\operatorname{ind}(x) - \operatorname{ind}(y) \neq 1$.

Consider the 'Laplace transform' of $\mathbb{I}_{x,y}$,

$$L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}, \qquad L(\mathbb{I}_{x,y})(\eta) := \sum_{\gamma\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma) e^{\eta(\gamma)} \tag{2}$$

where $\mathfrak{I}_{x,y} = \mathfrak{I}_{x,y}^X \subseteq \mathcal{Z}^1(M;\mathbb{C})$ denotes the subset of closed one forms η for which this sum converges absolutely. Moreover, set $\mathfrak{I} := \bigcap_{x,y\in\mathcal{X}} \mathfrak{I}_{x,y}$, and let $\mathring{\mathfrak{I}}_{x,y}$ resp. $\mathring{\mathfrak{I}} = \bigcap_{x,y\in\mathcal{X}} \mathring{\mathfrak{I}}_{x,y}$ denote the interior of $\mathfrak{I}_{x,y}$ resp. $\mathfrak{I}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞–topology.

Classically [24] the Laplace transform is a partially defined holomorphic function $z \mapsto \int_{\mathbb{R}} e^{-z\lambda} d\mu(\lambda)$, associated to a complex valued measure μ on the real line with support bounded from below. The Laplace transform has an abscissa of absolute convergence $\rho \leq \infty$ and will converge absolutely for $\Re(z) > \rho$. If the measure has discrete support this specializes to Dirichlet series, $z \mapsto \sum_i a_i e^{-z\lambda_i}$.

One easily derives the following proposition which summarizes some basic properties of $L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}$ analogous to basic properties of classical Laplace transforms [24]. The convexity follows from Hölder's inequality.

Proposition 6. *The set $\mathfrak{I}_{x,y}$ (and hence $\mathring{\mathfrak{I}}_{x,y}$) is convex and we have $\mathfrak{I}_{x,y} + \omega \subseteq \mathfrak{I}_{x,y}$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\Re(\omega) \leq 0$. Moreover, $\mathfrak{I}_{x,y}$ and (2) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{I}_{x,y}$ we have $\mathfrak{I}_{x,y} + dh \subseteq \mathfrak{I}_{x,y}$ and*

$$L(\mathbb{I}_{x,y})(\eta + dh) = L(\mathbb{I}_{x,y})(\eta) e^{h(y)-h(x)}.$$

The restriction $L(\mathbb{I}_{x,y}) : \mathring{\mathfrak{I}}_{x,y} \to \mathbb{C}$ is holomorphic.[1] *If ω is Lyapunov for X then $\mathfrak{I}_{x,y} + \omega \subseteq \mathring{\mathfrak{I}}_{x,y}$, and for all $\eta \in \mathfrak{I}_{x,y}$*

$$\lim_{t\to 0^+} L(\mathbb{I}_{x,y})(\eta + t\omega) = L(\mathbb{I}_{x,y})(\eta). \tag{3}$$

Particularly, $\mathring{\mathfrak{I}}_{x,y} \subseteq \mathfrak{I}_{x,y}$ is dense, and the function $L(\mathbb{I}_{x,y}) : \mathfrak{I}_{x,y} \to \mathbb{C}$ is completely determined by its restriction to $\mathring{\mathfrak{I}}_{x,y}$.

Remark 1. In view of the gauge invariance $L(\mathbb{I}_{x,y})$ can be regarded as a partially defined holomorphic function on the finite dimensional vector space $H^1(M;\mathbb{C}) \times \mathbb{C}$.

[1] For a definition of holomorphicity in infinite dimensions see [8].

For a vector field X which satisfies L, MS and NCT we define its *counting function of closed trajectories* by

$$\mathbb{P} = \mathbb{P}^X : [S^1, M] \to \mathbb{Q}, \qquad \mathbb{P}(\gamma) := \sum_{\sigma \in \gamma} \frac{\epsilon(\sigma)}{p(\sigma)}.$$

Here $[S^1, M]$ denotes the space of homotopy classes of maps $S^1 \to M$, and the sum is over all closed trajectories σ in the homotopy class $\gamma \in [S^1, M]$. Note that these sums are finite in view of Proposition 5. Moreover, define

$$h_*\mathbb{P} : H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z})) \to \mathbb{Q}, \qquad (h_*\mathbb{P})(a) := \sum_{h(\gamma)=a} \mathbb{P}(\gamma)$$

where $h : [S^1, M] \to H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))$, and the sum is over all $\gamma \in [S^1, M]$ for which $h(\gamma) = a$. Note that these are finite sums in view of Proposition 5.

Consider the 'Laplace transform' of $h_*\mathbb{P}$,

$$L(h_*\mathbb{P}) : \mathfrak{P} \to \mathbb{C}, \qquad L(h_*\mathbb{P})(\eta) := \sum_{a \in H_1(M;\mathbb{Z})/\operatorname{Tor}(H_1(M;\mathbb{Z}))} (h_*\mathbb{P})(a) e^{\eta(a)} \tag{4}$$

where $\mathfrak{P} = \mathfrak{P}^X \subseteq \mathcal{Z}^1(M;\mathbb{C})$ denotes the subset of closed one forms η for which this sum converges absolutely.[2] Let $\mathring{\mathfrak{P}}$ denote the interior of $\mathfrak{P}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞–topology. Analogously to Proposition 6 we have

Proposition 7. *The set $\mathfrak{P}$ (and hence $\mathring{\mathfrak{P}}$) is convex and we have $\mathfrak{P} + \omega \subseteq \mathfrak{P}$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\Re(\omega) \leq 0$. Moreover, $\mathfrak{P}$ and (4) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{P}$ we have*

$$L(h_*\mathbb{P})(\eta + dh) = L(h_*\mathbb{P})(\eta).$$

The restriction $L(h_*\mathbb{P}) : \mathring{\mathfrak{P}} \to \mathbb{C}$ is holomorphic. If ω is Lyapunov for X then $\mathfrak{P} + \omega \subseteq \mathring{\mathfrak{P}}$, and for all $\eta \in \mathfrak{P}$

$$\lim_{t \to 0^+} L(h_*\mathbb{P})(\eta + t\omega) = L(h_*\mathbb{P})(\eta). \tag{5}$$

Particularly, $\mathring{\mathfrak{P}} \subseteq \mathfrak{P}$ is dense, and the function $L(h_*\mathbb{P}) : \mathfrak{P} \to \mathbb{C}$ is completely determined by its restriction to $\mathring{\mathfrak{P}}$.

Remark 2. In view of the gauge invariance $L(h_*\mathbb{P})$ can be regarded as a partially defined holomorphic function on the finite dimensional vector space $H^1(M;\mathbb{C})$.

For $x \in \mathcal{X}$ let $L^1(W_x^-)$ denote the space of absolutely integrable functions $W_x^- \to \mathbb{C}$ with respect to the measure induced from the Riemannian metric $(i_x^-)^*g$, where g is a Riemannian metric on M. The space $L^1(W_x^-)$ does not depend on g. For a closed one form $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ let $h_x^\eta : W_x^- \to \mathbb{C}$ denote the unique smooth function which satisfies $h_x^\eta(x) = 0$ and $dh_x^\eta = (i_x^-)^*\eta$. For $x \in \mathcal{X}$ define

$$\mathfrak{R}_x = \mathfrak{R}_x^X := \left\{\eta \in \mathcal{Z}^1(M;\mathbb{C}) \;\middle|\; e^{h_x^\eta} \in L^1(W_x^-)\right\},$$

and set $\mathfrak{R} := \bigcap_{x \in \mathcal{X}} \mathfrak{R}_x$. Moreover, let $\mathring{\mathfrak{R}}_x$ resp. $\mathring{\mathfrak{R}} = \bigcap_{x \in \mathcal{X}} \mathring{\mathfrak{R}}_x$ denote the interior of $\mathfrak{R}_x$ resp. $\mathfrak{R}$ in $\mathcal{Z}^1(M;\mathbb{C})$ equipped with the C^∞–topology.

[2] We will see that $L(h_*\mathbb{P})(\eta)$ converges absolutely in some interesting cases, see Theorem 3 below. However, our arguments do not suffice to prove (absolute) convergence of $L(\mathbb{P})(\eta) := \sum_{\gamma \in [S^1,M]} \mathbb{P}(\gamma) e^{\eta(\gamma)}$. Of course $L(h_*\mathbb{P})(\eta) = L(\mathbb{P})(\eta)$, provided the latter converges absolutely.

For $\alpha \in \Omega^*(M;\mathbb{C})$ consider the 'Laplace transform' of $(i_x^-)^*\alpha \in \Omega^*(W_x^-;\mathbb{C})$,

$$L((i_x^-)^*\alpha) : \mathfrak{R}_x \to \mathbb{C}, \qquad L((i_x^-)^*\alpha)(\eta) := \int_{W_x^-} e^{h_x^\eta} \cdot (i_x^-)^*\alpha. \tag{6}$$

Note that these integrals converge absolutely for $\eta \in \mathfrak{R}_x$. Analogously to Propositions 6 and 7 we have

Proposition 8. *The set $\mathfrak{R}_x$ (and hence $\mathring{\mathfrak{R}}_x$) is convex and we have $\mathfrak{R}_x + \omega \subseteq \mathfrak{R}_x$ for all $\omega \in \mathcal{Z}^1(M;\mathbb{C})$ with $\Re(\omega) \leq 0$. Moreover, $\mathfrak{R}_x$ and (6) are gauge invariant, i.e. for $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{R}_x$ we have*

$$L((i_x^-)^*\alpha)(\eta + dh) = L((i_x^-)^*(e^h\alpha))(\eta)e^{-h(x)}.$$

The restriction $L((i_x^-)^*\alpha) : \mathring{\mathfrak{R}}_x \to \mathbb{C}$ is holomorphic. If ω is Lyapunov for X then $\mathfrak{R}_x + \omega \subseteq \mathring{\mathfrak{R}}_x$, and for all $\eta \in \mathfrak{R}_x$

$$\lim_{t \to 0^+} L((i_x^-)^*\alpha)(\eta + t\omega) = L((i_x^-)^*\alpha)(\eta). \tag{7}$$

Particularly, if X satisfies L, then $\mathring{\mathfrak{R}}_x \subseteq \mathfrak{R}_x$ is dense, and the function $L((i_x^-)^*\alpha) : \mathfrak{R}_x \to \mathbb{C}$ is completely determined by its restriction to $\mathring{\mathfrak{R}}_x$.

Be aware however, that without further assumptions the sets $\mathfrak{I}$, $\mathfrak{P}$ and $\mathfrak{R}$ might very well be empty.

1.5. **Morse complex and integration.** Let $\mathbb{C}^{\mathcal{X}} = \mathrm{Maps}(\mathcal{X};\mathbb{C})$ denote the vector space generated by $\mathcal{X}$. Note that $\mathbb{C}^{\mathcal{X}}$ is $\mathbb{Z}$–graded by $\mathbb{C}^{\mathcal{X}} = \bigoplus_q \mathbb{C}^{\mathcal{X}_q}$. For $\eta \in \mathfrak{I}$ define a linear map

$$\delta_\eta = \delta_\eta^{X,\mathcal{O}} : \mathbb{C}^{\mathcal{X}} \to \mathbb{C}^{\mathcal{X}}, \qquad \delta_\eta(f)(x) := \sum_{y \in \mathcal{X}} L(\mathbb{I}_{x,y})(\eta) \cdot f(y)$$

where $f \in \mathbb{C}^{\mathcal{X}}$ and $x \in \mathcal{X}$. In section 4.1 we will prove

Proposition 9. *We have $\delta_\eta^2 = 0$, for all $\eta \in \mathfrak{I}$.*

For a vector field X which satisfies L and MS, a choice of orientations $\mathcal{O}$ and $\eta \in \mathfrak{I}$ we let $C_\eta^*(X;\mathbb{C}) = C_\eta^*(X,\mathcal{O};\mathbb{C})$ denote the complex with underlying vector space $\mathbb{C}^{\mathcal{X}}$ and differential δ_η. Moreover, for $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ let $\Omega_\eta^*(M;\mathbb{C})$ denote the deRham complex with differential $d_\eta\alpha := d\alpha + \eta \wedge \alpha$. For $\eta \in \mathfrak{R}$ define a linear map

$$\mathrm{Int}_\eta = \mathrm{Int}_\eta^{X,\mathcal{O}} : \Omega^*(M;\mathbb{C}) \to \mathbb{C}^{\mathcal{X}}, \qquad \mathrm{Int}_\eta(\alpha)(x) := L((i_x^-)^*\alpha)(\eta)$$

where $\alpha \in \Omega^*(M;\mathbb{C})$ and $x \in \mathcal{X}$.

The following two propositions will be proved in section 4.1.

Proposition 10. *For $\eta \in \mathfrak{R}$ the linear map $\mathrm{Int}_\eta : \Omega^*(M;\mathbb{C}) \to \mathbb{C}^{\mathcal{X}}$ is onto.*

Proposition 11. *For $\eta \in \mathfrak{I} \cap \mathfrak{R}$ the integration is a homomorphism of complexes*

$$\mathrm{Int}_\eta : \Omega_\eta^*(M;\mathbb{C}) \to C_\eta^*(X;\mathbb{C}). \tag{8}$$

To make the gauge invariance more explicit, suppose $h \in C^\infty(M;\mathbb{C})$ and $\eta \in \mathfrak{I}\cap\mathfrak{R}$. Then $\eta+dh \in \mathfrak{I}\cap\mathfrak{R}$, and we have a commutative diagram of homomorphisms of complexes:

$$\begin{tikzcd} \Omega^*_\eta(M;\mathbb{C}) \arrow[r, "\mathrm{Int}_\eta"] & C^*_\eta(X;\mathbb{C}) \\ \Omega^*_{\eta+dh}(M;\mathbb{C}) \arrow[r, "\mathrm{Int}_{\eta+dh}"] \arrow[u, "e^h", "\simeq"'] & C^*_{\eta+dh}(X;\mathbb{C}) \arrow[u, "\simeq", "e^h"'] \end{tikzcd} \tag{9}$$

Let $\Sigma \subseteq \mathfrak{I}\cap\mathfrak{R}$ denote the subset of closed one forms η for which (8) does not induce an isomorphism in cohomology. Note that Σ is gauge invariant, i.e. $\Sigma + dh \subseteq \Sigma$ for $h \in C^\infty(M;\mathbb{C})$.

Suppose U is an open subset of a Fréchet space and let $S \subseteq U$ be a subset. We say S is an *analytic subset* of U if for every point $z \in U$ there exists a neighborhood V of z and finitely many holomorphic functions $f_1,\dots,f_N : V \to \mathbb{C}$ so that $S\cap V = \{v \in V \mid f_1(v) = \cdots = f_N(v) = 0\}$, see [25].

Theorem 1. *Suppose X satisfies* L *and* MS. *Then $\mathfrak{R} \subseteq \mathfrak{I}$. Moreover, $\Sigma \cap \mathring{\mathfrak{R}}$ is an analytic subset of $\mathring{\mathfrak{R}}$. If ω is a Lyapunov form for X and $\eta \in \mathfrak{R}$, then there exists t_0 such that $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for all $t > t_0$. Particularly, the integration* (8) *induces an isomorphism in cohomology for generic $\eta \in \mathfrak{R}$.*

In general (8) will not induce an isomorphism in cohomology for all $\eta \in \mathfrak{R}$. For example one can consider mapping cylinders and a nowhere vanishing X. In this case $\mathfrak{R} = \mathfrak{I} = \mathcal{Z}^1(M;\mathbb{C})$, and the complex $C^*_\eta(X;\mathbb{C})$ is trivial. However, the deRham cohomology is non-trivial for some η, e.g. $\eta = 0$.

1.6. **Exponential growth.** In order to guaranty that $\mathfrak{R}$ is non-trivial we introduce

Definition 4 (Exponential growth, EG)**.** A vector field X is said to have the *exponential growth property at a rest point x* if for some (and then every) Riemannian metric g on M there exists $C \geq 0$ so that $\mathrm{Vol}(B_x(r)) \leq e^{Cr}$, for all $r \geq 0$. Here $B_x(r) \subseteq W^-_x$ denotes the ball of radius r centered at $x \in W^-_x$ with respect to the induced Riemannian metric $(i^-_x)^*g$ on W^-_x. A vector field X is said to have the *exponential growth property,* EG for short, if it has the exponential growth property at all rest points.

For rather trivial reasons every vector field with $\mathfrak{R} \neq \emptyset$ satisfies EG, see Proposition 16. We are interested in the exponential growth property because of the following converse statement which will be proved in section 2.1.

Proposition 12. *If X satisfies* L *and* EG, *then $\mathring{\mathfrak{R}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$, such that $\eta + t\omega \in \mathring{\mathfrak{R}}$ for all $t > t_0$.*

Remark 3. Suppose X satisfies MS, L and EG. Let ω be a Lyapunov form for X, and let $x, y \in \mathcal{X}$. In view of Proposition 12 and Theorem 1 we have $t\omega \in \mathfrak{I}_{x,y}$ for sufficiently large t. Hence

$$\sum_{\gamma \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma) e^{t\omega(\gamma)} \tag{10}$$

converges absolutely for sufficiently large t. Particularly, there exists $C \geq 0$ such that $|\mathbb{I}_{x,y}(\gamma)| \leq e^{-C\omega(\gamma)}$, for all $\gamma \in \mathcal{P}_{x,y}$. Since the sum (10) is over homotopy classes, this is significantly stronger than what was conjectured in [16] and proved in [4] or [17].

Using a result of Pajitnov [17, 18, 19] we will prove the following weak genericity result in section 2.3.

Theorem 2. *Suppose X satisfies* L*. Then, in every C^0–neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies* L*,* MS*,* NCT *and* EG*.*

Conjecture 1. *If X satisfies* L*, then in every C^1–neighborhood of X there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$, and which satisfies* L*,* MS*,* NCT *and* EG*.*

For the sake of Theorem 3 below we have to introduce the *strong exponential growth property.* Consider the bordism $W := M \times [-1, 1]$. Set $\partial_\pm W := M \times \{\pm 1\}$. Let Y be a vector field on W. Assume that there are vector fields $X_\pm$ on M so that $Y(z, s) = X_+(z) + (s - 1)\partial/\partial s$ in a neighborhood of $\partial_+ W$ and so that $Y(z, s) = X_-(z) + (-s - 1)\partial/\partial s$ in a neighborhood of $\partial_- W$. Particularly, Y is tangential to ∂W. Moreover, assume that $ds(Y) < 0$ on $M \times (-1, 1)$. Particularly, there are no zeros or closed trajectories of Y contained in the interior of W. The properties MS, NCT, L and EG make sense for these kind of vector fields on W too.

If X satisfies MS, NCT and L, then it is easy to construct a vector field Y on W as above satisfying MS, NCT and L such that $X_+ = X$ and $X_- = -\operatorname{grad}_{g_0} f$ for a Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$, see Proposition 23 in appendix B. However, even if we assume that X satisfies EG, it is not clear that such a Y can be chosen to have EG. We thus introduce the following, somewhat asymmetric,

Definition 5 (Strong exponential growth, SEG)**.** A vector field X on M is said to have *strong exponential growth*, SEG for short, if there exists a vector field Y on $W = M \times [-1, 1]$ as above satisfying MS, NCT, L and EG such that $X_+ = X$ and $X_- = -\operatorname{grad}_{g_0} f$ for a Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$. Note that SEG implies MS, NCT, L and EG.

Example 1. A vector field without zeros satisfying NCT and L satisfies SEG.

Using the same methods as for Theorem 2 we will in section 2.3 prove

Theorem 2'. *Suppose X satisfies* L*. Then, in every C^0–neighborhood of X, there exists a vector field which coincides with X in a neighborhood of $\mathcal{X}$ and satisfies* SEG*.*

1.7. **Torsion.** Choose a Riemannian metric g on M. Equip the space $\Omega^*(M; \mathbb{C})$ with a weakly non-degenerate bilinear form $b(\alpha, \beta) := \int_M \alpha \wedge \star \beta$. For $\eta \in \Omega^1(M; \mathbb{C})$ let $d^t_\eta : \Omega^*(M; \mathbb{C}) \to \Omega^{*-1}(M; \mathbb{C})$ denote the formal transpose of d_η with respect to this bilinear form. Explicitly, we have $d^t_\eta \alpha = d^* + i_{\sharp\eta}\alpha$, where $\sharp\eta \in \Gamma(TM \otimes \mathbb{C})$ is defined by $g(\sharp\eta, \cdot) = \eta$. Consider the operator $B_\eta = d_\eta \circ d^t_\eta + d^t_\eta \circ d_\eta$. This is a zero order perturbation of the Laplace–Beltrami operator and depends holomorphically on η. Note that the adjoint of B_η with respect to the standard Hermitian structure on $\Omega^*(M; \mathbb{C})$ coincides with $B_{\bar\eta}$, where $\bar\eta$ denotes the complex conjugate of η.[3] Assume from now on that η is closed. Then B_η commutes with d_η and d^t_η.

[3] This is called a 'self adjoint holomorphic' family in [12].

For $\lambda \in \mathbb{C}$ let $E_\eta^*(\lambda)$ denote the generalized λ–eigen space of B_η. Recall from elliptic theory that $E_\eta^*(\lambda)$ is finite dimensional graded subspace $E_\eta^*(\lambda) \subseteq \Omega^*(M;\mathbb{C})$. The differentials d_η and d_η^t preserve $E_\eta^*(\lambda)$ since they commute with B_η. Note however that the restriction of $B_\eta - \lambda$ to $E_\eta^*(\lambda)$ will in general only be nilpotent. If $\lambda_1 \neq \lambda_2$ then $E_\eta^*(\lambda_1)$ and $E_\eta^*(\lambda_2)$ are orthogonal with respect to b since B_η is symmetric with respect to b. It follows that b restricts to a non-degenerate bilinear form on every $E_\eta^*(\lambda)$. In section 4.3 we will prove

Proposition 13. *Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$. Then $E_\eta^*(\lambda)$ is acyclic for all $\lambda \neq 0$, and the inclusion $E_\eta^*(0) \to \Omega_\eta^*(M;\mathbb{C})$ is a quasi isomorphism.*

If $\eta \in \mathfrak{R} \setminus \Sigma$ then, in view of Proposition 13, the restriction of the integration

$$\mathrm{Int}_\eta|_{E_\eta^*(0)} : E_\eta^*(0) \to C_\eta^*(X;\mathbb{C}) \tag{11}$$

is a quasi isomorphism. Recall that an endomorphism preserving a non-degenerate bilinear form has determinant ± 1. Therefore b determines an equivalence class of graded bases [15] in $E_\eta^*(0)$. Moreover, the indicator functions provide a graded basis of $C_\eta^*(X;\mathbb{C})$. Let $\pm T(\mathrm{Int}_\eta|_{E_\eta^*(0)}) \in \mathbb{C} \setminus 0$ denote the relative torsion of (11) with respect to these bases, see [15]. Moreover, define a complex valued Ray–Singer [21] kind of torsion

$$(T_{\eta,g}^{\mathrm{an}})^2 := \prod_q \big(\det{}' B_\eta^q\big)^{(-1)^{q+1}q} \in \mathbb{C} \setminus 0$$

where $\det{}' B_\eta^q$ denotes the zeta regularized product [10, 3] of all non-zero eigen values of $B_\eta^q : \Omega^q(M;\mathbb{C}) \to \Omega^q(M;\mathbb{C})$, computed with respect to the Agmon angle π. In section 3 we will provide a regularization $R(\eta, X, g)$ of the possibly divergent integral

$$\int_{M \setminus \mathcal{X}} \eta \wedge X^*\Psi(g),$$

where $\Psi(g) \in \Omega^{n-1}(TM \setminus M; \mathcal{O}_M)$ denotes the global angular form. Finally, set

$$(T\,\mathrm{Int}_\eta)^2 = (T\,\mathrm{Int}_{\eta,g}^{X,\mathcal{O}})^2 := (T(\mathrm{Int}_\eta|_{E_\eta^*(0)}))^2 \cdot (T_{\eta,g}^{\mathrm{an}})^2 \cdot (e^{-R(\eta,X,g)})^2. \tag{12}$$

In section 5.1 we will show

Proposition 14. *The quantity (12) does not depend on g. It defines a function*

$$(T\,\mathrm{Int})^2 : \mathfrak{R} \setminus \Sigma \to \mathbb{C} \setminus 0 \tag{13}$$

which satisfies $(T\,\mathrm{Int}_{\bar\eta})^2 = \overline{(T\,\mathrm{Int}_\eta)^2}$, and which is gauge invariant, i.e. for $\eta \in \mathfrak{R}\setminus\Sigma$ and $h \in C^\infty(M;\mathbb{C})$ we have

$$(T\,\mathrm{Int}_{\eta+dh})^2 = (T\,\mathrm{Int}_\eta)^2.$$

The restriction $(T\,\mathrm{Int})^2 : \mathring{\mathfrak{R}} \setminus \Sigma \to \mathbb{C} \setminus 0$ is holomorphic. If ω is Lyapunov for X and $\eta \in \mathfrak{R} \setminus \Sigma$ then for sufficiently small $t > 0$ we have $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$, and

$$\lim_{t \to 0^+} (T\,\mathrm{Int}_{\eta+t\omega})^2 = (T\,\mathrm{Int}_\eta)^2. \tag{14}$$

Remark 4. In view of the gauge invariance $(T\,\mathrm{Int})^2$ can be regarded as a partially defined holomorphic function on the finite dimensional vector space $H^1(M;\mathbb{C})$.

The rest of section 5 is dedicated to the proof of

Theorem 3. *Suppose X satisfies* SEG*. Then $\mathring{\mathfrak{P}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$ such that $\eta + t\omega \in \mathring{\mathfrak{P}}$ for all $t > t_0$. Moreover, for $\eta \in (\mathfrak{R} \setminus \Sigma) \cap \mathfrak{P}$*

$$(e^{L(h_*\mathbb{P})(\eta)})^2 = (T\,\mathrm{Int}_\eta)^2.$$

Particularly, the zeta function $\eta \mapsto (e^{L(h_*\mathbb{P})(\eta)})^2$ admits an analytic continuation to $\mathring{\mathfrak{R}}$ with zeros and singularities contained in the proper analytic subset $\mathring{\mathfrak{R}} \cap \Sigma$.

Example 2. Let $f : N \to N$ be a diffeomorphism, and let M denote the mapping cylinder obtained by glueing the boundaries of $N \times [0,1]$ with the help of f. Let $X = \partial/\partial t$, where t denotes the coordinate in $[0,1]$. Since it has no zeros at all X satisfies MS and SEG. Moreover, X satisfies NCT iff all fixed points of f^k are non-degenerate for all $k \in \mathbb{N}$. In this case we have

$$e^{L(h_*\mathbb{P})(zdt)} = \exp \sum_{k=1}^{\infty} \sum_{x \in \mathrm{Fix}(f^k)} \frac{\mathrm{ind}_x(f^k)}{k} (e^z)^k = \zeta_f(e^z)$$

where ζ_f denotes the Lefschetz zeta function associated with f. Theorem 3 implies that for generic z we have $\pm T^{\mathrm{an}}_{zdt,g} = e^{zR(dt,X,g)}\zeta_f(e^z)$. This was already established by Marcsik in his thesis [13].

In the acyclic case it suffices to assume EG.

Theorem 3'. *Suppose X satisfies* L, MS, NCT *and* EG*. Assume that there exists $\eta_0 \in \mathcal{Z}^1(M;\mathbb{C})$ such that $H^*_{\eta_0}(M;\mathbb{C}) = 0$. Then $\mathring{\mathfrak{P}}$ is non-empty. More precisely, if ω is a Lyapunov form for X and $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, then there exists $t_0 \in \mathbb{R}$ such that $\eta + t\omega \in \mathring{\mathfrak{P}}$ for all $t > t_0$. Moreover, for $\eta \in (\mathfrak{R} \setminus \Sigma) \cap \mathfrak{P}$*

$$(e^{L(h_*\mathbb{P})(\eta)})^2 = (T\,\mathrm{Int}_\eta)^2.$$

Particularly, the zeta function $\eta \mapsto (e^{L(h_*\mathbb{P})(\eta)})^2$ admits an analytic continuation to $\mathring{\mathfrak{R}}$ with zeros and singularities contained in the proper analytic subset $\mathring{\mathfrak{R}} \cap \Sigma$.

Conjecture 2. *Theorem 3' remains true without the acyclicity assumption.*

Remark 5. Suppose X satisfies SEG. Let ω be a Lyapunov form for X. In view of Theorem 3

$$\sum_{a \in H_1(M;\mathbb{Z})/\mathrm{Tor}(H_1(M;\mathbb{Z}))} (h_*\mathbb{P})(a) e^{t\omega(a)}$$

converges absolutely for sufficiently large t. Particularly, there exists $C \geq 0$ such that $|(h_*\mathbb{P})(a)| \leq e^{-C\omega(a)}$ for all $a \in H_1(M;\mathbb{Z})/\mathrm{Tor}(H_1(M;\mathbb{Z}))$. Note that for Pajitnov's class of vector fields the Laplace transform $L(h_*\mathbb{P})$ actually is a rational function [19].

1.8. **Interpretation via classical Dirichlet series.** Restricting to affine lines $\eta + z\omega$ in $\mathcal{Z}^1(M;\mathbb{C})$ we can interpret the above results in terms of classical Laplace transforms.

More precisely, let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$, and suppose ω is a Lyapunov form for X. If X satisfies EG then there exists $\rho < \infty$ so that for all $x \in \mathcal{X}$ and all $\alpha \in \Omega^*(M;\mathbb{C})$ the Laplace transform

$$\mathrm{Int}_{\eta+z\omega}(\alpha)(x) = \int_0^\infty e^{-z\lambda} d\big((-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)\big)(\lambda) \tag{15}$$

has abscissa of absolute convergence at most ρ, i.e. (15) converges absolutely for all $\Re(z) > \rho$, see Proposition 12. Here $(-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)$ denotes the push forward of $e^{h^\eta_x}(i^-_x)^*\alpha$ considered as measure on W^-_x via the map $-h^\omega_x : W^-_x \to [0,\infty)$. The integral in (15) is supposed to denote the Laplace transform of $(-h^\omega_x)_*(e^{h^\eta_x}(i^-_x)^*\alpha)$.

Assume in addition that X satisfies MS, and let $x, y \in \mathcal{X}$. Consider the mapping $-\omega : \mathcal{P}_{x,y} \to \mathbb{R}$ and define a measure with discrete support, $(-\omega)_*(\mathbb{I}_{x,y}e^\eta) : [0,\infty) \to \mathbb{C}$ by

$$\big((-\omega)_*(\mathbb{I}_{x,y}e^\eta)\big)(\lambda) := \sum_{\{\gamma \in \mathcal{P}_{x,y} \mid -\omega(\gamma)=\lambda\}} \mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)}. \tag{16}$$

In view of Theorem 1 its Laplace transform, i.e. the Dirichlet series

$$L(\mathbb{I}_{x,y})(\eta + z\omega) = \sum_{\lambda \in [0,\infty)} e^{-z\lambda}\big((-\omega)_*(\mathbb{I}_{x,y}e^\eta)\big)(\lambda), \tag{17}$$

has abscissa of absolute convergence at most ρ, i.e. (17) converges absolutely for all $\Re(z) > \rho$. Particularly, we see that from the germ at $+\infty$ of the holomorphic function $z \mapsto \delta_{\eta+z\omega}$ one can recover, via inverse Laplace transform, a good amount of the counting functions $\mathbb{I}_{x,y}$, namely the numbers (16) for all $\lambda \in \mathbb{R}$ and all $x, y \in \mathcal{X}$.

Assume in addition that X satisfies NCT and SEG. Consider the mapping $-\omega : [S^1, M] \to \mathbb{R}$ and define a measure with discrete support, $(-\omega)_*(\mathbb{P}e^\eta) : [0,\infty) \to \mathbb{C}$ by

$$\big((-\omega)_*(\mathbb{P}e^\eta)\big)(\lambda) := \sum_{\{\gamma \in [S^1,M] \mid -\omega(\gamma)=\lambda\}} \mathbb{P}(\gamma)e^{\eta(\gamma)}. \tag{18}$$

In view of Theorem 3 its Laplace transform, i.e. the Dirichlet series

$$L(h_*\mathbb{P})(\eta + z\omega) = \sum_{\lambda \in [0,\infty)} e^{-z\lambda}\big((-\omega)_*(\mathbb{P}e^\eta)\big)(\lambda), \tag{19}$$

has finite abscissa of convergence, i.e. for sufficiently large $\Re(z)$ the series (19) converges absolutely. Moreover, $z \mapsto e^{L(h_*\mathbb{P})(\eta+z\omega)}$ admits an analytic continuation with isolated singularities to $\{z \in \mathbb{C} \mid \Re(z) > \rho\}$. Particularly, we see that from the germ at $+\infty$ of the holomorphic function $z \mapsto T\,\mathrm{Int}_{\eta+z\omega}$ one can recover, via inverse Laplace transform, a good amount of the counting functions $\mathbb{P}$, namely the numbers (18) for all $\lambda \in \mathbb{R}$.

1.9. **Relation with Witten–Helffer–Sjöstrand theory.** The above results provide some useful additions to Witten–Helffer–Sjöstrand theory. Recall that Witten–Helffer–Sjöstrand theory on a closed Riemannian manifold (M, g) can be extended from a Morse function $f : M \to \mathbb{R}$ to a closed Morse one form $\omega \in \mathcal{Z}^1(M;\mathbb{R})$, see [4]. Precisely, let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ and consider the one parameter family of elliptic complexes $\Omega^*_{\eta+t\omega}(M;\mathbb{C}) := (\Omega^*(M;\mathbb{C}), d_{\eta+t\omega})$, equipped with the Hermitian scalar product induced by the Riemannian metric g. Then, for sufficiently large t, we have a canonic orthogonal decomposition of cochain complexes

$$\Omega^*_{\eta+t\omega}(M;\mathbb{C}) = \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \oplus \Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C}).$$

If X is a smooth vector field with all the above properties including exponential growth and having ω as a Lyapunov closed one form then the restriction of the integration

$$\mathrm{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C})$$

is an isomorphism for sufficiently large t. In particular the canonical base of $\mathbb{C}^{\mathcal{X}}$ provides a canonical base $\{E_x(t)\}_{x\in\mathcal{X}}$ for the small complex $\Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C})$, and the differential $d_{\eta+t\omega}$, when written in this base is a matrix whose components are the Laplace transforms $L(\mathbb{I}_{x,y})(\eta+t\omega)$. One can formulate this fact as: *The counting of instantons is taken care of by the small complex.*

The large complex $\Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C})$ is acyclic and has Ray–Singer torsion which in the case of a Morse function $\omega = df$ is exactly $tR(\omega,X,g)+\log\mathrm{Vol}(t)$ where $\log\mathrm{Vol}(t) := \sum(-1)^q\log\mathrm{Vol}_q(t)$ and $\mathrm{Vol}_q(t)$ denotes the volume of the canonical base $\{E_x(t)\}_{x\in\mathcal{X}_q}$. If ω is a non-exact form, the above expression has an additional term $\Re(L(h_*\mathbb{P})(\eta+t\omega))$. One can formulate this fact as: *The counting of closed trajectories is taken care of by the large complex.*

2. Exponential growth

In section 2.1 we will reformulate the exponential growth condition, see Proposition 15, and show that EG implies $\mathring{\mathfrak{R}} \neq \emptyset$, i.e. prove Proposition 12. In section 2.2 we will present a criterion which when satisfied implies exponential growth, see Proposition 17. This criterion is satisfied by a class of vector fields introduced by Pajitnov. A theorem of Pajitnov tells that his class is C^0–generic. Using this we will give a proof of Theorem 2 in section 2.3.

2.1. **Exponential growth.** Let g be a Riemannian metric on M, and let $x\in\mathcal{X}$ be a zero of X. Let $g_x := (i_x^-)^*g$ denote the induced Riemannian metric on the unstable manifold W_x^-. Let $r_x^g := \mathrm{dist}_{g_x}(x,\cdot) : W_x^- \to [0,\infty)$ denote the distance to x. Let $B_x^g(s) := \{y\in W_x^- \mid r_x^g(y)\leq s\}$ denote the ball of radius s, and let $\mathrm{Vol}_{g_x}(B_x^g(s))$ denote its volume. Recall from Definition 4 that X has the exponential growth property at x if there exists $C\geq 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s))\leq e^{Cs}$ for all $s\geq 0$. This does not depend on g although C does.

Proposition 15. *Let X be a vector field and suppose $x\in\mathcal{X}$. Then X has exponential growth property at x iff for one (and hence every) Riemannian metric g on M there exists a constant $C\geq 0$ such that $e^{-Cr_x^g}\in L^1(W_x^-)$.*

This proposition is an immediate consequence of the following two lemmas.

Lemma 1. *Suppose there exists $C\geq 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s))\leq e^{Cs}$ for all $s\geq 0$. Then $e^{-(C+\epsilon)r_x^g}\in L^1(W_x^-)$ for every $\epsilon>0$.*

Proof. Clearly

$$\int_{W_x^-} e^{-(C+\epsilon)r_x^g} = \sum_{n=0}^{\infty}\int_{B_x^g(n+1)\setminus B_x^g(n)} e^{-(C+\epsilon)r_x^g}. \tag{20}$$

On $B_x^g(n+1)\setminus B_x^g(n)$ we have $e^{-(C+\epsilon)r_x^g}\leq e^{-(C+\epsilon)n}$ and thus

$$\begin{aligned}\int_{B_x^g(n+1)\setminus B_x^g(n)} e^{-(C+\epsilon)r_x^g} &\leq \mathrm{Vol}_{g_x}(B_x^g(n+1))e^{-(C+\epsilon)n}\\ &\leq e^{C(n+1)}e^{-(C+\epsilon)n} = e^Ce^{-\epsilon n}\end{aligned}$$

So (20) implies

$$\int_{W_x^-} e^{-(C+\epsilon)r_x^g}\leq e^C\sum_{n=0}^{\infty}e^{-\epsilon n} = e^C(1-e^{-\epsilon})^{-1}<\infty$$

and thus $e^{-(C+\epsilon)r_x^g} \in L^1(W_x^-)$. $\square$

Lemma 2. *Suppose we have $C \geq 0$ such that $e^{-Cr_x^g} \in L^1(W_x^-)$. Then there exists a constant $A > 0$ such that $\mathrm{Vol}_{g_x}(B_x^g(s)) \leq Ae^{Cs}$ for all $s \geq 0$.*

Proof. We start with the following estimate for $N \in \mathbb{N}$:

$$\begin{aligned}
&\mathrm{Vol}_{g_x}(B_x^g(N+1))e^{-C(N+1)} = \\
&\qquad = \sum_{n=0}^{N} \mathrm{Vol}_{g_x}(B_x^g(n+1))e^{-C(n+1)} - \mathrm{Vol}_{g_x}(B_x^g(n))e^{-Cn} \\
&\qquad \leq \sum_{n=0}^{\infty} \big(\mathrm{Vol}_{g_x}(B_x^g(n+1)) - \mathrm{Vol}_{g_x}(B_x^g(n))\big)e^{-C(n+1)} \\
&\qquad = \sum_{n=0}^{\infty} \mathrm{Vol}_{g_x}\big(B_x^g(n+1) \setminus B_x^g(n)\big)e^{-C(n+1)} \\
&\qquad \leq \sum_{n=0}^{\infty} \int_{B_x^g(n+1)\setminus B_x^g(n)} e^{-Cr_x^g} = \int_{W_x^-} e^{-Cr_x^g}
\end{aligned}$$

Given $s \geq 0$ we choose an integer N with $N \leq s \leq N+1$. Then

$$\mathrm{Vol}_{g_x}(B_x^g(s))e^{-Cs} \leq \mathrm{Vol}_{g_x}(B_x^g(N+1))e^{-CN} = e^C \,\mathrm{Vol}_{g_x}(B_x^g(N+1))e^{-C(N+1)},$$

and thus $\mathrm{Vol}(B_x^g(s))e^{-Cs} \leq e^C \int_{W_x^-} e^{-Cr_x^g} =: A < \infty$. We conclude $\mathrm{Vol}_{g_x}(B_x^g(s)) \leq Ae^{Cs}$ for all $s \geq 0$. $\square$

Let $\eta \in \mathcal{Z}^1(M;\mathbb{C})$ be a closed one form. Recall that $h_x^\eta : W_x^- \to \mathbb{C}$ denotes the unique smooth function which satisfies $dh_x^\eta = (i_x^-)^*\eta$ and $h_x^\eta(x) = 0$. Recall from section 1.4 that $\eta \in \mathfrak{R}_x$ if $e^{h_x^\eta} \in L^1(W_x^-)$.

Proposition 16. *Let X be a vector field and suppose $x \in \mathcal{X}$. If $\mathfrak{R}_x \neq \emptyset$ then X has exponential growth at x. Particularly, if $\mathfrak{R} \neq \emptyset$ then X satisfies* EG.

This proposition follows immediately from Proposition 15 and the following

Lemma 3. *There exists a constant $C = C_{g,\eta} \geq 0$ such that $|h_x^\eta| \leq Cr_x^g$.*

Proof. Suppose $y \in W_x^-$. For every path $\gamma : [0,1] \to W_x^-$ with $\gamma(0) = x$ and $\gamma(1) = y$ we find

$$|h_x^\eta(y)| = \Big|\int_0^1 (dh_x^\eta)(\gamma'(t))dt\Big| \leq \|\eta\| \int_0^1 |\gamma'(t)|dt = \|\eta\| \,\mathrm{length}(\gamma)$$

where $\|\eta\| := \sup_{z\in M} |\eta_z|_g$. We conclude $|h_x^\eta(y)| \leq \|\eta\| r_x^g(y)$. Hence we can take $C := \|\eta\|$. $\square$

Let us recall the following crucial estimate from [4, Lemma 3].

Lemma 4. *Suppose ω is a Lyapunov for X, and suppose $x \in \mathcal{X}$. Then there exist $\epsilon = \epsilon_{g,\omega} > 0$ and $C = C_{g,\omega} \geq 0$ such that $r_x^g \leq -Ch_x^\omega$ on $W_x^- \setminus B_x^g(\epsilon)$.*

Proof of Proposition 12. Suppose $x \in \mathcal{X}$. In view of Lemma 3 and Lemma 4 there exists $C > 0$ so that $\Re(h_x^\eta + th_x^\omega) \leq (C - t/C)r_x^g$ on $W_x^- \setminus B_x^g(\epsilon)$. Since X has exponential growth at x we have $|e^{h_x^{\eta+t\omega}}| = e^{\Re(h_x^\eta+th_x^\omega)} \leq e^{(C-t/C)r_x^g} \in L^1(W_x^-)$, and hence $\eta+t\omega \in \mathfrak{R}_x$, for sufficiently large t. We conclude that $\eta+t\omega \in \mathring{\mathfrak{R}} = \bigcap_{x\in\mathcal{X}} \mathring{\mathfrak{R}}_x$ for sufficiently large t, see Proposition 8. $\square$

2.2. **Virtual interactions.** Suppose $N \subseteq M$ is an immersed submanifold of dimension q. Let $\mathrm{Gr}_q(TM)$ denote the Grassmannified tangent bundle of M, i.e. the compact space of q–planes in TM. The assignment $z \mapsto T_zN$ provides an immersion $N \subseteq \mathrm{Gr}_q(TM)$. We let $\overline{\mathrm{Gr}(N)} \subseteq \mathrm{Gr}_q(TM)$ denote the closure of its image. Moreover, for a zero $y \in \mathcal{X}$ we let $\mathrm{Gr}_q(T_yW_y^-) \subseteq \mathrm{Gr}_q(TM)$ denote the Grassmannian of q–planes in $T_yW_y^-$ considered as subset of $\mathrm{Gr}_q(TM)$.

Definition 6 (Virtual interaction)**.** For a vector field X and two zeros $x \in \mathcal{X}_q$ and $y \in \mathcal{X}$ we define their *virtual interaction* to be the compact set

$$K_x(y) := \mathrm{Gr}_q(T_yW_y^-) \cap \overline{\mathrm{Gr}(W_x^- \setminus B)}$$

where $B \subseteq W_x^-$ is a compact ball centered at x. Note that $K_x(y)$ does not depend on the choice of B.

Note that $K_x(y)$ is non-empty iff there exists a sequence $z_k \in W_x^-$ so that $\lim_{k\to\infty} z_k = y$ and so that $T_{z_k}W_x^-$ converges to a q–plane in T_yM which is contained in $T_yW_y^-$.

Although we removed B from W_x^- the set $K_x(x)$ might be non-empty. However, if we would not have removed B the set $K_x(x)$ would never be empty for trivial reasons. Because of dimensional reasons we have $K_x(y) = \emptyset$ whenever $\mathrm{ind}(x) > \mathrm{ind}(y)$. Moreover, it is easy to see that $K_x(y) = \emptyset$ whenever $\mathrm{ind}(y) = n$.

We are interested in virtual interactions because of the following

Proposition 17. *Suppose X satisfies L, let $x \in \mathcal{X}$, and assume that the virtual interactions $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$. Then X has exponential growth at x.*

To prove Proposition 17 we will need the following

Lemma 5. *Let (V, g) be an Euclidean vector space and $V = V^+ \oplus V^-$ an orthogonal decomposition. For $\kappa \geq 0$ consider the endomorphism $A_\kappa := \kappa\,\mathrm{id} \oplus -\,\mathrm{id} \in \mathrm{End}(V)$ and the function*

$$\delta^{A_\kappa} : \mathrm{Gr}_q(V) \to \mathbb{R}, \quad \delta^{A_\kappa}(W) := \mathrm{tr}_{g|_W}(p_W^\perp \circ A_\kappa \circ i_W),$$

where $i_W : W \to V$ denotes the inclusion and $p_W^\perp : V \to W$ the orthogonal projection. Suppose we have a compact subset $K \subseteq \mathrm{Gr}_q(V)$ for which $\mathrm{Gr}_q(V^+) \cap K = \emptyset$. Then there exists $\kappa > 0$ and $\epsilon > 0$ with $\delta^{A_\kappa} \leq -\epsilon$ on K.

Proof. Consider the case $\kappa = 0$. Let $W \in \mathrm{Gr}_q(V)$ and choose a $g|_W$ orthonormal base $e_i = (e_i^+, e_i^-) \in V^+ \oplus V^-$, $1 \leq i \leq q$, of W. Then

$$\delta^{A_0}(W) = \sum_{i=1}^{q} g(e_i, A_0 e_i) = -\sum_{i=1}^{q} g(e_i^-, e_i^-).$$

So we see that $\delta^{A_0} \leq 0$ and $\delta^{A_0}(W) = 0$ iff $W \in \mathrm{Gr}_q(V^+)$. Thus $\delta^{A_0}|_K < 0$. Since δ^{A_κ} depends continuously on κ and since K is compact we certainly find $\kappa > 0$ and $\epsilon > 0$ so that $\delta^{A_\kappa}|_K \leq -\epsilon$. □

Proof of Proposition 17. Let $S \subseteq W_x^-$ denote a small sphere centered at x. Let $\tilde{X} := (i_x^-)^* X$ denote the restriction of X to W_x^- and let Φ_t denote the flow of $\tilde{X}$ at time t. Then

$$\varphi : S \times [0, \infty) \to W_x^-, \quad \varphi(x, t) = \varphi_t(x) = \Phi_t(x)$$

parameterizes W_x^- with a small neighborhood of x removed.

Let $\kappa > 0$. For every $y \in \mathcal{X}$ choose a chart $u_y : U_y \to \mathbb{R}^n$ centered at y so that

$$X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} u_y^i \frac{\partial}{\partial u_y^i}.$$

Let g be a Riemannian metric on M which restricts to $\sum_i du_y^i \otimes du_y^i$ on U_y and set $g_x := (i_x^-)^* g$. Then

$$\nabla X|_{U_y} = \kappa \sum_{i \leq \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i} - \sum_{i > \mathrm{ind}(y)} du_y^i \otimes \frac{\partial}{\partial u_y^i}.$$

In view of our assumption $K_x(y) = \emptyset$ for all $y \in \mathcal{X}$ Lemma 5 permits us to choose $\kappa > 0$ and $\epsilon > 0$ so that after possibly shrinking U_y we have

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla \tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)) \cap (i_x^-)^{-1}\Big(\bigcup_{y \in \mathcal{X}} U_y\Big). \tag{21}$$

Let ω be a Lyapunov form for X. Since $\omega(X) < 0$ on $M \setminus \mathcal{X}$, we can choose $\tau > 0$ so that

$$\tau\omega(X) + \mathrm{ind}(x)||\nabla X||_g \leq -\epsilon < 0 \quad \text{on} \quad M \setminus \bigcup_{y \in \mathcal{X}} U_y. \tag{22}$$

Using $\tau \tilde{X} \cdot h_x^\omega \leq 0$ and

$$\mathrm{div}_{g_x}(\tilde{X}) = \mathrm{tr}_{g_x}(\nabla \tilde{X}) \leq \mathrm{ind}(x)||\nabla \tilde{X}||_{g_x} \leq \mathrm{ind}(x)||\nabla X||_g$$

(21) and (22) yield

$$\tau \tilde{X} \cdot h_x^\omega + \mathrm{div}_{g_x}(\tilde{X}) \leq -\epsilon < 0 \quad \text{on} \quad \varphi(S \times [0, \infty)). \tag{23}$$

Choose an orientation of W_x^- and let μ denote the volume form on W_x^- induced by g_x. Consider the function

$$\psi : [0, \infty) \to \mathbb{R}, \quad \psi(t) := \int_{\varphi(S \times [0,t])} e^{\tau h_x^\omega} \mu \geq 0.$$

For its first derivative we find

$$\psi'(t) = \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu > 0$$

and for the second derivative, using (23),

$$\begin{aligned} \psi''(t) &= \int_{\varphi_t(S)} \big(\tau \tilde{X} \cdot h_x^\omega + \mathrm{div}_{g_x}(\tilde{X})\big) e^{\tau h_x^\omega} i_{\tilde{X}} \mu \\ &\leq -\epsilon \int_{\varphi_t(S)} e^{\tau h_x^\omega} i_{\tilde{X}} \mu = -\epsilon \psi'(t). \end{aligned}$$

So $(\ln \circ \psi')'(t) \leq -\epsilon$ hence $\psi'(t) \leq \psi'(0) e^{-\epsilon t}$ and integrating again we find

$$\psi(t) \leq \psi(0) + \psi'(0)(1 - e^{-\epsilon t})/\epsilon \leq \psi'(0)/\epsilon.$$

So we have $e^{\tau h_x^\omega} \in L^1\big(\varphi(S \times [0, \infty)\big)$ and hence $e^{\tau h_x^\omega} \in L^1(W_x^-)$ too. We conclude $\tau\omega \in \mathfrak{R}_x$. From Proposition 16 we see that X has exponential growth at x. $\square$

2.3. **Proof of Theorem 2.** Let X be a vector field satisfying L. Using Proposition 3 we find a Lyapunov form ω for X with integral cohomology class. Hence there exists a smooth function $\theta : M \to S^1$ so that $\omega = d\theta$ is Lyapunov for X.

Choose a regular value $s_0 \in S^1$ of θ. Set $V := \theta^{-1}(s_0)$ and let W denote the bordism obtained by cutting M along V, i.e. $\partial_\pm W = V$. This construction provides a diffeomorphism $\Phi : \partial_- W \to \partial_+ W$. Such a pair (W, Φ) is called a cyclic bordism in [18]. When referring to Pajitnov's work below we will make precise references to [18] but see also [17] and [19].

We continue to denote by X the vector field on W induced from X, and by $\theta : W \to [0, 1]$ the map induced from θ. We are exactly in the situation of Pajitnov: $-X$ is a θ–gradient in the sense of [18, Definition 2.3]. In view of [18, Theorem 4.8] we find, arbitrarily C^0–close to X, a smooth vector field Y on W which coincides with X in a neighborhood of $\mathcal{X} \cup \partial W$, and so that $-Y$ is a θ–gradient satisfying condition $(\mathfrak{C}\mathcal{Y})$ from [18, Definition 4.7]. For the reader's convenience we will below review Pajitnov's condition $(\mathfrak{C}\mathcal{Y})$ in more details.

Since X and Y coincide in a neighborhood of ∂W, Y defines a vector field on M which we denote by Y too. Clearly, $\omega = d\theta$ is Lyapunov for Y. Using the C^0–openness statement in [18, Theorem 4.8] and Proposition 1 we may, by performing a C^1–small perturbation of Y, assume that Y in addition satisfies MS and NCT. Obviously, condition $(\mathfrak{C}\mathcal{Y})$ implies that $K_x^Y(y) = \emptyset$ whenever $\mathrm{ind}(x) \leq \mathrm{ind}(y)$, see below. For trivial reasons we have $K_x^Y(y) = \emptyset$ whenever $\mathrm{ind}(x) > \mathrm{ind}(y)$. It now follows from Proposition 17 that Y satisfies EG too. This completes the proof of Theorem 2.

We will now turn to Pajitnov's condition $(\mathfrak{C}\mathcal{Y})$, see [18, Definition 4.7]. Recall first that a smooth vector field $-X$ on a closed manifold M which satisfies MS and is an f–gradient in the sense of [18, Definition 2.3] for some Morse function f, provides a partition of the manifold in cells, the unstable sets of the rest points of $-X$. We will refer to such a partition as a *generalized triangulation.* The union of the unstable sets of $-X$ of rest points of index at most k represents the k–skeleton and will be denoted [18, section 2.1.4] by

$$D(\mathrm{ind} \leq k, -X).$$

¿From this perspective the dual triangulation is associated to the vector field X which has the same properties with respect to $-f$.

Given a Riemannian metric g on M we will also write

$$B_\delta(\mathrm{ind} \leq k, -X) \quad \text{resp.} \quad D_\delta(\mathrm{ind} \leq k, -X)$$

for the open resp. closed δ–thickening of $D(\mathrm{ind} \leq k, -X)$. They are the sets of points which lie on trajectories of $-X$ which depart from the open resp. closed ball of radius δ centered at the rest points of Morse index at most k. It is not hard to see [18, Proposition 2.30] that when $\delta \to 0$ the sets $B_\delta(\mathrm{ind} \leq k, -X)$ resp. $D_\delta(\mathrm{ind} \leq k, -X)$ provide a fundamental system of open resp. closed neighborhoods of $D(\mathrm{ind} \leq k, -X)$. We also write

$$C_\delta(\mathrm{ind} \leq k, -X) := M \setminus B_\delta(\mathrm{ind} \leq n - k - 1, X).$$

Note that for sufficiently small $\delta > 0$

$$B_\delta(\mathrm{ind} \leq k, -X) \subseteq C_\delta(\mathrm{ind} \leq k, -X).$$

These definitions and notations can be also used in the case of a bordism, see [18] and [15]. Denote by $U_\pm \subseteq \partial_\pm W$ the set of points $y \in \partial_\pm W$ so that the trajectory of the vector field $-X$ trough y arrives resp. departs from $\partial W_\mp$ at some positive resp. negative time t. They are open sets. Following Pajitnov's notation we denote by $(-X)^{\rightsquigarrow}: U_+ \to U_-$ resp. $X^{\rightsquigarrow}: U_- \to U_+$ the obvious diffeomorphisms induced by the flow of X which are inverse one to the other. If $A \subseteq \partial_\pm W$ we write for simplicity $(\mp X)^{\rightsquigarrow}(A)$ instead of $(\mp X)^{\rightsquigarrow}(A \cap U_\pm)$.

Definition 7 (Property $(\mathfrak{C}\mathcal{Y})$, see [18, Definition 4.7])**.** A gradient like vector field $-X$ on a cyclic bordism (W, Φ) satisfies $(\mathfrak{C}\mathcal{Y})$ if there exist generalized triangulations $X_\pm$ on $\partial_\pm W$ and sufficiently small $\delta > 0$ so that the following hold:

$$\Phi(X_-) = X_+$$

$$X^{\rightsquigarrow}\big(C_\delta(\mathrm{ind} \leq k, X_-)\big) \cup \big(D_\delta(\mathrm{ind} \leq k+1, X) \cap \partial_+ W\big) \subseteq B_\delta(\mathrm{ind} \leq k, X_+) \tag{B+}$$

$$(-X)^{\rightsquigarrow}\big(C_\delta(\mathrm{ind} \leq k, -X_+)\big) \cup \big(D_\delta(\mathrm{ind} \leq k+1, -X) \cap \partial_- W\big) \subseteq B_\delta(\mathrm{ind} \leq k, -X_-) \tag{B−}$$

If the vector field Y on (W, Φ) constructed by the cutting off construction satisfies $(\mathfrak{C}\mathcal{Y})$ then, when regarded on M, it has the following property: Every zero y admits a neighborhood which does not intersect the unstable set of a zero x with $\mathrm{ind}(y) \geq \mathrm{ind}(x)$. Hence the virtual interaction $K_x^Y(y)$ is empty. This is exactly what we used in the derivation of Theorem 2 above.

Using Proposition 23 in appendix B and Proposition 3 it is a routine task to extend the considerations above to the manifold $M \times [-1, 1]$ and prove Theorem 2' along the same lines.

3. The regularization $R(\eta, X, g)$

In this section we discuss the numerical invariant $R(\eta, X, g)$ associated with a vector field X, a closed one form $\eta \in \mathcal{Z}^1(M; \mathbb{C})$ and a Riemannian metric g. The invariant is defined by a possibly divergent but regularizable integral. It is implicit in the work of Bismut–Zhang [1]. More on this invariant is contained in [5].

Throughout this section we assume that M is a closed manifold of dimension n, and X is a smooth vector field with zero set $\mathcal{X}$. We assume that the zeros are non-degenerate but not necessarily of the form (1). It is not difficult to generalize the regularization to vector fields with isolated singularities, see [5].

3.1. **Euler, Chern–Simons, and the global angular form.** Let $\pi : TM \to M$ denote the tangent bundle, and let $\mathcal{O}_M$ denote the orientation bundle, a flat real line bundle over M. For a Riemannian metric g let

$$\mathrm{e}(g) \in \Omega^n(M; \mathcal{O}_M)$$

denote its *Euler form*. For two Riemannian metrics g_1 and g_2 let

$$\mathrm{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$$

denote their *Chern–Simons class.* The definition of both quantities is implicit in the formulae (27) and (28) below. They have the following properties which follow immediately from (27) and (28) below.

$$d\,\mathrm{cs}(g_1,g_2) = \mathrm{e}(g_2)-\mathrm{e}(g_1) \tag{24}$$
$$\mathrm{cs}(g_2,g_1) = -\,\mathrm{cs}(g_1,g_2) \tag{25}$$
$$\mathrm{cs}(g_1,g_3) = \mathrm{cs}(g_1,g_2)+\mathrm{cs}(g_2,g_3) \tag{26}$$

Let ξ denote the Euler vector field on TM which assigns to a point $x\in TM$ the vertical vector $-x\in T_xTM$. A Riemannian metric g determines the Levi–Civita connection in the bundle $\pi: TM\to M$. There is a canonic $\mathrm{vol}(g)\in\Omega^n(TM;\pi^*\mathcal{O}_M)$, which vanishes when contracted with horizontal vectors and which assigns to an n–tuple of vertical vectors *their volume times their orientation.* The global angular form, see for instance [2], is the differential form

$$\Psi(g) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}|\xi|^n} i_\xi \,\mathrm{vol}(g)\in\Omega^{n-1}(TM\setminus M;\pi^*\mathcal{O}_M).$$

This form was also considered by Mathai and Quillen [14], and was referred to as the Mathai–Quillen form in [1]. Note that $\Psi(g)$ is the pull back of a form on $(TM\setminus M)/\mathbb{R}_+$. Moreover, we have the equalities:

$$d\Psi(g) = \pi^*\,\mathrm{e}(g). \tag{27}$$
$$\Psi(g_2)-\Psi(g_1) = \pi^*\,\mathrm{cs}(g_1,g_2) \mod \pi^* d\Omega^{n-2}(M;\mathcal{O}_M) \tag{28}$$

Further, if $x\in\mathcal{X}$ then

$$\lim_{\epsilon\to 0}\int_{\partial(M\setminus B_x(\epsilon))} X^*\Psi(g) = \mathrm{IND}(x), \tag{29}$$

where $\mathrm{IND}(x)$ denotes the *Hopf index* of X at x, and $B_x(\epsilon)$ denotes the ball of radius ϵ centered at x.

3.2. **Euler and Chern–Simons class for vector fields.** Let $C_k(M;\mathbb{Z})$ denote the complex of smooth singular chains in M. Define a singular zero chain

$$\mathrm{e}(X) := \sum_{x\in\mathcal{X}}\mathrm{IND}(x)x\in C_0(M;\mathbb{Z}).$$

For two vector fields X_1 and X_2 we are going to define

$$\mathrm{cs}(X_1,X_2)\in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z}) \tag{30}$$

with the following properties analogous to (24)–(26).

$$\partial\,\mathrm{cs}(X_1,X_2) = \mathrm{e}(X_2)-\mathrm{e}(X_1) \tag{31}$$
$$\mathrm{cs}(X_2,X_1) = -\,\mathrm{cs}(X_1,X_2) \tag{32}$$
$$\mathrm{cs}(X_1,X_3) = \mathrm{cs}(X_1,X_2)+\mathrm{cs}(X_2,X_3) \tag{33}$$

It is constructed as follows. Consider the vector bundle $p^*TM\to I\times M$, where $I:=[1,2]$ and $p: I\times M\to M$ denotes the natural projection. Choose a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i on $\{i\}\times M$, $i=1,2$. The zero set of $\mathbb{X}$ is a canonically oriented one dimensional submanifold with boundary. Its fundamental class, when pushed forward via p, gives rise to $c(\mathbb{X})\in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$. Clearly $\partial c(\mathbb{X}) = \mathrm{e}(X_2)-\mathrm{e}(X_1)$.

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two non-degenerate homotopies from X_1 to X_2. Then $c(\mathbb{X}_1)=c(\mathbb{X}_2)\in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$. Indeed, consider the vector bundle $q^*TM\to I\times I\times M$, where $q: I\times I\times M\to M$ denotes the natural projection. Choose a section of q^*TM which is transversal to the zero section, restricts to $\mathbb{X}_i$ on $\{i\}\times I\times M$,

$i=1,2$, and which restricts to X_i on $\{s\}\times\{i\}\times M$ for all $s\in I$ and $i=1,2$. The zero set of such a section then gives rise to σ satisfying $c(\mathbb{X}_2)-c(\mathbb{X}_1)=\partial\sigma$. Hence we may define $\mathrm{cs}(X_1,X_2):=c(\mathbb{X})$.

3.3. **The regularization.** Let g be a Riemannian metric, and let $\eta\in\mathcal{Z}^1(M;\mathbb{C})$. Choose a smooth function $f:M\to\mathbb{C}$ so that $\eta':=\eta-df$ vanishes on a neighborhood of $\mathcal{X}$. Then the following expression is well defined:

$$R(\eta,X,g;f):=\int_{M\setminus\mathcal{X}}\eta'\wedge X^*\Psi(g)-\int_M f\,\mathrm{e}(g)+\sum_{x\in\mathcal{X}}\mathrm{IND}(x)f(x)\tag{34}$$

Lemma 6. *The quantity $R(\eta,X,g;f)$ is independent of f.*

Proof. Suppose f_1 and f_2 are two functions such that $\eta_i':=\eta-df_i$ vanishes in a neighborhood of $\mathcal{X}$, $i=1,2$. Then f_2-f_1 is locally constant near $\mathcal{X}$. Using (29) and Stokes' theorem we therefore get

$$\int_{M\setminus\mathcal{X}}d\big((f_2-f_1)X^*\Psi(g)\big)=\sum_{x\in\mathcal{X}}\big(f_2(x)-f_1(x)\big)\,\mathrm{IND}(x).$$

Together with (27) this immediately implies $R(\eta,X,g;f_1)=R(\eta,X,g;f_2)$. □

Definition 8. For a vector field X with non-degenerate zeros, a Riemannian metric g and a closed one form $\eta\in\Omega^1(M;\mathbb{C})$ we define $R(\eta,X,g)$ by (34). In view of Lemma 6 this does not depend on the choice of f. We think of $R(\eta,X,g)$ as regularization of the possibly divergent integral $\int_{M\setminus\mathcal{X}}\eta\wedge X^*\Psi(g)$.

Proposition 18. *For a smooth function $h:M\to\mathbb{C}$ we have*

$$R(\eta+dh,X,g)-R(\eta,X,g)=-\int_M h\,\mathrm{e}(g)+\sum_{x\in\mathcal{X}}\mathrm{IND}(x)h(x).\tag{35}$$

Proof. This is trivial, h can be absorbed in the choice of f. □

Proposition 19. *For two Riemannian metrics g_1 and g_2 we have*

$$R(\eta,X,g_2)-R(\eta,X,g_1)=\int_M\eta\wedge\mathrm{cs}(g_1,g_2).\tag{36}$$

Proof. This follows easily from (28), Stokes' theorem and (24). □

Proposition 20. *For two vector fields X_1 and X_2 we have*

$$R(\eta,X_2,g)-R(\eta,X_1,g)=\eta(\mathrm{cs}(X_1,X_2)).\tag{37}$$

Proof. In view of (35) and (31) we may w.l.o.g. assume that η vanishes on a neighborhood of $\mathcal{X}_1\cup\mathcal{X}_2$. Choose a non-degenerate homotopy $\mathbb{X}$ from X_1 to X_2. Perturbing the homotopy, cutting it into several pieces and using (33) we may further assume that the zero set $\mathbb{X}^{-1}(0)\subseteq I\times M$ is actually contained in a simply connected $I\times V$. Again, we may assume that η vanishes on V. Then the right hand side of (37) obviously vanishes. Moreover, in this situation Stokes' theorem implies

$$\begin{aligned} R(\eta, X_2, g) - R(\eta, X_1, g) &= \int_{M\setminus V} \eta \wedge X_2^* \Psi(g) - \int_{M\setminus V} \eta \wedge X_1^* \Psi(g) \\ &= \int_{I\times(M\setminus V)} d(p^*\eta \wedge \mathbb{X}^* \tilde{p}^* \Psi(g)) \\ &= -\int_{I\times(M\setminus V)} p^*(\eta \wedge \mathrm{e}(g)) = 0. \end{aligned}$$

Here $p : I \times M \to M$ denotes the natural projection, and $\tilde{p} : p^*TM \to TM$ denotes the natural vector bundle homomorphism over p. For the last calculation note that $d\mathbb{X}^*\tilde{p}^*\Psi(g) = p^* \mathrm{e}(g)$ in view of (27), and that $\eta \wedge \mathrm{e}(g) = 0$ because of dimensional reasons. □

4. Completion of trajectory spaces and unstable manifolds

If a vector field satisfies MS and L, then the space of trajectories as well as the unstable manifolds can be completed to manifolds with corners. In section 4.1 we recall these results, see Theorem 4 below, and use them to prove Propositions 9 and 11. The rest of this section is dedicated to the proof of Theorem 1.

4.1. **The completion.** Let X be vector field on the closed manifold M and suppose that X satisfies MS. Let $\pi : \tilde{M} \to M$ denote the universal covering. Denote by $\tilde{X}$ the vector field $\tilde{X} := \pi^*X$ and set $\tilde{\mathcal{X}} = \pi^{-1}(\mathcal{X})$.

Given $\tilde{x} \in \tilde{\mathcal{X}}$ let $i^\pm_{\tilde{x}} : W^\pm_{\tilde{x}} \to \tilde{M}$ denote the one-to-one immersions whose images define the stable and unstable sets of $\tilde{x}$ with respect to the vector field $\tilde{X}$. For any $\tilde{x}$ with $\pi(\tilde{x}) = x$ one can canonically identify $W^\pm_{\tilde{x}}$ to $W^\pm_x$ so that $\pi \circ i^\pm_{\tilde{x}} = i^\pm_x$. Define $\mathcal{M}(\tilde{x}, \tilde{y}) := W^-_{\tilde{x}} \cap W^+_{\tilde{y}}$ if $\tilde{x} \neq \tilde{y}$, and set $\mathcal{M}(\tilde{x}, \tilde{x}) := \emptyset$. As the maps $i^-_{\tilde{x}}$ and $i^+_{\tilde{y}}$ are transversal, $\mathcal{M}(\tilde{x}, \tilde{y})$ is a submanifold of $\tilde{M}$ of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y})$. It is equipped with a free $\mathbb{R}$–action defined by the flow generated by $\tilde{X}$. Denote the quotient $\mathcal{M}(\tilde{x}, \tilde{y})/\mathbb{R}$ by $\mathcal{T}(\tilde{x}, \tilde{y})$. The quotient $\mathcal{T}(\tilde{x}, \tilde{y})$ is a smooth manifold of dimension $\mathrm{ind}(\tilde{x}) - \mathrm{ind}(\tilde{y}) - 1$, possibly empty. If $\mathrm{ind}(\tilde{x}) \leq \mathrm{ind}(\tilde{y})$, in view the transversality required by the hypothesis MS, the manifolds $\mathcal{M}(\tilde{x}, \tilde{y})$ and $\mathcal{T}(\tilde{x}, \tilde{y})$ are empty.

An *unparameterized broken trajectory* from $\tilde{x} \in \tilde{\mathcal{X}}$ to $\tilde{y} \in \tilde{\mathcal{X}}$, is an element of the set $\hat{\mathcal{T}}(\tilde{x}, \tilde{y}) := \bigcup_{k\geq 0} \hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k$, where

$$\hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_k, \tilde{y}_{k+1}) \tag{38}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$ and $\tilde{y}_{k+1} = \tilde{y}$.

For $\tilde{x} \in \tilde{\mathcal{X}}$ introduce the *completed unstable set* $\hat{W}^-_{\tilde{x}} := \bigcup_{k\geq 0} (\hat{W}^-_{\tilde{x}})_k$, where

$$(\hat{W}^-_{\tilde{x}})_k := \bigcup \mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k} \tag{39}$$

and the union is over all (tuples of) critical points $\tilde{y}_i \in \tilde{\mathcal{X}}$ with $\tilde{y}_0 = \tilde{x}$.

Let $\hat{i}^-_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \tilde{M}$ denote the map whose restriction to $\mathcal{T}(\tilde{y}_0, \tilde{y}_1) \times \cdots \times \mathcal{T}(\tilde{y}_{k-1}, \tilde{y}_k) \times W^-_{\tilde{y}_k}$ is the composition of the projection on $W^-_{\tilde{y}_k}$ with $i^-_{\tilde{y}_k}$.

Recall that an n–dimensional *manifold with corners* P, is a paracompact Hausdorff space equipped with a maximal smooth atlas with charts $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \mid x_i \geq 0\}$. The collection of points of P which correspond by some (and hence every) chart to points in $\mathbb{R}^n$ with exactly k coordinates equal to zero is a well defined subset of P called the *k–corner of P* and it will be denoted by P_k. It has a structure of a smooth $(n-k)$–dimensional manifold. The union $\partial P = P_1 \cup P_2 \cup \cdots \cup P_n$ is a closed subset which is a topological manifold and $(P, \partial P)$ is a topological manifold with boundary ∂P.

The following theorem can easily be derived from [4, Theorem 1] by lifting everything to the universal covering, see Proposition 2.

Theorem 4. *Let M be a closed manifold, and suppose X is a smooth vector field which satisfies MS and L.*

(i) *For any two rest points $\tilde{x}, \tilde{y} \in \tilde{\mathcal{X}}$ the set $\hat{\mathcal{T}}(\tilde{x}, \tilde{y})$ admits a natural structure of a compact smooth manifold with corners, whose k–corner coincides with $\hat{\mathcal{T}}(\tilde{x}, \tilde{y})_k$ from* (38).

(ii) *For every rest point $\tilde{x} \in \tilde{\mathcal{X}}$, the set $\hat{W}^-_{\tilde{x}}$ admits a natural structure of a smooth manifold with corners, whose k–corner coincides with $(\hat{W}^-_{\tilde{x}})_k$ from* (39).

(iii) *The function $\hat{i}^-_{\tilde{x}} : \hat{W}^-_{\tilde{x}} \to \tilde{M}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

(iv) *If ω is Lyapunov for X and $h : \tilde{M} \to \mathbb{R}$ is a smooth function with $dh = \pi^*\omega$ then the function $h \circ \hat{i}^-_{\tilde{x}}$ is smooth and proper, for all $\tilde{x} \in \tilde{\mathcal{X}}$.*

As a first folklore application of Theorem 4 we will give a

Proof of Proposition 9. Let $x, z \in \mathcal{X}$. Theorem 4(i) implies

$$\sum_{y \in \mathcal{X}} \sum_{\gamma_1 \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1) \cdot \mathbb{I}_{y,z}(\gamma_1^{-1}\gamma) = 0$$

for all $\gamma \in \mathcal{P}_{x,z}$. If $\eta \in \mathfrak{I}$ we can reorder sums and find

$$\sum_{y \in \mathcal{X}} \sum_{\gamma_1 \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1) e^{\eta(\gamma_1)} \sum_{\gamma_2 \in \mathcal{P}_{y,z}} \mathbb{I}_{y,z}(\gamma_2) e^{\eta(\gamma_2)} = \sum_{\gamma \in \mathcal{P}_{x,z}} \Big(\sum_{y \in \mathcal{X}} \sum_{\gamma_1 \in \mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma_1) \cdot \mathbb{I}_{y,z}(\gamma_1^{-1}\gamma) \Big) e^{\eta(\gamma)} = 0.$$

This implies $\delta_\eta^2 = 0$. □

As a second application of Theorem 4 we will give a

Proof of Proposition 11. We follow the approach in [4]. Let $\chi : \mathbb{R} \to [0,1]$ be smooth, and such that $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) = 1$ for $t \geq 1$. Choose a Lyapunov form ω for X. For $y \in \mathcal{X}$ and $s \in \mathbb{R}$ define $\hat{\chi}^s_y := \chi \circ (\hat{h}^\omega_y + s) : \hat{W}^-_y \to [0,1]$. Note that $\operatorname{supp}(\hat{\chi}^s_y)$ is compact in view of Theorem 4(iv). Suppose $x \in \mathcal{X}$, $\alpha \in \Omega^*(M; \mathbb{C})$, and $\eta \in \mathfrak{R}$. Absolute convergence implies

$$\operatorname{Int}_\eta(d_\eta \alpha)(x) = \lim_{s \to \infty} \int_{\hat{W}^-_x} \hat{\chi}^s_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* d_\eta \alpha.$$

Moreover,

$$\hat{\chi}^s_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* d_\eta \alpha = d\big(\hat{\chi}^s_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* \alpha\big) - \big(\chi' \circ (\hat{h}^\omega_x + s)\big) \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* \omega \wedge \alpha.$$

Since $\eta \in \mathfrak{R}$ and since χ' is bounded we have

$$\lim_{s \to \infty} \int_{\hat{W}^-_x} \big(\chi' \circ (\hat{h}^\omega_x + s)\big) \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* \omega \wedge \alpha = 0.$$

Using Theorem 4(ii) and Stokes' theorem for the compactly supported smooth form $\hat\chi^s_x \cdot e^{\hat h^\eta_x} \cdot (\hat i^-_x)^*\alpha \in \Omega^*(\hat W^-_x;\mathbb{C})$ we get

$$\int_{\hat W^-_x} d\big(\hat\chi^s_x \cdot e^{\hat h^\eta_x} \cdot (\hat i^-_x)^*\alpha\big) = \sum_{y\in\mathcal{X}}\sum_{\gamma\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)}\int_{\hat W^-_y} \hat\chi^{s+\omega(\gamma)}_y \cdot e^{\hat h^\eta_y}\cdot(\hat i^-_y)^*\alpha.$$

Since $\eta \in \mathfrak{I}\cap\mathfrak{R}$ the form $\mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)} \cdot e^{\hat h^\eta_y}\cdot(\hat i^-_y)^*\alpha$ is absolutely integrable on $\mathcal{P}_{x,y}\times \hat W^-_y$. Hence we may interchange limits and find

$$\begin{aligned}\lim_{s\to\infty}\int_{\hat W^-_x} & d\big(\hat\chi^s_x \cdot e^{\hat h^\eta_x} \cdot (\hat i^-_x)^*\alpha\big) \\ &= \sum_{y\in\mathcal{X}}\sum_{\gamma\in\mathcal{P}_{x,y}} \mathbb{I}_{x,y}(\gamma)e^{\eta(\gamma)}\lim_{s\to\infty}\int_{\hat W^-_y} \hat\chi^{s+\omega(\gamma)}_x \cdot e^{\hat h^\eta_y}\cdot(\hat i^-_y)^*\alpha \\ &= \sum_{y\in\mathcal{X}} L(\mathbb{I}_{x,y})(\eta)\cdot \mathrm{Int}_\eta(\alpha)(y) = \delta_\eta(\mathrm{Int}_\eta(\alpha))(x).\end{aligned}$$

We conclude $\mathrm{Int}_\eta(d_\eta\alpha)(x) = \delta_\eta(\mathrm{Int}_\eta(\alpha))(x)$. □

We close this section with a lemma which immediately implies Proposition 10.

Lemma 7. *Suppose $\eta\in\mathfrak{R}$, $x\in\mathcal{X}$, and let $\epsilon>0$. Then there exists $\alpha\in\Omega^*(M;\mathbb{C})$ so that $|\mathrm{Int}_\eta(\alpha)(y)-\delta_{x,y}|\leq\epsilon$, for all $y\in\mathcal{X}$.*

Proof. We follow the approach in [4]. Let U be a neighborhood of x on which X has canonical form (1). Let $B\subseteq W^-_x$ denote the connected component of $W^-_x\cap U$ containing x. Choose $\alpha\in\Omega^*(M;\mathbb{C})$ with $\mathrm{supp}(\alpha)\subseteq U$ and such that $\int_B e^{h^\eta_x}(i^-_x)^*\alpha = 1$. For every $y\in\mathcal{X}$ choose a compact $K_y\subseteq W^-_y$ such that $\int_{W^-_y\setminus K_y}|e^{h^\eta_y}(i^-_y)^*\alpha|\leq\epsilon$. Assume $B\subseteq K_x$. By multiplying α with a bump function which is 1 on B and whose support is sufficiently concentrated around B, we may in addition assume $\mathrm{supp}(\alpha)\cap(K_x\setminus B)=\emptyset$, and $\mathrm{supp}(\alpha)\cap K_y=\emptyset$ for all $x\neq y\in\mathcal{X}$. Then

$$\big|\mathrm{Int}_\eta(\alpha)(y)-\delta_{x,y}\big| = \Big|\int_{W^-_y\setminus K_y} e^{h^\eta_y}(i^-_y)^*\alpha\Big| \leq \int_{W^-_y\setminus K_y}|e^{h^\eta_y}(i^-_y)^*\alpha|\leq\epsilon. \qquad □$$

4.2. **Proof of the first part of Theorem 1.** Suppose X satisfies MS and L. Let $\Gamma:=\pi_1(M)$ denote the fundamental group acting from the left on the universal covering $\pi:\tilde M\to M$ in the usual manner. Equip $\mathbb{C}^{\mathcal{X}}$ with a norm. Equip $\mathbb{A}:=\mathrm{end}(\mathbb{C}^{\mathcal{X}})$ with the corresponding operator norm. Choose a Lyapunov form ω for X. Let N denote the vector space of maps $a:\Gamma\to\mathbb{A}$ for which $\{\gamma\in\Gamma\mid -\omega(\gamma)\leq K, a(\gamma)\neq 0\}$ is finite, for all $K\in\mathbb{R}$. Equipped with the convolution product N becomes an algebra with unit. Let $L^1:=L^1(\Gamma;\mathbb{A})$ denote the Banach space of functions $a:\Gamma\to\mathbb{A}$ for which $\|a\|_{L^1}:=\sum_{\gamma\in\Gamma}\|a(\gamma)\|<\infty$. Recall that the convolution product makes L^1 a Banach algebra with unit.

Lemma 8. *Let $I,a\in N$. Assume $\|1-a\|_{L^1}<1$, $I*a\in L^1$, and assume that $a(\gamma)\neq 0$ implies $-\omega(\gamma)\geq 0$. Then $I\in L^1$.*

Proof. Since L^1 is a Banach algebra $\|1-a\|_{L^1}<1$ implies that $a\in L^1$ is invertible. Clearly it suffices to show $(I*a)*a^{-1} = I*(a*a^{-1})$. That is, for fixed $\rho\in\Gamma$, we have to show

$$\sum_{\sigma\in\Gamma}\sum_{\tau\in\Gamma} I(\sigma)a(\sigma^{-1}\tau)a^{-1}(\tau^{-1}\rho) = \sum_{\tau\in\Gamma}\sum_{\sigma\in\Gamma} I(\sigma)a(\sigma^{-1}\tau)a^{-1}(\tau^{-1}\rho). \qquad (40)$$

Using $a^{-1} = \sum_{k=0}^{\infty}(1-a)^k$ it is not difficult to show that $a^{-1}(\gamma) \neq 0$ implies $-\omega(\gamma) \geq 0$. Using $I, a \in N$ we thus conclude that

$$\big\{\sigma \in \Gamma \;\big|\; \exists \tau \in \Gamma : I(\sigma)a(\sigma^{-1}\tau)a^{-1}(\tau^{-1}\rho) \neq 0\big\}$$

is finite. Equation (40) follows immediately. □

Let us now turn to the proof of $\mathfrak{R} \subseteq \mathfrak{I}$. Let $\eta \in \mathfrak{R}$. Choose a lift $s(x) \in \tilde{\mathcal{X}}$ for every zero $x \in \mathcal{X}$, i.e. $\pi(s(x)) = x$. For $x, y \in \mathcal{X}$ and $\gamma \in \Gamma$ let $\rho^\gamma_{x,y} \in \mathcal{P}_{x,y}$ denote the homotopy class of paths determined by the lifts $s(x)$ and $\gamma \cdot s(y)$. Moreover, set

$$I_{x,y}(\gamma) := \mathbb{I}_{x,y}(\rho^\gamma_{x,y}) \cdot e^{\eta(\rho^\gamma_{x,y})}. \tag{41}$$

We write $I : \Gamma \to \mathbb{A}$ for the matrix valued map defined by (41). Note that $\eta \in \mathfrak{I}$ iff $I \in L^1$. It thus suffices to construct $a : \Gamma \to \mathbb{A}$ for which Lemma 8 is applicable. Note that $I \in N$ in view of Proposition 4.

In order to construct a choose a smooth function $\chi : \tilde{M} \to [0,1]$ so that $\chi = 1$ in a neighborhood of $s(\mathcal{X})$, so that $\operatorname{supp}(\chi)$ is compact, and so that $\operatorname{supp}(\chi) \cap \operatorname{supp}(\gamma^*\chi) = \emptyset$ for all non-trivial $\gamma \in \Gamma$. For $x \in \mathcal{X}$ and $\gamma \in \Gamma$ define a function $\hat\chi^\gamma_x : \hat{W}^-_x \to [0,1]$ by $\hat\chi^\gamma_x := (\gamma^{-1})^*\chi \circ \hat{i}^-_{s(x)}$. Note that $\operatorname{supp}(\hat\chi^\gamma_x)$ is compact in view of Theorem 4(iii). Possibly shrinking the support of χ we may assume that $\hat\chi^\gamma_x \neq 0$ implies $-\omega(\gamma) \geq 0$.

The construction of a will also depend on the choice of $\beta_x \in \Omega^*(M;\mathbb{C})$, $x \in \mathcal{X}$, which will be specified below. For $x, y \in \mathcal{X}$ and $\gamma \in \Gamma$ define

$$a_{x,y}(\gamma) := \int_{\hat{W}^-_x} \hat\chi^\gamma_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^*\beta_y. \tag{42}$$

We write $a : \Gamma \to \mathbb{A}$ for the matrix valued function defined by (42). Note that $a \in N$. Moreover, $a(\gamma) \neq 0$ implies $-\omega(\gamma) \geq 0$.

We will choose β_x so that its support is concentrated near x. More precisely, we assume $\operatorname{supp}(d\chi) \cap \operatorname{supp}(\pi^*\beta_x) = \emptyset$ for all $x \in \mathcal{X}$. Clearly we may also assume that $a_{x,y}(e) = \delta_{x,y}$, i.e. $a(e) = 1$. Note that the mutual disjointness of $\operatorname{supp}(\chi^\gamma_x)$, $\gamma \in \Gamma$, implies

$$\sum_{e \neq \gamma \in \Gamma} |a_{x,y}(\gamma)| \leq \sum_{e \neq \gamma \in \Gamma} \int_{\operatorname{supp}(\hat\chi^\gamma_x)} |e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^*\beta_y| \leq \int_{\hat{W}^-_x \setminus \operatorname{supp}(\hat\chi^e_x)} |e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^*\beta_y|.$$

Using $\eta \in \mathfrak{R}$ and arguing as in the proof of Lemma 7, we may therefore assume that, given $\epsilon > 0$, the β_x are chosen so that $\sum_{e \neq \gamma \in \Gamma} |a_{x,y}(\gamma)| < \epsilon$ for all $x, y \in \mathcal{X}$. Obviously this implies $\|1 - a\|_{L^1} < 1$.

Using $\operatorname{supp}(d\chi) \cap \operatorname{supp}(\pi^*\beta_y) = \emptyset$ and applying Stokes' theorem for the compactly supported form $\hat\chi^\gamma_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^*\beta_y \in \Omega^*(\hat{W}^-_x;\mathbb{C})$, see Theorem 4(ii), we find

$$\begin{aligned}\int_{\hat{W}^-_x} \hat\chi^\gamma_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* d_\eta\beta_y &= \int_{\hat{W}^-_x} d\big(\hat\chi^\gamma_x \cdot e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^*\beta_y\big) \\ &= \sum_{z \in \mathcal{X}} \sum_{\sigma \in \Gamma} \mathbb{I}_{x,z}(\alpha^\sigma_{xz}) e^{\eta(\alpha^\sigma_{xz})} \int_{\hat{W}^-_z} \hat\chi^{\sigma^{-1}\gamma}_z \cdot e^{\hat{h}^\eta_z} \cdot (\hat{i}^-_z)^*\beta_y = (I * a)_{x,y}(\gamma).\end{aligned}$$

Since $\eta \in \mathfrak{R}$ we therefore get

$$\sum_{\gamma \in \Gamma} |(I * a)_{x,y}(\gamma)| \leq \sum_{\gamma \in \Gamma} \int_{\operatorname{supp}(\hat\chi^\gamma_x)} |e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x)^* d_\eta\beta_y| \leq \int_{\hat{W}^-_x} |e^{\hat{h}^\eta_x} \cdot (\hat{i}^-_x) d_\eta\beta_y| < \infty$$

for all $x, y \in \mathcal{X}$. We conclude $\|I * a\|_{L^1} < \infty$. Hence we can apply Lemma 8, obtain $I \in L^1$ and thus $\eta \in \mathfrak{I}$. This completes the proof of $\mathfrak{R} \subseteq \mathfrak{I}$.

4.3. **Proof of the second part of Theorem 1.** We will start with a lemma whose first part, when applied to the eigen spaces of B_η, implies Proposition 13.

Lemma 9. *Let C^* be a finite dimensional graded complex over $\mathbb{C}$ with differential d. Let b be a non-degenerate graded bilinear form on C^*. Let d^t denote the formal transpose of d, i.e. $b(dv, w) = b(v, d^t w)$ for all $v, w \in C^*$. Set $B := dd^t + d^t d$ and suppose* $\ker B = 0$. *Then $C^* = \operatorname{img} d \oplus \operatorname{img} d^t$, and this decomposition is orthogonal with respect to b. Particularly, the cohomology of C^* vanishes. For its torsion, with respect to the equivalence class of graded bases determined by b, we have*

$$\tau(C^*, b)^2 = \prod_q (\det B^q)^{(-1)^{q+1} q}$$

where $B^q : C^q \to C^q$ denotes the part of B acting in degree q.

Proof. Clearly $\operatorname{img} d \subseteq (\ker d^t)^\perp$, and hence $\operatorname{img} d = (\ker d^t)^\perp$ since C^* is finite dimensional. Similarly we get $\operatorname{img} d^t = (\ker d)^\perp$. Therefore

$$(\operatorname{img} d + \operatorname{img} d^t)^\perp = (\operatorname{img} d)^\perp \cap (\operatorname{img} d^t)^\perp = \ker d^t \cap \ker d \subseteq \ker B = 0,$$

and thus $C^* = \operatorname{img} d + \operatorname{img} d^t$. Moreover, since $\operatorname{img} d^t \subseteq (\ker d)^\perp \subseteq (\operatorname{img} d)^\perp$ this decomposition is orthogonal. The cohomology vanishes for we have $\ker d = (\operatorname{img} d^t)^\perp = \operatorname{img} d$. Using

$$\begin{aligned} \det B^q &= \det(d^t d|_{\operatorname{img} d^t \cap C^q}) \cdot \det(dd^t|_{\operatorname{img} d \cap C^q}) \\ &= \det(d^t d|_{\operatorname{img} d^t \cap C^q}) \cdot \det(d^t d|_{\operatorname{img} d^t \cap C^{q-1}}) \end{aligned}$$

a trivial telescoping calculation shows

$$\prod_q (\det B^q)^{(-1)^{q+1} q} = \prod_q \big(\det d^t d|_{\operatorname{img} d^t \cap C^q}\big)^{(-1)^q} = \tau(C^*, b)^2. \qquad \square$$

Let us next prove that $\mathring{\mathfrak{R}} \cap \Sigma$ is an analytic subset of $\mathring{\mathfrak{R}}$. Let $\eta_0 \in \mathring{\mathfrak{R}}$. Choose a simple closed curve K around $0 \in \mathbb{C}$ which avoids the spectrum of B_{η_0}. Let U be an open neighborhood of η_0 so that K avoids the spectrum of every B_η, $\eta \in U$. Assume U is connected and $U \subseteq \mathring{\mathfrak{R}}$. Let $E^*_\eta(K)$ denote the image of the spectral projection associated with K, i.e. $E^*_\eta(K)$ is the sum of all eigen spaces of B_η corresponding to eigen values contained in the interior of K. Since the spectral projection depends holomorphically on η, wee see that $E^*_\eta(K)$ is a holomorphic family of finite dimensional complexes parametrized by $\eta \in U$. From Proposition 13 we see that the inclusion $E^*_\eta(K) \to \Omega^*_\eta(M; \mathbb{C})$ is a quasi isomorphism for all $\eta \in U$.

Consider the restriction of the integration $\operatorname{Int}_\eta : E^*_\eta(K) \to C^*_\eta(X; \mathbb{C})$, and let $\mathcal{C}^*_\eta(K)$ denote its mapping cone. More precisely, as graded vector space $\mathcal{C}^*_\eta(K) = C^{*-1}_\eta(X; \mathbb{C}) \oplus E^*_\eta(K)$, and the differential is given by $(f, \alpha) \mapsto (-\delta_\eta f + \operatorname{Int}_\eta \alpha, d_\eta \alpha)$. This is a family of finite dimensional complexes, holomorphically parametrized by $\eta \in U$. Note that the dimension of $\mathcal{C}^*_\eta(K)$ is even, $\dim \mathcal{C}^*_\eta(K) = 2k$. Possibly shrinking U we may assume that we have a base $\{v^1_\eta, \dots, v^{2k}_\eta\}$ of $\mathcal{C}^*_\eta(K)$ holomorphically parametrized by $\eta \in U$. Let $f^1_\eta, \dots, f^N_\eta \in \mathbb{C}$ denote the $k \times k$–minors of the differential of $\mathcal{C}^*_\eta(K)$ with respect to this base, $N = \binom{2k}{k}^2$. This provides N holomorphic functions $f^i : U \to \mathbb{C}$, $1 \le i \le N$. For $\eta \in U$ the integration will induce an isomorphism in cohomology iff $\mathcal{C}^*_\eta(K)$ is acyclic. Moreover, $\mathcal{C}^*_\eta(K)$ is acyclic iff its differential has rank k. This in turn is equivalent to $f^i(\eta) \neq 0$ for some $1 \leq i \leq N$. We conclude $\Sigma \cap U = \{\eta \in U \mid f^i(\eta) = 0, 1 \leq i \leq N\}$. Hence $\mathring{\mathfrak{R}} \cap \Sigma$ is an analytic subset of $\mathring{\mathfrak{R}}$.

Suppose ω is a Lyapunov form for X, and let $\eta \in \mathfrak{R}$. Recall that we have an integration homomorphism

$$\mathrm{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C}) \tag{43}$$

for $t \geq 0$, and that $\eta + t\omega \in \mathring{\mathfrak{R}}$ for $t > 0$, see Proposition 8. We have to show that (43) induces an isomorphism in cohomology for sufficiently large t. In view of the gauge invariance, see (9), we may assume that η vanishes in a neighborhood of $\mathcal{X}$, and that there exists a Riemannian metric g, such that $\omega = -g(X, \cdot)$ as in Proposition 2.

Consider the one parameter family of complexes $\Omega^*_{\eta+t\omega}(M;\mathbb{C})$. Let $\Delta_{\eta+t\omega}$ denote the corresponding Laplacians with respect to the standard Hermitian structure on $\Omega^*(M;\mathbb{C})$. Witten–Helffer–Sjöstrand theory [4] tells that as $t \to \infty$ the spectrum of $\Delta_{\eta+t\omega}$ develops a gap, providing a canonic orthogonal decomposition

$$\Omega^*_{\eta+t\omega}(M;\mathbb{C}) = \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \oplus \Omega^*_{\eta+t\omega,\mathrm{la}}(M;\mathbb{C}).$$

Moreover, for sufficiently large t the restriction of the integration

$$\mathrm{Int}_{\eta+t\omega} : \Omega^*_{\eta+t\omega,\mathrm{sm}}(M;\mathbb{C}) \to C^*_{\eta+t\omega}(X;\mathbb{C})$$

is an isomorphism [4]. It follows that (43) induces an isomorphism in cohomology, and hence $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for sufficiently large t.

5. Proof of Theorem 3

The proof of Theorem 3 is based a result of Bismut–Zhang [1] and formula of Hutchings–Lee [9] and Pajitnov [19]. The Bismut–Zhang theorem implies that Theorem 3 holds for Morse–Smale vector fields, see section 5.2. The Hutchings–Lee formula permits to establish an anomaly formula in X for $(T\,\mathrm{Int}^X_\eta)^2$, see Proposition 22 in section 5.4. Putting this together we will obtain Theorem 3, see section 5.7.

5.1. **Proof of Proposition 14.** Let us first show $(T\,\mathrm{Int}_{\bar\eta})^2 = \overline{(T\,\mathrm{Int}_\eta)^2}$. Clearly we have $R(\bar\eta, X, g) = \overline{R(\eta, X, g)}$. Note that complex conjugation on $\Omega^*(M;\mathbb{C})$ intertwines d_η with $d_{\bar\eta}$, d^t_η with $d^t_{\bar\eta}$, and B_η with $B_{\bar\eta}$. Therefor the spectrum of B_η is conjugate to the spectrum of $B_{\bar\eta}$. It follows that $(T^{\mathrm{an}}_{\bar\eta,g})^2 = \overline{(T^{\mathrm{an}}_{\eta,g})^2}$. Moreover, complex conjugation restricts to an anti-linear isomorphism of complexes $E^*_\eta(0) \simeq E^*_{\bar\eta}(0)$ which is easily seen to intertwine the equivalence class of bases determined by b. Complex conjugation also defines an anti-linear isomorphism of complexes $C^*_\eta(X;\mathbb{C}) \simeq C^*_{\bar\eta}(X;\mathbb{C})$ which intertwines the equivalence class of bases determined by the indicator functions. These isomorphisms intertwine Int_η with $\mathrm{Int}_{\bar\eta}$. Hence they provide an anti-linear isomorphism of mapping cones, and therefore $\pm T(\mathrm{Int}_{\bar\eta}|_{E^*_{\bar\eta}(0)}) = \overline{\pm T(\mathrm{Int}_\eta|_{E^*_\eta(0)})}$. Putting everything together we find $(T\,\mathrm{Int}_{\bar\eta})^2 = \overline{(T\,\mathrm{Int}_\eta)^2}$.

Let us next show that $(T\,\mathrm{Int}_\eta)^2$ depends holomorphically on $\eta \in \mathring{\mathfrak{R}} \setminus \Sigma$. Let $\eta_0 \in \mathring{\mathfrak{R}} \setminus \Sigma$. As in the proof of Theorem 1 in section 4.3 let U be a connected open neighborhood of η_0 so that K avoids the spectrum of B_η for all $\eta \in U$. Assume $U \subseteq \mathring{\mathfrak{R}} \setminus \Sigma$. For $\eta \in U$ let us write $\prod_q(\det{}^K B^q_\eta)^{(-1)^{q+1}q}$ for the zeta regularized product of eigen values of B_η not contained in the interior of K. This depends holomorphically on $\eta \in U$. Let us write $\mathcal{C}^*_\eta(K)$ for the mapping cone of $\mathrm{Int}_\eta : E^*_\eta(K) \to C^*_\eta(X;\mathbb{C})$. This is a finite dimensional family of complexes holomorphically parametrized by $\eta \in U$, see section 4.3. Note that these complexes are acyclic since $U \cap \Sigma = \emptyset$. We equip $\mathcal{C}^*_\eta(K)$ with the basis determined by the restriction of the bilinear form b and the indicator functions in $C^*_\eta(X;\mathbb{C})$. These equivalence classes of bases depend holomorphically on $\eta \in U$. Hence the torsion $(T(\mathrm{Int}_\eta|_{E^*_\eta(K)}))^2 = (T\mathcal{C}^*_\eta(K))^2$ depends holomorphically on $\eta \in U$. Using Lemma 9 it is easy to see that

$$(T(\mathrm{Int}_\eta|_{E^*_\eta(0)}))^2 \cdot \prod_q (\det{}' B^q_\eta)^{(-1)^{q+1}q} = (T(\mathrm{Int}_\eta|_{E^*_\eta(K)}))^2 \cdot \prod_q (\det{}^K B^q_\eta)^{(-1)^{q+1}q}.$$

Hence $(T\,\mathrm{Int}_\eta)^2$ depends holomorphically on η too.

Similarly, using (3) and (7) one shows that $\lim_{t\to 0^+}(T\,\mathrm{Int}_{\eta+t\omega})^2 = (T\,\mathrm{Int}_\eta)^2$ for a Lyapunov form ω and $\eta \in \mathfrak{R}\setminus\Sigma$.

Next we will show that $(T\,\mathrm{Int}_\eta)^2$ does not depend on g. For real valued $\eta \in \mathcal{Z}^1(M;\mathbb{R}) \cap (\mathfrak{R}\setminus\Sigma)$ the operator B_η coincides with the Laplacian associated with g and η, and hence $T^{\mathrm{an}}_{\eta,g}$ coincides with the Ray–Singer torsion. For two Riemannian metrics g_1 and g_2 on M, the anomaly formula in [1, Theorem 0.1] then implies

$$\log\frac{(T(\mathrm{Int}_\eta|_{E^*_{\eta,g_2}(0)}))^2\cdot(T^{\mathrm{an}}_{\eta,g_2})^2}{(T(\mathrm{Int}_\eta|_{E^*_{\eta,g_1}(0)}))^2\cdot(T^{\mathrm{an}}_{\eta,g_1})^2} = 2\int_M \eta\wedge \mathrm{cs}(g_1,g_2).$$

Together with (36) this yields $(T\,\mathrm{Int}_{\eta,g_1})^2 = (T\,\mathrm{Int}_{\eta,g_2})^2$. Since both sides depend holomorphically on η, see Proposition 14, this relation is true for $\eta \in \mathring{\mathfrak{R}}\setminus\Sigma$ too. In view of (14) it continues to hold for $\eta \in \mathfrak{R}\setminus\Sigma$.

Let us finally turn to the gauge invariance. Again, for real $\eta \in \mathcal{Z}^1(M;\mathbb{R})\cap(\mathfrak{R}\setminus\Sigma)$ and real $h \in C^\infty(M;\mathbb{R})$ the anomaly formula in [1, Theorem 0.1.] implies

$$\log\frac{(T(\mathrm{Int}_\eta|_{E^*_{\eta+dh,g}(0)}))^2\cdot(T^{\mathrm{an}}_{\eta+dh,g})^2}{(T(\mathrm{Int}_\eta|_{E^*_{\eta,g}(0)}))^2\cdot(T^{\mathrm{an}}_{\eta,g})^2} = 2\Big(-\int_M h\,\mathrm{e}(g) + \sum_{x\in\mathcal{X}}\mathrm{IND}(x)h(x)\Big).$$

Together with (35) this implies $(T\,\mathrm{Int}_{\eta+dh})^2 = (T\,\mathrm{Int}_\eta)^2$. Since both sides depend holomorphically on η and h, see Proposition 14, this relation continues to hold for $\eta \in \mathring{\mathfrak{R}}\setminus\Sigma$ and $h \in C^\infty(M;\mathbb{C})$. In view of (14) it remains true for $\eta\in\mathfrak{R}\setminus\Sigma$. This completes the proof of Proposition 14.

5.2. **The Bismut–Zhang theorem.** Suppose our vector field is of the form $X = -\operatorname{grad}_{g_0} f$ for some Riemannian metric g_0 on M and a Morse function $f : M \to \mathbb{R}$. Then df is Lyapunov for X, hence X satisfies L. There are no closed trajectories. Hence X satisfies NCT, $\mathfrak{P} = \mathcal{Z}^1(M;\mathbb{C})$ and $e^{L(h_*\mathbb{P})(\eta)} = 1$. In view of Theorem 4(iv) the completion of the unstable manifolds are compact, hence $\mathfrak{R} = \mathfrak{I} = \mathcal{Z}^1(M;\mathbb{C})$. It is well known that $\Sigma = \emptyset$, i.e. the integration induces an isomorphism for all η. A theorem of Bismut–Zhang [1, Theorem 0.2] tells that in this case

$$(T\,\mathrm{Int}_\eta)^2 = 1 \tag{44}$$

for all $\eta \in \mathcal{Z}^1(M;\mathbb{R})$. Since $(T\,\mathrm{Int}_\eta)^2$ depends holomorphically on η, see Proposition 14, the relation (44) continues to hold for all $\eta\in\mathcal{Z}^1(M;\mathbb{C})$. To make a long story short, Theorem 3 is true for vector fields of the form $X = -\operatorname{grad}_{g_0} f$.

5.3. **An anomaly formula.** Consider the bordism $W := M\times[-1,1]$. Set $\partial_\pm W := M\times\{\pm1\}$. Let Y be a vector field on W. Assume that there are vector fields $X_\pm$ on M so that $Y(z,s)=X_+(z)+(s-1)\partial/\partial s$ in a neighborhood of ∂_+W and so that $Y(z,s)=X_-(z)+(-s-1)\partial/\partial s$ in a neighborhood of ∂_-W. Particularly, Y is tangential to ∂W. Moreover, assume that $ds(Y)<0$ on $M\times(-1,1)$. Particularly, there are no zeros or closed trajectories of Y contained in the interior of W. Let $\mathcal{X}_\pm$ denote the zeros of $X_\pm$. For $x\in\mathcal{X}_-$ we have $\mathrm{ind}_Y(x)=\mathrm{ind}_{X_-}(x)$, but note that for $x\in\mathcal{X}_+$ we have $\mathrm{ind}_Y(x)=\mathrm{ind}_{X_+}(x)+1$. We choose the orientations of the unstable manifolds of Y so that $W^-_{Y,x}=W^-_{X_-,x}$ is orientation preserving for $x\in\mathcal{X}_-$, and so that $\partial W^-_{Y,x}=W^-_{X_+,x}$ is orientation reversing for $x\in\mathcal{X}_+$.

Suppose Y satisfies MS and L. Note that this implies that $X_\pm$ satisfy MS and L too. Then Proposition 4, Theorem 4(i) and Proposition 9 continue to hold for Y. Hence we get a complex $C^*_{\tilde\eta}(Y;\mathbb{C})$ for all $\tilde\eta\in\mathfrak{I}^Y$. Note that for $\tilde\eta\in\mathfrak{I}^Y$ we have $\eta_\pm:=\iota_\pm^*\tilde\eta\in\mathfrak{I}^{X_\pm}$, where $\iota_\pm:M\to\partial_\pm W$, $\iota_\pm(z)=(z,\pm1)$. Clearly

$$C^*_{\tilde\eta}(Y;\mathbb{C})=C^{*-1}_{\eta_+}(X_+;\mathbb{C})\oplus C^*_{\eta_-}(X_-;\mathbb{C}),\qquad \delta^Y_{\tilde\eta}=\begin{pmatrix}-\delta^{X_+}_{\eta_+} & u^Y_\eta\\ 0 & \delta^{X_-}_{\eta_-}\end{pmatrix}\tag{45}$$

for some

$$u^Y_{\tilde\eta}:C^*_{\eta_-}(X_-;\mathbb{C})\to C^*_{\eta_+}(X_+;\mathbb{C}).\tag{46}$$

¿From $(\delta^Y_{\tilde\eta})^2=0$ we see that (46) is a homomorphism of complexes.

Theorem 4(ii) needs a minor adjustment in the case with boundary. More precisely, for $x\in\mathcal{X}_+$ the completion of the unstable manifold W^-_x has additional boundary parts stemming from the fact that W^-_x intersects ∂_+W transversally. For $\tilde\eta\in\mathfrak{R}^Y$ we get a linear mapping $\mathrm{Int}^Y_{\tilde\eta}:\Omega^*(W;\mathbb{C})\to C^*_{\tilde\eta}(Y;\mathbb{C})$ satisfying

$$\mathrm{Int}^Y_{\tilde\eta}\circ d_{\tilde\eta}=\delta^Y_{\tilde\eta}\circ\mathrm{Int}^Y_{\tilde\eta}-(\iota_+)_*\circ\mathrm{Int}^{X_+}_{\eta_+}\circ\iota_+^*.\tag{47}$$

Here $\iota_+^*:\Omega^*_{\tilde\eta}(W;\mathbb{C})\to\Omega^*_{\eta_+}(M;\mathbb{C})$ is the pull back of forms, and $(\iota_+)_*:C^*_{\eta_+}(X_+;\mathbb{C})\to C^{*+1}_{\tilde\eta}(Y;\mathbb{C})$ is the obvious inclusion stemming from $\mathcal{X}_+\subseteq\mathcal{Y}$. But note that while ι_+^* is a homomorphism of complexes, we have $(\iota_+)_*\circ\delta^{X_+}_{\eta_+}+\delta^Y_{\tilde\eta}\circ(\iota_+)_*=0$. Moreover, note that $\tilde\eta\in\mathfrak{R}^Y$ implies $\eta_\pm\in\mathfrak{R}^{X_\pm}$. For η_- this is trivial. For η_+ it follows from $W^-_{Y,x}\supseteq W^-_{X_+,x}\times(1-\epsilon,1]$ for some $\epsilon>0$. Moreover, $\mathfrak{R}^Y\subseteq\mathfrak{I}^Y$, cf. Theorem 1. So (47) indeed makes sense for $\tilde\eta\in\mathfrak{R}^Y$. Splitting $\mathrm{Int}^Y_{\tilde\eta}$ according to (45) we find $\mathrm{Int}^Y_{\tilde\eta}=(h^Y_{\tilde\eta},\mathrm{Int}^{X_-}_{\eta_-}\circ\iota_-^*)$ for some

$$h^Y_{\tilde\eta}:\Omega^*_{\tilde\eta}(W;\mathbb{C})\to C^{*-1}_{\eta_+}(X_+;\mathbb{C}),$$

and (47) tells that for all $\tilde\eta\in\mathfrak{R}^Y$

$$h^Y_{\tilde\eta}\circ d_{\tilde\eta}=-\delta^{X_+}_{\eta_+}\circ h^Y_{\tilde\eta}+u^Y_{\tilde\eta}\circ\mathrm{Int}^{X_-}_{\eta_-}\circ\iota_-^*-\mathrm{Int}^{X_+}_{\eta_+}\circ\iota_+^*.\tag{48}$$

Let $p:W\to M$ denote the projection. For $\eta\in\mathcal{Z}^1(M;\mathbb{C})$ we write $u^Y_\eta:=u^Y_{p^*\eta}$ and $h^Y_\eta:=h^Y_{p^*\eta}\circ p^*$. Then $u^Y_\eta:C^*_\eta(X_-;\mathbb{C})\to C^*_\eta(X_+;\mathbb{C})$ is a homomorphism of complexes, and h^Y_η is a homotopy between $u^Y_\eta\circ\mathrm{Int}^{X_-}_\eta$ and $\mathrm{Int}^{X_+}_\eta$.

Proposition 21. *Let Y be a vector field on $W=M\times[-1,1]$ as above. Suppose $\eta\in(\mathfrak{R}^{X_-}\setminus\Sigma^{X_-})\cap(\mathfrak{R}^{X_+}\setminus\Sigma^{X_+})$ and assume $p^*\eta\in\mathfrak{R}^Y$. Then $u^Y_\eta:C^*_\eta(X_-;\mathbb{C})\to$*

$C^*_\eta(X_+;\mathbb{C})$ *is a quasi isomorphism, and*

$$\frac{(T\,\mathrm{Int}^{X_+}_\eta)^2}{(T\,\mathrm{Int}^{X_-}_\eta)^2} = (Tu^Y_\eta)^2 \cdot (e^{-\eta(\mathrm{cs}(X_-,X_+))})^2. \tag{49}$$

Here the torsion $\pm Tu^Y_\eta$ *is computed with respect to the base determined by the indicator functions on* $\mathcal{X}_\pm$.

Proof. ¿From the discussion above we know that $\mathrm{Int}^{X_+}_\eta$ is homotopic to $u^Y_\eta \circ \mathrm{Int}^{X_-}_\eta$. Hence u^Y_η is a quasi isomorphism and

$$\frac{\pm T(\mathrm{Int}^{X_+}_\eta |_{E^*_\eta(0)})}{\pm T(\mathrm{Int}^{X_-}_\eta |_{E^*_\eta(0)})} = \pm Tu^Y_\eta.$$

Together with (37) this yields (49). □

5.4. **Hutchings–Lee formula.** Let X be a vector field which satisfies MS and L. Let $\Gamma := \mathrm{img}\big(\pi_1(M) \to H_1(M;\mathbb{R})\big)$. Let $\omega \in \Omega^1(M;\mathbb{R})$ be Lyapunov for X and such that $\omega : \Gamma \to \mathbb{R}$ is injective. Note that such Lyapunov forms exist in view of Proposition 3. Let Λ_ω denote the corresponding Novikov field consisting of all functions $\lambda : \Gamma \to \mathbb{C}$ for which $\{\gamma \in \Gamma \mid \lambda(\gamma) \neq 0, -\omega(\gamma) \leq K\}$ is finite for all $K \in \mathbb{R}$, equipped with the convolution product. Let us write Λ^+_ω for the subring of functions λ for which $\lambda(\gamma) \neq 0$ implies $-\omega(\gamma) > 0$.

The vector field X gives rise to a Novikov complex $C^*(X;\Lambda_\omega)$. This complex can be described as follows. Let $\pi : \tilde{M} \to M$ denote the covering corresponding to the kernel of $\pi_1(M) \to H_1(M;\mathbb{R})$. This is a principal Γ–covering. Let $\tilde{\mathcal{X}} := \pi^{-1}(\mathcal{X})$ denote the zero set of the vector field $\tilde{X} := \pi^* X$. Choose a function $h : \tilde{M} \to \mathbb{R}$ such that $dh = \pi^*\omega$. Now $C^*(X;\Lambda_\omega)$ is the space of all functions $c : \tilde{\mathcal{X}} \to \mathbb{C}$ for which $\{\tilde{x} \in \tilde{\mathcal{X}} \mid c(\tilde{x}) \neq 0, -h(\tilde{x}) \leq K\}$ is finite for all $K \in \mathbb{R}$. This is a finite dimensional vector space over Λ_ω, independent of the choice of h. Note that for a section $s : \mathcal{X} \to \tilde{\mathcal{X}}$ the indicator functions for $s(x)$, $x \in \mathcal{X}$, define a basis of $C^*(X;\Lambda_\omega)$.

To describe the differential let us call two elements $\gamma_1, \gamma_2 \in \mathcal{P}_{x,y}$ equivalent if $\gamma_2^{-1}\gamma_1$ vanishes in $H_1(M;\mathbb{R})$. Let $p_{x,y} : \mathcal{P}_{x,y} \to \mathcal{P}'_{x,y}$ denote the projection onto the space of equivalence classes. Γ acts free and transitively on $\mathcal{P}'_{x,y}$. The differential on $C^*(X;\Lambda_\omega)$ is determined by the counting functions

$$\mathbb{I}'_{x,y} := (p_{x,y})_* \mathbb{I}_{x,y} : \mathcal{P}'_{x,y} \to \mathbb{Z}, \qquad \mathbb{I}'_{x,y}(a) := \sum_{p_{x,y}(\gamma)=a} \mathbb{I}_{x,y}(\gamma). \tag{50}$$

Note that these sums are finite in view of Proposition 4.

Now suppose Y is a vector field on $W = M \times [-1,1]$ as in section 5.3. Assume Y satisfies MS and L. Suppose $p^*\omega$ is Lyapunov for Y where $p : W \to M$ denotes the projection. As in section 5.3 the differential of the Novikov complex $C^*(Y;\Lambda_{p^*\omega})$ gives rise to a homomorphism of Novikov complexes

$$u^Y : C^*(X_-;\Lambda_\omega) \to C^*(X_+;\Lambda_\omega). \tag{51}$$

It is well known that (51) is a quasi isomorphism. Let $s_\pm : \mathcal{X}_\pm \to \tilde{\mathcal{X}}_\pm$ be sections and equip $C^*(X_\pm;\Lambda_\omega)$ with the corresponding base. Assume (X_-, s_-) and (X_+, s_+)

determine the same Euler structure [23, 5]. Recall that this implies

$$\sum_{x\in\mathcal{X}_+} h^\eta(s(x)) - \sum_{x\in\mathcal{X}_-} h^\eta(s(x)) = \eta(\mathrm{cs}(X_-,X_+)) \tag{52}$$

for all $\eta\in\mathcal{Z}^1(M;\mathbb{C})$ and all smooth functions $h^\eta:\tilde M\to\mathbb{C}$ with $dh^\eta=\pi^*\eta$.

A result of Hutchings–Lee [9] and Pajitnov [19] tells that if Y in addition satisfies NCT, then the torsion of (51) is

$$\pm T(u^Y) = \pm\exp(h_*\mathbb{P}^{X_+} - h_*\mathbb{P}^{X_-}) \in 1+\Lambda^+_\omega. \tag{53}$$

5.5. **Two lemmas.** Let $\Gamma := \operatorname{img}\big(\pi_1(M)\to H_1(M;\mathbb{R})\big)$, let $\omega\in\mathcal{Z}^1(M;\mathbb{R})$ be a closed one form, suppose $\omega:\Gamma\to\mathbb{R}$ is injective and let Λ_ω denote the Novikov field as introduced in section 5.4. For a closed one form $\eta\in\Omega^1(M;\mathbb{C})$ we let L^1_η denote the Banach algebra of all functions $\lambda:\Gamma\to\mathbb{C}$ with $\|\lambda\|_\eta := \sum_{\gamma\in\Gamma}|\lambda(\gamma)e^{\eta(\gamma)}|<\infty$ equipped with the convolution product. Moreover, let us write $\mathrm{ev}_\eta: L^1_\eta\to\mathbb{C}$ for the homomorphism given by $\mathrm{ev}_\eta(\lambda) := L(\lambda)(\eta) = \sum_{\gamma\in\Gamma}\lambda(\gamma)e^{\eta(\gamma)}$.

Lemma 10. *Suppose $0\neq\lambda\in\Lambda_\omega\cap L^1_\eta$. Then there exists $t_0\in\mathbb{R}$ so that $\lambda^{-1}\in\Lambda_\omega\cap L^1_{\eta+t\omega}$ for all $t\geq t_0$.*

Proof. Using the Novikov property of λ and the injectivity of $\omega:\Gamma\to\mathbb{R}$ it is easy to see that we may w.l.o.g. assume $1-\lambda\in\Lambda^+_\omega$. Since $\lambda\in L^1_\eta$ we have $\|\lambda\|_\eta<\infty$. Using the Novikov property of λ and the injectivity of $\omega:\Gamma\to\mathbb{R}$ again we find $t_0\in\mathbb{R}$ so that $\|1-\lambda\|_{\eta+t\omega}<1$, for all $t\geq t_0$. Since $L^1_{\eta+t\omega}$ is a Banach algebra $\sum_{k\geq0}(1-\lambda)^k$ will converge and $\lambda^{-1}\in\Lambda_\omega\cap L^1_{\eta+t\omega}$, for all $t\geq t_0$. □

Recall that we have a bijection $\exp:\Lambda^+_\omega\to1+\Lambda^+_\omega$.

Lemma 11. *Suppose $\lambda\in\Lambda^+_\omega$ and $\exp(\lambda)\in L^1_\eta$. Then there exists $t_0\in\mathbb{R}$ so that $\lambda\in\Lambda_\omega\cap L^1_{\eta+t\omega}$, for all $t\geq t_0$.*

Proof. Similar to the proof of Lemma 10 using $\log(1-\mu) = -\sum_{k>0}\frac{\mu^k}{k}$. □

5.6. **Computation of the anomaly.** With the help of the Hutchings–Lee formula it is possible to compute the right hand side of (49) in terms of closed trajectories under some assumptions.

Proposition 22. *Suppose Y is a vector field on $M\times[-1,1]$ as in Proposition 21 which satisfies MS, L, NCT and EG. Let $\eta\in\mathcal{Z}^1(M;\mathbb{C})$ be a closed one form. Suppose $\omega\in\mathcal{Z}^1(M;\mathbb{R})$ such that $\omega:\Gamma\to\mathbb{R}$ is injective and such that $[p^*\omega]$ is a Lyapunov class for Y. Then there exists t_0 such that for $t>t_0$ we have $\eta+t\omega\in\big(\mathring{\mathfrak{R}}^{X_+}\setminus\Sigma^{X_+}\big)\cap\big(\mathring{\mathfrak{R}}^{X_-}\setminus\Sigma^{X_-}\big)$, $L(h_*\mathbb{P}^{X_+}-h_*\mathbb{P}^{X_-})(\eta+t\omega)$ converges absolutely, and*

$$\frac{(T\operatorname{Int}^{X_+}_{\eta+t\omega})^2}{(T\operatorname{Int}^{X_-}_{\eta+t\omega})^2} = \big(e^{L(h_*\mathbb{P}^{X_+}-h_*\mathbb{P}^{X_-})(\eta+t\omega)}\big)^2.$$

Proof. Since $X_\pm$ satisfies EG, and since the cohomology class of ω contains a Lyapunov form for $X_\pm$ we obtain from Proposition 12, Theorem 1 and Proposition 8 that $\eta+t\omega\in\big(\mathring{\mathfrak{R}}^{X_+}\setminus\Sigma^{X_+}\big)\cap\big(\mathring{\mathfrak{R}}^{X_-}\setminus\Sigma^{X_-}\big)$ for sufficiently large t. Arguing similarly for Y we see that $p^*(\eta+t\omega)\in\mathfrak{R}^Y$ for sufficiently large t. Particularly, Proposition 21 is applicable and we get

$$\frac{(T\operatorname{Int}^{X_+}_{\eta+t\omega})^2}{(T\operatorname{Int}^{X_-}_{\eta+t\omega})^2}=(Tu^Y_{\eta+t\omega})^2\cdot\big(e^{-(\eta+t\omega)(\operatorname{cs}(X_-,X_+))}\big)^2.$$

Since $\mathfrak{R}^Y\subseteq\mathfrak{I}^Y$, see Theorem 1, the Novikov complex of Y is defined over the ring $\Lambda_t:=\Lambda_\omega\cap L^1_{\eta+t\omega}$ for sufficiently large t. More precisely, for sufficiently large t the counting functions (50) actually define a complex $C^*(Y;\Lambda_t)$ over Λ_t with

$$C^*(Y;\Lambda_\omega)=C^*(Y;\Lambda_t)\otimes_{\Lambda_t}\Lambda_\omega.$$

Since the basis determined by sections $s_\pm:\mathcal{X}_\pm\to\tilde{\mathcal{X}}_\pm$ obviously consist of elements in $C^*(Y;\Lambda_t)$ we conclude that the torsion $T(u^Y)$ is contained in the quotient field $Q(\Lambda_t)\subseteq\Lambda_\omega$. In view of (53) Lemma 10 and Lemma 11 we thus have

$$h_*\mathbb{P}^{X_+}-h_*\mathbb{P}^{X_-}\in\Lambda_t,$$

and hence $L(h_*\mathbb{P}^{X_+}-h_*\mathbb{P}^{X_-})(\eta+t\omega)$ converges absolutely for sufficiently large t.

For sufficiently large t, let us write $\operatorname{ev}_t:\Lambda_t\to\mathbb{C}$ for the homomorphism given by $\operatorname{ev}_t(\lambda):=L(\lambda)(\eta+t\omega)=\sum_{\gamma\in\Gamma}\lambda(\gamma)e^{(\eta+t\omega)(\gamma)}$. Clearly,

$$C^*_{\eta+t\omega}(Y;\mathbb{C})=C^*(Y;\Lambda_t)\otimes_{\operatorname{ev}_t}\mathbb{C}.$$

Moreover, using (52) and Lemma 10, it is easy to see that this implies

$$\pm Tu^Y_{\eta+t\omega}\cdot e^{-(\eta+t\omega)(\operatorname{cs}(X_-,X_+))}=\pm L(Tu^Y)(\eta+t\omega),$$

and (53) yields

$$\pm Tu^Y_{\eta+t\omega}\cdot e^{-(\eta+t\omega)(\operatorname{cs}(X_-,X_+))}=\pm e^{L(h_*\mathbb{P}^{X_+}-h_*\mathbb{P}^{X_-})(\eta+t\omega)}.\qquad\square$$

5.7. **Proof of Theorem 3.** Let X be a vector field satisfying SEG. Choose a vector field Y on $M\times[-1,1]$ as in Definition 5. Note that $h_*\mathbb{P}^{X_+}=h_*\mathbb{P}^X$, $h_*\mathbb{P}^{X_-}=0$, and $(T\operatorname{Int}^{X_-}_{\tilde\eta})^2=1$ for all $\tilde\eta\in\mathcal{Z}^1(M;\mathbb{C})$, see (44). Let $\Gamma:=\operatorname{img}\big(\pi_1(M)\to H_1(M;\mathbb{R})\big)$. Let ω_0 be a Lyapunov form for X such that $\omega_0:\Gamma\to\mathbb{R}$ is injective, see Proposition 3. ¿From Propositions 23 (Appendix B) and 3 we see that $[p^*\omega_0]$ is a Lyapunov class for Y.

Let $\eta\in\mathcal{Z}^1(M;\mathbb{C})$ be a closed one form. In view of Proposition 22 we have $\eta+t\omega_0\in\mathring{\mathfrak{R}}^X$ for sufficiently large t. Let ω be an arbitrary Lyapunov form for X. Using Propositions 3 and 8 we conclude that $\eta+t\omega\in\mathring{\mathfrak{R}}^X$ for sufficiently large t. Applying Proposition 22 to various η we see that

$$(T\operatorname{Int}^X_{\tilde\eta})^2=(e^{L(h_*\mathbb{P}^X)(\tilde\eta)})^2\tag{54}$$

holds for an open subset of $\tilde\eta\in(\mathring{\mathfrak{R}}^X\setminus\Sigma^X)\cap\mathring{\mathfrak{P}}^X$. By analyticity, see Propositions 7 and 14, equality (54) holds for all $\tilde\eta\in(\mathring{\mathfrak{R}}^X\setminus\Sigma^X)\cap\mathring{\mathfrak{P}}^X$. Using (5) and (14) we see that the relation (54) continues to hold for all $\tilde\eta\in(\mathfrak{R}^X\setminus\Sigma^X)\cap\mathfrak{P}^X$. This completes the proof of Theorem 3.

5.8. **Proof of Theorem 3'.** Let ω be a Lyapunov form for X and assume $\omega:\Gamma\to\mathbb{R}$ is injective, see Proposition 3. Let $\eta\in\mathcal{Z}^1(M;\mathbb{C})$. Note that since $H_{\eta_0}(M;\mathbb{C})=0$ the deRham cohomology will be acyclic, generically. More precisely, $H^*_{\eta+t\omega}(M;\mathbb{C})=0$ for sufficiently large t. In view of Theorem 1 we also have $H^*_{\eta+t\omega}(X;\mathbb{C})=0$ for sufficiently large t.

As in section 5.6 one shows that for sufficiently large t the Novikov complex $C^*(M;\Lambda_\omega)$ is actually defined over the ring $\Lambda_t := \Lambda_\omega \cap L^1_{\eta+t\omega}$,

$$C^*(X;\Lambda_\omega) = C^*(X;\Lambda_t) \otimes_{\Lambda_t} \Lambda_\omega.$$

Moreover, for sufficiently large t

$$C^*(X;\Lambda_t) \otimes_{\mathrm{ev}_t} \mathbb{C} = C^*_{\eta+t\omega}(X;\mathbb{C}).$$

We conclude that the Novikov complex $C^*(X;\Lambda_\omega)$ is acyclic.

Let $Y = -\operatorname{grad}_{g_0} f$ be a Morse–Smale vector field with zero set $\mathcal{Y}$. Let $s_X : \mathcal{X} \to \tilde{\mathcal{X}}$ and $s_Y : \mathcal{Y} \to \tilde{\mathcal{Y}}$ be a sections and assume that they define the same Euler structure, i.e.

$$\sum_{x\in\mathcal{X}} h^{\tilde\eta}(s_X(x)) - \sum_{y\in\mathcal{Y}} h^{\tilde\eta}(s_Y(y)) = \tilde\eta(\mathrm{cs}(X,Y))$$

for all $\tilde\eta \in \mathcal{Z}^1(M;\mathbb{C})$ and all smooth functions $h : \tilde M \to \mathbb{C}$ with $dh^{\tilde\eta} = \pi^*\tilde\eta$, see section 5.4. Equip the complexes $C^*(X;\Lambda_\omega)$ and $C^*(Y;\Lambda_\omega)$ with the corresponding graded bases. For the torsion we have [9, 19]

$$\frac{\pm T\big(C^*(X;\Lambda_\omega)\big)}{\pm T\big(C^*(Y;\Lambda_\omega)\big)} = \pm\exp(h_*\mathbb{P}^X) \in 1 + \Lambda^+_\omega.$$

As in section 5.6 one shows that this torsion must be contained in the quotient field $Q(\Lambda_t) \subseteq \Lambda_\omega$, hence $(h_*\mathbb{P})(\eta+t\omega)$ converges absolutely, and thus $\eta + t\omega \in \mathring{\mathfrak{R}} \setminus \Sigma$ for sufficiently large t. Again, this remains true for arbitrary Lyapunov ω in view of Proposition 3.

Equip the complexes $C^*_{\eta+t\omega}(X;\mathbb{C})$ and $C^*_{\eta+t\omega}(Y;\mathbb{C})$ with the graded bases determined by the indicator functions. As in section 5.6 we conclude that

$$\frac{\pm T\big(C^*_{\eta+t\omega}(X;\mathbb{C})\big)}{\pm T\big(C^*_{\eta+t\omega}(Y;\mathbb{C})\big)} e^{-(\eta+t\omega)(\mathrm{cs}(X,Y))} = \pm e^{L(h_*\mathbb{P}^X)(\eta+t\omega)}$$

for sufficiently large t. Using (37) this implies

$$\frac{(T\,\mathrm{Int}^X_{\eta+t\omega})^2}{(T\,\mathrm{Int}^Y_{\eta+t\omega})^2} = (e^{L(h_*\mathbb{P}^X)(\eta+t\omega)})^2.$$

In view of (44) we have $(T\,\mathrm{Int}^Y_{\eta+t\omega})^2 = 1$. We conclude that

$$(T\,\mathrm{Int}^X_{\tilde\eta})^2 = (e^{L(h_*\mathbb{P}^X)(\tilde\eta)})^2$$

holds for an open set of $\tilde\eta \in (\mathring{\mathfrak{R}}^X \setminus \Sigma^X) \cap \mathring{\mathfrak{P}}^X$. By analyticity, see Propositions 7 and 14, this relation holds for all $\tilde\eta \in (\mathring{\mathfrak{R}}^X \setminus \Sigma^X) \cap \mathring{\mathfrak{P}}^X$. Using (14) and (5) it remains true for all $\eta \in (\mathfrak{R}^X \setminus \Sigma^X) \cap \mathfrak{P}^X$.

Appendix A. Proof of Proposition 2

We will make use of the following lemma whose proof we leave to the reader.

Lemma 12. *Let N be a compact smooth manifold, possibly with boundary, and let $K \subseteq N$ be a compact subset. Let $L := N \times \partial I \cup K \times I$ where $I := [0,1]$. Suppose F is a smooth function defined in a neighborhood of L so that $\partial F/\partial t < 0$ whenever defined, and so that $F(x,0) > F(x,1)$ for all $x \in N$. Then there exists a smooth function $G : N \times I \to \mathbb{R}$ which agrees with F on a neighborhood of L and satisfies $\partial G/\partial t < 0$.*

For $\rho \geq \epsilon > 0$ define

$$\mathbb{D}_{\rho,\epsilon} := \bigl\{(y,z) \in \mathbb{R}^q \times \mathbb{R}^{n-q} \bigm| -\rho \leq -\tfrac{1}{2}|y|^2 + \tfrac{1}{2}|z|^2 \leq \rho, |y| \cdot |z| \leq \epsilon\bigr\}.$$

Lemma 13. *Suppose $F : \mathbb{D}_{\rho,\rho} \to \mathbb{R}$ is a smooth function with $F(0) = 0$ which is strictly decreasing along non-constant trajectories of X, see* (1)*. Then there exists $\rho > \epsilon > 0$ and a smooth function $G : \mathbb{D}_{\rho,\rho} \to \mathbb{R}$ which is strictly decreasing along non-constant trajectories of X, which coincides with F on a neighborhood of $\partial\mathbb{D}_{\rho,\rho}$ and which coincides with $-\frac{1}{2}|y|^2 + \frac{1}{2}|z|^2$ on $\mathbb{D}_{\epsilon,\epsilon}$.*

Proof. Consider the partially defined function which coincides with F in a neighborhood of $\partial\mathbb{D}_{\rho,\rho}$ and which coincides with $-\frac{1}{2}|y|^2 + \frac{1}{2}|z|^2$ on a neighborhood of $\mathbb{D}_{\epsilon,\epsilon}$. We will extend this to a globally defined smooth function $\mathbb{D}_{\rho,\rho} \to \mathbb{R}$ which is strictly decreasing along non-constant trajectories of X. This will be accomplished in two steps.

For the first step notice that $\overline{\mathbb{D}_{\rho,\epsilon} \setminus \mathbb{D}_{\epsilon,\epsilon}}$ is diffeomorphic to $N \times I$ where $N = S^{q-1} \times D^{n-q} \cup D^q \times S^{n-q-1}$. Here S^{k-1} and D^k denote unit sphere and unite ball in $\mathbb{R}^k$, respectively. Choosing ϵ sufficiently small we can apply Lemma 12 with $K = \emptyset$, and obtain an extension to $\mathbb{D}_{\rho,\epsilon}$.

For the second step notice that $\overline{\mathbb{D}_{\rho,\rho} \setminus \mathbb{D}_{\rho,\epsilon}}$ is diffeomorphic to $N \times I$ where $N = C^q \times S^{n-q-1}$ and $C^q := \{y \in \mathbb{R}^q \mid 1 \leq |y| \leq 2\}$. Applying Lemma 12 with $K = \partial C^q$, provides the desired extension to $\mathbb{D}_{\rho,\rho}$. □

Proof of Proposition 2. Let $\omega \in \Omega^1(M;\mathbb{R})$ be a closed one form such that $\omega(X) < 0$ on $M \setminus \mathcal{X}$. By adding a small closed one form with support contained in $M \setminus \mathcal{X}$ we may in addition assume that the cohomology class of ω is rational. Multiplying with a positive number we may assume that the cohomology class of ω is integral. Moreover, in view of Lemma 13 we may assume that ω has canonical form in a neighborhood of $\mathcal{X}$. More precisely, for every $x \in \mathcal{X}_q$ there exist coordinates $(x_1, \dots, x_n)$ centered at x in which

$$X = \sum_{i \leq q} x_i \frac{\partial}{\partial x_i} - \sum_{i > q} x_i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \omega = -\sum_{i \leq q} x_i dx^i + \sum_{i > q} x_i dx^i. \tag{55}$$

Define a Riemannian metric g on M as follows. On a neighborhood of $\mathcal{X}$ on which X and ω have canonic form define $g := \sum_i (dx^i)^2$. Note that this implies $\omega = -g(X, \cdot)$ where defined. Since $\omega(X) < 0$ we have $TM = \ker\omega \oplus [X]$ over $M \setminus \mathcal{X}$. Extend $g|_{\ker\omega}$ smoothly to a fiber metric on $\ker\omega$ over $M \setminus \mathcal{X}$, and let the restriction of g to $\ker\omega$ be given by this extension. Moreover, set $g(X,X) := -\omega(X)$ and $g(X, \ker\omega) := 0$. This defines a smooth Riemannian metric on M, and certainly $\omega = -g(X, \cdot)$. □

Appendix B. Vector fields on $M \times [-1, 1]$

Proposition 23. *Let $X_\pm$ be two vector fields on M. Then there exists a vector field Y on $M \times [-1,1]$ such that $Y(z,s) = X_+(z) + (s-1)\partial/\partial s$ in a neighborhood of ∂_+W, such that $Y(z,s) = X_-(z) + (-s-1)\partial/\partial s$ in a neighborhood of ∂_-W, and such that $ds(Y) < 0$ on $M \times (-1,1)$. Moreover, every such vector field has the following property: If $\xi \in H^1(M;\mathbb{R})$ is a Lyapunov class for X_+ and X_-, then $p^*\xi \in H^1(M \times [-1,1];\mathbb{R})$ is a Lyapunov class for Y, where $p : M \times [-1,1] \to M$ denotes the projection.*

Proof. The existence of such a vector field Y is obvious. Suppose $\xi \in H^1(M;\mathbb{R})$ is a Lyapunov class for X_- and X_+. It is easy to construct a closed one form $\omega \in \mathcal{Z}^1(M\times[-1,1];\mathbb{R})$ representing $p^*\xi$ such that $\omega_\pm(X_\pm)<0$ on $M\setminus\mathcal{X}_\pm$, where $\omega_\pm := \iota_\pm^*\omega \in \mathcal{Z}^1(M;\mathbb{R})$, and $\iota_\pm : M \to M\times\{\pm1\}\subseteq M\times[-1,1]$ denotes the canonic inclusions. We may moreover assume that $i_{\partial_s}\omega$ vanishes in a neighborhood of $M\times\{\pm1\}$. For sufficiently large t the form $\omega+tds\in\mathcal{Z}^1(M\times[-1,1];\mathbb{R})$ will be a Lyapunov form for Y representing $p^*\xi$. $\square$

References

[1] J.M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller.* Astérisque **205**, Société Mathématique de France, 1992.

[2] R. Bott and L.W. Tu, *Differential forms in algebraic topology.* Graduate Texts in Mathematics **82**. Springer-Verlag, New York–Berlin, 1982.

[3] D. Burghelea, L. Friedlander, T. Kappeler and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. **6**(1996), 751–859.

[4] D. Burghelea and S. Haller, *On the topology and analysis of closed one form. I (Novikov theory revisited)*, Monogr. Enseign. Math. **38**(2001), 133–175.

[5] D. Burghelea and S. Haller, *Euler structures, the variety of representations and the Milnor–Turaev torsion,* preprint `arXiv.math.DG/0310154`.

[6] D. Burghelea and S. Haller, *Laplace transform, dynamics and spectral geometry,* preprint `arXiv.math.DG/0405037`.

[7] D. Fried, *Lefschetz formulas for flows.* The Lefschetz centennial conference, Part III (Mexico City, 1984), 19–69, Contemp. Math. **58**, III, Amer. Math. Soc., Providence, RI, 1987.

[8] E. Hille and R.S. Phillips, *Functional analysis and semi-groups.* Third printing of the revised edition of 1957. American Mathematical Society Colloquium Publications **XXXI**. American Mathematical Society, Providence, R. I., 1974.

[9] M. Hutchings, Y.J. Lee, *Circle–valued Morse Theory, Reidemeister torsion and Seiberg–Witten invariants of* 3 *manifolds*, Topology **38**(1999), 861–888.

[10] G. Illies, *Regularized products and determinants,* Comm. Math. Phys. **220**(2001), 69–94.

[11] I. Kupka, *Contribution à la théorie des champs génénriques,* Contributions to Differential Equations **2**(1963), 457–484.

[12] T. Kato, *Perturbation theory for linear operators.* Second edition. Grundlehren der Mathematischen Wissenschaften **132**. Springer-Verlag, Berlin-New York, 1976.

[13] J. Marcsik, *Analytic torsion and closed one forms.* Thesis OSU **8**(1998).

[14] V. Mathai and D. Quillen, *Superconnections, Thom Classes, and Equivariant differential forms*, Topology **25**(1986), 85–110.

[15] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72**(1966), 358–426.

[16] S.P. Novikov, *Quasiperiodic structures in topology*, in Topological methods in modern mathematics, Proceedings of the symposium in honor of John Milnor's sixtieth birthday held at the State University of New York, Stony Brook, New York, 1991, eds L. R. Goldberg and A. V. Phillips, Publish or Perish, Houston, TX, 1993, 223–233.

[17] V.A. Pajitnov, *The incidence coefficients in the Novikov complex are generically rational functions,* St. Petersburg Math. J. **9**(1998), 969–1006. Translation from Algebra i Analiz **9**(1997), 92–139.

[18] V.A. Pajitnov, *C^0–generic properties of boundary operators in the Novikov complex.* Pseudoperiodic topology, 29–115, Amer. Math. Soc. Transl. **197**, Amer. Math. Soc., Providence, RI, 1999.

[19] V.A. Pajitnov, *On closed orbits of gradient flows of circle-valued mappings,* St. Petersburg Math. J. **14**(2003), 499–534. Translation from Algebra i Analiz **14**(2002), 186–240.

[20] M.M. Peixoto, *On an Approximation Theorem of Kupka and Smale*, J. Differential Equations **3**(1967), 214–227.

[21] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds,* Adv. Math. **7**(1971), 145–210.

[22] S. Smale, *Stable manifolds for differential equations and diffeomorphisms,* Ann. Scuola Norm. Sup. Pisa **17**(1963), 97–116.

[23] V.Turaev, *Euler structures, nonsingular vector fields, and Reidemeister-type torsions*, Math. USSR–Izv. **34**(1990), 627–662.

[24] D.V. Widder, *The Laplace Transform.* Princeton Mathematical series **6**, Princeton University Press, Princeton, N.J., 1941.

[25] H. Whitney, *Complex analytic varieties.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1972.

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