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Abstract

Let X be a complex projective toric manifold. We associated to X, a positive and closed $(1,1)$-current called the canonical toric Kähler current of X denoted by $\omega_{X,can}$, and a new invariant called the canonical spectrum of X. This spectrum is obtained as the set of the eigenvalues of a singular Laplacian defined by $\omega_{X,can}$ and which is described uniquely by the combinatorial structure of X. The construction of this Laplacian and the study of its spectral properties are the consequence of a generalized spectral theory of Laplacians on compact Kähler manifolds that we develop in this article.

Contents

It is known that several properties and quantities of toric varieties can be translated into the language of convex geometry. For example, the degree of a toric variety wrt an equivariant line bundle is given as the volume of a convex polytope associated to the line bundle. Toric varieties are often used as simple test cases for difficult geometric problem. The proof of the Grothendieck-Riemann-Roch theorem is easier in the context of toric varieties. The combinatorial structure of toric manifolds gives rise to an interesting class of metrics on equivariant line bundles introduced first by Batyrev and Tschinkel in this setting and called canonical metrics (see [1]). These metrics are irregular but can be approximated uniformly by a sequence of positive and smooth metrics, we say that the canonical metrics are admissible. The notion of admissible metrics plays an important role in the generalization of the Arakelov geometry in [12], see also [8]. One of the motivations of this article is to give a satisfactory notion of admissible metrics on holomorphic vector bundles of higher rank.

Our first result is the following theorem:

Theorem 0.1 (Theorem 1.6)**.** *Let X be a compact Kähler manifold of dimension n and ω_0 is a Kähler form on X. Let F be a continuous function on X such that $\int_X e^F\omega_0^n = 1$. Then there exists $\varphi \in \mathcal{C}^{1,1}(X)$ which is a solution to the following singular Monge-Ampère equation:*

$$\big(\omega_0 + dd^c\varphi\big)^n = e^F\omega_0^n. \tag{1}$$

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Our main result can be stated as follows: Given (X, ω_0) a compact Kähler manifold of dimension n and let $(F_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions on X converging uniformly to a function F such that

1. For any $k \in \mathbb{N}$, we assume that $dd^c F_k \geq -C\omega_0$ for a positive constant C.
2. $\int_X e^{F_k}\omega_0^n = 1$ and we denote by $\omega_k = \omega_0 + dd^c\varphi_k$ the solution of the Monge-Ampère equation

$$\omega_k^n = e^{F_k}\omega_0^n.$$

Then the sequence $(\omega_k)_{k\in\mathbb{N}}$ converges wrt the L^2-norm to a closed and positive $(1,1)$-current ω_∞ which is solution to the singular Monge-Ampère equation

$$\omega_\infty^n = e^F\omega_0^n, \tag{2}$$

see Theorem 1.6 and Corollary 1.16. The current ω_∞ will be called a *singular Kähler current* on X (see Definition 1.7). This notion should be seen as a generalization of the notion of Kähler forms on compact Kähler manifolds. We associate to any singular Kähler current a singular Laplacian Δ_∞ which generalizes the notion of Laplacian on compact Kähler manifolds and we prove that Δ_∞ possesses a infinite, discrete and non-negative spectrum (see Theorem 1.19).

In Section 2, we apply this theory to toric manifolds. Let X be complex toric projective manifold of dimension n. The holomorphic tangent bundle TX is an equivariant vector bundle and hence $\det(TX)$ (see Section 2). The canonical metric $\|\cdot\|_{\det(TX),\infty}$ of $\det(TX)$ induces a canonical continuous volume form on X, that we normalize and we denote by $\mu_{X,can}$. For example, in the case of $\mathbb{P}^n$ the complex projective space of dimension n , we have on a standard affine open subset of $\mathbb{P}^n$,

$$\omega_{\mathbb{P}^n,can} = (\frac{i}{2\pi})^n \frac{\prod_{i=1}^n dz_i \wedge d\overline{z}_i}{(n+1)\max(1,|z_1|,\ldots,|z_n|)^{2(n+1)}}. \tag{3}$$

The first Chern class of X has a particular representative given by

$$\tau_{X,can} := c_1((\det(TX), \|\cdot\|_{\det(TX),\infty}))$$

which is a closed $(1,1)$-current and canonically associated to X. We define the canonical toric Kähler current of X denoted by $\omega_{X,can}$ as the solution to the following singular Monge-Ampère equation,

$$\omega_{X,can}^n = \mu_{X,can}. \tag{4}$$

$\omega_{X,\infty}$ exists and unique by Theorem 1.6. An equivalent definition for the notion of canonical toric Kähler current can be given as follows: $\omega_{X,can}$ is the unique solution to $\tau_{X,can} = \mathrm{Ricc}(\omega_{X,can})$ (the Ricci current of $\omega_{X,can}$). From this equation and by Theorem 1.19, it is natural to consider the spectrum of the singular Laplacian defined by $\omega_{X,can}$ and to call it the canonical spectrum of the toric manifold X, we denote it by Spec_X (see Definition 2.4). Examples of computation of the canonical spectrum are given in Example 2.5. Some properties of this spectrum in dimension 1 are studied in [4].

One ofthe fundamental problems in spectral geometry is to ask to what extent the eigenvalues determine the geometry of a given manifold. A natural question arises from the theory developed in this article. In this setting, to what extent the canonical spectrum determine the geometry of the toric manifold? Given two toric projectives manifolds X and X' and having the same canonical spectrum, is it true that X and X' are isometric?

The main tools of this article are the famous result of Yau, [11] and a theorem of Eyssidieux-Guedj-Zeriahi-Berman-Boucksom [10]. This theorem will allows us to generalize the result of Yau. First let us recall the main theorem of Yau in [11]. Let (X, ω_0) be a Kähler compact manifold with ω_0 is a Kähler form on X. For a positive function F which belongs to $\mathcal{C}^k(X)$ with $k \geq 3$, there exists $\varphi \in \mathcal{C}^{k+1,\alpha}(X)$ with $0 \leq \alpha \leq 1$ which is a solution for the following Monge-Ampère equation

$$\big(\omega_0 + dd^c\varphi\big)^n = e^F \omega_0^n. \tag{5}$$

We consider a sequence $(F_k)_{k\in\mathbb{N}}$ as above. Using a result of [10] (see Appendix 3), we establish that there exists a constant A such that

$$|\Delta_0 \varphi_k| \leq A \quad \forall k \in \mathbb{N},$$

where Δ_0 is the Laplacian associated to ω_0 and acting on $\mathcal{C}^\infty(X)$, (see Theorem 1.3). Using this result, we can prove a generalization of Yau's result (see Theorem 1.6).

1 Toward spectral properties of toric manifolds

Let X be a compact Kähler manifold of dimension n and ω_0 a Kähler form on X. Let F be a continuous function on X. We assume that $\int_X e^F \omega_0^n = \int_X \omega_0^n$.

Let $(F_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions on X converging uniformly to F such that $dd^c F_k \geq -C\omega_0$ for any $k \in \mathbb{N}$ where C is a positive constant and $\int_X e^{F_k}\omega_0^n = \int_X \omega_0^n$ for any $k \in \mathbb{N}$. We suppose moreover that $F_0 = 0$. By [11], we know that there exists a unique smooth function φ_k on X with $\int_X \varphi_k \omega_0^n = 0$ such that

$$\big(\omega_0 + dd^c\varphi_k\big)^n = e^{F_k}\omega_0^n \quad \forall k \in \mathbb{N}. \tag{6}$$

We set $\omega_k := \omega_0 + dd^c\varphi_k$ and we denote by h_k the associated metric to ω_k for any $k \in \mathbb{N}$.

1.1 A singular Monge-Ampère equation

By [6], we know that there exists a unique continuous function φ on X with $\int_X \varphi\omega_0^n = 0$ such that φ is a solution for the following singular Monge-Ampère equation

$$\big(\omega_0 + dd^c\varphi\big)^n = e^F \omega_0^n. \tag{7}$$

The main tool for the proof of the existence uses the notion of capacity.

In this paragraph, we give a new proof for this result by generalising the proof of [11]. We denote by $\|\cdot\|_{L^2,k}$ (resp. $\|\cdot\|_{L^2}$) the inner product on $\mathcal{C}^\infty(X)$ (the space of smooth functions on X) induced by ω_k^n (resp. $e^F\omega_0^n$) and by $\mathcal{H}_0(X)$ the completion of the space $\mathcal{C}^\infty(X)$ wrt $\|\cdot\|_{L^2}$. Let ∇_k be the covariant derivative associated to ω_k. We denote by Δ_k the Laplacian acting on $\mathcal{C}^\infty(X)$ associated to ω_k for any $k \in \mathbb{N}$.

Theorem 1.1. *The sequence $(\varphi_k)_{k\in\mathbb{N}}$ is bounded wrt the sup-norm.*

The proof of this theorem will be a special case of the following result

Proposition 1.2. *Let $(G_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions on X. We assume that $\|G_k\|_{\sup}$ is bounded from above by a positive constant C for any $k \in \mathbb{N}$. Suppose that $\int_X e^{G_k}\omega_0^n = \int_X \omega_0^n$ for any $k \in \mathbb{N}$. We denote by φ_k the solution of the Monge-Ampere equation*

$$\big(\omega_0 + dd^c\varphi_k\big)^n = e^{G_k}\omega_0^n, \tag{8}$$

with $\int_X \varphi_k\omega_0^n = 0$. Then $(\varphi_k)_{k\in\mathbb{N}}$ is bounded wrt the sup*-norm.*

Proof. From [11, (2.27)], we can see that there exists a constant C' which depends only on (X,ω_0) such that

$$\psi_k := \varphi_k - C' - 1 \leq -1 \quad \forall k \in \mathbb{N},$$

Applying the Sobolev inequality, we can find two constants C_1 and C_2 depending only on (X,ω_0) such that for any $p \geq 2$

$$\Big(\int_X |\psi_k|^{2p}\omega_0^n\Big)^{\frac{1}{2}} \leq C_1 \int_X |\psi_k|^p\omega_0^n + C_2 \int_X |\nabla_0(-\psi_k)^{\frac{p}{2}}|^2\omega_0^n \quad \forall k \in \mathbb{N}.$$

From [11, p.353 (2.32)] we get

$$\int_X |\nabla_0(-\psi_k)^{\frac{p}{2}}|^2\omega_0^n \leq p \sup_{x\in X} |1-e^{G_k(x)}| \int_X |\psi_k|^{p-1}\omega_0^n.$$

Since $\psi_k \leq -1$ we obtain the following inequality

$$\Big(\int_X |\psi_k|^{2p}\omega_0^n\Big)^{\frac{1}{2}} \leq \big(C_1 + p \sup_{x\in X} |1-e^{G(x)}|C_2\big) \int_X |\psi_k|^p\omega_0^n.$$

That is,

$$\|\psi_k\|^p_{\frac{pn}{n-1}} \leq \big(C_1 + p \sup_{x\in X} |1-e^{G_k(x)}|C_2\big)\|\psi_k\|_p^p \quad \forall k \in \mathbb{N}.$$

As in [11], this inequality gives an upper bound for $\|\psi_k\|_{\sup}$ in terms of C_1, C_2 and $\|1-e^{G_k}\|_{\sup}$. Then $(\|\varphi_k\|_{\sup})_{k\in\mathbb{N}}$ is a bounded sequence. □

Theorem 1.3. *We keep the same hypothesis as before. There exists a constant K depending only on (X,ω_0) such that*

$$|\Delta_0\varphi_k| \leq K, \quad \forall k \in \mathbb{N}.$$

Proof. This will follows from Proposition (3.2) which is a slight generalization of a result in [10]. □

Corollary 1.4. *The metrics h_k are uniformly equivalent for any $k \in \mathbb{N}$.*

Proof. Recall that h_k is the metric on TX defined by ω_k. This result follows automatically from Theorem 1.3. □

Proposition 1.5. *There exists a constant C such that*

$$\sup_X |\nabla_0\varphi_k| \leq C \quad \forall k \in \mathbb{N}.$$

Proof. Let $k \in \mathbb{N}$. We let $f_k = \Delta_0\varphi_k$. Then by Theorem (1.3), $-K \leq f_k \leq K$. We can deduce as in [11, pp 355-356], that there exists a constant C which depends only on (M,ω_0) and K such that

$$\sup_X |\nabla_0\varphi_k| \leq C\big(K + \int_X |\varphi_k|\omega_0^n\big) \quad \forall k \in \mathbb{N}.$$

But the sequence $\big(\int_X |\varphi_k|\omega_0^n\big)_{k\in\mathbb{N}}$ is bounded as an easy application of [11, p.352 (2.28)]. This concludes the proof of the proposition. □

Theorem 1.6. *The equation (7) admits a solution $\varphi \in \mathcal{C}^{1,1}(X)$.*

Proof. By Theorem 1.1 and Proposition 1.5, we can apply the Ascoli's theorem to deduce the existence of a subsequence of $(\varphi_k)_{k\in\mathbb{N}}$ converging uniformly to a continuous function φ. It is clear that φ is a solution of Equation (7). Since $\Delta_0\varphi_k$ is uniformly bounded, then by the same argument we deduce that $\varphi \in \mathcal{C}^{1,1}(X)$. □

Definition 1.7. *Let ω be a positive and closed $(1,1)$-current on X. We say that ω is a singular Kähler current on X if the current ω^n is continuous.*

By Theorem 1.3, we know that for any normalized and continuous volume form Vol on X there exists a singular Kähler metric on X such that $\omega^n = \mathrm{Vol}$.

1.2 The Spectral theory of singular Kähler metrics

Proposition 1.8. *Let U be a relatively compact open local chart of X, and $\{z_1, \dots, z_n\}$ a system of local coordinates over U. Let $k \in \mathbb{N}$. We set $\varphi_{k,\alpha\overline{\beta}\overline{\gamma}} := \frac{\partial^3 \varphi_k}{\partial z_\alpha \partial \overline{z}_\beta \partial \overline{z}_\gamma}$ for any $\alpha, \beta, \gamma \in \{1, \dots, n\}$. Then the following sequence*

$$\Big((\frac{i}{2\pi})^n \int_U |\varphi_{k,\alpha\overline{\beta}\overline{\gamma}}|^2 \prod_{j=1}^n dz_j \wedge d\overline{z}_j \Big)_{k\in\mathbb{N}},$$

is bounded.

Proof. The proof is a combination of [11, Equation 3.3, p.360] and Corollary (1.4). □

Lemma 1.9. *We set $\varphi_{kl} = \varphi_k - \varphi_l$ for any $k, l \in \mathbb{N}$. There exists a constant K' which depends only on (X, ω_0) such that we have*

$$\int_X |\nabla_k \varphi_{kl}|^2 \omega_k^n \leq K' C_2(F_{j,k}), \quad k, l \in \mathbb{N}, \tag{9}$$

where $C_2(F_{k,l}) := \sup_X \big|1 - \exp(F_l - F_k)\big|$ and ∇_k is the covariant derivative w.r.t ω_k.

Proof. Let $k, l \in \mathbb{N}$. Obviously, $\varphi_{kl} - \int_X \varphi_l \omega_k^n$ is the unique smooth function solution to the following Monge-Ampère equation

$$\omega_j^n = \big(\omega_k + dd^c \varphi_{kl}\big)^n = e^{F_j - F_k} \omega_k^n,$$

which satisfies $\int_X (\varphi_{kl} - \int_X \varphi_l \omega_k^n) \omega_k^n = 0$.

From [11, p.353 (2.32)], we have for any k and j in $\mathbb{N}$, the following inequality

$$\int_X |\nabla_k \varphi_{k,l}|^2 \omega_k^n \leq 2C_2(F_{j,k}) \int_X |\varphi_{j,k} - \int_X \varphi_l \omega_k^n | \omega_k^n.$$

Recall that $(\int_X |\varphi_k| \omega_0^n)_{k\in\mathbb{N}}$ is bounded, and $(F_k)_{k\in\mathbb{N}}$ converges uniformly to F. We can deduce that there exists a constant K' which depends only on (X, ω_0) such that

$$\int_X |\nabla_k \varphi_{kl}|^2 \omega_0^n \leq K' C_2(F_{j,k}) \quad \forall k, l \in \mathbb{N}. \tag{10}$$

□

Proposition 1.10. *There exists a constant K'' which depends only on (X, ω_0) such that we have*

$$\int_X |\nabla_0 \varphi_{kl}|^2 \omega_0^n \leq K'' C_2(F_{j,k}) \quad \forall k, l \in \mathbb{N}.$$

Proof. The proof is a consequence of Corollary 1.4 and Lemma 1.9. □

Proposition 1.11. *There exist m and M two positive constants such that*

$$m \|\Delta_l \xi\|^2_{L^2,l} \leq \|\Delta_k \xi\|^2_{L^2,k} \leq M \|\Delta_l \xi\|^2_{L^2,l},$$

for any $k, l \in \mathbb{N}$ and any $\xi \in \mathcal{C}^\infty(X)$.

Proof. Let $k \in \mathbb{N}$. Let $\xi \in \mathcal{C}^\infty(X)$. We have

$$\|\Delta_k \xi\|^2_{L^2,0} = \int_X \sum_{(i,j),(i',j')} h_k^{ij} h_k^{i'j'} \frac{\partial^2 \xi}{\partial z_i \partial \overline{z}_j} \frac{\partial^2 \overline{\xi}}{\partial z_{j'} \partial \overline{z}_{i'}} \omega_0^n \tag{11}$$

($\{z_i\}_i$ are local holomorphic coordinates.) Let $h_k^{-1} \otimes \overline{h_k^{-1}}$ be the Kronecker product of h_k^{-1} and $\overline{h_k^{-1}}$. Obviously, we have

$$\langle h_k^{-1} \otimes \overline{h_k^{-1}} \frac{\partial^2 \xi}{\partial z_* \partial \overline{z}_*}, \frac{\partial^2 \xi}{\partial z_* \partial \overline{z}_*} \rangle^a = \sum_{(i,j),(i',j')} h_k^{ij} h_k^{i'j'} \frac{\partial^2 \xi}{\partial z_i \partial \overline{z}_j} \frac{\partial^2 \overline{\xi}}{\partial z_{j'} \partial \overline{z}_{i'}}. \tag{12}$$

Recall that if A and B are two $n \times n$ complex matrices, then the eigenvalues of $A \otimes B$ (the Kronecker product of A and B) is a product of eigenvalues of A and B. Then using Corollay (1.4), we can find two positive constants $0 < m < M$ depending only on (X, ω_0) such that

$$m\|\Delta_l \xi\|^2_{L^2,l} \leq \|\Delta_k \xi\|^2_{L^2,k} \leq M'\|\Delta_l \xi\|^2_{L^2,l} \quad \forall k,l \in \mathbb{N}, \forall \xi \in \mathcal{C}^\infty(X).$$

□

Theorem 1.12. *For any $\xi \in \mathcal{C}^\infty(X)$, the following sequence*

$$\big(\Delta_k \xi\big)_{k \in \mathbb{N}}$$

converges to a limit in $\mathcal{H}_0(X)$ wrt $\|\cdot\|_{L^2,0}$. Moreover, this limit does not depend on the choice of the sequence $(F_k)_{k\in\mathbb{N}}$.

Proof. Let $k, l \in \mathbb{N}$. For any $1 \leq i,j \leq n$, the difference $h_k^{ij} - h_l^{ij}$ of the minors h_k^{ij} and h_l^{ij} of h_k^{-1} and h_l^{-1} respectively, can be written as the sum of terms where each one is a product of elements of h_l and h_k and of $(\frac{\partial^2 \varphi_{k,l}}{\partial z_\alpha \partial \overline{z}_\beta})_{1\leq \alpha,\beta\leq n}$. Hence, for $i,j = 1,\dots,n$ we can find a function S_{ij} such that $|\Delta_k \xi - \Delta_l \xi|^2 = (\Delta_k \xi - \Delta_l \xi) \cdot (\overline{\Delta_k \xi - \Delta_l \xi}) = \sum_{ij} \frac{\partial^2 \varphi_{k,l}}{\partial z_i \partial \overline{z}_j} S_{ij}$.

Let $(U_\alpha)_{\alpha\in\Omega}$ be a finite open covering of X such that $\omega_0^n = v_\alpha \prod_{i=1}^n dz_{\alpha,i} \wedge d\overline{z}_{\alpha,i}$ on U_α with v_α is a smooth function on U_α and $\{z_{\alpha,i}\}_i$ is a system of local coordinates on U_α for any $\alpha \in \Omega$. Let $(\rho_\alpha)_{\alpha\in\Omega}$ be a partition of unity subordinate to $(U_\alpha)_{\alpha\in\Omega}$. We have

$$\begin{aligned} \big\|\Delta_k \xi - \Delta_l \xi\big\|^2_{L^2,0} &= \sum_{ij} \int_X \frac{\partial^2 \varphi_{k,l}}{\partial z_i \partial \overline{z}_j} S_{ij}\, \omega_0^n \\ &= \sum_{ij} \sum_{\alpha\in\Omega} \int_{U_\alpha} \frac{\partial \varphi_{kl}}{\partial \overline{z}_{\alpha,j}} \frac{d}{dz_{\alpha,i}} \big(S_{ij} \rho_\alpha v_\alpha\big) \prod_{i=1}^n dz_{\alpha,i} \wedge d\overline{z}_{\alpha,i} \quad \text{by the Stokes theorem.} \end{aligned} \tag{13}$$

Observe that for any $1 \leq i,j \leq n$, the term $\frac{d}{dz_i}\big(S_{ij}\rho_\alpha v_\alpha\big)$ contains a derivative of order 3 of some φ_{kl} or $\overline{\varphi_{kl}}$, or $\frac{\partial v_\alpha}{\partial z_i}$ and the other coefficients depend only on h_l and $(\frac{\partial \varphi_{kl}}{\partial z_\alpha \partial \overline{z}_\beta})_{1\leq\alpha,\beta\leq n}$.

By Theorem 1.3 and Proposition 1.8, we can find a constant C which depends only on the L^2-norms of ξ and its derivatives of order less than 3 and on (X, ω_0) such that

$$\big\|\Delta_k \xi - \Delta_l \xi\big\|^2_{L^2,0} \leq C\Big(\int_X |\nabla_0 \varphi_{kl}|^2 \omega_0^n\Big)^{\frac{1}{2}} \quad \forall k,l \in \mathbb{N}. \tag{14}$$

[a] Here $\langle , \rangle$ is the standard scalar product on $\mathbb{C}^{n^2}$.

Now by Proposition 1.10, we conclude that there exists K a constant depending only on ξ, its derivatives and on (X, ω_0) such that

$$\left\|\Delta_k \xi - \Delta_l \xi\right\|_{L^2,0}^2 \leq K C_2(F_{j,k})^{\frac{1}{2}} \quad \forall k, l \in \mathbb{N}.$$

This ends the proof of the theorem. □

Definition 1.13. *Let ω_∞ be the singular Kähler current solution to the Monge-Ampère equation 7. For any $\xi \in \mathcal{C}^\infty(X)$, we denote by $\Delta_\infty \xi$ the limit of the sequence in Theorem 1.12. The following operator denoted by Δ_∞*

$$\mathcal{C}^\infty(X) \to \mathcal{H}_0(X), \quad \xi \mapsto \Delta_\infty \xi,$$

is clearly linear. We call it the Laplacian associated to the singular Kähler current ω_∞.

Proposition 1.14. *There exist two positive constants m' and M' such that*

$$m' \|\Delta_k \xi\|_{L^2,k} \leq \|\Delta_\infty \xi\|_{L^2,\infty} \leq M' \|\Delta_k \xi\|_{L^2,k} \quad \forall k \in \mathbb{N}, \forall \xi \in \mathcal{C}^\infty(X).$$

Proof. The proof follows from Proposition 1.11 and Theorem 1.12. □

Let φ be a smooth function on X. For any $k \in \mathbb{N}$, we denote by $\|\varphi\|_{H_j,k}$ the j-norm of Sobolev given as follows

$$\|\varphi\|_{H_{j,k}}^2 = \sum_{l=0}^{j} \|\nabla_k^{(l)} \varphi\|_{L^2}^2$$

with $\nabla_k^{(l)} \varphi := \nabla_k \cdots \nabla_k \varphi$ such that ∇_k appears l time. We denote by $\mathcal{H}_2(X)$ the Sobolev space endowed with $\|\cdot\|_{H_2,k}$.

Proposition 1.15. *There exists a constant K which depends only on (X, ω_0) such that*

$$\|\varphi_{kl}\|_{H_2,k} \leq K C_2(F_{j,k})^{\frac{1}{4}} \quad \forall k, l \in \mathbb{N}.$$

Proof. By the elliptic estimate (see for instance [7, Appendix A]), we have for any $k, l \in \mathbb{N}$

$$\|\xi\|_{H_2,k} \leq C_1 \|\Delta_j \xi\|_{L^2} + C_2 \|\xi\|_{L^2} \quad \forall \xi \in \mathcal{C}^\infty(X),\ \forall j, k \in \mathbb{N}, \tag{15}$$

where C_1 and C_2 two constants which do not depend on j by Theorem 1.3. In fact, it is known that C_1 and C_2 depend on the eigenvalues of the Kähler metric. In particular, we have

$$\|\varphi_{kl}\|_{H_2} \leq C_1 \|\Delta_0 \varphi_{kl}\|_0 + C_2 \|\varphi_{kl}\|_{L^2}. \tag{16}$$

As in the proof of (13), we can use Theorem 1.3 and Proposition 1.8 to show the following inequality

$$\left\|\Delta_0 \varphi_{kl}\right\|_{L^2,0}^2 \leq K' C_2(F_{j,k})^{\frac{1}{2}} \quad \forall k, l \in \mathbb{N}, \tag{17}$$

where K' depends only on (X, ω_0). By 16 and 17, we deduce the following

$$\|\varphi_{kl}\|_{H_2,k} \leq K' C_2(F_{j,k})^{\frac{1}{4}} \quad \forall k, l \in \mathbb{N}.$$

□

Corollary 1.16. *We keep the same notations as above. The sequence $(\omega_k)_{k \in \mathbb{N}}$ converges to ω_∞ wrt the L^2-norm.*

Proof. This is an easy application of 17. □

Corollary 1.17. *Let $\xi \in \mathcal{C}^\infty(X)$, we have*

$$\left\|\Delta_k\xi - \Delta_l\xi\right\|_{L^2,0} \leq K'' C_2(F_{j,k})^{\frac{1}{4}}\left(\|\Delta_l\xi\|_{L^2,0} + \|\xi\|_{L^2,0}\right) \quad \forall k,l \in \mathbb{N}, \tag{18}$$

where K'' is constant which depends only on (X,ω_0).

Proof. We can show that there exists a constant C such that

$$\left\|\Delta_k\xi - \Delta_l\xi\right\|_{L^2,0} \leq C\|\varphi_{kl}\|_{H_2,k}\|\xi\|_{H_2,k}$$

If we combine 15 and Proposition 1.15 with the previous inequality, we conclude the proof of the corollary. □

Theorem 1.18. *The operator Δ_∞ admits a maximal, positive and selfadjoint extension to $\mathcal{H}_2(X)$.*

Proof. From Theorem 1.12 we obtain easily the following

$$(\Delta_\infty\xi,\eta)_{L^2,\infty} = (\xi,\Delta_\infty\eta)_{L^2,\infty}, \quad \forall \xi,\eta \in \mathcal{C}^\infty(X). \tag{19}$$

Let $(\Delta_\infty)_{\max}$ be the maximal extension of $\Delta_\infty : \mathcal{C}^\infty(X) \to \mathcal{H}_0(X)$. By definition,

$$\mathrm{Dom}((\Delta_\infty)_{\max}) = \left\{s \in \mathcal{H}_0(X) \middle| \Delta_\infty s \in \mathcal{H}_0(X)\right\}.$$

(see for instance [7, Appendix D]) Then it is the Hilbert space adjoint of Δ_∞, so $(\Delta_\infty)_{\max}$ is closed. Let us prove that $(\Delta_\infty)_{\max}$ is self-adjoint, and $\mathrm{Dom}((\Delta_\infty)_{\max}) = \mathcal{H}_2(X)$. Clearly, $\mathcal{H}_2(X) \subset \mathrm{Dom}((\Delta_\infty)_{\max})$. Let $s \in \mathrm{Dom}((\Delta_\infty)_{\max})$. By Proposition 1.14, there exists $k \in \mathbb{N}$ such that

$$\|\Delta_k s\|_{L^2,k} < \infty.$$

Then by the elliptic estimate ([7, A.3.2]) we get $s \in \mathcal{H}_2(X)$. From 19, we obtain that $(\Delta_\infty)_{\max}$ is selfadjoint. □

Theorem 1.19. *The operator $\Delta_\infty + I : \mathcal{H}_2(X) \to \mathcal{H}_0(X)$ is invertible and Δ_∞ has an infinite, non-negative and discrete spectrum and it admits a heat kernel, we denote its heat kernel by $e^{-t\Delta_\infty}$, for any $t > 0$.*

Proof. Let $s \in \mathcal{C}^\infty(X)$. As $(\Delta_\infty s, s)_{L^2,\infty} \geq 0$, we get

$$\|(I+\Delta_\infty)s\|_{L^2} \geq \|s\|_{L^2}.$$

Then, for any $s \in \mathcal{H}_2$

$$\|(I+\Delta_\infty)s\|_{L^2} \geq \|s\|_{L^2}. \tag{20}$$

Thus $I + \Delta_\infty : \mathcal{H}_2(X) \to \mathcal{H}_0(X)$ is bijective, and $(I+\Delta_\infty)^{-1} : \mathcal{H}_0 \to \mathcal{H}_2$ is well-defined as a linear operator. Fix $k \in \mathbb{N}$, we know that there exists a constant C_1 such that $\|\Delta_\infty s\|_{L^2,\infty} \leq C_1\|\Delta_\infty s\|_{L^2,k}$ for any $s \in \mathcal{H}_2(X)$. Thus

$$\|(I+\Delta_\infty)s\|_{L^2,\infty} \leq C_1\|s\|_{\mathcal{H}_2,k} \quad \forall s \in \mathcal{H}_2(X).$$

Let $s \in \mathcal{H}_0(X)$, by the elliptic estimate, we can find a constant C_2 independent on s such that

$$\begin{aligned}\|(I+\Delta_\infty)^{-1}s\|_{H_2,k} &\leq C_2\Big(\|(I+\Delta_k)(I+\Delta_\infty)^{-1}s\|_{L^2,k} + \|(I+\Delta_\infty)^{-1}s\|_{L^2,k}\Big)\\ &\leq C_2\Big(\|\Delta_k(I+\Delta_\infty)^{-1}s\|_{L^2,k} + 2\|(I+\Delta_\infty)^{-1}s\|_{L^2,k}\Big)\end{aligned}$$

From Corollary 1.17 which we apply to $\xi := (I+\Delta_\infty)^{-1}s$, we get the following

$$\|\Delta_k(I+\Delta_\infty)^{-1}s\|_{L^2,k} \leq K'\|s\|_{L^2,\infty}$$

for some constant K' which depends only on (X,ω_0). We conclude that there exists a constant C such that

$$\|(I+\Delta_\infty)^{-1}s\|_{H_2,k} \leq C\|s\|_{L^2,k} \quad \forall s \in \mathcal{H}_0(X). \tag{21}$$

From the previous inequalities and the Rellich's theorem (cf. for example [7, Theorem A.3.1]), we know that $(I+\Delta_\infty)^{-1} : (\mathcal{H}_0(X), \|\cdot\|_{L^2,\infty}) \to (\mathcal{H}_0(X), \|\cdot\|_{L^2,\infty})$ is a compact operator, and it is self-adjoint. Therefore, Δ_∞ has an infinite discrete and positive spectrum, and it admits a heat kernel denoted by $(e^{-t\Delta_\infty})_{t>0}$. ☐

Corollary 1.20. *There exist two positive constants m'' and M'' such that*

$$m''\lambda_{k,j} \leq \lambda_{\infty,j} \leq M''\lambda_{k,j} \quad \forall k \in \mathbb{N}\ \forall j \in \mathbb{N}_{\geq 1}, \tag{22}$$

where $\lambda_{k,j}$ is the j-th eigenvalue of Δ_k for $k \in \mathbb{N}\cup\{\infty\}$.

Proof. This follows from Proposition 1.14, Theorem 1.19, and the monotonicity principle. ☐

Theorem 1.21. *The sequence $\big((I+\Delta_k)^{-1}\big)_{k\in\mathbb{N}}$ converges to $(I+\Delta_\infty)^{-1}$ with respect to the norm $\|\cdot\|_{L^2}$ and hence wrt $\|\cdot\|_{L^2,k}$ for any $k \in \mathbb{N}$.*

Proof. Note that for any $k,l \in \mathbb{N}$,

$$(I+\Delta_l)^{-1} - (I+\Delta_k)^{-1} = (I+\Delta_l)^{-1}(\Delta_k - \Delta_l)(I+\Delta_k)^{-1},$$

and we use Corollary 1.17, to deduce the existence of a constant K such that

$$\big\|(I+\Delta_l)^{-1} - (I+\Delta_k)^{-1}\big\|_{L^2} \leq KC_2(F_{k,l})^{\frac{1}{4}} \quad \forall k,l \in \mathbb{N}. \tag{23}$$

This ends the proof of the theorem. ☐

Theorem 1.22. *For any $t>0$, the sequence $\big(e^{-t\Delta_k}\big)_{k\in\mathbb{N}}$ converges to $e^{-t\Delta_\infty}$ wrt the norm $\|\cdot\|_{L^2}$.*

Proof. We fix $t>0$. Let γ be the curve in $\mathbb{C}$ given by: $+\infty+i \to -1+i \to -1-i \to +\infty-i$, then for any $k\in\mathbb{N}$, we have

$$e^{-t\Delta_k} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(\lambda-\Delta_k)^{-1}d\lambda. \tag{24}$$

Observe that we can obtain a similar estimation to (23) for $(\lambda-\Delta_k)^{-1}$ with λ is a non-zero complex number. By (24), we get

$$\|e^{-t\Delta_k} - e^{-t\Delta_l}\|_{L^2,\infty} \leq KC_2(F_{k,l})^{\frac{1}{4}} \quad \forall k,l \in \mathbb{N}. \tag{25}$$

with K is a constant. It follows that $(e^{-t\Delta_k})_{k\in\mathbb{N}}$ converges to $e^{-t\Delta_\infty}$ wrt the L^2-norm. ☐

To the operator Δ_∞ we associate the following function ζ_{Δ_∞} given as follows

$$\zeta_{\Delta_\infty}(s) = \sum_{j=1}^{\infty}\frac{1}{\lambda_{\infty,j}^s}$$

for $s\in\mathbb{C}$. When Δ is the Laplacian operator associated to a smooth Kähler metric, the function ζ_Δ is called the zeta function of Δ which known to be holomorphic on $\{s\in\mathbb{C}|\Re(s) > \dim_{\mathbb{C}}(X)\}$ and admits a meromorphic continuation to the whole complex plane with poles at $s=1,2,\dots,\dim_{\mathbb{C}}(X)$. We have the following result

Theorem 1.23. *For any $\Re(s) > n$, $\zeta_{\Delta_\infty}(s)$ converges absolutely and ζ_{Δ_∞} is holomorphic on $\{s \in \mathbb{C} | \Re(s) > n\}$ with a pole at $s = n$.*

Proof. Let $k \in \mathbb{N}$. Since ζ_{Δ_k} is finite on $\{s \in \mathbb{C} | \Re(s) > n\}$, it follows from Corollary 1.20, that $\zeta_{\Delta_\infty}(s)$ converges absolutely on $\{s \in \mathbb{C} | \Re(s) > n\}$. A result of Hörmander (see [5]) asserts that the $N_k(\lambda)$, the counting function of the eigenvalues of a Laplacian Δ_k, has the following asymptotic expansion

$$N_k(\lambda) = C\lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}) \quad \forall \lambda \gg 1.$$

Thus,

$$j = C\lambda_{k,j}^{\frac{n}{2}} + O(\lambda_j^{\frac{n-1}{2}}).$$

Then $\lambda_{k,j} \sim j^n$ for any $j \gg 1$. By Corollary 1.20, it follows that ζ_{Δ_∞} is holomorphic on $\{s \in \mathbb{C} | \Re(s) > n\}$ and $s = n$ is a pole. □

Definition 1.24. *We call ζ_{Δ_∞} the zeta function associated to Δ_∞.*

1.3 An example

In the sequel of this paragraph, we present an exemple of this theory. Let $\mathbb{P}^1$ be the complex projective space of dimension 1. We consider the following hermitian continuous metric $\|\cdot\|_{can}$ on the line bundle $\mathcal{O}(1)$ defined as follows

$$\|s\|_{can}^2(z) := \frac{|s(z)|^2}{\max(1, |z|)^2} \exp(-k(z)), \quad z \in \mathbb{C},$$

where s is a local holomorphic section of $\mathcal{O}(1)$ and $k(z) = \frac{1}{2} \min(|z|^2, \frac{1}{|z|^2})$, for any $z \in \mathbb{C}$.

Let $(\mathbb{P}^1)^n$ be the product of n copies of $\mathbb{P}^1$. We consider the following continuous volume form on $(\mathbb{P}^1)^n$, given on $\mathbb{C}^n$ as follows

$$\nu_{can} := \left(\frac{i}{4\pi}\right)^n \prod_{j=1}^n \frac{dz_j \wedge d\bar{z}_j}{\max(1, |z_j|)^4}.$$

One checks easily that $\int_{(\mathbb{P}^1)^n} \nu_{can} = 1$. By Theorem 1.6, there exists a positive and closed $(1,1)$-current $\omega_{n,\infty}$ on $(\mathbb{P}^1)^n$ such that

$$\omega_{n,\infty}^n = \nu_{can}.$$

Proposition 1.25. *The current $c_1(\mathcal{O}(1), \|\cdot\|_{can})$ on $\mathbb{P}^1$ is positive and*

$$\omega_{n,\infty} = \sum_{j=1}^n p_j^* c_1(\mathcal{O}(1), \|\cdot\|_{can}),$$

where $p_j : (\mathbb{P}^1)^n \to \mathbb{P}^1$ is the j-th projection.

Proof. This proposition follows from [4, Proposition 3.4] □

We denote by $\Delta_{n,\infty}$ the Laplacian associated to the singular Kähler metric $\omega_{n,\infty}$. By Theorem 1.19, $\Delta_{n,\infty}$ has a non-negative, infinite and discrete spectrum. The following theorem describes explicitly the spectrum.

Theorem 1.26. *We denote by* $\mathrm{Spec}(\Delta_{n,\infty})$ *the spectrum of* $\Delta_{n,\infty}$. *We have*

$$\mathrm{Spec}(\Delta_{n,\infty}) = \Big\{\sum_{j=1}^n \frac{\lambda_j^2}{2} \,\Big|\, \textit{for}\, j = 1, \ldots, n,\ J_k(\lambda_j) J_k'(\lambda_j) = 0\, \textit{for some}\, k \in \mathbb{Z}\Big\},$$

where J_k is the k-th Bessel function of order k.

Proof. We can generalize easily [2, Proposition A.II.3] to the case of the product of compact Kähler manifolds endowed with singular Kähler metrics. Then, the theorem is a consequence of [3, Theorem 1.4]. Note that the volume form in [3, Theorem 1.4] is not normalized. $\square$

2 Projective toric manifolds

Toric varieties are often used as simple test cases for difficult geometric problems, and many interesting quantities and theorems are easy to express and to prove on toric varieties. A toric variety is defined in terms of a $\mathbb{Z}$-module N and a fan Σ on $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. Several algebraic and geometric properties and quantities can be expressed in terms of the combinatorics of N.

Let N be a free $\mathbb{Z}$-module of rank n, M its dual $\mathbb{Z}$-module and $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$. A subset σ of $N_{\mathbb{R}}$ is called a *strongly convex rational polyhedral cone* if there exist a finite number of elements $n_1, n_2, \ldots, n_s$ in N such that

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \ldots + \mathbb{R}_{\geq 0} n_s,$$

and $\sigma \cap (-\sigma) = \{0\}$. A subset τ of σ is called a *face* if

$$\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma |\ \langle m_0, y\rangle = 0\},$$

for an $m_0 \in \sigma^{\vee} := \{x \in M_{\mathbb{R}} |\ \langle x, y\rangle \geq 0 \, \forall y \in \sigma\}$. A *fan* in N is a nonempty collection Σ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:

(i) Every face of any $\sigma \in \Sigma$ is contained in Σ.

(ii) For any $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The union $|\Sigma| := \sum_{\sigma \in \Sigma} \sigma$ is called the support of Σ.

Let Σ be a fan on N. For $\sigma \in \Sigma$, we let $\mathscr{S}_{\sigma} := M \cap \sigma^{\vee}$. We can show that $\mathscr{S}_{\sigma}$ is a finitely generated subsemigroup of M, and $\mathscr{S}_{\sigma}$ defines an irreductible normal algebraic subset U_{σ} of some $\mathbb{C}^p$ (see [9, Propositions 1.1, 1.2, pp 3-4]). The properties of Σ allow us to glue naturally $\{U_{\sigma} |\ \sigma \in \Sigma\}$ together to obtain an irreductible and normal complex analytic space of dimension $n = \mathrm{rank}(N)$. We denote this complex analytic space by $X_N(\Sigma)$ or simply X and we call it a toric variety associated to (N, Σ). After Batyrev and Tschinkel, there exists a canonical covering of X into compact subsets $(C_{\sigma})_{\sigma \in \Sigma_{\max}}$ where each C_{σ} is diffeomorphic to a product of unit balls and unit circles (see for instance [8, §3.3]).

It is known that X is a toric variety of dimension n, if and only if X is an irreductible normal algebraic variety which contains an open subset isomorphic to the algebraic torus $T := (\mathbb{C}^*)^n$, such that T acts algebraically on X (see [9, Theorem 1.5, p.10]).

Recall that a holomorphic vector bundle $p : E \to X$ is a T-equivariant bundle on X if it has an action of the torus T on E which is linear on the fibers and makes the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d, "p"] & E \arrow[d, "p"] \\ X \arrow[r, "t"] & X \end{tikzcd}$$

for any $t \in \mathbb{T}$. When L is a T-equivariant line bundle, one can associate, in a canonical way, to L a continuous hermitian metric $\| \cdot \|_{L,\infty}$ called the canonical metric of L. This metric is given in terms of the combinatorial structure of X (see [1] or [8]) and we can show that $\| \cdot \|_{L,\infty}$ is a solution to the first Calabi problem for the singular volume form $\delta_{S_N} \wedge d\mu_N^+$ (see [8, Remark 6.3.6]). More precisely, we have

$$c_1(L, \| \cdot \|_{L,\infty})^n = \deg(c_1(L)^n) \delta_{S_N} \wedge d\mu_N^+. \tag{26}$$

We assume that X is a non-singular projective toric variety over $\mathbb{C}$ and we consider a Kähler form ω on X. The holomorphic tangent bundle TX is clearly a T-equivariant vector bundle, and so is $\det(TX)$. The first Chern class of X has a particular representative given by $\tau_{X,can} := c_1((\det(TX), \|\cdot\|_{\det(TX),\infty}))$ which is a $(1,1)$-current closed and canonically associated to X. The canonical metric of $\det(TX)$ induces a canonical continuous volume form on X, that we normalize and we denote it by $\mu_{X,can}$.

Example 2.1. *Let $\mathbb{P}^n$ be the complex projective space of dimension n. On a standard affine open subset of $\mathbb{P}^n$, we have*

$$\omega_{\mathbb{P}^n,can} = (\frac{i}{2\pi})^n \frac{\prod_{i=1}^n dz_i \wedge d\overline{z}_i}{(n+1)\max(1,|z_1|,\ldots,|z_n|)^{2(n+1)}}. \tag{27}$$

In view of the theory developed in this article, we introduce the following definitions,

Definition 2.2. *Let X be a projective toric complex manifold of dimension n. We call a canonical toric Kähler current of X and we denote it by ω_{can}, a positive and closed $(1,1)$-current which is solution to the following singular Monge-Ampère equation,*

$$\omega_{X,can}^n = \mu_{X,can} \tag{28}$$

By Theorem 1.6, we know that $\omega_{X,can}$ exists and unique. An equivalent definition for the notion of canonical toric Kähler current might be given as follows $\omega_{X,can}$ is the unique solution to $\tau_{X,can} = \mathrm{Ricc}(\omega_{X,can})$ (the Ricci current of $\omega_{X,can}$).

Remark 2.3. *Let X be a projective toric complex manifold of dimension n. We have the following*

$$\omega_{\mathbb{P}^n,can_{|C_j}} = \frac{1}{n+1}(\frac{i}{2\pi})^n \prod_{k=1}^n dy_k \wedge d\overline{y}_k, \tag{29}$$

where $C_j = \{z \in \mathbb{C}^n | |z_k/z_j| \leq 1, |1/z_j| \leq 1, k = 1,\ldots,n\}$ for $j \geq 1$ and $C_0 = \{z \in \mathbb{C}^n | |z_k| \leq 1, k = 1,\ldots,n\}$ and $\{y_k\}_{k=1,\ldots,n}$ are the canonical coordinates on C_j which define a diffeomorphism from C_j into C_0. In fact, the C_j correspond to the C_σ for $\sigma \in \Sigma_{\max}$. More generally, given X a projective toric manifold over $\mathbb{C}$ of dimension n, we can show that for any $\sigma \in \Sigma_{\max}$ the following local form $\omega_{X,can_{|C_\sigma}}$, has a similar expression as in 29. More precisely, for any $\sigma \in \Sigma_{\max}$, there exists a system of local coordinates $\{y_1,\ldots,y_n\}$ in $\mathrm{Int}(C_\sigma)$ and a constant c_σ such that

$$\omega_{X,can_{|C_\sigma}} = c_\sigma \prod_{k=1}^n dy_k \wedge d\overline{y}_k. \tag{30}$$

We use Theorem 1.19 to associate to any projective toric complex manifold a new combinatorial invariant, by introducing the following definition

Definition 2.4. *Let X be a projective toric complex manifold of dimension n. Let $\Delta_{X,can}$ be the Laplacian associated to the canonical toric Kähler current ω_{can}. We consider $\mathrm{Spec}_X := \mathrm{Spec}(\Delta_{X,can})$ and we call it the canonical spectrum of the toric variety X.*

From (28), we can see that the canonical spectrum Spec_X depends only on the combinatorial structure of X.

Example 2.5. *Let $n \in \mathbb{N}_{\geq 1}$. We consider the following toric variety $(\mathbb{P}^1)^n$. We have,*

$$\mathrm{Spec}_{(\mathbb{P}^1)^n} = \Big\{\sum_{j=1}^n \frac{\lambda_j^2}{2} \,|\, for\, j = 1,\ldots,n,\ J_k(\lambda_j)J_k'(\lambda_j) = 0\, for\ some\, k \in \mathbb{Z}\Big\}, \tag{31}$$

This equality follows from Theorem 1.26.

3 Appendix

Let (X, ω) be a compact Kähler manifold of dimension n, Δ the Laplacian operator acting on smooth functions on X defined by ω, and $\theta \geq 0$ is a semi-positive closed $(1,1)$-form such that $\int_X \theta^n > 0$. We let $\mathrm{Amp}(\theta)$ denote the ample locus of the cohomology class of θ.

Theorem 3.1. *Let μ be a positive measure on X of the form $\mu = e^{\psi^+ - \psi^-} dV$ with $\psi^\pm$ quasi-psh and $e^{-\psi^-} \in L^p$ for some $p > 1$. Assume that φ is a bounded θ-psh function such that $(\theta + dd^c\varphi)^n = \mu$. Then we have $\Delta\varphi = O(e^{-\psi^-})$ locally on $\mathrm{Amp}(\theta)$.*

More precisely, assume given a constant $C > 0$ such that

1. *$dd^c\psi^+ \geq -C\omega$ and $\sup_X \psi^+ \leq C$.*
2. *$dd^c\psi^- \geq -C\omega$ and $\|e^{-\psi^-}\|_{L^p} \leq C$.*

Let also $U \Subset \mathrm{Amp}(\theta)$ be a relatively compact open subset. Then there exists $A > 0$ only depending on θ, p, C and U such that

$$0 \leq \theta + dd^c\varphi \leq Ae^{-\psi^-}\omega$$

on U.

Proof. See [10, Theorem 10.1]. □

Proposition 3.2. *We keep the same hypothesis as in Theorem (3.1), and we assume that $\theta = \omega$.*

1. *Then, there exists a constant A which depends only C and ω such that*

$$|\Delta\varphi| \leq Ae^{-\psi^-},$$

on X.

2. *Suppose that $\psi^+ = F$, with F is a continuous function on X and $\psi^- = 0$ and let $(F_k)_{k\in\mathbb{N}}$ be a sequence of smooth functions converging uniformly to F on X such that $dd^cF_k \geq -C\omega$ for any $k \in \mathbb{N}$, where C is a positive constant[b]. Then, there exists $A > 0$ only depending on C, ω and X such that*

$$|\Delta\varphi_k| \leq A \quad \forall k \in \mathbb{N},$$

on X.

Proof. 1. The proof of the first part of the theorem is a special case of the proof of Theorem (3.1) which we recall here. Our goal is to analyse the constant A. We keep the same notations as in the proof of [10, Theorem 10.1]. Since $\theta = \omega$ then $\mathrm{Amp}(\theta) = X$ and we have

$$\widetilde{\omega} = \omega, \quad \psi = 0.$$

We can take $\delta = 1$. For $0 < \varepsilon \leq 1$, $\omega_\varepsilon = \widetilde{\omega} + \varepsilon\omega = (1+\varepsilon)\omega \geq \omega$, so that

$$\mathrm{tr}_{\omega_\varepsilon}(\alpha) \leq \mathrm{tr}_\omega(\alpha), \tag{32}$$

for every positive $(1,1)$-form α.

Assume ψ^+ and ψ^- are smooth functions satisfying (1) and (2) of Theorem (3.1), and assume given a smooth normalized θ_ε-psh function φ_ε such that

$$(\theta + \varepsilon\omega + dd^c\varphi_\varepsilon)^n = e^{\psi^+ - \psi^-} dV. \tag{33}$$

[b] For example, we consider $(\mathcal{O}, e^F)$ and we decompose it into two admissible line bundles (L_1, h_1) and (L_2, h_2) ...

Since $\omega_\varepsilon \le (\varepsilon+1)\omega$ on X, it will be enough to prove that

$$\omega'_\varepsilon := \theta + \epsilon\omega + dd^c\varphi_\epsilon (= (1+\varepsilon)\omega + dd^c\varphi_\epsilon)$$

satisfies $\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \le Ae^{-\psi^-}$ on X.

Following [10, p. 45], there exists B a constant which depends only on a lower bound for the holomorphic bisectional curvature of ω_ϵ such that

$$\Delta_{\omega'_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \ge -\frac{\mathrm{tr}_{\omega_\epsilon}(\mathrm{Ric}(\omega'_\epsilon))}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - B\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon). \tag{34}$$

Applying $dd^c \log$ to $(\omega'_\epsilon)^n = e^{-\psi^+ - \psi^-}\omega^n$, yields

$$-\mathrm{Ric}(\omega'_\epsilon) = -\mathrm{Ric}(\omega) + dd^c\psi^+ - dd^c\psi^- \ge A_1\omega - dd^c\psi^-, \tag{35}$$

where A_1 is constant which depends only on ω and C.

Using $\mathrm{tr}_{\omega_\epsilon}(\omega) \le n$ and the trivial inequality

$$n \le \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) \tag{36}$$

we infer from (34),

$$\Delta_{\omega'_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \ge -\frac{nA_1 + \Delta_{\omega_\epsilon}\psi^-}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - B\mathrm{tr}_{\omega'_\epsilon}(\omega_\varepsilon) \ge -\frac{\Delta_{\omega_\epsilon}\psi^-}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - (A_1+B)\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon). \tag{37}$$

That is

$$\Delta_{\omega'_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \ge -\frac{\Delta_{\omega_\epsilon}\psi^-}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - A_2\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon). \tag{38}$$

where $A_2 := A_1 + B$, which depends only on ω, C.

By the assumption (2) of Theorem (3.1), we have $A_3\omega_\epsilon + dd^c\psi^- \ge 0$, where $A_3 = C(1+\epsilon)^{-1}$. Applying $\mathrm{tr}_{\omega_\epsilon}$ to the trivial inequality

$$0 \le A_3\omega_\epsilon + dd^c\psi^- \le \mathrm{tr}_{\omega'_\epsilon}(A_3\omega_\epsilon + dd^c\psi^-)\omega'_\epsilon$$

yields

$$0 \le A_3 n + \Delta_{\omega_\epsilon}\psi^- \le (A_3\mathrm{tr}_{\omega_\epsilon}(\omega_\epsilon) + \Delta_{\omega'_\epsilon}\psi^-)\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)$$

Plugging this into (38) and using again (36) we obtain

$$\begin{aligned}
\Delta_{\omega'_\epsilon} \log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) &\ge -\frac{\Delta_{\omega_\epsilon}\psi^-}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - A_2\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon)\\
&\ge -\frac{\Delta_{\omega_\epsilon}\psi^- + A_3 n}{\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon)} - (A_2+A_3)\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon)\\
&\ge -A_3\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) - \Delta_{\omega'_\epsilon}\psi^- - (A_2+A_3)\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon).
\end{aligned}$$

That is

$$\Delta_{\omega'_\epsilon}\big(\log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) + \psi^-\big) \ge -A_4\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon), \tag{39}$$

where $A_4 := 2A_2 + A_3$.

Following the notations of [10, p. 45], and under our assumptions,

$$\rho_\epsilon := \varphi_\epsilon$$

(since $\psi = 0$). We then have $n = \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) + \Delta_{\omega'_\epsilon}\rho_\epsilon$, and we finally deduce from (39)

$$\begin{aligned}\Delta_{\omega'_\epsilon}\big(\log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) + \psi^-\big) &\geq -(A_4 - 1)\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) + \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) \\ &= -(A_4 - 1)(n - \Delta_{\omega'_\epsilon}\rho_\epsilon) + \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon)\end{aligned}$$

Thus,

$$\Delta_{\omega'_\epsilon}\big(\log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) + \psi^- - A_5\rho_\epsilon\big) \geq \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) - A_6, \tag{40}$$

where $A_5 := A_4 - 1$, $A_6 := (A_4 - 1)n$.

The function

$$H := \log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) + \psi^- - A_5\rho_\epsilon$$

on X is bounded. We apply the maximum principle, H achieves its maximum at a point x_0 and (40) yields $\mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon)(x_0) \leq A_6$. On other hand, trivial eigenvalue considerations show that

$$\mathrm{tr}_{\tau_1}(\tau_2) \leq n(\tau_2^n/\tau_1^n)\mathrm{tr}_{\tau_2}(\tau_1)^{n-1}$$

for any two Kähler forms τ_1, τ_2, whence

$$\log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \leq \psi^+ - \psi^- + \log\Big(\frac{\omega^n}{\omega_\epsilon^n}\Big) + (n-1)\log \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) + \log n$$

by Equation (3.1). Since $\omega \leq \omega_\epsilon (= (1+\epsilon)\omega)$ it follows that

$$H \leq \psi^+ - A_5\rho_\epsilon - n\log(1+\epsilon) + (n-1)\log \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) \leq (n-1)\log \mathrm{tr}_{\omega'_\epsilon}(\omega_\epsilon) + A_7 - A_5\rho_\epsilon$$

where $A_7 := C - n\log(1+\epsilon)$.

We obtain

$$\sup_X H = H(x_0) \leq (n-1)e^{-A_6} + A_7 - A_5 \inf_X \rho_\epsilon \leq (n-1)e^{-A_6} + A_7 - A_5 \inf_X \varphi_\epsilon,$$

That is

$$\sup_X H = H(x_0) \leq A_8 - A_5 \inf_X \varphi_\epsilon \tag{41}$$

where $A_8 := (n-1)e^{-A_6} + A_7$. By a result of Kolodziej, see [6], or by Lemma (1.2) in the case when $\psi^+ - \psi^-$ is continuous, there exists a constant $A_9 < 0$ depending only on X, ω and the norm L^p (resp. on the sup-norm) of $e^{\psi^+ - \psi^-}$, such that $|\varphi_\varepsilon| \leq -A_9$.

Then,

$$\log \mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) + \psi^- - A_5\rho_\epsilon = H \leq A_8 - A_5A_9,$$

We infer that

$$\mathrm{tr}_{\omega_\epsilon}(\omega'_\epsilon) \leq A_{10}e^{-\psi^-} \tag{42}$$

where $A_{10} := \exp(A_8 - 2A_5A_9)$ which depends only on ω, C and the norm L^p of $e^{\psi^+ - \psi^-}$.

When ψ^+ and ψ^- are not smooth we proceed as in [10] by using Demailly's regularization. We conclude there exists A_{11} a constant which depend only on C and ω and the L^p-norm (resp. the sup norm in the continuous case) of $e^{\psi^+ - \psi^-}$ such that

$$|\Delta\varphi| \leq A_{11}e^{-\psi^-}.$$

2. Let $(F_k)_{k\in\mathbb{N}}$ be a sequence satisfying assumptions of 2. From the discussion in 1. One can see that there exists a positive constant A such that

$$|\Delta\varphi_k| \leq A, \quad \forall k \in \mathbb{N}.$$

Let F be a continuous function on X. Clearly, there exists a constant $C > 0$ such that $dd^cF \geq -C\omega$ and $\sup_X F \leq C$. We set on X

$$\psi^+ := F, \quad \psi^- := 0.$$

then ψ^+ and ψ^- satisfy condition of Theorem (3.1). Thus

$$|\Delta\varphi| \leq A_{11}$$

on X.

□

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