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by

Jae-Hyun Yang

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Jae-Hyun Yang

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Inha University
Incheon 22212
Korea

MPIM 18-17

STABLE SCHOTTKY-JACOBI FORMS

JAE-HYUN YANG

ABSTRACT. In this article, we prove that there do not exist stable Schottky-Jacobi forms for the universal Jacobian locus and also prove that there exists non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

1. Introduction

For a positive integer g, we let

$$\mathbb{H}_g = \left\{ \tau \in \mathbb{C}^{(g,g)} \mid \tau = {}^t\tau,\ \operatorname{Im}\tau > 0 \right\}$$

be the Siegel upper half plane of degree g and let

$$Sp(2g,\mathbb{R}) = \{ M \in \mathbb{R}^{(2g,2g)} \mid {}^tM J_g M = J_g \}$$

be the symplectic group of degree g, where $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F for two positive integers k and l, ${}^tM$ denotes the transposed matrix of a matrix M and

$$J_g = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then $Sp(2g,\mathbb{R})$ acts on $\mathbb{H}_g$ transitively by

$$M \cdot \tau = (A\tau + B)(C\tau + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{R})$ and $\Omega \in \mathbb{H}_n$. Let

$$\Gamma_g = Sp(2g,\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{R}) \,\Big|\, A, B, C, D \text{ integral} \right\}$$

be the Siegel modular group of degree g. This group acts on $\mathbb{H}_g$ properly discontinuously.

Let $\mathcal{A}_g := \Gamma_g \backslash \mathbb{H}_g$ be the Siegel modular variety of degree g, that is, the moduli space of g-dimensional principally polarized abelian varieties, and let $\mathcal{M}_g$ be the the moduli space of projective curves of genus g. Then according to Torelli's theorem, the Jacobi mapping

$$T_g : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

defined by

$$C \longmapsto J(C) := \text{the Jacobian of } C$$

is injective. The Jacobian locus $J_g := T_g(\mathcal{M}_g)$ is a $(3g-3)$-dimensional subvariety of $\mathcal{A}_g$

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The Schottky problem is to characterize the Jacobian locus or its closure $\bar{J}_g$ in $\mathcal{A}_g$. At first this problem had been investigated from the analytical point of view : to find explicit equations of J_g (or $\bar{J}_g$) in $\mathcal{A}_g$ defined by Siegel modular forms on $\mathbb{H}_g$, for example, polynomials in the theta constant $\theta\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}(\tau,0)$ and their derivatives. The first result in this direction was due to Friedrich Schottky [23] who gave the simple and beautiful equation satisfied by the theta constants of Jacobians of dimension 4. Much later the fact that this equation characterizes the Jacobian locus J_4 was proved by J. Igusa [14] (see also [9], [11] and [13]). Past decades there has been some progress on the characterization of Jacobians by some mathematicians.

For two positive integers g and h, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(g,h)}=\left\{\,(\lambda,\mu;\kappa)\mid \lambda,\mu\in\mathbb{R}^{(h,g)},\ \kappa\in\mathbb{R}^{(h,h)},\ \kappa+\mu\,{}^t\lambda \text{ symmetric }\right\}$$

endowed with the following multiplication law

$$\big(\lambda,\mu;\kappa\big)\circ\big(\lambda',\mu';\kappa'\big)=\big(\lambda+\lambda',\mu+\mu';\kappa+\kappa'+\lambda\,{}^t\mu'-\mu\,{}^t\lambda'\big)$$

with $\big(\lambda,\mu;\kappa\big),\big(\lambda',\mu';\kappa'\big)\in H_{\mathbb{R}}^{(g,h)}$. We define the *Jacobi group* G^J of degree g and index h that is the semidirect product of $Sp(2g,\mathbb{R})$ and $H_{\mathbb{R}}^{(g,h)}$

$$G^J=Sp(2g,\mathbb{R})\ltimes H_{\mathbb{R}}^{(g,h)}$$

endowed with the following multiplication law

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot\big(M',(\lambda',\mu';\kappa'\,)\big)=\,\big(MM',(\tilde{\lambda}+\lambda',\tilde{\mu}+\mu';\kappa+\kappa'+\tilde{\lambda}\,{}^t\mu'-\tilde{\mu}\,{}^t\lambda'\,)\big)$$

with $M,M'\in Sp(2g,\mathbb{R}),(\lambda,\mu;\kappa),\,(\lambda',\mu';\kappa')\in H_{\mathbb{R}}^{(g,h)}$ and $(\tilde{\lambda},\tilde{\mu})=(\lambda,\mu)M'$. Then G^J acts on $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ transitively by

$$\big(M,(\lambda,\mu;\kappa)\big)\cdot(\Omega,Z)=\Big(M\cdot\Omega,(Z+\lambda\Omega+\mu)(C\Omega+D)^{-1}\Big),$$

where $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(2g,\mathbb{R}),\ (\lambda,\mu;\kappa)\in H_{\mathbb{R}}^{(g,h)}$ and $(\Omega,Z)\in\mathbb{H}_g\times\mathbb{C}^{(h,g)}$. We note that the Jacobi group G^J is *not* a reductive Lie group and the homogeneous space $\mathbb{H}_g\times\mathbb{C}^{(h,g)}$ is not a symmetric space. From now on, for brevity we write $\mathbb{H}_{g,h}=\mathbb{H}_g\times\mathbb{C}^{(h,g)}$. The homogeneous space $\mathbb{H}_{g,h}$ is called the *Siegel-Jacobi space* of degree g and index h.

Let $\Gamma_g^J:=\Gamma_g\ltimes H_{\mathbb{Z}}^{(g,h)}$ be the Jacobi modular group. Let

$$\mathcal{A}_{g,h}:=\Gamma_g^J\backslash\mathbb{H}_{g,h}$$

be the universal abelian variety. Consider the natural projection map

$$\pi_{g,h}:\mathcal{A}_{g,h}\longrightarrow\mathcal{A}_g.$$

Let

$$J_{g,h}:=\pi_{g,h}^{-1}(J_g)$$

be the universal Jacobian locus and let

$$Hyp_{g,h}:=\pi_{g,h}^{-1}(Hyp_g)$$

be the universal hyperelliptic locus, where Hyp_g is the hyperelliptic locus in $\mathcal{A}_g$.

Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree h. According to Theorem 3.6 in [28], if $g+h>2k+1$ with a nonnegative integer k, the Siegel-Jacobi operator

$$\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

is an isomorphism (see also Theorem 2.2). Using this fact, we define the notion of stable Jacobi forms of weight k and index $\mathcal{M}$. A Jacobi form $F \in J_{k,\mathcal{M}}(\Gamma_g)$ is said to be a *Schottky-Jacobi form* for $J_{g,h}$ (resp. $Hyp_{g,h}$) if it vanishes along $J_{g,h}$ (resp. $Hyp_{g,h}$). In a natural way, we can define the notion of *stable Schottky-Jacobi forms* of index $\mathcal{M}$. For precise definitions, we refer to Definition 2.3 and Definition 4.2.

The aim of this paper is to prove the non-existence of stable Schottky-Jacobi forms for the universal Jacobian locus and also to prove that there exist non-trivial stable Schottky-Jacobi forms for the universal hyperelliptic locus.

This article is organized as follows. In Section 2, we review some properties of the Siegel-Jacobi operator and the notion of stable Jacobi forms introduced by J.-H. Yang [31]. In Section 3, we review the notion of stable Schottky-Siegel forms and the works that were done recently by G. Codogni and N. I. Shepherd-Barron [3, 4]. In Section 4, we introduce the notion of stable Schottky-Jacobi forms and prove the following two theorems.

Theorem 1.1. *Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree h. Then there do not exist stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal Jacobian locus.*

Theorem 1.2. *Let $2\mathcal{M}$ be a positive definite, even unimodular integral symmetric matrix of degree h. Then there exist non-trivial stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal hyperelliptic locus.*

In the final section, we make some comments and present several questions.

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Notations: We denote by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the ring of integers and the set of all positive integers respectively. $\mathbb{R}^+$ denotes the set of all positive real numbers. $\mathbb{Z}_+$ and $\mathbb{R}_+$ denote the set of all nonnegative integers and the set of all nonnegative real numbers respectively. The symbol ":=" means that the expression on the right is the definition of that on the left. For two positive integers k and l, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring F. For a square matrix $A \in F^{(k,k)}$ of degree k, $\sigma(A)$ denotes the trace of A. For any $M \in F^{(k,l)}$, ${}^tM$ denotes the transpose of a matrix M. I_n denotes the identity matrix of degree n. We put $i=\sqrt{-1}$.

2. Stable Jacobi Forms

For a non-negative integer k, we denote by $[\Gamma_g, k]$ the vector space of all Siegel modular forms of weight k. The Siegel Φ-operator

$$\Phi_g : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k]$$

is an important linear map defined by

$$(\Phi_g f)(\tau) := \lim_{t \longrightarrow \infty} f \begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, \qquad f \in [\Gamma_g, k],\ \tau \in \mathbb{H}_{g-1}. \tag{2.1}$$

H. Maass [18, 19] proved that if k is even and $k > 2g$, then Φ_g is surjective. E. Freitag [7] proved that if $g > 2k$, then Φ_g is injective. Using the theory of singular modular forms developed by Freitag [7, 8, 10], he showed the following:

(SO1) $[\Gamma_g, k] = 0$ for $g > 2k,\ k \not\equiv 0 \,(\mathrm{mod}\, 4)$.

(SO2) Φ_g is an isomorphism if $g > 2k + 1$.

Definition 2.1. *A collection $(f_g)_{g \geq 0}$ is called a* stable modular form *of weight k if it satisfies the following conditions (SM1) and (SM2):*

(SM1) $f_g \in [\Gamma_g, k]$ *for all* $g \geq 0$.

(SM2) $\Phi_g f_g = f_{g-1}$ *for all* $g > 0$.

Let ρ be a rational representation of $GL(g, \mathbb{C})$ on a finite dimensional complex vector space V_ρ. Let $\mathcal{M} \in \mathbb{R}^{(h,h)}$ be a symmetric half-integral semi-positive definite matrix of degree h. The canonical automorphic factor

$$J_{\rho,\mathcal{M}} : G^J \times \mathbb{H}_{g,h} \longrightarrow GL(V_\rho)$$

for G^J on $\mathbb{H}_{g,h}$ is given as follows:

$$\begin{aligned} J_{\rho,\mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z)) &= e^{2\pi i \sigma\left(\mathcal{M}(z+\lambda\tau+\mu)(C\tau+D)^{-1}C\,{}^t(z+\lambda\tau+\mu)\right)} \\ &\quad \times e^{-2\pi i \sigma\left(\mathcal{M}(\lambda\tau\,{}^t\lambda + 2\lambda\,{}^tz + \kappa + \mu\,{}^t\lambda)\right)} \rho(C\tau + D), \end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\tau, z) \in \mathbb{H}_{g,h}$. We refer to [30] for a geometrical construction of $J_{\rho,\mathcal{M}}$.

Let $C^\infty(\mathbb{H}_{g,h}, V_\rho)$ be the algebra of all C^∞ functions on $\mathbb{H}_{g,h}$ with values in V_ρ. For $f \in C^\infty(\mathbb{H}_{g,h}, V_\rho)$, we define

$$\begin{aligned} \left(f|_{\rho,\mathcal{M}}[(M, (\lambda, \mu; \kappa))]\right)(\tau, z) &= J_{\rho,\mathcal{M}}((M, (\lambda, \mu; \kappa)), (\tau, z))^{-1} \\ &\quad f\left((A\tau + B)(C\tau + D)^{-1}, (z + \lambda\tau + \mu)(C\tau + D)^{-1}\right), \end{aligned}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{R}),\ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)}$ and $(\tau, z) \in \mathbb{H}_{g,h}$.

Definition 2.2. *Let ρ and $\mathcal{M}$ be as above. Let*

$$H_{\mathbb{Z}}^{(g,h)} := \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(g,h)} \mid \ \lambda, \mu, \kappa \text{ integral } \right\}$$

be the discrete subgroup of $H_{\mathbb{R}}^{(g,h)}$. *A* Jacobi form *of index* $\mathcal{M}$ *with respect to* ρ *on a subgroup* Γ *of* Γ_g *of finite index is a holomorphic function* $f \in C^{\infty}(\mathbb{H}_{g,h}, V_\rho)$ *satisfying the following conditions* (A) *and* (B)*:*

(A) $f|_{\rho,\mathcal{M}}[\tilde{\gamma}] = f$ *for all* $\tilde{\gamma} \in \widetilde{\Gamma} := \Gamma \ltimes H_{\mathbb{Z}}^{(g,h)}$.

(B) *For each* $M \in \Gamma_g$, $f|_{\rho,\mathcal{M}}[M]$ *has a Fourier expansion of the following form :*

$$(f|_{\rho,\mathcal{M}}[M])(\tau,z) = \sum_{\substack{T={}^tT\geq 0 \\ \text{half-integral}}} \sum_{R\in\mathbb{Z}^{(g,h)}} c(T,R)\cdot e^{\frac{2\pi i}{\lambda_\Gamma}\sigma(T\tau)} \cdot e^{2\pi i\,\sigma(Rz)}$$

with $\lambda_\Gamma(\neq 0) \in \mathbb{Z}$ *and* $c(T,R) \neq 0$ *only if* $\begin{pmatrix} \frac{1}{\lambda_\Gamma}T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} \geq 0.$

If $g \geq 2$, the condition (B) is superfluous by Köcher principle (cf. [34] Lemma 1.6). We denote by $J_{\rho,\mathcal{M}}(\Gamma)$ the vector space of all Jacobi forms of index $\mathcal{M}$ with respect to ρ on Γ. Ziegler (cf. [34] Theorem 1.8 or [6] Theorem 1.1) proved that the vector space $J_{\rho,\mathcal{M}}(\Gamma)$ is finite dimensional. In the special case $\rho(A) = (\det(A))^k$ with $A \in GL(g,\mathbb{C})$ and a fixed $k \in \mathbb{Z}$, we write $J_{k,\mathcal{M}}(\Gamma)$ instead of $J_{\rho,\mathcal{M}}(\Gamma)$ and call k the *weight* of the corresponding Jacobi forms. For more results about Jacobi forms with $g > 1$ and $h > 1$, we refer to [27, 28, 29, 30, 31, 32, 33] and [34]. Jacobi forms play an important role in lifting elliptic cusp forms to Siegel cusp forms of degree $2g$ (cf. [16, 17]).

Now we consider the special case $\rho = \det^k$ with $k \in \mathbb{Z}_+$. We define the Siegel-Jacobi operator

$$\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$$

by

$$(\Psi_{g,\mathcal{M}}F)(\tau,z) := \lim_{t\longrightarrow\infty} F\left(\begin{pmatrix} \tau & 0 \\ 0 & it \end{pmatrix}, (z,0)\right), \tag{2.2}$$

where $F \in J_{k,\mathcal{M}}(\Gamma_g)$, $\tau \in \mathbb{H}_{g-1}$ and $z \in \mathbb{C}^{(h,g-1)}$. We observe that the above limit exists and $\Psi_{g,\mathcal{M}}$ is a well-defined linear map (cf. [34]).

J.-H. Yang [28] proved the following theorems.

Theorem 2.1. *Let* $2\mathcal{M}$ *be a positive even unimodular symmetric integral matrix of degree* h *and let* k *be an even nonnegative integer. If* $g+h > 2k$*, then the Siegel-Jacobi operator* $\Psi_{g,\mathcal{M}}$ *is injective.*

Proof. See Theorem 3.5 in [28]. □

Theorem 2.2. *Let* $2\mathcal{M}$ *be as above in Theorem 2.1 and let* k *be an even nonnegative integer. If* $g+h > 2k+1$*, then the Siegel-Jacobi operator* $\Psi_{g,\mathcal{M}}$ *is an isomorphism.*

Proof. See Theorem 3.6 in [28]. □

Remark 2.1. *A Jacobi form in* $J_{k,\mathcal{M}}(\Gamma_g)$ *is said to be* singular *if it admits a Fourier expansion such that a Fourier coefficient* $c(T,R)$ *vanishes unless*

$$\det\begin{pmatrix} T & \frac{1}{2}R \\ \frac{1}{2}{}^tR & \mathcal{M} \end{pmatrix} = 0.$$

Let $2\mathcal{M}$ be as above in Theorem 2.1. Yang proved that if k is an even nonnegative integer and $g + \mathrm{rank}(\mathcal{M}) > 2k$, then any non-zero Jacobi form in $J_{k,\mathcal{M}}(\Gamma_g)$ is singular (cf. [29, Theorem 4.5]*).*

Theorem 2.3. *Let $2\mathcal{M}$ be as above in Theorem 2.1 and let k be an even nonnegative integer. If $2k > 4g + h$, then the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}$ is surjective.*

Proof. See Theorem 3.7 in [28]. □

Remark 2.2. *Yang* [28, Theorem 4.2] *proved that the action of the Hecke operatos on Jacobi forms is compatible with that of the Siegel-Jacobi operator.*

Definition 2.3. *A collection $(F_g)_{g\geq 0}$ is called a* stable Jacobi form *of weight k and index $\mathcal{M}$ if it satisfies the following conditions (SJ1) and (SJ2):*

(SJ1) $F_g \in J_{k,\mathcal{M}}(\Gamma_g)$ *for all* $g \geq 0$.

(SJ2) $\Psi_{g,\mathcal{M}} F_g = F_{g-1}$ *for all* $g \geq 1$.

Remark 2.3. *The concept of a stable Jacobi forms was introduced by Yang* [31].

Example. Let S be a positive even unimodular symmetric integral matrix of degree $2k$ and let $c \in \mathbb{Z}^{(2k,h)}$ be an integral matrix. We define the theta series $\vartheta_{S,c}^{(g)}$ by

$$\vartheta_{S,c}^{(g)}(\tau,z) := \sum_{\lambda \in \mathbb{Z}^{(2k,g)}} e^{\pi i \left\{ \sigma(S\lambda\tau\, {}^t\lambda) + 2\sigma({}^t c\, S \lambda\, {}^t z) \right\}}, \quad (\tau,z) \in \mathbb{H}_{g,h}.$$

It is easily seen that $\vartheta_{S,c}^{(g)} \in J_{k,\mathcal{M}}(\Gamma_g)$ with $\mathcal{M} := \frac{1}{2}\, {}^t c S c$ for all $g \geq 0$ and $\Psi_{g,\mathcal{M}} \vartheta_{S,c}^{(g)} = \vartheta_{S,c}^{(g-1)}$ for all $g \geq 1$. Thus the collection

$$\Theta_{S,c} := \left(\vartheta_{S,c}^{(g)} \right)_{g \geq 0}$$

is a stable Jacobi form of weight k and index $\mathcal{M}$.

3. Stable Schottky-Siegel Forms

Let $\mathcal{A}_g^{\mathrm{Sat}}$ be the Satake compactification of the Siegel modular variety $\mathcal{A}_g$ (cf. [22]).

$$\mathcal{A}_g^{\mathrm{Sat}} = \mathcal{A}_g \cup \mathcal{A}_{g-1} \cup \cdots \mathcal{A}_1 \cup \mathcal{A}_0.$$

W. Baily [1] proved that $\mathcal{A}_g^{\mathrm{Sat}}$ is a normal projective variety in which $\mathcal{A}_g$ is Zariski open. In particular, we have a closed embedding

$$\iota_g : \mathcal{A}_{g-1}^{\mathrm{Sat}} \hookrightarrow \mathcal{A}_g^{\mathrm{Sat}}.$$

The collection $(\mathcal{A}_g^{\mathrm{Sat}})_{g\geq 0}$ and the above embeddings $(\iota_g)_{g\geq 0}$ define the inductive limit

$$\mathcal{A}_\infty^{\mathrm{Sat}} := \bigcup_{g\geq 0} \mathcal{A}_g^{\mathrm{Sat}} = \varinjlim_g \mathcal{A}_g^{\mathrm{Sat}}$$

which is called the *stable Satake compactification*. Let $\mathcal{L}_g$ be the determinant line bundle of the Hodge bundle over $\mathcal{A}_g$. The we have the isomorphism

$$H^0(\mathcal{A}_g, \mathcal{L}_g^{\otimes k}) \cong [\Gamma_g, k].$$

Let J_g^{Sat} (resp. Hyp_g^{Sat}) be the closure of J_g (resp. Hyp_g) inside $\mathcal{A}_g^{\text{Sat}}$. We define

$$J_\infty := \bigcup_{g\geq 0} J_g^{\text{Sat}} \quad \text{and} \quad Hyp_\infty := \bigcup_{g\geq 0} Hyp_g^{\text{Sat}}.$$

Definition 3.1. *A pair (Λ, Q) is called a quadratic form if Λ is a lattice and Q is an integer-valued bilinear symmetric form on Λ. The rank of (Λ, Q) is defined to be the rank of Λ. For $v \in \Lambda$, the integer $Q(v, v)$ is called the norm of v. A quadratic form (Λ, Q) is said to be even if $Q(v, v)$ is even for all $v \in \Lambda$. A quadratic form (Λ, Q) is said to be unimodular if* $\det(Q) = 1$.

Definition 3.2. *Let (Λ, Q) be an even unimodular positive definite quadratic form of rank m. For a positive integer g, the theta series $\theta_{Q,g}$ associated to (Λ, Q) is defined to be*

$$\theta_{Q,g}(\tau) := \sum_{x_1,\cdots,x_g\in\Lambda} \exp\left(\pi i \sum_{p,q=1}^{g} Q(x_p, x_q)\tau_{pq}\right), \qquad \tau = (\tau_{pq}) \in \mathbb{H}_g.$$

It is well known that $\theta_{Q,g}(\tau)$ is a Siegel modular form on $\mathbb{H}_g$ of weight $\frac{m}{2}$. We easily see that

$$\Phi_{g+1}(\theta_{Q,g+1}) = \theta_{Q,g} \qquad \text{for all } g \geq 0.$$

Therefore the collection of all theta series associated to (Λ, Q)

$$\Theta_Q := (\theta_{Q,g})_{g\geq 0} \tag{3.1}$$

is a stable modular form of weight $\frac{m}{2}$.

Freitag [8] proved the following theorem.

Theorem 3.1. *The ring of stable modular forms is a polynomial ring in countably many theta series Θ_Q $(\theta_{Q,g})_{g\geq 0}$ associated to irreducible positive even unimodular quadratic forms.*

Proof. See Theorem 2.5 in [8]. □

Definition 3.3. *A modular form $f \in [\Gamma_g, k]$ is called a* Schottky-Siegel form *of weight k for J_g (resp. Hyp_g) if it vanishes along J_g (resp. Hyp_g). A collection $(f_g)_{g\geq 0}$ is called a* stable Schottky-Siegel form *of weight k for the Jacobian locus (resp. the hyperelliptic locus) if $(f_g)_{g\geq 0}$ is a stable modular form of weight k and f_g vanishes along J_g (resp. Hyp_g) for every $g \geq 0$.*

G. Codogni and N. I. Shepherd-Barron [4] proved the following.

Theorem 3.2. *There do not exist stable Schottky-Siegel form for the Jacobian locus.*

Proof. See Theorem 1.3 and Corollary 1.4 in [4]. □

Remark 3.1. *Let*

$$\varphi_g(\tau) := \theta_{E_8\oplus E_8,g}(\tau) - \theta_{D_{16}^+,g}(\tau), \qquad \tau \in \mathbb{H}_g \tag{3.2}$$

be the Igusa modular form, that is, the difference of the theta series in genus g associated to the two distinct positive even unimodular quadratic forms $E_8 \oplus E_8$ and D_{16}^+ of rank 16. *We see that $\varphi_g(\tau)$ is a Siegel modular form on $\mathbb{H}_g$ of weight* 8. *Since $\Phi_g\varphi_g = \varphi_{g-1}$ for all $g \geq 1$, a collection $(\varphi_g)_{g\geq 0}$ is a stable modular form of weight* 8. *Igusa* [14, 15] *showed that the Schottky-Siegel form discovered by Schottky* [23] *is an explicit rational multiple of φ_4.*

In [14]*, he also showed that the Jacobian locus J_4 is reduced and irreducible, and so cuts out exactly J_4 in $\mathcal{A}_4$. Indeed, $\varphi_4(\tau)$ is a degree* 16 *polynomial in the Thetanullwerte of genus* 4*. On the other hand, Grushevsky and Salvati Manni* [12] *showed that the Igusa modular form φ_5 of genus* 5 *cuts out exactly the trigonal locus in J_5 and so does not vanish along J_5. Thus $(\varphi_g)_{g\geq 0}$ is not a stable Schottky-Siegel form.*

G. Codogni [3] proved the following.

Theorem 3.3. *There exist non-trivial stable Schottky-Siegel form for the hyperelliptic locus. Precisely the ideal of stable Schottky-Siegel forms for the hyperelliptic locus is generated by differences of theta series*

$$\Theta_P - \Theta_Q,$$

where P and Q are positive definite even unimodular quadratic forms of the same rank.

Proof. See Theorem 1.2 in [3]. □

Remark 3.2. *Let P and Q be two positive even unimodular quadratic forms of the same rank. We let*

$$\Theta_P := (\theta_{P,g})_{g\geq 0} \quad \text{and} \quad \Theta_Q := (\theta_{Q,g})_{g\geq 0}$$

be two stable modular forms. Codogni [3, Theorem 1.4] *showed that the difference of theta series*

$$\Theta_P - \Theta_Q$$

is a stable Schottky-Siegel form for the hyperelliptic locus when one of the following conditions (1)–(3) *is satisfied:*

(1) rank*(P)*=rank*(Q)=24 and the two quadratic forms have the same number of vectors of norm* 2*;*

(2) rank*(P)*=rank*(Q)=32 and the two quadratic forms do not have any vectors of norm* 2*;*

(3) rank*(P)*=rank*(Q)=48 and the two quadratic forms do not have any vectors of norm* 2 *and* 4*.*

4. Stable Schottky-Jacobi Forms and Proofs of Main Theorems

In this section, we introduce the notion of stable Schottky-Jacobi forms and prove the main theorems.

We let

$$\mathcal{A}_{g,h} := \Gamma^J_{g,h} \backslash \mathbb{H}_{g,h}$$

be the universal abelian variety and let

$$\mathcal{A}^{\text{Sat}}_{g,h} := \mathcal{A}_{g,h} \cup \mathcal{A}_{g-1,h} \cup \cdots \cup \mathcal{A}_{1,h} \cup \mathcal{A}_{0,h}$$

be the Satake compactification of $\mathcal{A}_{g,h}$. We consider the natural projection map

$$\pi_{g,h} : \mathcal{A}_{g,h} \longrightarrow \mathcal{A}_g$$

of $\mathcal{A}_{g,h}$ onto $\mathcal{A}_g$. Let

$$J_{g,h} := \pi^{-1}_{g,h}(J_g)$$

be the universal Jacobian locus and let

$$Hyp_{g,h} := \pi_{g,h}^{-1}(Hyp_g)$$

be the universal hyperelliptic locus. Let $J_{g,h}^S$ (resp. $Hyp_{g,h}^S$) be the closure of $J_{g,h}$ (resp. $Hyp_{g,h}$) in $\mathcal{A}_{g,h}^{\mathrm{Sat}}$. We put

$$\mathcal{A}_{\infty,h} := \bigcup_{g\geq 0} \mathcal{A}_{g,h}^{\mathrm{Sat}},$$

$$J_{\infty,h} := \bigcup_{g\geq 0} J_{g,h}^{\mathrm{S}}$$

and

$$Hyp_{\infty,h} := \bigcup_{g\geq 0} Hyp_{g,h}^{\mathrm{S}}.$$

Definition 4.1. *Let $\mathcal{M}$ be a half-integral semi-positive symmetric matrix of degree h and $k \in \mathbb{Z}_+$. A Jacobi form $F \in J_{k,\mathcal{M}}(\Gamma_g)$ is called a* Schottky-Jacobi form *of weight k and index $\mathcal{M}$ for the universal Jacobian (resp. hyperelliptic) locus if it vanishes along $J_{g,h}$ (resp. $Hyp_{g,h}$).*

Definition 4.2. *Let $\mathcal{M}$ be a half-integral semi-positive symmetric matrix of degree h and $k \in \mathbb{Z}_+$. A collection $(F_g)_{g\geq 0}$ is called a* stable Schottky-Jacobi form *of weight k and index $\mathcal{M}$ for the universal Jacobian (resp. hyperelliptic) locus if it satisfies the following conditions* (SSJ1) *and* (SSJ2)*:*

(SSJ1) $F_g \in J_{k,\mathcal{M}}(\Gamma_g)$ *is a Schottky-Jacobi form of weight k and index $\mathcal{M}$ for the universal Jacobian locus $J_{g,h}$ (resp. the universal hyperelliptic locus $Hyp_{g,h}$) for all $g \geq 0$.*

(SSJ2) $\Psi_{g,\mathcal{M}} F_g = F_{g-1}$ *for all $g \geq 1$.*

Theorem 4.1. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h. Then there do not exist stable Schottky-Jacobi forms of index $\mathcal{M}$ for the universal Jacobian locus.*

Proof. We first observe that $h \equiv 0 \pmod 8$ (cf. [24]). Assume that there exists a non-trivial stable Schottky-Jacobi form $(F_g)_{g\geq 0}$ of weight k and index $\mathcal{M}$ for the universal Jacobian locus.

Case 1: k is even.

Using the Shimura isomorphsm (cf. [25, 26]), we obtain the following

$$J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k_*] \cdot \vartheta_{2\mathcal{M}}^{[g]}(\tau, z), \tag{4.1}$$

where $k_* := k - \frac{h}{2}$ and

$$\vartheta_{2\mathcal{M}}^{[g]}(\tau, z) := \sum_{\lambda \in \mathbb{Z}^{(h,g)}} e^{2\pi i\, \sigma\left(\mathcal{M}(\lambda \tau\, {}^t\lambda + 2\lambda\, {}^t z)\right)}. \tag{4.2}$$

We refer to [34, Theorem 3.3] for the proof of the formula (4.1). We see from (SO1) in Section 2 that $[\Gamma_g, k_*] = 0$ if $g + h > 2k$ and $k \not\equiv 0 \pmod 4$. So $k \equiv 0 \pmod 4$. We observe that the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}} : J_{k,\mathcal{M}}(\Gamma_g) \longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$ is an isomorphism if $g + h > 2k + 1$ (see Theorem 2.2 in Section 2). It is easy to see that

$$\Psi_{g,\mathcal{M}} \vartheta_{2\mathcal{M}}^{[g]} = \vartheta_{2\mathcal{M}}^{[g-1]} \quad \text{for all } g \geq 1.$$

According to the formula (4.1), we may write

$$F_g(\tau,z)=f_g(\tau)\cdot\vartheta^{[g]}_{2\mathcal{M}}(\tau,z),\quad f\in[\Gamma,k_*].$$

Now we have, for $(\tau,z)\in\mathbb{H}_{g-1,h}$,

$$\begin{aligned}\left(\Psi_{g,\mathcal{M}}F_g\right)(\tau,z) &= \lim_{t\longrightarrow\infty}F_g\left(\begin{pmatrix}\tau & 0\\ 0 & it\end{pmatrix},(z,0)\right)\\ &= \lim_{t\longrightarrow\infty}f_g\begin{pmatrix}\tau & 0\\ 0 & it\end{pmatrix}\cdot\vartheta^{[g]}_{2\mathcal{M}}\left(\begin{pmatrix}\tau & 0\\ 0 & it\end{pmatrix},(z,0)\right)\\ &= \left(\Phi_g f_g\right)(\tau)\cdot\vartheta^{[g-1]}_{2\mathcal{M}}(\tau,z).\end{aligned}$$

Here Φ_g is the Siegel Φ-operator defined by (2.1).

On the other hand, by the assumption that $(F_g)_{g\geq 0}$ is a stable Schottky-Jacobi form, we have

$$\Psi_{g,\mathcal{M}}F_g=F_{g-1}=f_{g-1}\cdot\vartheta^{[g-1]}_{2\mathcal{M}}\quad\text{for some } f_{g-1}\in[\Gamma_{g-1},k]$$

for all $g\geq 1$. Therefore

$$\Phi_g f_g=f_{g-1}\quad\text{for all } g\geq 1.$$

Obviously f_g vanishes along J_g for all $g\geq 0$. Thus $(f_g)_{g\geq 0}$ is a non-trivial stable Schottky-Siegel form of weight k_*. This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus (see Theorem 3.2).

Case 2: k is odd.

Using the Shimura isomorphism, we may write

$$F_g(\tau,z)=\psi_g(\tau)\cdot\vartheta^{[g]}_{2\mathcal{M}}(\tau,z)\qquad\text{for all } g\geq 1,$$

where $\vartheta^{[g]}_{2\mathcal{M}}(\tau,z)$ is the theta series defined by Formula (4.2) and $f_g(\tau)$ satisfies the following behaviours (4.3) and (4.4):

$$\psi_g(\tau+S)=\psi_g(\tau)\qquad\text{for all } S={}^tS\in\mathbb{Z}^{(g,g)};\tag{4.3}$$

$$\psi_g(-\tau^{-1})=\det(-\tau)^k\det\left(\frac{\tau}{i}\right)^{-\frac{h}{2}}\psi_g(\tau),\quad\tau\in\mathbb{H}_g.\tag{4.4}$$

We put

$$\xi_g(\tau):=\{\psi_g(\tau)\}^2\qquad\text{for all } g\geq 1.$$

Then we see easily that a collection $\xi=(\xi_g)_{g\geq 0}$ is a non-trivial stable Schottky-Siegel form of weight $2k-h$ for the Jacobian locus. This contradicts the non-existence of a non-trivial stable Schottky-Siegel form for the Jacobian locus. Hence we complete the proof of the above theorem(=Theorem 1.1). □

Theorem 4.2. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h. Then there exist non-trivial stable Schottky-Jacobi forms of $\mathcal{M}$ for the universal hyperelliptic locus.*

Proof. According to Theorem 3.3, there exists a non-tivial stable Schottky-Siegel form $(f_g)_{g\geq 0}$ of weight k for the hyperelliptic locus. We see from (SO1) in Section 2 that $k\equiv 0\,(\mathrm{mod}\ 4)$ and $k\in\mathbb{Z}^+$. We put $\ell := k+\frac{h}{2}$. Then using the Shimura isomorphism, we have

$$J_{\ell,\mathcal{M}}(\Gamma_g)=[\Gamma_g,k]\cdot\vartheta^{[g]}_{2\mathcal{M}}(\tau,z),$$

where $\vartheta^{[g]}_{2\mathcal{M}}(\tau,z)$ is the theta series defined by Formula (4.2). We define the Jacobi forms

$$F_g(\tau,z):=f_g(\tau)\cdot\vartheta^{[g]}_{2\mathcal{M}}(\tau,z),\quad g\geq 0.$$

Since $f_g\in[\Gamma_g,k]$ is a Jacobi form of weight k and index 0, we get $F_g\in J_{\ell,\mathcal{M}}(\Gamma_g)$ for all $g\geq 0$. For $[(\tau,z])\in Hyp_{g,h}$,

$$F_g(\tau,z)=0,\quad g\geq 0.$$

By a simple calculation, we obtain

$$\Psi_{g,\mathcal{M}}F_g=F_{g-1}\qquad\text{for all } g\geq 1.$$

Thus $(F_g)_{g\geq 0}$ is a non-trivial stable Schottky-Jacobi form of weight ℓ and index $\mathcal{M}$ for the universal hyperelliptic locus. This completes the proof of the above theorem(=Theorem 1.2). □

We define the invariant $\mu(Q)$ of a quadratic form (Q,Λ) by

$$\mu(Q):=\min\{Q(v,v)\mid v\in\Lambda, v\neq 0\}.$$

Theorem 4.3. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h. Let (Q,Λ) and (P,Γ) be two positive even unimodular quadratic forms of rank m. Assume that*

$$\frac{m}{\mu}\leq 8,\qquad \textit{where } \mu:=\min\{\mu(Q),\mu(P)\}.$$

We put

$$F_g(\tau,z):=\{\theta_{Q,g}(\tau)-\theta_{P,g}(\tau)\}\cdot\vartheta^{(g)}_{2\mathcal{M}}(\tau,z),\quad g\geq 0.$$

Then $(F_g)_{g\geq 0}$ is a stable Schottky-Jacobi form of weight $\frac{1}{2}(m+h)$ and index $\mathcal{M}$ for the universal hyperelliptic locus.

Proof. It is easily seen that

$$(4.5)\qquad \theta_{Q,g},\,\theta_{P,g}\in J_{\frac{m}{2},0}(\Gamma_g)\quad\text{and}\quad \theta_{Q,g}\cdot\vartheta^{[g]}_{2\mathcal{M}},\ \theta_{P,g}\cdot\vartheta^{[g]}_{2\mathcal{M}}\in J_{\frac{1}{2}(m+h),\mathcal{M}}(\Gamma_g)$$

for all $g\geq 0$. The proof follows immediately from Theorem 5.5 in [3] and the above facts (4.5). □

5. **Final Remarks**

In the final section, we make some remarks and present several open questions.

Remark 5.1. *Let $2\mathcal{M}$ be a positive even unimodular symmetric integral matrix of degree h. Assume that*

$$g+\mathrm{rank}(\mathcal{M})>2k+1\quad\text{and } k\in\mathbb{Z}_+ \text{ is even.}$$

We denote by $\mathcal{C}_{k,\mathcal{M}}$ be the vector space of stable Jacobi forms of weight k and index $\mathcal{M}$. According to Theorem 2.2, the Siegel-Jacobi operator $\Psi_{g,\mathcal{M}}:J_{k,\mathcal{M}}(\Gamma_g)\longrightarrow J_{k,\mathcal{M}}(\Gamma_{g-1})$ is an isomorphism, and hence we obtain

$$\dim\mathcal{C}_{k,\mathcal{M}}=\dim J_{k,\mathcal{M}}(\Gamma_g).$$

From Formula (4.1), we see that

$$J_{k,\mathcal{M}}(\Gamma_g) = [\Gamma_g, k_*] \cdot \vartheta^{[g]}_{2\mathcal{M}}(\tau, z), \qquad \text{where } k_* := k - \frac{h}{2}.$$

Therefore from (SO1) in Section 2, we get the vanishing result:

$$J_{k,\mathcal{M}}(\Gamma_g) = 0 \qquad \text{if } 2k \not\equiv h \pmod 8.$$

Thus $k \equiv 0 \pmod 4$ *if* $J_{k,\mathcal{M}}(\Gamma_g) \neq 0$. *According to Yang* [28], *any Jacobi form in* $J_{k,\mathcal{M}}(\Gamma_g)$ *is* singular. *We note that any element in* $[\Gamma_g, k - \frac{h}{2}]$ *is a singular modular form (see* [8, 10]*). Hence we conclude that* $\mathcal{C}_{k,\mathcal{M}}$ *is spanned by stable Jacobi forms of the form*

$$\Big(\theta_{P,g}(\tau)\vartheta^{[g]}_{2\mathcal{M}}(\tau, z)\Big)_{g\geq 0},$$

where P *runs over the set of positive even unimodular quadratic forms of rank* $2k - h$.

Remark 5.2. *Let* $\varphi_g(\tau)$ *be the Igusa modular form defined by the formula (3.2). We denote by* $[\Gamma_g, k]_0$ *be the space of all Siegel cuspidal Hecke eigenforms on* $\mathbb{H}_g$ *of weight* k. *It is known that* $[\Gamma_4, 8]_0 = \mathbb{C} \cdot \varphi_4$ *(for a nice proof of this, we refer to* [5]*). Poor* [21] *showed that* $\varphi_g(\tau)$ *vanishes on the hyperelliptic locus* Hyp_g *for all* $g \geq 1$, *and the divisor of* $\varphi_g(\tau)$ *in* $\mathcal{A}_g$ *is proper and irreducible for all* $g \geq 4$. *And Ikeda* [16, 17] *proved that if* $g \equiv k \pmod 2$, *there exists a canonical lifting*

$$I_{g,k} : [\Gamma_1, 2k]_0 \longrightarrow [\Gamma_{2g}, g + k]_0.$$

Considering the special cases of the Ikeda lift maps $I_{2,6}$ *and* $I_{6,6}$, *Breulman and Kuss* [2] *showed that*

$$I_{2,6}(\Delta) = c\,\varphi_4, \quad c(\neq 0) \in \mathbb{C},$$

and constructed a nonzero Siegel cusp form of degree 12 *and weight* 12 *which is the image of* $\Delta(\tau)$, *where*

$$\Delta(\tau) = (2\pi i)^{12}\ q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q := e^{2\pi i\tau}, \quad \tau \in \mathbb{H}_1$$

is a cusp form of weight 12. We refer to [20] *for some results related to the Igusa modular form* $\varphi_4(\tau)$.

Remark 5.3. *We consider a half-integral semi-positive symmetric integral matrix* $\mathcal{M}$ *such that* $2\mathcal{M}$ *is not unimodular. The natural questions arise:*

Question 1. *Are there non-trivial stable Schottky-Jacobi forms of index* $\mathcal{M}$ *for the universal Jacobian locus ?*

Question 2. *Are there non-trivial stable Schottky-Jacobi forms of index* $\mathcal{M}$ *for the universal hyperelliptic locus ?*

Remark 5.4. *Let* $2\mathcal{M}$ *be a positive even unimodular symmetric integral matrix of degree* h. *For a nonnegative integer* k *and a positive integer* ℓ, *we let* $A_{k,\ell\mathcal{M}}$ *be the vector space of stable Jacobi forms of weight* k *and index* $\ell\mathcal{M}$. *We put*

$$A(\mathcal{M}) := \bigoplus_{\ell=0}^{\infty}\bigoplus_{k=0}^{\infty} A_{k,\ell\mathcal{M}}.$$

Then we see easily that

$$A_{k,\ell\mathcal{M}} \bullet A_{p,q\mathcal{M}} \subset A_{k+p,(\ell+q)\mathcal{M}}$$

with respect to the natural multiplication $\bullet$. *Thus* $A(\mathcal{M})$ *is a bi-graded ring. Let* $I(\mathcal{M})$ *be the space of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in* $A(\mathcal{M})$. *Then* $I(M)$ *is an ideal of* $A(\mathcal{M})$.

According to Theorem 3.1, the subset

$$A(\mathcal{M})_0 := \bigoplus_{k=0}^{\infty} A_{k,0}$$

of $A(\mathcal{M})$ *has a polynomial ring structure.*

Let

$$A^{[4]}(\mathcal{M})_1 := \bigoplus_{k\equiv 0 \,(\mathrm{mod}\, 4)} A_{k,\mathcal{M}}$$

and let $B^{[4]}(\mathcal{M})_1$ *be the subspace of all stable Schottky-Jacobi forms for the universal hyperelliptic locus contained in* $A^{[4]}(\mathcal{M})_1$. *Using Theorem 4.2 in* [3], *we can show that*

$$(\Theta_P - \Theta_Q)\ \Theta_{2\mathcal{M}} \tag{5.1}$$

is a stable Schottky-Jacobi form for the universal hyperelliptic locus of weight $\frac{1}{2}(m+h)$ *and index* $\mathcal{M}$, *that is,* $(\Theta_P - \Theta_Q)\ \Theta_{2\mathcal{M}} \in B^{[4]}(\mathcal{M})_1$. *Here* P *and* Q *are two positive even unimodular quadratic forms of the same rank* m $(m \in \mathbb{Z}^+)$, *and* Θ_P, Θ_Q *are stable modular forms that are defined in Formula (3.1). The subspace* $B^{[4]}(\mathcal{M})_1$ *of* $A^{[4]}(\mathcal{M})_1$ *is spanned by all the stable Jacobi forms of type (5.1), where* m *runs over the set of all positive integers* $8n\,(n \in \mathbb{Z}^+)$.

Question 3. *What kinds of structures does* $A(\mathcal{M})$ *have ?*

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Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Present address: Department of Mathematics, Inha University, Incheon 22212, Korea
E-mail address: jhyang@inha.ac.kr