Quantum cohomology of blowups of surfaces and its functoriality property

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Abstract

In this paper, using the WDVV equation, one first proved that all Gromov-Witten invariants of blowups of surfaces can be computed from the Gromov-Witten invariants of itself by some recursive relations. Furthermore, one may determine the quantum product on blowups. One also proved that there is some degree of functoriality of the big quantum cohomology for a blowup.

1 Introduction

Historically, the idea of quantum cohomology first appeared in the physical literature in the guise of the chiral ring of a Calabi-Yau 3-fold. The chiral ring was understood as a deformation of the ordinary cohomology ring [LVW], agreeing with the ordinary cohomology ring at large radius limit points. The quantum product can be understood more generally in the context of topological quantum field theories [W1, W2]. The mathematical foundation of quantum cohomology was first established for semi-positive symplectic manifold by Ruan and Tian [RT]. There actually are two types of quantum cohomology: the small and big quantum cohomology. In this paper, we will only consider the later.

One can use Gromov-Witten(GW) invariants to define the big quantum cohomology ring $QH^*(X)$ of a compact symplectic manifold X. It is well-known that the quantum product is associative. Moreover, the associativity is equivalent to a certain partial differential equation satisfied by the generating function Φ, the so-called WDVV equation [W2, DVV1, DVV2]. WDVV

[1]supported by Zhongshan university advanced research center foundation
[2]2000 AMS Classification: Primary 53D45. Secondary 14N35

equation gives rise to some relations among the GW invariants which often are sufficient to recursively determine all GW invariants from a few basic ones. This approach was used to determine the numbers N_d of nodal rational curves of degree d in projective plane in [KM] and [RT]. In section 3, we will first use the WDVV equation to obtain some recursive relation among GW invariants of blowups of nonsingular algebraic surfaces. Therefore, we may compute all GW invariants of the blowup of X from ones of X. Furthermore, we may determine the quantum product on the blowups. Quantum cohomology of blowups of surfaces was considered by Crauder-Miranda [CM] and Göttsche-Pandharipande[GP]. The method of [CM] is fully algebro-geometric. Our method is the same as that of [GP]. They only consider blowups of $\mathbf{P}^2$, but we consider general surfaces. We actually generalize the reaults of [GP] to blowups of general surfaces.

So far, we know little about the naturality of quantum cohomology [R1, R2]. To understand its naturality property, we also need to understand the change of GW invariant under some symplectic morphisms. It is wellknown that blowup is a type of fundamental surgeries in the category of algebraic and symplectic geometry. In section 4, we will show that there is some degree (i. e. module the effect of homology) of functoriality property of the big quantum cohomology for a blowup.

Acknowledgement: The author would like to thank Prof. Yongbin Ruan for his many suggestive discussion and encouragement.

2 Quantum cohomology and WDVV equation

Let X be a smooth projective manifold of dimension $2n$, ω be a complexified Kahler class on X, $\beta \in H_2(X)$ and k be a positive integer. Denote by $\overline{\mathcal{M}}_k(X,\beta)$ the moduli space of genus zero k-pointed stable maps $f: C \longrightarrow X$ with the homology class $f_*[C] = \beta$. It is wellknown that $\overline{\mathcal{M}}_k(X,\beta)$ has a virtual fundamental class, denoted by $[\overline{\mathcal{M}}_k(X,\beta)]^{Vir}$, with the expected dimension.

There are k evaluation maps

$$ev_i : \overline{\mathcal{M}}_k(X,\beta) \longrightarrow X$$

for each marked point $x_1, \cdots, x_k$. Given cohomology classes $\gamma_1, \cdots, \gamma_k \in H^*(X)$, we can define the **Gromov-Witten invariant** as

$$\Psi_\beta(\gamma_1, \cdots, \gamma_k) = \int_{[\overline{\mathcal{M}}_k(X,\beta)]^{Vir}} \Pi_i ev_i^* \gamma_i.$$

We fix a basis $T_0 = 1, T_1, \cdots, T_m$ of the rational cohomology $H^*(X, \mathbf{Q})$. Let $g_{ij} = \int_X T_i \cup T_j$ and then $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix.

Put $\gamma = \sum_{i=0}^m t_i T_i$ where t_i is a supercommutative variable with $\deg t_i = \deg T_i$. Here supercommutativity means that

$$t_i t_j = (-1)^{\deg t_i \deg t_j} t_j t_i$$

and $t_i^2 = 0$ if t_i has odd degree.

We define the **Gromov-Witten potential** Φ as the formal series

$$\begin{aligned} \Phi(t_0, t_1, \cdots, t_m) &:= \sum_{n \geq 3} \sum_{\beta \in H_2(X, \mathbf{Z})} \frac{1}{n!} \Psi_\beta(\gamma^n) q^\beta \\ &= \frac{1}{6} \int_X \gamma^3 + \sum_{n \geq 3} \sum_{\beta \in H_2(X, \mathbf{Z}) - \{0\}} \frac{1}{n!} \Psi_\beta(\gamma^n) q^\beta \end{aligned}$$

where $q^\beta = e^{2\pi i \int_\beta \omega}$.

To define the quantum product in terms of GW potential Φ, we will need to avoid the convergence problem of some formal series like Φ. We will regard q^β as a formal variable rather than a number. More precisely, by the effectivity axiom, we can assume that β lies in the integral Mori cone $M(X)_{\mathbf{Z}}$. Then we consider

$$\mathcal{R} = \mathbf{Q}[[q^\beta; \beta \in M(X)_{\mathbf{Z}}]],$$

where the double brackets indicate that we allow infinite sums. The product of two such elements is

$$\sum_{M(X)_{\mathbf{Z}}} a_\beta q^\beta \cdot \sum_{M(X)_{\mathbf{Z}}} b_\beta q^\beta = \sum_{M(X)_{\mathbf{Z}}} \Big(\sum_{\beta = \beta_1 + \beta_2} a_{\beta_1} b_{\beta_2} \Big) q^\beta.$$

This sum makes sense because for any $\beta \in M(X)_{\mathbf{Z}}$, the sum is parentheses is finite since the free part of the Mori cone consists of lattice points in a strongly convex cone. Thus $\mathcal{R}$ is a ring.

Let $T^k = \sum_l g^{kl} T_l$. Then we define the **quantum product** of T_i and T_j as follows:

$$T_i * T_j = \sum_k \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} T^k = \sum_{k,l} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_k} g^{kl} T_l.$$

Extending this linearly gives the big quantum product on the cohomology $H^*(X, \mathcal{R})$. Indeed, $H^*(X, \Lambda(X, \mathcal{R})$ is a ring under the big quantum product. Moreover, the quantum product is associative. We call $H^*(X, \mathcal{R})$ with the quantum product as the big quantum cohomology of X.

Kontsevich-Manin [KM] and Ruan-Tian [RT] proved that the associative property of big quantum cohomology is equivalent to a certain partial differential equation satisfied by Φ, the so-called WDVV equation. That means that the Gromov-Witten potential Φ satisfies the **WDVV equation**

$$\sum_{a,b} \frac{\partial^3 \Phi}{\partial t_i \partial t_j \partial t_a} g^{ab} \frac{\partial^3 \Phi}{\partial t_b \partial t_k \partial t_l}$$
$$= (-1)^{\deg t_i (\deg t_j + \deg t_k)} \sum_{a,b} \frac{\partial^3 \Phi}{\partial t_j \partial t_k \partial t_a} g^{ab} \frac{\partial^3 \Phi}{\partial t_b \partial t_i \partial t_l}$$

for all i, j, k, l. This WDVV equation will give rise to some recursive relations among GW invariants.

3 Quantum cohomology of blowups of surfaces

In the following sections, we will assume X is a nonsingular semipositive projective surface with the Kahler form as the symplectic structure ω. For simplicity we also assume that the Chow and homology rings of X coincide. Let X_r be the blowup of X in r generic points $x_1, \cdots, x_r$. Let $m = r + p + 1$. Let $T_0 = 1, T_1, \cdots, T_p, T_{i+p} (i = 1, \cdots, r), T_m$ be a basis of $H^*(X_r, \mathbf{Q})$, where $T_1, \cdots, T_p$ is a basis of $H^2(X)$. T_{i+p}, $(i = 1, \cdots, r)$ are respectively the Poincare dual cohomology classes of the exceptional divisors E_i. T_m is the class of a point.

For an r-tuple $\alpha = (a_1, \cdots, a_r)$ of integers, denote by (β, α) the homology class $\beta - \sum_{i=1}^r a_i E_i$, where β is the pullback of a homology class of X. In the following, we will use the same notation to denote the homology class and its pullback. Let $|\alpha| = \sum_{i=1}^r a_i$. Let $n_{\beta,\alpha} = C_1(\beta) - |\alpha| - 1$ be the expected dimension of the moduli space $\overline{\mathcal{M}}_0(X_r, (\beta, \alpha))$, where C_1 is the first Chern class of X. If $n_{\beta,\alpha} \geq 0$, let

$$N_{\beta,\alpha} = \Psi_{(\beta,\alpha)}(T_m^{n_{\beta,\alpha}})$$

be the corresponding GW invariant. When writing $N_{\beta,\alpha}$ for a sequence α of length r, we will always mean the GW invariant on X_r. For any r, we write $[i]_r = (j_1, \cdots, j_r)$ with $j_k = \delta_{ik}$. For any $\alpha = (a_1, \cdots, a_{r-1})$, we denote by (α, k) the sequence $(a_1, \cdots, a_{r-1}, k)$. For a permutation σ of $\{1, \cdots, r\}$, denote by α_σ the sequence $(a_{\sigma(1)}, \cdots, a_{\sigma(r)})$. For an integer k, we write $\alpha \geq k$ to mean that $a_i \geq k$ for all i.

About the change of GW invariants under blowup, the author[H1] proved the following lemma.

Lemma 3.1: Let X be a compact symplectic manifold of dimension $2n$. Suppose $\beta \in H_2(X)$, $\gamma_1, \cdots, \gamma_k \in H^*(X)$. Denote by $p : \tilde{X} \longrightarrow X$ the projection of the blowup, Then we have

(i) $\Psi^X_\beta(\gamma_1, \cdots, \gamma_k) = \Psi^{\tilde{X}}_\beta(p^*\gamma_1, \ldots, p^*\gamma_k)$.

(ii) $\Psi^X_\beta(\gamma_1, \cdots, \gamma_k, [pt]) = \Psi^{\tilde{X}}_{\beta-e}(p^*\gamma_1, \cdots, p^*\gamma_k)$.

where e the homology class of a line in the exceptional divisor.

¿From Lemma 3.1, we have $N_{\beta,\alpha}$ satisfy the following properties:

(1) $N_{\beta,\alpha} = N_{\beta,(\alpha,0)}$

(2) If $n_{\beta,\alpha} > 0$, then $N_{\beta,\alpha} = N_{\beta,(\alpha,1)}$.

It is wellknown that All effective classes (β, α) on X_r satisfy $\alpha \leq C_1(\beta)$.We hope to obtain some recursion formulae determining the $N_{\beta,\alpha}$ from the associativity of the quantum product.

Let $\gamma = t_0T_0 + \delta + \epsilon$ where $\delta = \sum_{i=1}^{r+p} t_iT_i$ and $\epsilon = t_mT_m$. Then the GW potential of X_r is

$$\begin{aligned}\Phi &= \frac{1}{6}\int_{X_r}\gamma^3 + \sum_{n\geq 3}\sum_{(\beta,\alpha)\neq 0}\frac{1}{n!}\Psi_{(\beta,\alpha)}((\delta+\epsilon)^n)q^{(\beta,\alpha)} \\ &= \frac{1}{6}\int_{X_r}\gamma^3 + \sum_{(\beta,\alpha)\neq 0} N_{\beta,\alpha}e^{\sum_{i=1}^{p} t_i\int_\beta T_i}e^{\sum_{i=1}^{r} a_it_{i+p}}\frac{t_m^{n_{\beta,\alpha}}}{n_{\beta,\alpha}!}q^\beta q_1^{a_1}\cdots q_r^{a_r}\end{aligned}$$

where the summation runs over all $(\beta, \alpha) \neq 0$ satisfying $n_{\beta,\alpha} \geq 0$, $C_1(\beta) \geq 0$ and $\alpha \leq C_1(\beta)$.

Let $\Phi_{ijk} = \partial_i\partial_j\partial_k\Phi$ where $\partial_i = \frac{\partial}{\partial t_i}$. Then the **quantum product** of T_i and T_j is given by

$$T_i * T_j = (T_i \cdot T_j)T_m + \sum_{k,l=1}^{p}\Phi_{ijl}g^{lk}T_k - \sum_{k=p+1}^{p+r}\Phi_{ijk}T_k + \Phi_{ijm}T_0. \qquad (3.1)$$

Therefore, from the WDVV equation, it is easy to prove the following lemma.

Lemma 3.2: For $i,j,k,l \in \{1,\cdots,m\}$, there is a relation:

$$(\mathbf{R_{i,j,k,l}}) \qquad (T_i\cdot T_j)\Phi_{mkl} + (T_k\cdot T_l)\Phi_{ijm} - (T_j\cdot T_k)\Phi_{mil} - (T_i\cdot T_l)\Phi_{jkm}$$
$$= \sum_{a,b=1}^{p} (\Phi_{jka}g^{ab}\Phi_{bil} - \Phi_{ija}g^{ab}\Phi_{bkl}) + \sum_{a=p+1}^{p+r} (\Phi_{ija}\Phi_{akl} - \Phi_{jka}\Phi_{ail}).$$

For the purpose of the recursive determination of the $N_{\beta,\alpha}$, we first obtained that for $i,j \in \{1,\cdots,p\}$

$$(\mathbf{R_{i,j,m,m}}) \qquad (T_i\cdot T_j)\Phi_{mmm} =$$
$$\sum_{a,b=1}^{p} (\Phi_{jma}g^{ab}\Phi_{bim} - \Phi_{ija}g^{ab}\Phi_{bmm}) + \sum_{s=p+1}^{p+r} (\Phi_{ijs}\Phi_{smm} - \Phi_{jms}\Phi_{sim})$$

and for all $k = p+1,\cdots,p+r$, $i,j \in \{1,\cdots,p\}$,

$$(\mathbf{R_{i,j,k,k}}) \qquad (T_i\cdot T_j)\Phi_{mkk} - \Phi_{ijm} =$$
$$\sum_{a,b=1}^{p} (\Phi_{jka}g^{ab}\Phi_{bik} - \Phi_{ija}g^{ab}\Phi_{bkk}) + \sum_{a=p+1}^{p+r} (\Phi_{ija}\Phi_{akk} - \Phi_{jka}\Phi_{aik}).$$

Remark: It is easy to see that in case $X = \mathbf{P}^2$, the relation $R_{i,j,k,l}$ coincides with $R_{i,j,k,l}$ of [GP].

Let the symbol $\vdash (\beta,\alpha)$ denote the set of pairs $((\beta_1,\alpha_1),(\beta_2,\alpha_2))$ satisfying

(i) (β_1,α_1), $(\beta_2,\alpha_2) \neq 0$.

(ii) $(\beta_1,\alpha_1) + (\beta_2,\alpha_2) = (\beta,\alpha)$.

(iii) n_{β_1,α_1}, $n_{\beta_2,\alpha_2} \geq 0$, $C_1(\beta_1)$, $C_1(\beta_2) \geq 0$, $\alpha_1 \leq C_1(\beta_1)$, $\alpha_2 \leq C_1(\beta_2)$.

The notation $\vdash (\beta,\alpha) \neq 0$ will be used to denote the subset of $\vdash (\beta,\alpha)$ satisfying $\beta_1 \neq 0$, $\beta_2 \neq 0$. The binomial coefficient $\begin{pmatrix} p \\ q \end{pmatrix}$ is defined to be zero if $q < 0$ or $p < q$.

Theorem 3.3: *The $N_{\beta,\alpha}$ are determined by the initial values*

(i) $N_{(\beta,0)} = \Psi_\beta^X(T_m^{n_\beta})$ *for all* r.

(ii) $N_{0,-[i]_r} = 1$, *for* $i \in \{1,\cdots,r\}$.

and the following recursion relations.

If $n_{\beta,\alpha} \geq 3$*, choosing* $i, j \in \{1, \cdots, p\}$ *such that* $g_{ij} \neq 0$*, then relation* $R(m)$ *holds:*

$$(\mathbf{R}(\mathbf{m})) \qquad N_{\beta,\alpha} = \frac{1}{g_{ij}} \sum_{\vdash(\beta,\alpha)\neq 0} N_{\beta_1,\alpha_1} N_{\beta_2,\alpha_2} \left(\sum_{a,b=1}^{p} T_a(\beta_1) g^{ab} T_b(\beta_2) - \sum_{s=1}^{r} b_s c_s \right) \left[T_i(\beta_1) T_j(\beta_2) \binom{n_{\beta,\alpha} - 3}{n_{\beta_1,\alpha_1} - 1} - T_i(\beta_1) T_j(\beta_1) \binom{n_{\beta,\alpha} - 3}{n_{\beta_1,\alpha_1}} \right].$$

If $n_{\beta,\alpha} \geq 0$*, then for any* $k \in \{1, \cdots, r\}$ *the relation* $R(k)$ *holds:*

$$(\mathbf{R}(\mathbf{k})) \qquad T_i(\beta) T_j(\beta) a_k N_{\beta,\alpha} = \left(T_i(\beta) T_j(\beta) - g_{ij}(a_k - 1)^2 \right) N_{\beta,\alpha-[k]} + \sum_{\vdash(\beta,\alpha-[k])\neq 0} N_{\beta_1,\alpha_1} N_{\beta_2,\alpha_2} \left(\sum_{a,b=1}^{p} T_a(\beta_1) g^{ab} T_b(\beta_2) - \sum_{s=1}^{r} b_s c_s \right) \left(T_j(\beta_1) T_i(\beta_2) b_k c_k - T_i(\beta_1) T_j(\beta_1) c_k^2 \right) \binom{n_{\beta,\alpha-[k]} - 1}{n_{\beta_1,\alpha_1}}.$$

Furthermore, the properties (P1), (P2) and (P3) hold.

(P1) $N_{0,\alpha} = 0$ *unless* $\alpha = -[i]$ *for some* i*.*

(P2) $N_{\beta,\alpha} = 0$ *if* β *is effective and any of the* a_i *is negative.*

(P3) $N_{\beta,\alpha} = N_{\beta,\alpha_\sigma}$ *for any permutation* σ *of* $\{1, \cdots, r\}$*.*

Proof: Lemma 3.1 implies the equation (i). Bceause $N_{0,-[i]_r}$ counts the number of curves in the exceptional class E_i, $N_{0,-[i]_r} = 1$. So it is sufficient to prove the two recursive relations and the properties (P1), (P2) and (P3). One first want to prove the recursive relations.

To exploit $(R_{i,j,m,m})$, note that $N_{\beta,\alpha}$ appears in Φ_{mmm} in the term

$$N_{\beta,\alpha} \frac{t_m^{n_{\beta,\alpha}-3}}{(n_{\beta,\alpha} - 3)!} q^\beta q_1^{a_1} \cdots q_r^{a_r}.$$

This suggests looking at terms containing $t_m^{n_{\beta,\alpha}-3}\ q^\beta q_1^{a_1} \cdots q_r^{a_r}$ in RHS of $(R_{i,j,m,m})$. In $\sum_{a,b=1}^{p} (\Phi_{jma} g^{ab} \Phi_{bim} - \Phi_{ija} g^{ab} \Phi_{bmm})$, the terms without $t_1, \cdots,$ t_{m-1} are

$$\sum_{a,b=1}^{p} \sum_{\vdash(\beta,\alpha)} \left[\frac{T_j(\beta_1) T_a(\beta_1) g^{ab} T_i(\beta_2) T_b(\beta_2)}{(n_{\beta_1,\alpha_1} - 1)!(n_{\beta_2,\alpha_2} - 2)!} - \frac{T_i(\beta_1) T_j(\beta_1) T_a(\beta_1) g^{ab} T_b(\beta_2)}{n_{\beta_1,\alpha_1}!(n_{\beta_2,\alpha_2} - 3)!} \right]$$

$$N_{\beta_1,\alpha_1}N_{\beta_2,\alpha_2}q^{\beta}q_1^{a_1}\cdots q_r^{a_r}.$$

In $\sum_{s=1}^r(\Phi_{ij(s+p)}\Phi_{(s+p)mm}-\Phi_{jm(s+p)}\Phi_{(s+p)im})$, the terms without $t_1,\cdots,\ t_{m-1}$ are

$$\sum_{s=1}^r\sum_{\vdash(\beta,\alpha)}\left[\frac{T_i(\beta_1)T_j(\beta_1)b_sc_s}{n_{\beta_1,\alpha_1}!(n_{\beta_2,\alpha_2}-3)!}-\frac{T_i(\beta_1)T_j(\beta_2)b_sc_s}{(n_{\beta_1,\alpha_1}-1)!(n_{\beta_2,\alpha_2}-2)!}\right]$$

$$N_{\beta_1,\alpha_1}N_{\beta_2,\alpha_2}q^{\beta}q_1^{a_1}\cdots q_r^{a_r}.$$

Therefore, if we choose $i,j\in\{1,\cdots,p\}$ such that $g_{ij}\neq 0$, then we have

$$(R(m)^*)\qquad N_{\beta,\alpha}= \frac{1}{g_{ij}}\sum_{\vdash(\beta,\alpha)\neq 0}N_{\beta_1,\alpha_1}N_{\beta_2,\alpha_2}\left(\sum_{a,b=1}^{p}T_a(\beta_1)g^{ab}T_b(\beta_2)-\sum_{s=1}^{r}b_sc_s\right) \left[T_i(\beta_1)T_j(\beta_2)\binom{n_{\beta,\alpha}-3}{n_{\beta_1,\alpha_1}-1}-T_i(\beta_1)T_j(\beta_1)\binom{n_{\beta,\alpha}-3}{n_{\beta_1,\alpha_1}}\right].$$

For $k\in\{1,\cdots,r\}$, from the relation $(R_{i,j,k+p,k+p})$ above, we get immediately (for $n_{\beta,\alpha}\geq 1$) the following recursion formula

$$(R(k)^*)\qquad (g_{ij}a_k^2-T_i(\beta)T_j(\beta))N_{\beta,\alpha} =\sum_{\vdash(\beta,\alpha)}N_{\beta_1,\alpha_1}N_{\beta_2,\alpha_2}\left(\sum_{a,b=1}^{p}T_a(\beta_1)g^{ab}T_b(\beta_2)-\sum_{s=1}^{r}b_sc_s\right) \left(T_j(\beta_1)T_i(\beta_2)b_kc_k-T_i(\beta_1)T_j(\beta_1)c_k^2\right)\binom{n_{\beta,\alpha}-1}{n_{\beta_1,\alpha_1}}$$

Now we want first to prove property (P1). If $N_{0,\alpha}\neq 0$, then (β,α) is effective and therefore $\alpha\leq 0$. If $n_{0,\alpha}=0$, we get $\alpha=-[i]$ for some $i\in\{1,\cdots,r\}$. If $n_{0,\alpha}>0$, we apply $(R(k)^*)$ for an k with $a_k\neq 0$. We will see that all summands on RHS are divisible by $T_j(\beta_1)=0$, and thus (P1) follows.

Now we want to prove that the sum in $(R(m)^*)$ only runs over $\vdash(\beta,\alpha)>0$. It will be shown that the terms with $\beta_1=0$ or $\beta_2=0$ vanish. Since all summands are divisible by $T_i(\beta_1)$, only the case $\beta_2=0$ needs be considered. By (P1), either $N_{0,\gamma}=0$ or $\gamma=-[i]$. In the second case, both binomial coefficients vanish. Thus relation (R(m)) follows.

Now we want to show relation (R(k)) holds. We first apply relation $(R(k)^*)$ to $N_{\beta,\alpha-[k]}$. Then we have

$$(R(k)^{**})\qquad (g_{ij}(a_k-1)^2-T_i(\beta)T_j(\beta))N_{\beta,\alpha-[k]}$$

$$= \sum_{\vdash(\beta,\alpha-[k])} N_{\beta_1,\alpha_1} N_{\beta_2,\alpha_2} \left(\sum_{a,b=1}^{p} T_a(\beta_1) g^{ab} T_b(\beta_2) - \sum_{s=1}^{r} b_s c_s \right)$$
$$\left(T_j(\beta_1)T_i(\beta_2)b_k c_k - T_i(\beta_1)T_j(\beta_1)c_k^2\right) \left(\begin{array}{c} n_{\beta,\alpha-[k]} - 1 \\ n_{\beta_1,\alpha_1} \end{array} \right)$$

All summands on RHS of $(R(k)^{**})$ are divisible by $T_j(\beta_1)$, thus all nonvanishing summands have $C_1(\beta_1) > 0$. By (P1), $N_{0,\gamma}$ can be zero only if $\gamma = -[l]$ for some $l \in \{1, \cdots, r\}$. Since RHS of $(R(k)^{**})$ is divisible by c_k, the only nonzero summand on RHS of $(R(k)^{**})$ with $\beta_2 = 0$ occurs for $(\beta_2, \alpha_2) = (0, -[k])$ and is $-T_i(\beta)T_j(\beta)a_k N_{\beta,\alpha}$. Therefore, it is easy to see from $(R(k)^{**})$ that (R(k)) holds.

The remain proof is similar to that of Theorem 3.6 of [GP]. We omit it now.

4 Functoriality property for a blowup

Quantum cohomology is an admixture of homology and cohomology. Therefore, it is too much for us to expect that quantum cohomology has the same functoriality property as ordinary cohomology. Crauder and Miranda[CM] first consider the case of rational surfces and showed that there is some degree of functoriality for a general blowup. In this section, we will consider blowups of smooth projective manifolds of dimension $2n$ instead of surfaces and show that the big quantum cohomology has some degree of functoriality for a blowup.

The fundamental reason why cohomology is a very useful is its naturality. Namely, a continuous map induces a ring homomorphism on cohomology. Therefore, the fundamental problem in quantum cohomology is the **quantum naturality problem**[R2]: *Define "morphism" of symplectic manifolds so that quantum cohomology is natural.* Qin and Ruan [QR] showed that quantum cohomology is not natural for fibrations. Their results also shows that possible "morphisms" must be very rigid. The existence of these rigid morphisms will set apart quntum cohomology from ordinary cohomology and gives it its own identity.

To define a homomorphism of the quantum product, we need to match Novikov rings and hence symplectic classes. Here we view quantum cohomology as a theory of GW invariants instead of just quantum product. Since the GW potential Φ contains the same information as quantum product, therefore, it is more convenient to work directly with the GW potential Φ.

Definition4.1: Suppose X, Y are symplectic manifolds and that f :

$X \longrightarrow Y$ is a map and induces group homomorphism

$$f_*: H_2(X, \mathbf{Z}) \longrightarrow H_2(Y, \mathbf{Z}) \quad \text{and} \quad f^*: H^*(Y, \mathbf{R}) \longrightarrow H^*(X, \mathbf{R}).$$

We say f is weakly natural with repect to the big quantum cohomology if $\Phi^X(f^*\gamma) = \Phi^Y(\gamma)$ after change of formal variable $\beta \longrightarrow f_*(\beta)$. We say f is natural with repect to the big quantum cohomology if $\Phi^X(f^*\gamma) = \Phi^Y(\gamma)$ after change of formal variable $q^\beta \longrightarrow q^{f_*(\beta)}$. If f_* and f^* are also isomorphisms and f is natural, we say f induces an isomorphism on big quantum cohomology or X and Y have the same big quantum cohomology.

As in [R2], if necessary, we will consider two power series F, G as the same if $F = H + F'$, $G = H + G'$ are power series expansion for the same rational function at the different points. For example, we can expand $\frac{1}{1-t} = \sum_{i=0} t^i$ at $t = 0$ or $\frac{1}{1-t} = \frac{1}{-t(1-t^{-1})} = -\sum_{i=0} t^{-i-1}$ at $t = \infty$. Hence, we need to treat $\sum_{i=0} t^i$, $\sum_{i=0} t^{-i-1}$ as the same power series.

Theorem 4.2: Suppose that X is a smooth projective manifold of dimension $2n$ and $p: \tilde{X}_r \longrightarrow X$ is a blowup at r general points. Then p is weakly natural with respect to big quantum cohomology.

Proof: Without loss of generalization, we assume $r = 1$. For simplicity, we also assume that $H^{odd}(X) = 0$. Let $T_0 = 1, \cdots, T_p, \xi, \cdots, \xi^{n-1}, T_m$ be a basis of $H^*(\tilde{X})$, where $T_0 = 1, T_1, \cdots, T_m$ is a basis of $H^*(X)$ and ξ is a generator of the cohomology $H^*(E)$ of the exceptional divisor E. Let $\gamma^{\tilde{X}} = \sum_{i=0}^m t_i T_i + \sum_{i=1}^{n-1} y_i \xi^i$ and $\gamma^X = \sum_{i=0}^m t_i T_i$. Denote by (β, α) the homology class $\beta - \alpha e \in H_2(\tilde{X})$, where e is the class of a line in the exceptional divisor E. Then we have GW potentials of $\tilde{X}$ and X as follows:

$$\begin{aligned}
&\Phi^{\tilde{X}}(t_0, t_1, \cdots, t_m, y_1, \cdots, y_{n-1}) \\
&\quad = \frac{1}{6}\int_{\tilde{X}} \left(\gamma^{\tilde{X}}\right)^3 + \sum_{n \geq 3} \sum_{(\beta,\alpha) \in H_2(\tilde{X}, \mathbf{Z}) - \{0\}} \frac{1}{n!} \Psi^{\tilde{X}}_{(\beta,\alpha)}((\gamma^{\tilde{X}})^n) q^\beta q_1^\alpha. \\
&\Phi^X(t_0, t_1, \cdots, t_m) = \frac{1}{6}\int_X \left(\gamma^X\right)^3 + \sum_{n \geq 3} \sum_{\beta \in H_2(X, \mathbf{Z}) - \{0\}} \frac{1}{n!} \Psi^X_\beta \left((\gamma^X)^n\right) q^\beta.
\end{aligned}$$

Therefore, we have

$$\Phi^{\tilde{X}}(p^*\gamma^X) = \frac{1}{6}\int_{\tilde{X}} \left(p^*\gamma^X\right)^3 + \sum_{n \geq 3} \sum_{(\beta,\alpha) \in H_2(\tilde{X}, \mathbf{Z}) - \{0\}} \frac{1}{n!} \Psi^{\tilde{X}}_{(\beta,\alpha)} \left((p^*\gamma^X)^n\right) q^\beta q_1^\alpha.$$

By the degree reason and Lemma 3.1, after replacing (β, α) by $p_*(\beta, \alpha) = \beta$, we actually have the terms with q_1^α, $(\alpha \neq 0)$ must be zero. Therefore, we have $\Phi^{\tilde{X}}(p^*\gamma^X) = \Phi^X(\gamma^X)$. This completes the proof of the theorem.

Remark 4.3: Replacing $\gamma^{\tilde{X}}$ by $p^*\gamma^X$ is equivalent to seting $y_1 = \cdots = y_{n-1} = 0$ in $\Phi^{\tilde{X}}(t_i, y_j)$. The change of formal variable $(\beta, \alpha) \longrightarrow p_*(\beta, \alpha) = \beta$ is also equivalent to truncating the terms with $q_1^\alpha (\alpha \neq 0)$ in $\Phi^{\tilde{X}}(p^*\gamma^X)$. The real meaning of our theorem means that there is some degree of functoriality of the big quantum cohomology.

Example: Suppose that X is a smooth projective surfaces as in section 3. If we set $\epsilon = t_0 T_0 + t_m T_m$, by the equality (3.1), then the big quantum product of p^*T_i and p^*T_j is given by

$$\begin{aligned}
p^*T_i * p^*T_j &= \sum_{l,k=1}^{p} \sum_{n=0}^{\infty} \sum_{(\beta,\alpha)} \frac{1}{n!} \Psi^{\tilde{X}}_{(\beta,\alpha)}(T_i, T_j, T_l, \epsilon^n) e^{\sum_{i=1}^{p} t_i \int_\beta T_i} e^{t_{p+1}\alpha} q^\beta q_1^\alpha g^{lk} T_k \\
&- \sum_{k=p+1}^{p+r} \sum_{n=0}^{\infty} \sum_{(\beta,\alpha)} \frac{1}{n!} \Psi^{\tilde{X}}_{(\beta,\alpha)}(T_i, T_j, T_k, \epsilon^n) e^{\sum_{i=1}^{p} t_i \int_\beta T_i} e^{t_{p+1}\alpha} q^\beta q_1^\alpha T_k \\
&+ (T_i \cdot T_j) T_m + \sum_{n=0}^{\infty} \frac{1}{n!} \Psi^{\tilde{X}}_{(\beta,\alpha)}(T_i, T_j, T_m, \epsilon^n) e^{\sum_{i=1}^{p} t_i \int_\beta T_i} e^{t_{p+1}\alpha} q^\beta q_1^\alpha T_0.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
T_i * T_j &= \sum_{l,k=1}^{p} \sum_{n=0}^{\infty} \sum_{\beta} \frac{1}{n!} \Psi^X_\beta(T_i, T_j, T_l, \epsilon^n) e^{\sum_{i=1}^{r} t_i \int_\beta T_i} q^\beta g^{lk} T_k \\
&+ (T_i \cdot T_j) T_m + \sum_{n=0}^{\infty} \sum_{\beta} \frac{1}{n!} \Psi^X_\beta(T_i, T_j, T_m, \epsilon^n) e^{\sum_{i=1}^{r} t_i \int_\beta T_i} q^\beta T_0.
\end{aligned}$$

Therefore, if we identify T_i with p^*T_i, it is easy to see that

$$p^*T_i *_{\tilde{X}} p^*T_j|_{q_1 = y_1 = 0} = T_i *_X T_j.$$

Therefore, we may consider the quantum product of T_i and T_j on X is the restriction of the quantum product on $\tilde{X}$ of their pullbacks to $H^*(X, \mathcal{R})$. Our theorem also shows that there is some degree (i.e. module the effect of homology) of functoriality property of the big quantum cohomology for a blowup. In the case of rational surfaces, our conlusion coincides with that of [CM].

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