ON THE GEOMETRY OF COHOMOGENEITY ONE MANIFOLDS WITH POSITIVE CURVATURE

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There are very few known examples of manifolds with positive sectional curvature. Apart from the compact rank one symmetric spaces, they exist only in dimensions 24 and below and are all obtained as quotients of a compact Lie group equipped with a biinvariant metric under an isometric group action. They consist of certain homogeneous spaces in dimensions $6, 7, 12, 13$ and 24 due to Berger [Be], Wallach [Wa], and Aloff-Wallach [AW], and of biquotients in dimensions $6, 7$ and 13 due to Eschenburg [E1],[E2] and Bazaikin [Ba].

When trying to find new examples, it is natural to search among manifolds with large isometry group, a program initiated by K.Grove in the 90's, see [Wi] for a recent survey. Homogeneous spaces with positive curvature were classified in [Wa],[BB] in the 70's. The next natural case to study is therefore manifolds on which a group acts isometrically with one dimensional quotient, so called cohomogeneity one manifolds. L.Verdiani [V1, V2] showed that in even dimensions, positively curved cohomogeneity one manifolds are equivariantly diffeomorphic to an isometric action on a rank one symmetric space. In odd dimensions K.Grove and the author observed in 1998 that there are infinite families among the known non-symmetric positively curved manifolds which admit isometric cohomogeneity one actions, and suggested a family of potential 7 dimensional candidates P_k. In [GWZ] a classification in odd dimensions was carried out and another family Q_k and an isolated manifold R emerged in dimension 7. It is not yet known whether these manifolds admit a cohomogeneity one metric with positive curvature, although they all admit one with non-negative curvature as a consequence of the main result in [GZ].

In [GWZ] the authors also discovered an intriguing connection that the manifolds P_k and Q_k have with a family of self dual Einstein orbifold metrics constructed by Hitchin [Hi1] on $\mathbb{S}^4$. They naturally give rise to 3-Sasakian metrics on P_k and Q_k, which by definition have lots of positive curvature already.

The purpose of this survey is three fold. In Section 2 we study the positively curved cohomogeneity one metrics on known examples with positive curvature including the explicit functions that define the metric. In Section 3 we describe the classification theorem in [GWZ]. It is remarkable that among 7-manifolds where $G = \mathrm{S}^3 \times \mathrm{S}^3$ acts by cohomogeneity one, one has the known positively curved Eschenburg spaces E_p, the Berger space B^7, the Aloff-Wallach space W^7, and the sphere $\mathbb{S}^7$, and that the candidates P_k, Q_k and R all carry such an action as well. We thus carry out the proof in this most intriguing case where $G = \mathrm{S}^3 \times \mathrm{S}^3$ acts by cohomogeneity one on a compact 7-dimensional simply connected manifold. In Section 4 we describe the relationship to Hitchin's self dual Einstein metrics. We also discuss some curvature properties of these Einstein metrics and the

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metrics they define on P_k and Q_k. The behavior of these metrics, as well as the known metrics with positive curvature, are illustrated in a series of pictures.

1. Preliminaries

In this section we discuss the basic structure of cohomogeneity one actions and the significance of the Weyl group. For more details we refer the reader to [AA, Br, GZ, Mo]. We assume from now on that the manifold M and the group G that acts on M are compact and will only consider the most interesting case, where $M/G = I = [0, L]$. If $\pi\colon M \to M/G$ is the orbit projection, the inverse images of the interior points are the regular orbits and $B_- = \pi^{-1}(0)$ and $B_+ = \pi^{-1}(L)$ are the two non-regular orbits. Choose a point $x_- \in B_-$ and let $\gamma : [0, L] \to M$ be a minimal geodesic from B_- to B_+, parameterized by arc length, which we can assume starts at x_-. The geodesic is orthogonal to B_- and hence to all orbits. Define $x_+ = \gamma(L) \in B_+$, $x_0 = \gamma(\frac{L}{2})$ and let $K^\pm = G_{x_\pm}$ be the isotropy groups at $x_\pm$ and $H = G_{x_0} = G_{\gamma(t)}$, $0 < t < L$, the principal isotropy group. Thus $B_\pm = G \cdot x_\pm = G/K^\pm$ and $G \cdot \gamma(t) = G/H$ for $0 < t < L$. For simplicity we denote the tangent space of $B_\pm$ at $x_\pm$ by $T_\pm$ and its normal space by $T_\pm^\perp$.

By the slice theorem, we have the following description of the tubular neighborhoods $D(B_-) = \pi^{-1}([0, \frac{L}{2}])$ and $D(B_+) = \pi^{-1}([\frac{L}{2}, L])$ of the nonprincipal orbits:

$$D(B_\pm) = G \times_{K_\pm} D^{\ell_\pm},$$

where $D^{\ell_\pm}$ are disks of radius $\frac{L}{2}$ in $T_\pm^\perp$. Here the action of $K_\pm$ on $G \times D^{\ell_\pm}$ is given by $k \star (g, p) = (gk^{-1}, kp)$ where k acts on $D^{\ell_\pm}$ via the slice representation, i.e., the restriction of the isotropy representation to $T_\pm^\perp$. Hence we have the decomposition

$$M = D(B_-) \cup_E D(B_+) \ ,$$

where $E = G \cdot x_0 = G/H$ is a principal orbit which is canonically identified with the boundaries $\partial D(B_\pm) = G \times_{K_\pm} \mathbb{S}^{\ell_\pm - 1}$, via the maps $g \cdot H \to [g, \dot\gamma(0)]$ respectively $g \cdot H \to [g, -\dot\gamma(L)]$. Note also that $\partial D^{\ell_\pm} = \mathbb{S}^{\ell_\pm - 1} = K^\pm/H$ since the boundary of the tubular neighborhoods must be a G orbit and hence $\partial D^{\ell_\pm}$ is a $K^\pm$ orbit.

All in all we see that we can recover M from G and the subgroups H and $K^\pm$. We caution though that the isotropy types, i.e., the conjugacy classes of the isotropy groups $K^\pm$ and H do not determine M. The isotropy groups depend on the choice of a minimal geodesic between the two non-regular orbits, and thus on the metric as well. A different choice of a minimal geodesic corresponds to conjugating all isotropy groups by an element of G. A change of the metric corresponds to changing K^+ to nK^+n^{-1} for some $n \in N(H)_0$, the identity component of the normalizer (cf. [AA, GWZ]).

An important fact about cohomogeneity one actions is that there is a converse to the above construction. Suppose G is a compact Lie group with subgroups $H \subset K^\pm \subset G$ and assume furthermore that $K^\pm/H = \mathbb{S}^{\ell_\pm - 1}$ are spheres. We sometimes denote this situation by $H \subset \{K^-, K^+\} \subset G$ and call it a group diagram. It is well known that a transitive action of a compact Lie group K on a sphere $\mathbb{S}^{\ell-1}$ is conjugate to a linear action. We can thus assume that $K^\pm$ acts linearly on $\mathbb{S}^{\ell_\pm}$ with isotropy group the chosen subgroup

$H \subset K^{\pm}$ at some point $p_{\pm} \in \mathbb{S}^{\ell_{\pm}-1}$. It hence extends to a linear action on the bounding disk $D^{\ell_{\pm}}$ and we can thus define a manifold

$$M = G \times_{K^-} D^{\ell_-} \cup_{G/H} G \times_{K^+} D^{\ell_+},$$

where we glue the two boundaries by sending $[g, p_-]$ to $[g, p_+]$. The group G acts on M via $g \star [g', p] = [gg', p]$ on each half and one easily checks that the gluing is G-equivariant, and that the action has isotropy groups $K^{\pm}$ at $[e, 0]$ and H at $[e, p_{\pm}]$. One may also choose an equivariant map $G/H \to G/H$ to glue the two boundaries together. But such equivariant maps are given by $gH \to gnH$ for some $n \in N(H)$ and the new manifold is alternatively obtained by replacing K^+ with nK^+n^{-1} in the group diagram. But we caution that this new manifold may not be equivariantly diffeomorphic to the old one if n does not lie in the identity component of $N(H)$.

Another important ingredient for understanding the geometry of a cohomogeneity one manifold is given by the Weyl group. The Weyl group W of the action is by definition the stabilizer of the geodesic γ modulo its kernel, which by construction is equal to H. It is easy to see (cf. [AA]) that W is a dihedral subgroup of $N(H)/H$ with $M/G = \mathrm{Im}(\gamma)/W$, and is generated by involutions $w_{\pm} \in W$ with $w_-(\gamma(t)) = \gamma(-t)$ and $w_+(\gamma(-t+L)) = \gamma(t+L)$. Thus w_+w_- is a translation by $2L$, and has order $|W|/2$ when W is finite. The involutions $w_{\pm}$ can be represented by the unique element $a \in K_0^{\pm}$ mod H with $av = -v$, where $K_v^{\pm} = H$. If $\ell_{\pm} = 1$, they are also the unique element $a \in K_0^{\pm}$ mod H such that a^2 but not a lies in H. For simplicity we denote such representatives $a \in K_0^{\pm}$ again by $w_{\pm}$.

Note that W is finite if and only if γ is a closed geodesic, and in that case the order $|W|$ is the number of minimal geodesic segments intersecting the regular part. In [GWZ] it was shown that a cohomogeneity one manifold with an invariant metric of positive curvature necessarily has finite Weyl group. Note also that any non-principal isotropy group along γ is of the form $wK^{\pm}w^{-1}$ for some $w \in N(H)$ representing an element of W, and that the isotropy types $K^{\pm}$ alternate along γ.

We now discuss how to describe cohomogeneity one metrics on M. For $0 < t < L$, $\gamma(t)$ is a regular point with constant isotropy group H and the metric on the principal orbits $G\gamma(t) = G/H$ is a family of homogeneous metrics g_t. Thus on the regular part the metric is determined by

$$g_{\gamma(t)} = d\,t^2 + g_t,$$

and since the regular points are dense it also describes the metric on M. Using a fixed biinvariant inner product Q on $\mathfrak{g}$ we define the Q-orthogonal splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ which thus satisfies $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. We identify the tangent space to G/H at $\gamma(t), t \in (0, L)$ with $\mathfrak{m}$ via action fields: $X \in \mathfrak{m} \to X^*(\gamma(t))$, which also identifies the isotropy representation with the action of $\mathrm{Ad}(H)_{|\mathfrak{m}}$. We can choose a Q-orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$ of $\mathfrak{m}$ into $\mathrm{Ad}(H)$ invariant irreducible subspaces and thus $g_t|\mathfrak{m}_i = f_i(t)Q|\mathfrak{m}_i$ for some functions $f_1, \ldots, f_k$. If the modules $\mathfrak{m}_i$ are inequivalent to each other, they are automatically orthogonal and the functions f_i describe the metric completely. In positive curvature, it typically happens, as we will see, that the modules are not orthogonal to each other and further functions are necessary to describe their inner products. In order for the metric on M to be smooth, these functions must satisfy certain smoothness conditions at the endpoints $t = 0$ and $t = L$, which in general can be complicated. The action of $w_{\pm}$ on $T_{x_{\pm}}M$ (well determined only up to $\mathrm{Ad}(H)$), preserves $T_{\pm}$ and $T_{\pm}^{\perp}$ and the action on

$T_\pm$, given by $\mathrm{Ad}(w_\pm)$, relates the functions describing the metric: $(\mathrm{Ad}(w_-)(X))^*(\gamma(t)) = X^*(\gamma(-t))$. If, e.g., $\mathrm{Ad}(w_-)(\mathfrak{m}_i) \subset \mathfrak{m}_i$, the function f_i must be even, and if $\mathrm{Ad}(w_-)(\mathfrak{m}_i) \subset \mathfrak{m}_j$, then $f_i(t) = f_j(-t)$. In fact most, but not all, of the smoothness conditions at the endpoints can be explained in this fashion by the action of the Weyl group.

2. Known examples of cohomogeneity one manifolds with positive curvature

In this section we describe the cohomogeneity one actions on the known cohomogeneity one manifolds with positive curvature which were discovered by K.Grove and the author in 1998. Apart from a cohomogeneity one action by $\mathrm{SU}(4)$ on the infinite family of 13 dimensional Bazaikin spaces B^{13}_p, which we will not discuss in this survey, they are all cohomogeneity one under an action of $\mathrm{S}^3 \times \mathrm{S}^3$ or one of its finite quotients. We start with the well known action of $\mathrm{SO}(3)$ on $\mathbb{S}^4$ and $\mathrm{SO}(3)$ on $\mathbb{CP}^2$ since they are important in understanding the remaining examples and determine much of their geometry. We then study the action of $\mathrm{SO}(4)$ on $\mathbb{S}^7$, and of $\mathrm{SO}(4)$ on the Berger space $B^7 = \mathrm{SO}(5)/\,\mathrm{SO}(3)$. This latter action was also discovered by Verdiani-Podesta in [PV2]. Of a different nature is the action of $\mathrm{SU}(2) \times \mathrm{SO}(3)$ on the infinite family of Eschenburg biquotients E_p, which contains as a special case a homogeneous Aloff-Wallach space. We finish with a second cohomogeneity one action on the same Aloff-Wallach space which shares some features of both actions. To distinguish them, we denote the first one by $W^7_{(1)}$ and the second by $W^7_{(2)}$. For a survey of the known examples of positive curvature, see [Zi2].

All actions described here are by groups locally isomorphic to S^3 or $\mathrm{S}^3 \times \mathrm{S}^3$. For comparison we will describe them ineffectively so that $G = \mathrm{S}^3$ or $G = \mathrm{S}^3 \times \mathrm{S}^3$ acts on the manifold. The effective version of the action, which we denote by $\bar{G}$, is obtained by dividing G by the ineffective kernel, which is the intersection of the center of G with the principal isotropy group H.

$$M = \mathbb{S}^4 \text{ or } \mathbb{CP}^2 \quad \text{with} \quad \bar{G} = \mathrm{SO}(3).$$

We begin by describing the well known cohomogeneity one action by $\mathrm{SO}(3)$ on $\mathbb{S}^4$. Let V be the 5-dimensional vector space of real 3×3 matrices with $A = A^t$, $\mathrm{tr}(A) = 0$ and with inner product $\langle A, B\rangle = \mathrm{tr}\, AB$. The group $\mathrm{SO}(3)$ acts on V via conjugation $g \cdot A = gAg^{-1}$ preserving the inner product and hence acts on $\mathbb{S}^4(1) \subset V$. Every point in $\mathbb{S}^4(1)$ is conjugate to a matrix in the great circle $F = \{\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \mid \sum \lambda_i = 0, \sum \lambda_i^2 = 1\}$ and hence the quotient space is one dimensional. For the purpose of computations, we choose an orthonormal basis

$$e_1 = \frac{\mathrm{diag}(1,1,-2)}{\sqrt{6}}\ ,\ e_2 = \frac{\mathrm{diag}(1,-1,0)}{\sqrt{2}}\ ,\ e_3 = \frac{S_{12}}{\sqrt{2}}\ ,\ e_4 = \frac{S_{13}}{\sqrt{2}}\ ,\ e_5 = \frac{S_{23}}{\sqrt{2}}, \tag{2.1}$$

where S_{ij} is a symmetric matrix with a one in entries ij and ji and 0 everywhere else. If we choose $x_- = e_1$, then clearly $K^- = \mathrm{S}(\mathrm{O}(2)\,\mathrm{O}(1))$ and the orbit G/K^- is the set of all symmetric matrices with 2 equal positive eigenvalues. Furthermore, $T_- = \mathrm{span}(e_4, e_5)$ with $T_-^\perp = \mathrm{span}(e_2, e_3)$ and thus $\gamma(t) = \cos(t)e_1 + \sin(t)e_2$ is a geodesic orthogonal to B_- and hence to all orbits. Clearly $x_+ = \gamma(\pi/3) = \mathrm{diag}(2,-1,-1)/\sqrt{6}$ is the first point along the geodesic γ which lies on the second singular orbit, consisting of the set of all symmetric matrices with 2 equal negative eigenvalues. Thus $L = \pi/3$ and $K^+ = \mathrm{S}(\mathrm{O}(1)\,\mathrm{O}(2))$. For $\gamma(t)$ with $0 < t < \frac{\pi}{3}$, all eigenvalues λ_i are distinct and hence the principal isotropy group is $H = \mathrm{S}(\mathrm{O}(1)\mathrm{O}(1)\mathrm{O}(1)) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

If we denote by E_{ij} the usual basis of the set of skew symmetric matrices, the above action of SO(3) on $\mathbb{S}^4(1)$ induces 3 action fields E_{ij}^*. A computation shows that:

$$E_{12}^* = 2\sin(t)e_3\ ,\ E_{23}^* = (\sqrt{3}\cos(t) - \sin(t))e_5\ ,\ E_{13}^* = (\sqrt{3}\cos(t) + \sin(t))e_4.$$

For the functions $f_1 = |E_{12}^*|^2,\ f_2 = |E_{23}^*|^2,\ f_3 = |E_{13}^*|^2$, which describe the metric, we thus obtain:

$$f_1 = 4\sin^2(t),\ f_2 = (\sqrt{3}\cos(t) - \sin(t))^2,\ f_3 = (\sqrt{3}\cos(t) + \sin(t))^2,$$

and all other inner products are 0.

For later purposes it will be convenient to lift the isotropy groups into S^3 under the two-fold cover $\mathrm{S}^3 = \mathrm{Sp}(1) \to \mathrm{SO}(3)$, given by conjugation on $\mathbb{H}$, which sends $q \in \mathrm{Sp}(1)$ into a rotation in the 2-plane $\mathrm{Im}(q)^\perp \subset \mathrm{Im}(\mathbb{H})$ with angle 2θ, where θ is the angle between q and 1 in S^3. After renumbering the coordinates, the group K^- lifts to $\mathrm{Pin}(2) = \{e^{i\theta} \mid \theta \in \mathbb{R}\} \cup \{je^{i\theta} \mid \theta \in \mathbb{R}\}$ which we abbreviate to $e^{i\theta} \cup je^{i\theta}$. Similarly, K^+ lifts to $\mathrm{Pin}(2) = e^{j\theta} \cup ie^{j\theta}$, and $H = \mathrm{S}(\mathrm{O}(1)\mathrm{O}(1)\mathrm{O}(1)) \subset \mathrm{SO}(3)$ lifts to the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. Thus the cohomogeneity one manifold $\mathbb{S}^4$ can also be represented by the group diagram

$$Q \subset \{e^{i\theta} \cup je^{i\theta}, e^{j\theta} \cup ie^{j\theta}\} \subset \mathrm{S}^3\,.$$

We will now discuss the Weyl symmetry using this group picture. Clearly $w_- = e^{i\frac{\pi}{4}}$ since $e^{i\frac{\pi}{2}} = i$ lies in H and similarly $w_+ = e^{j\frac{\pi}{4}}$. Thus $a = w_+ w_- = \frac{1}{2}(1 + i + j + k)$ represents a translation by $2L$ along the geodesic and since $a^3 = -1 \in H$, the Weyl group is $W = D_3$. This is consistent with the fact that the angle between x_- and x_+ is $\pi/3$ and hence $\gamma(t)$ intersects the regular part in $6 = |W|$ components. Notice now that $aia^{-1} = j, aja^{-1} = k$, and $aka^{-1} = i$. Thus $f_1(t + 2L) = f_2(t)$ and $f_1(t + 4L) = f_3(t)$. By applying only w_+ at $t = L$, we also obtain $f_3(t) = f_1(-t + 2L)$. Thus the function $f_1(t)$ on the interval $[0, 3L]$ determines the full geometry of the cohomogeneity one manifold:

$$f_2(t) = f_1(t + 2L)\ ,\ f_3(t) = f_1(t + 4L) = f_1(-t + 2L) \quad ,\quad 0 < t < L \tag{2.2}$$

There is a related cohomogeneity one action by $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ on $\mathbb{CP}^2$ which has singular orbits the real points $B_- = \mathbb{RP}^2 \subset \mathbb{CP}^2$ and the quadric $B_+ = \mathbb{S}^2 = \{[z_0, z_1, z_2] \mid \sum z_i^2 = 0\}$. Here we use the metric on $\mathbb{CP}^2$ induced by the biinvariant metric $\langle A, B\rangle = -\frac{1}{2}\operatorname{Re}\operatorname{tr} AB$ on SU(3), which has curvature $1 \le \sec \le 4$. One easily shows that the unit speed geodesic $\gamma(t)$, given in homogeneous coordinates by $[(\cos(t), i\sin(t), 0)]$, is orthogonal to all orbits and that the isotropy group at $x_- = \gamma(0)$ is $K^- = \mathrm{S}(\mathrm{O}(1)\mathrm{O}(2))$, at $x_+ = \gamma(\pi/4)$ is $K^+ = \mathrm{SO}(2)$, embedded in the first two coordinates, and that $H = G_{\gamma(t)} = \mathbb{Z}_2 = \langle \mathrm{diag}(-1, -1, 1)\rangle$ for $0 < t < \pi/4$. Hence $L = \pi/4$ and the group diagram, lifted to S^3, is given by:

$$\{\pm 1, \pm j\} \subset \{e^{i\theta} \cup je^{i\theta}, e^{j\theta}\} \subset \mathrm{S}^3\,.$$

Thus a projection along the orbits gives rise to a two fold branched cover $\mathbb{CP}^2 \to \mathbb{S}^4$ with branching locus the singular orbit $\mathbb{RP}^2 = \mathrm{S}^3/(e^{i\theta} \cup je^{i\theta})$. One easily shows that the functions describing the metric are given by

$$f_1 = \sin^2(t),\ f_2 = \cos^2(2t),\ f_3 = \cos^2(t).$$

The Weyl group symmetry changes since $w_- = i$ but $w_+ = e^{j\frac{\pi}{4}}$ and hence $W = D_2$. The functions are thus related by

$$f_3(t) = f_1(t + 2L) = f_1(-t + 2L)\ ,\ f_2(t) = f_2(-t) = f_2(-t + 2L). \tag{2.3}$$

Hence in this case the functions f_1 and f_2 on $[0, 2L]$ determines the full geometry of the cohomogeneity one manifold.

We finally mention the cohomogeneity one action by $\mathrm{SU}(2)$ on $\mathbb{CP}^2$ which fixes a point p_0. The second singular orbit is then the cut locus of p_0 and hence $L = \pi/2$ in this case.

$$M = \mathbb{S}^7 \ \text{ with }\ \bar{G} = \mathrm{SO}(4).$$

The action of $\mathrm{SO}(4)$ on $\mathbb{R}^8$ which induces a cohomogeneity one action on $\mathbb{S}^7$ is given by the isotropy representation of the rank 2 symmetric space $\mathrm{G}_2 / \mathrm{SO}(4)$ on its tangent space. As a complex representation, it is the representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ obtained by taking the tensor product of the unique 2-dimensional irreducible representation of $\mathrm{SU}(2)$ with the 4-dimensional one on the second factor. Thus the first $\mathrm{SU}(2)$ factor acts as the Hopf action on $\mathbb{S}^7$ and the second factor induces an action by $\mathrm{SO}(4)/\mathrm{SU}(2) \simeq \mathrm{SO}(3)$ on $\mathbb{S}^7(1)/\mathrm{SU}(2) = \mathbb{S}^4(\frac{1}{2})$. This action of $\mathrm{SO}(3)$ on $\mathbb{R}^5$ is irreducible since the representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbb{R}^8$ is irreducible. Thus it agrees with the cohomogeneity one action of $\mathrm{SO}(3)$ in the previous example and hence the $\bar{G}$ action on $\mathbb{S}^7$ is cohomogeneity as well. Since $\mathrm{SU}(2)$ acts freely, this implies in particular that both actions have isomorphic isotropy groups, i.e. $K^{\pm} \simeq \mathrm{O}(2)$ and $H \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ considered as subgroups of $\mathrm{SO}(4)$. We now need to determine their explicit embeddings into $\mathrm{SO}(4)$ respectively $\mathrm{S}^3 \times \mathrm{S}^3$.

If we let V_k be the vector space of homogeneous polynomials of degree k in two complex variables z, w, then $\mathrm{SU}(2)$ acting on vectors (z, w) via matrix multiplication, induces an irreducible representation on V_k of (complex) dimension $k + 1$ and preserves the inner product which makes $z^m w^n$ into an orthogonal basis with $|z^m w^n|^2 = m!n!$. The isotropy representation (complexified) is thus $V_1 \otimes V_3$. The map $(z, w) \to (w, -z)$, extended to be a complex antilinear map $J_k \colon V_k \to V_k$, satisfies $J_k^2 = (-1)^k \,\mathrm{Id}$ and hence $(J_1 \otimes J_3)^2 = \mathrm{Id}$. Thus $J_1 \otimes J_3$ induces a real structure on $\mathbb{C}^8$ and hence its $+1$ eigenspace W is invariant under the action of $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$, and is spanned by:

$$\begin{aligned} &xz^3 + yw^3\ ,\ i(xz^3 - yw^3)\ ,\ xzw^2 + ywz^2\ ,\ i(xzw^2 - ywz^2), \\ &yz^3 - xw^3\ ,\ i(yz^3 + xw^3)\ ,\ xz^2w - yw^2z\ ,\ i(xz^2w + yw^2z), \end{aligned}$$

which is our desired representation of G on $\mathbb{R}^8$.

Now let ΔQ be the diagonal embedding of the quaternion group into $\mathrm{SU}(2) \times \mathrm{SU}(2) = \mathrm{S}^3 \times \mathrm{S}^3$, i.e. $\Delta Q = \{\pm(1, 1), \pm(i, i), \pm(j, j), \pm(k, k)\}$. Here we identify $a + bj \in \mathrm{S}^3$ with $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \in \mathrm{SU}(2)$. One easily checks that ΔQ fixes the two plane spanned by the orthonormal vectors $a = (xz^3 + yw^3)/2\sqrt{3}$, $b = (xzw^2 + ywz^2)/2$. By the above, this is then also the principal isotropy group H since the image of ΔQ in $\mathrm{SO}(4)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The great circle $\gamma(t) = \cos(t)a + \sin(t)b$ in this 2-plane meets all orbits orthogonally since it agrees with the fixed point set of H on $\mathbb{S}^7$. Now one easily checks that $x_- = \gamma(0)$ is fixed by the circle $(e^{-3it}, e^{it}) \subset \mathrm{S}^3 \times \mathrm{S}^3$ and hence $K^- = (e^{-3it}, e^{it}) \cdot H = (e^{-3it}, e^{it}) \cup (j, j) \cdot (e^{-3it}, e^{it})$. The first singular point along γ occurs at $x_+ = \gamma(\pi/6)$ since the projection of γ is a normal geodesic in the cohomogeneity one manifold $\mathbb{S}^4(\frac{1}{2})$. Thus $L = \pi/6$ and a computation shows that $\gamma(\pi/6)$ is fixed by the circle (e^{jt}, e^{jt}) and hence $K^+ = (e^{jt}, e^{jt}) \cup (i, i) \cdot (e^{jt}, e^{jt})$. Thus the group picture is given by

$$H = \Delta Q \subset \{(e^{-3it}, e^{it}) \cdot H \ , \ (e^{jt}, e^{jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3 \, .$$

We identify the Lie algebra $\mathfrak{g}$ with $\mathrm{Im}\,\mathbb{H} \oplus \mathrm{Im}\,\mathbb{H}$ and will use the basis $X_1 = (i, 0)$, $X_2 = (j, 0)$, $X_3 = (k, 0)$ and $Y_1 = (0, i)$, $Y_2 = (0, j)$, $Y_3 = (0, k)$ of $\mathfrak{g}$ and define $f_i(t) = |X_i^*(\gamma(t))|^2$, $g_i(t) = |Y_i^*(\gamma(t))|^2, h_i(t) = \langle X_i^*(\gamma(t)), Y_i^*(\gamma(t))\rangle$. Using the above action of $\mathrm{S}^3 \times \mathrm{S}^3$ applied to $\gamma(t)$, we obtain the action fields

$$X_1^*(\gamma(t)) = i\cos(t)\frac{xz^3 - yw^3}{2\sqrt{3}} + i\sin(t)\frac{xzw^2 - ywz^2}{2},$$
$$Y_1^*(\gamma(t)) = i\cos(t)\frac{3xz^3 - 3yw^3}{2\sqrt{3}} + i\sin(t)\frac{-xzw^2 + ywz^2}{2},$$

and thus

$$f_1 = 1 \ , \ g_1 = 8\cos^2(t) + 1 \ , \ h_1 = 4\cos^2(t) - 1.$$

As in the case of $\mathbb{S}^4$, the remaining functions are now determined via Weyl symmetry. We have $w_- = e^{i\frac{\pi}{4}}(-1, 1)$ and $w_+ = e^{j\frac{\pi}{4}}(1, 1)$ and thus $a = w_+w_- = \frac{1}{2}(1 + i + j + k)(-1, 1)$ with $a^3 = (-1, 1)$. Hence $W = D_6$ corresponding to the fact that γ meets B_+ at $t = \pi/6$ and hence intersects the regular part in 12 components. Conjugation with a behaves as in the case of $\mathbb{S}^4$ on each component and hence f_i, as well g_i and h_i, satisfy the symmetry relations (2.2). Finally, we observe that all remaining inner products are necessarily 0 since the actions of the isotropy group ΔQ on the 3 subspaces $\mathrm{span}\{X_i, Y_i\}, i = 1, 2, 3$ are inequivalent to each other.

$$M = B^7 \ \text{ with } \ \bar{G} = \mathrm{SO}(4).$$

As we saw in our first example, $\mathrm{SO}(3)$ acts orthogonally on the vector space V, consisting of the set of traceless symmetric 3×3 matrices, via conjugation and hence isometrically on $\mathbb{S}^4$. This gives rise to an embedding $\phi\colon \mathrm{SO}(3) \to \mathrm{SO}(5)$ and defines a homogeneous space $B^7 = \mathrm{SO}(5)/\mathrm{SO}(3)$, also known as the Berger space. Berger showed in [Be] that a biinvariant metric on $\mathrm{SO}(5)$ induces a metric on B^7 with positive sectional curvature.

The subgroup $\mathrm{SO}(4) \subset \mathrm{SO}(5)$ acts on B_7 via left multiplication and we claim it is cohomogeneity one. Using the basis (2.1) from Example 1, we let $\mathrm{SO}(4) = \mathrm{SO}(5)_{e_1}$ be the subgroup fixing e_1. The isotropy groups are then given by $\mathrm{SO}(4)_{g\,\mathrm{SO}(3)} = \mathrm{SO}(4) \cap g\,\mathrm{SO}(3)g^{-1} = g(\mathrm{SO}(3)_{g^{-1}e_1})g^{-1}$. Hence it follows from our first example that $K^\pm \simeq \mathrm{O}(2)$ and $H \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and thus the action is cohomogeneity one. We now need to compute the explicit embeddings of $K^\pm$ in $\mathrm{SO}(4)$ respectively $\mathrm{S}^3 \times \mathrm{S}^3$.

To avoid confusion, we let E_{ij} be the basis of skew symmetric matrices in $\mathrm{SO}(3)$ and F_{ij} the one in $\mathrm{SO}(5)$. If we set $\phi_*(E_{12}) = H_1$, $\phi_*(E_{23}) = H_2$, $\phi_*(E_{13}) = H_3$, one easily shows, using the explicit description of the action of $\mathrm{SO}(3)$ on V, that

$$H_1 = 2F_{23} + F_{45} \ , \ H_2 = F_{34} + \sqrt{3}F_{15} - F_{25} \ , \ H_3 = F_{35} + \sqrt{3}F_{14} + F_{24}.$$

Thus H_i defines an orthogonal basis of the Lie algebra of $\phi(\mathrm{SO}(3))$ with $|H_i|^2 = 5$.

For the point x_- we choose $x_- = e \cdot \mathrm{SO}(3)$, the identity coset in $\mathrm{SO}(5)/\mathrm{SO}(3)$. The group $K_0^- \simeq \mathrm{SO}(2)$ then acts by rotation with angle 2θ in the e_2, e_3 plane and angle θ in the e_4, e_5 plane since $\phi_*(E_{12}) = 2F_{23} + F_{45}$, which determines its embedding into $\mathrm{SO}(4)$. For K^+ we need to follow a normal geodesic. Clearly, F_{12} and F_{13} are orthogonal to H_i and the orbit of $\mathrm{SO}(4)$ and hence lie in the normal space of B_-. In B^7, being normal homogeneous, a geodesic is the image of a one parameter group with initial vector orthogonal to H_i. Thus we can let $\gamma(t) = \exp(tF_{12}) \cdot \mathrm{SO}(3) = (\cos(t)e_1 + \sin(t)e_2) \cdot \mathrm{SO}(3)$ be the geodesic orthogonal to all orbits. From Example 1 it follows that the isotropy at $\gamma(t)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for $0 < t < \pi/3$ and to $\mathrm{O}(2)$ at $\gamma(\pi/3)$ and thus $L = \pi/3$. If $g = \exp(\frac{\pi}{3}F_{12})$, we have $g^{-1}e_1 = \frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2 = \mathrm{diag}(2, -1, -1)/\sqrt{6}$ and hence $K^+ = g(\mathrm{S}(\mathrm{O}(1)\,\mathrm{O}(2))g^{-1}$. From the embedding ϕ it is clear that $\mathrm{SO}(2) \subset \mathrm{S}(\mathrm{O}(1)\,\mathrm{O}(2) \subset \mathrm{SO}(3)$ fixes $g^{-1}e_1$ and rotates by θ in the e_3, e_4 plane and by 2θ in the plane spanned by $\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$ and e_5. Conjugating with g gives a rotation that fixes e_1, rotates by θ in the e_3, e_4 plane and by -2θ in the e_2, e_5 plane.

We now lift these groups into $G = \mathrm{S}^3 \times \mathrm{S}^3$ using the identification $e_2 \leftrightarrow 1$, $e_3 \leftrightarrow i$, $e_4 \leftrightarrow j$, $e_5 \leftrightarrow k$ and the 2-fold cover $\mathrm{S}^3 \times \mathrm{S}^3 \to \mathrm{SO}(4)$ given by left and right multiplication of unit quaternions. It sends $X_1 = (i, 0) \to F_{23} + F_{45}$, $Y_1 = (0, i) \to -F_{23} + F_{45}$ and similarly for $X_i, Y_i, i = 2, 3$. Thus, after renumbering the coordinates, it follows that $K_0^- = (e^{-3it}, e^{it})$ and $K_0^+ = (e^{jt}, e^{-3jt})$. For the group picture to be consistent, we are left with:

$$\Delta Q \subset \{(e^{-3it}, e^{it}) \cdot H \ , \ (e^{jt}, e^{-3jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3 \, .$$

In order to determine the functions describing the metric, let $\bar{X}_i, \bar{Y}_i$ be the action fields on $\mathrm{SO}(5)$ and X_i^*, Y_i^* those on $\mathrm{SO}(5)/\mathrm{SO}(3)$. To compute their length at $\gamma(t)$, we translate them back to the identity with the isometric left translation by $\exp(tF_{12})^{-1}$. We thus obtain:

$$\bar{X}_1(t) = \mathrm{Ad}(-\exp(tF_{12}))(F_{23} + F_{45}) = -\sin(t)F_{13} + \cos(t)F_{23} + F_{45},$$
$$\bar{Y}_1(t) = \mathrm{Ad}(-\exp(tF_{12}))(-F_{23} + F_{45}) = \sin(t)F_{13} - \cos(t)F_{23} + F_{45}.$$

Since $X_1^* = \bar{X}_1 - \frac{1}{5}\langle \bar{X}_1, H_1\rangle H_1 = \bar{X}_1 - \frac{1}{5}(2\cos(t) + 1)H_1$ and $Y_1^* = \bar{Y}_1 - \frac{1}{5}(-2\cos(t) + 1)H_1$ we have:

$$f_1 = \frac{1}{5}(5 + 4\sin^2(t) - 4\cos(t)) \ , \ g_1 = \frac{1}{5}(5 + 4\sin^2(t) + 4\cos(t)) \ , \ h_1 = -\frac{1}{5}(1 - 4\cos^2(t)).$$

The remaining functions are determined by Weyl group symmetry. Similarly to the example of $\mathbb{S}^7$, we see that $w_- = e^{i\frac{\pi}{4}}(1, 1)$ and $w_+ = e^{j\frac{\pi}{4}}(1, 1)$ and hence $(w_+w_-)^3 = (-1, -1) \in H$. Thus in this case $W = D_3$. But conjugation by w_+w_- behaves as before and hence all functions satisfy the same Weyl symmetry as in (2.2).

It is now interesting to compare the metrics in these two examples, which we do in a sequence of pictures. Figure 1 shows all 9 functions between two singular orbits, clearly not very instructive. Figure 2 illustrates the effects of Weyl symmetry in these pictures in a typical case of the 3 g functions for the Berger space. Thus Figure 3, which shows $f = f_1$, $g = g_1$, $h = h_1$ on $[0, 3L]$, encodes all the geometry of the space. As was discovered by K.Grove, B.Wilking and the author, the positivity of the sectional curvatures $\sec(\gamma'(t), X^*)$, $X \in \mathfrak{g}$ implies that the inverse of the metric tensor is a convex matrix. Figure 4 shows the functions F_1, G_1, H_1 in the inverse of $\begin{pmatrix} f_1 & h_1 \\ h_1 & g_1 \end{pmatrix}$ on the interval $[0, 3L]$.

Smoothness conditions are now encoded in the behavior of the functions as $t \to 0$ and $t \to 3L$.

$$M = E_p^7 \ \text{ with } \ \bar{G} = \mathrm{SO}(3) \times \mathrm{S}^3.$$

Next, we examine a family of biquotients among the 7-dimensional Eschenburg spaces. Define

$$E_p := \mathrm{SU}(3) /\!/ \mathrm{S}^1_p = \operatorname{diag}(z, z, z^p) \backslash \mathrm{SU}(3) / \operatorname{diag}(1, 1, z^{p+2})^{-1},$$

where it is understood that $\mathrm{S}^1 = \{z \mid |z| = 1\}$ acts on $\mathrm{SU}(3)$ simultaneously on the left and on the right. Up to equivalence, we can assume that $p \geq 0$ and Eschenburg showed that it admits a metric with positive sectional curvature if $p \geq 1$. The positively curved metric is obtained by scaling the biinvariant metric $\langle A, B\rangle = -\frac{1}{2} \operatorname{Re} \operatorname{tr} AB$ on $\mathrm{SU}(3)$ in direction of the subgroup $\operatorname{diag}(A, \det \bar{A})$, $A \in \mathrm{U}(2)$ by an amount $\epsilon < 1$. The group $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on E_p by multiplying on the left and on the right in the first two coordinates since it clearly commutes with the circle action, and we claim this action is cohomogeneity one. Indeed, we can first divide by the second $\mathrm{SU}(2)$ and the action of the first $\mathrm{SU}(2)$, since $(\{e\} \times \mathrm{SU}(2)) \cdot \mathrm{S}^1_p = \mathrm{U}(2)$, is then an action on $\mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{CP}^2$ which fixes the identity coset $e \cdot \mathrm{U}(2)$ and acts transitively on the normal sphere. Thus this action on $\mathbb{CP}^2$, and hence also the action of G on E_p, is cohomogeneity one. Notice that if p is even, the action becomes effectively an action by $\mathrm{SO}(3) \times \mathrm{SU}(2)$, whereas if p is odd, by $\mathrm{SU}(2) \times \mathrm{SO}(3)$.

One singular point is clearly the image of the identity matrix, $x_- = e \cdot \mathrm{S}^1_p$, with $K^- = \{(g, \pm g) \mid g \in \mathrm{SU}(2)\}$. One easily checks that in the modified metric on $\mathrm{SU}(3)$ the one parameter group $\exp(tE_{13})$, being orthogonal to $\mathrm{U}(2)$, is still a (unit speed) geodesic in $\mathrm{SU}(3)$ (see, e.g., [DZ]). Since it is also orthogonal to the orbit of G at x_-, its projection $\gamma(t)$ into $\mathrm{SU}(3)/\!/\mathrm{S}^1_p$ is a geodesic orthogonal to all orbits. Its projection to $\mathbb{CP}^2$ is also a normal geodesic and the induced $\mathrm{SU}(2)$ action has isotropy $\mathrm{SU}(2)$ at $t = 0$ and isotropy S^1 at $t = \pi/2$ and the principal isotropy is trivial. Thus the same holds for the action of $\bar{G}$ on E_p, in particular $L = \pi/2$. For the explicit embeddings, one easily checks ([Zi1],[GSZ]) that the isotropy group of G at $x_+ = \gamma(\pi/2)$ is equal to $K^+ = (e^{i(p+1)t}, e^{ipt})$. Hence we obtain the group diagram:

$$\mathbb{Z}_2 = ((-1)^{p+1}, (-1)^p) \subset \{\Delta \mathrm{S}^3 \cdot H \ , \ (e^{i(p+1)t}, e^{ipt})\} \subset \mathrm{S}^3 \times \mathrm{S}^3 \, .$$

To compute the functions describing the metric, we identify $\mathrm{S}^3 \times \mathrm{S}^3$ as before with $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and translate the action fields at $\gamma(t)$ back to the identity. We then obtain:

$$\begin{aligned}
\bar{X}_1^* &= \operatorname{Ad}(\exp(-tE_{13})) \operatorname{diag}(i, -i, 0) = \operatorname{diag}(i\cos^2(t), -i, i\sin^2(t)) + \cos(t)\sin(t) I_{13},\\
\bar{X}_2^* &= \operatorname{Ad}(\exp(-tE_{13})) E_{12} = \cos(t) E_{12} - \sin(t) E_{23},\\
\bar{X}_3^* &= \operatorname{Ad}(\exp(-tE_{13})) I_{12} = \cos(t) I_{12} + \sin(t) I_{23},\\
\bar{Y}_1^* &= -\operatorname{diag}(i, -i, 0) \ , \ \bar{Y}_2^* = -E_{12} \ , \ \bar{Y}_3^* = -I_{12},
\end{aligned}$$

where we used the notation I_{kl} for a matrix in $\mathfrak{su}(3)$ with i in entry kl and lk and 0 elsewhere. The vertical space of the Riemannian submersion $\pi\colon \mathrm{SU}(3) \to \mathrm{SU}(3)/\!/\mathrm{S}^1_p$

(translated back to the identity) is spanned by

$$
\begin{aligned}
v &= \mathrm{Ad}(\exp(-tE_{13}))i\,\mathrm{diag}(1,1,p) - i\,\mathrm{diag}(0,0,p+2)\\
&= i\,\mathrm{diag}(\cos^2(t)+p\sin^2(t),1,\sin^2(t)+p\cos^2(t)-p-2)+(1-p)\cos(t)\sin(t)I_{13},
\end{aligned}
$$

whose length in the Eschenburg metric is

$$|v|^2 = 3\epsilon + (1-p)^2(1-\epsilon)\cos^2(t)\sin^2(t) + \epsilon(p+2)(p-1)\sin^2(t).$$

Notice that $\bar X_2, \bar X_3, \bar Y_2, \bar Y_3$ are already horizontal with respect to the Riemannian submersion π but that we need to subtract the vertical component from $\bar X_1$ and $\bar Y_1$. A computation now shows that:

$$
\begin{aligned}
f_1 = \frac{\epsilon}{4\alpha}\Big[&\big(3\epsilon(p-2)^2-4p^2+8p-16\big)\cos^4(t)+\\
&\qquad + \big(4p^2-8p+16-6\epsilon p(p-2)\,\big)\cos^2(t)+3\epsilon p^2\Big]\,,\\
g_1 = \frac{\epsilon}{4\alpha}\Big[&(p-1)^2(3\epsilon-4)\cos^4(t)+(2p-2)(2p-2-3\epsilon(p+1)\,)\cos^2(t)+3\epsilon(p+1)^2\Big]\,,\\
h_1 = -\frac{\epsilon}{4\alpha}\Big[&(p-1)(3\epsilon(p-2)-4p+4)\cos^4(t)+\\
&\qquad + (4(p-1)^2-6\epsilon(p^2-p-1)\,)\cos^2(t)+3\epsilon p(p+1)\Big]\,,\\
f_2 = f_3 &= 1+(\epsilon-1)\cos^2(t)\ ,\ g_2=g_3=\epsilon\ ,\ h_2=h_3=-\epsilon\cos(t).
\end{aligned}
$$

where $\alpha = (p-1)^2(\epsilon-1)\cos^4(t)+(p-1)(p-1-\epsilon(2p+1)\,)\cos^2(t)+\epsilon(p^2+p+1)$ and all other inner products are 0.

Notice that $p=1$ is a special case since the Eschenburg space is now simply the homogeneous Aloff-Wallach space $W^7 = \mathrm{SU}(3)/\,\mathrm{diag}(z,z,\bar z^2)$ and the functions are given by:

$$
\begin{aligned}
&f_1 = \frac{1}{4}\Big[(\epsilon-4)\cos^4(t)+2(\epsilon+2)\cos^2(t)+\epsilon\Big]\ ,\ f_2=f_3=1+(\epsilon-1)\cos^2(t),\\
&g_1=g_2=g_3=\epsilon\ ,\ h_1=-\frac{\epsilon}{2}(\cos^2(t)+1)\ ,\ h_2=h_3=-\epsilon\cos(t).
\end{aligned}
$$

A major difference with the previous two cases lies in the Weyl group. Clearly $w_- = (-1,-1)$ and $w_+ = (i^{p+1}, i^p)$. Thus $(w_+w_-)^2 \in H$ and hence $W = D_2$. The Weyl group elements multiply each of the natural basis vectors in $T_{p_\pm}B_\pm$ with ± 1 and hence Weyl group symmetry simply says that all (non-collapsing) functions must be even at $t=0$ and at $t=L$, in particular their first derivatives must vanish. Thus any of the relationships between different functions that was so useful in the previous cases is lost. Also, notice that, unlike in the previous two examples, the vanishing of the remaining inner products is not forced anymore by the action of the isotropy group since it acts trivially on G/H. The basic behavior of the functions is illustrated in Figure 5 for the Aloff-Wallach space and the Eschenburg space with a typical value of $p=10$ and $\epsilon=\frac{1}{2}$. Here we have drawn the graphs on $[0,4L]$, i.e., once around the closed geodesic, for better comparison. As $p\to\infty$, the functions converge, but the limiting metric is not smooth at the singular orbits.

$W^7_{(2)}$ with $\bar G = \mathrm{SO}(3)\,\mathrm{SO}(3)$.

The Aloff-Wallach space $W^7 = \mathrm{SU}(3)/\operatorname{diag}(z,z,\bar{z}^2)$ has a second cohomogeneity one action by combining right multiplication by SU(2) as in the previous example with left multiplication by $\mathrm{SO}(3) \subset \mathrm{SU}(3)$. Observe that the right action by SU(2) is effectively an action of $\mathrm{SO}(3) = \mathrm{U}(2)/Z(\mathrm{U}(2))$ and that the action is free with quotient $\mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{CP}^2$. The left action by SO(3) then induces an action on the quotient which has to be the cohomogeneity one action by SO(3) on $\mathbb{CP}^2$ mentioned earlier, since there exists only one SO(3) in SU(3). In particular, the $\bar{G} = \mathrm{SO}(3)\,\mathrm{SO}(3)$ action on W^7 is cohomogeneity one, which also determines the isomorphism type of the isotropy groups: $K^- = \mathrm{O}(2)$, $K^+ = \mathrm{SO}(2)$ and $H = \mathbb{Z}_2$.

We can choose x_- again to be the identity coset since the isotropy is $\mathrm{SO}(3) \cap \mathrm{SU}(2) \cdot \operatorname{diag}(z,z,\bar{z}^2) = \mathrm{SO}(3) \cap \mathrm{U}(2) = \mathrm{O}(2)$. The tangent space to B_- is spanned by E_{ij}, I_{12} and $\operatorname{diag}(i,-i,0)$ and hence $\gamma(t) = \exp(tI_{13}) \cdot \operatorname{diag}(z,z,\bar{z}^2)$ is a unit speed geodesic in $\mathrm{SU}(3)/\operatorname{diag}(z,z,\bar{z}^2)$ orthogonal to all orbits. Since the projection to $\mathbb{CP}^2$ is a normal geodesic orthogonal to the orbits of the cohomogeneity one action on $\mathbb{CP}^2$, it follows that the singular isotropy groups occur at $t = 0$ and $t = \pi/4$ and thus $L = \pi/4$. Instead of trying to compute the embedding of these isotropy groups directly, we will use the functions describing the metric instead.

We first compute the action fields X_i^*. For comparison, we again consider the action as an (ineffective) action by $\mathrm{S}^3 \times \mathrm{S}^3$, and since the two fold cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ multiplies the natural basis vectors by 2, we find:

$$\begin{aligned}
\bar{X}_1(t) &= \operatorname{Ad}(-\exp(tI_{13}))(2E_{12}) = -2\cos(t)E_{12} - 2\sin(t)I_{23},\\
\bar{X}_2(t) &= \operatorname{Ad}(-\exp(tI_{13}))(2E_{13}) = -2\cos(2t)E_{13} + 2\sin(2t)\operatorname{diag}(i,0,-i),\\
\bar{X}_3(t) &= \operatorname{Ad}(-\exp(tI_{13}))(2E_{23}) = 2\sin(t)I_{12} - 2\cos(t)E_{23}.
\end{aligned}$$

Notice that the component of $\operatorname{diag}(i,0,-i)$ orthogonal to $\operatorname{diag}(z,z,\bar{z}^2)$ is $\frac{1}{2}\operatorname{diag}(i,-i,0)$ and that $\bar{X}_1(t)$ and $\bar{X}_3(t)$ are orthogonal already.

For the right action fields we have $\bar{Y}_1^* = -E_{12}$, $\bar{Y}_2^* = -\operatorname{diag}(i,-i,0)$, $\bar{Y}_3^* = -I_{12}$ and thus:

$$\begin{aligned}
f_1 &= 4\sin^2(t) + 4\epsilon\cos^2(t)\ ,\ g_1 = \epsilon\ ,\ h_1 = 2\epsilon\cos(t),\\
f_2 &= 4\cos^2(2t) + \epsilon\sin^2(2t)\ ,\ g_2 = \epsilon\ ,\ h_2 = -\epsilon\sin(2t),\\
f_3 &= 4\cos^2(t) + 4\epsilon\sin^2(t)\ ,\ g_3 = \epsilon\ ,\ h_3 = -2\epsilon\sin(t),
\end{aligned}$$

with all other inner products being 0. Since $X_1^*(0) - 2Y_1^*(0) = 0$ and $X_2^*(\pi/4) + Y_2^*(\pi/4) = 0$ it follows that $K_0^- = (e^{it}, e^{-2it}) \cdot H$ and $K_0^+ = (e^{jt}, e^{jt}) \cdot H$. Since the isomorphism type of the isotropy groups of the action is already determined, this leaves only the following possibility for its group diagram:

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(\pm 1, \pm 1), (\pm j, \pm j)\} \subset \{(e^{it}, e^{-2it}) \cdot H\ ,\ (e^{jt}, e^{jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3\,.$$

For the Weyl group we have $w_- = (i,-1)$, $w_+ = (e^{j\frac{\pi}{4}}, e^{j\frac{\pi}{4}})$ and $(w_+w_-)^2 = (-1, j)$ and hence $W = D_4$. The functions f_i and g_i satisfy the same Weyl symmetry as in (2.3), but for h_i we have:

$$h_3(t) = -h_1(-t+2L) = h_1(t+2L)\ ,\ h_2(t) = -h_2(-t) = h_2(-t+2L).$$

It is interesting to observe this modified Weyl symmetry behavior in Figure 6 (for a typical value of $\epsilon = \frac{1}{2}$).

The above metric with $\epsilon = 1$ is the one induced by the biinvariant metric, which has non-negative curvature, but not positive. For $\epsilon = 2$, we obtain a 3-Sasakian metric (see Section 4) after dividing the metric by 2, i.e., multiplying all functions by $\frac{1}{2}$ and replacing the parameter t by $\sqrt{2}\,t$. The second SU(2) factor is then the 3-Sasakian action.

For later purposes we note that, up to conjugation, this group diagram can also be written as:

$$\mathbb{Z}_4 \oplus \mathbb{Z}_2 = \{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{it}, e^{it}) \cdot H \ , \ (e^{jt}, e^{2jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3 \,.$$

In Figure 7 we show the graphs for the functions in the inverse matrix on $W^7_{(1)}$ and $W^7_{(2)}$. We only include the most interesting case of the one's with index 1, although these do not determine the remaining ones as was the case for the sphere and the Berger space.

3. Classification of cohomogeneity one manifolds with positive curvature

In this section we describe the classification result in [GWZ]. In even dimensions, positively curved cohomogeneity one manifolds were classified by L.Verdiani [V1, V2]. Here only rank one symmetric spaces occur. The actions for these spaces though are numerous and have been classified in [HL, Iw1, Iw2, Uc]. In odd dimensions, cohomogeneity one actions on spheres are even more numerous. The classification of course must also contain all of the examples described in the previous section, as well as the cohomogeneity one action by SU(4) on the Bazaikin spaces $B^{13}_p = \mathrm{diag}(z, z, z, z, z^p)\backslash \mathrm{SU}(5)/ \mathrm{diag}(z^{p+4}, A)^{-1}$, where $A \in \mathrm{Sp}(2) \subset \mathrm{SU}(4) \subset \mathrm{SU}(5)$. Here SU(4) acts by multiplication on the left in the first 4 coordinates. Encouragingly, a series of "candidates" P_k, Q_k and R emerges in dimension 7 for which it is not yet known whether they can carry a cohomogeneity one metric with positive curvature. They are cohomogeneity one under an action of $\mathrm{S}^3 \times \mathrm{S}^3$, and are defined as cohomogeneity one manifolds in terms of their isotropy groups. For P_k the group diagram is

$$\Delta Q \subset \{(e^{it}, e^{it}) \cdot H \ , \ (e^{j(1+2k)t}, e^{j(1-2k)t}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

and for Q_k it is

$$\{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{it}, e^{it}) \cdot H \ , \ (e^{jkt}, e^{j(k+1)t}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

whereas for R we have

$$\{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{it}, e^{2it}) \cdot H \ , \ (e^{3jt}, e^{jt}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3 \,.$$

Notice that the action of $\mathrm{S}^3 \times \mathrm{S}^3$ on P_k is effectively an action by SO(4), and the one on Q_k and R by $\mathrm{SO}(3) \times \mathrm{SO}(3)$.

Theorem 3.1 (L.Verdiani, K.Grove-B.Wilking-W.Ziller). *A simply connected compact cohomogeneity one manifold with an invariant metric of positive sectional curvature is equivariantly diffeomorphic to one of the following:*

- *A compact rank one symmetric space with an isometric action,*
- *One of E^7_p, B^{13}_p or B^7,*
- *One of the 7-manifolds P_k, Q_k, or R,*

with one of the actions described above.

The first in each sequence P_k, Q_k admit an invariant metric with positive curvature since from the group diagrams in Section 2 it follows that $P_1 = \mathbb{S}^7$ and $Q_1 = W^7_{(2)}$. By the main result in [GZ], the manifolds P_k, Q_k, R all carry an invariant metric with non-negative curvature since the cohomogeneity one actions have singular orbits of codimensions 2. Recall that the cohomogeneity one action on B^7 looks like those for P_k with slopes $(-3, 1)$ and $(1, -3)$. In some tantalizing sense then, the exceptional Berger manifold B^7 is associated with the P_k family in an analogues way as the exceptional candidate R is associated with the Q_k family.

The candidates also have interesting topological properties. In [GWZ] it was shown that the manifolds P_k are two-connected with $\pi_3(P_k) = \mathbb{Z}_k$ and that Q_k has the same cohomology ring as E_k. The fact that the manifolds P_k are 2-connected is particularly significant since by the finiteness theorem of Petrunin-Tuschmann [PT] and Fang-Rong [FR], there exist only finitely many diffeomorphism types of 2-connected positively curved manifolds, if one specifies the dimension and the pinching constant, i.e. δ with $\delta \leq sec \leq 1$. Thus, if P_k admit positive curvature metrics, the pinching constants δ_k necessarily go to 0 as $k \to \infty$, and P_k would be the first examples of this type.

It is remarkable that all non-linear actions in Theorem 3.1, apart from the Bazaikin spaces B^{13}_p, occur in dimension 7 and are cohomogeneity one under a group locally isomorphic to $\mathrm{S}^3 \times \mathrm{S}^3$. It is also remarkable that in positive curvature only the above slopes are allowed, whereas for arbitrary slopes one has an invariant metric with non-negative curvature by [GZ]. We will give a proof of Theorem 3.1 in this special case of $G = \mathrm{S}^3 \times \mathrm{S}^3$ since this case is clearly of particular interest.

Three important ingredients in the proof of the classification Theorem 3.1 are given by:

- The normal geodesic is closed, or equivalently, the Weyl group is finite.
- The action is linearly primitive, i.e. the Lie algebras of all singular isotropy groups along a fixed normal geodesic generate $\mathfrak{g}$ as vector spaces.
- The action is group primitive, i.e. the groups K^- and K^+ generate G as subgroups and so do K^- and nK^+n^{-1} for any $n \in N(H)_0$.

For $G = \mathrm{S}^3 \times \mathrm{S}^3$, the classification is based on the following Lemma.

LEMMA 3.2. *Let $G = \mathrm{S}^3 \times \mathrm{S}^3$ act by cohomogeneity one on the positively curved manifold M. If G/K is a singular orbit, $G/K_0 = \mathrm{S}^3 \times \mathrm{S}^3 / (e^{ipt}, e^{iqt})$ with $p, q \neq 0, (p, q) = 1$, and $H \cap K_0 = \mathbb{Z}_k$, we have:*

(a) *$k \geq 2$ and if $k = 2$, then $|p + q| = 1$ or $|p - q| = 1$,*
(b) *If $k > 2$, then $(p, q) = (\pm 1, \pm 1)$ or $|2p + 2q| = k$ or $|2p - 2q| = k$.*

If furthermore $G/K_0^- = \mathrm{S}^3 \times \mathrm{S}^3 / (e^{ip_-t}, e^{iq_-t})$, $G/K_0^+ = \mathrm{S}^3 \times \mathrm{S}^3 / (e^{jp_+t}, e^{jq_+t})$ and, up to conjugacy, $H = \Delta Q$ or $H = \{(\pm 1, \pm 1), (\pm i, \pm i)\}$, then $\min\{|p_+|, |p_-|\} = \min\{|q_+|, |q_-|\} = 1$.

Proof. The main ingredient in the proof of (a) and (b) is the equivariance of the second fundamental form of G/K regarded as a K equivariant linear map $B\colon S^2T \to T^\perp$. The non-trivial irreducible representations of $\mathrm{S}^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ consist of two dimensional representations given by multiplication by $e^{in\theta}$ on $\mathbb{C}$, called a weight n representation.

The action of K_0 on $T^\perp = \mathbb{R}^2$ will have weight k if $H \cap K_0 = \mathbb{Z}_k$, since $\mathbb{Z}_k$ is necessarily the ineffective kernel. The action of K_0 on T on the other hand has weights 0 on $W_0 = \operatorname{span}\{(-qi, pi)\}$, weight $2p$ on the two plane $W_1 = \operatorname{span}\{(j,0),(k,0)\}$ and weight $2q$ on $W_2 = \operatorname{span}\{(0,j),(0,k)\}$. The action on $S^2(W_1 \oplus W_2)$ has therefore weights 0 and $4p$ on S^2W_1, 0 and $4q$ on S^2W_2 and $2p+2q$ and $2p-2q$ on $W_1 \otimes W_2$.

Let us first assume that $(p,q) \neq (\pm 1, \pm 1)$. Then for any homogeneous metric on G/K_0 there exists a vector $w_1 \in W_1$ and $w_2 \in W_2$ such that the 2-plane spanned by w_1 and w_2 tangent to G/K has curvature 0 intrinsically. Indeed, since $p \neq \pm q$, $\operatorname{Ad}(K_0)$ invariance of the metric on G/K_0 implies that the two planes $\operatorname{span}\{(j,0),(0,j)\}$ and $\operatorname{span}\{(k,0),(0,k)\}$ and the line W_0 are orthogonal to each other. Hence $\operatorname{Ad}((j,j))$ induces an isometry on G/K_0, which implies that the two plane spanned by $w_1 = (j,0) \in W_1$ and $w_2 = (0,j) \in W_2$ is the tangent space of the fixed point set of $\operatorname{Ad}((j,j))$ and thus has curvature 0 since the fixed point set in G/K_0 is a 2-torus with a left invariant metric. Since $(p,q) \neq (\pm 1, \pm 1)$ at least one of the numbers $4p$ or $4q$ is not equal to the normal weight $k > 0$. The equivariance of the second fundamental form then implies that $B_{S^2W_i}$ vanishes for at least one i and hence by the Gauss equations $B(w_1, w_2) \neq 0$ for the above vectors w_1 and w_2. Using the equivariance of the second fundamental form once more we see that $W_1 \otimes W_2$ contains a subrepresentation whose weight is equal to the normal weight k. Hence, $|2p+2q| = k$ or $|2p-2q| = k$. In particular, $k \geq 2$.

It remains to show that if $(p,q) = (\pm 1, \pm 1)$, then $k > 2$. We first show that we still have a 2-plane as above with 0-curvature. Indeed, $\operatorname{Ad}(K_0)$ invariance implies that the inner products between W_1 and W_2 are given by $\langle (X,0),(0,Y) \rangle = \langle \phi(X), Y \rangle$ where $\phi\colon W_1 \to W_2$ is an $\operatorname{Ad}(K_0)$ equivariant map. Hence, if we choose $j' = \phi(j)$ and $k' = \phi(k)$, the two planes $\operatorname{span}\{(j,0),(0,j')\}$ and $\operatorname{span}\{(k,0),(0,k')\}$ are orthogonal to each other, so that by the same argument $w_1 = (j,0) \in W_1$ and $w_2 = (0,j') \in W_2$ span a 2-plane with curvature 0. We thus obtain again that $B_{S^2W_i} = 0$, hence $B(w_1,w_2) \neq 0$, which gives a contradiction if $k < 4$.

To prove part (c), we use the following general fact about an isometric G action on M. The strata, i.e., components in M/G of orbits of the same type (K), are (locally) totally geodesic (cf. [Gr]). Indeed, by the slice theorem, such a component near the image of $p \in M$ with $G_p = K$ is given by the fixed point set $D^K \subset D/K \subset M/G$ where D is a slice at p. In the case of the $\mathrm{S}^3 \times 1$ action on M, the isotropy groups are given by the intersections of $\mathrm{S}^3 \times 1$ with $K^\pm$ and H since $\mathrm{S}^3 \times 1$ is normal in G. Using the special form of the principal isotropy group H, it follows that on the regular part the isotropy groups are effectively trivial. On the other hand, if $\min\{|q_+|, |q_-|\} > 1$, they are non-trivial along $B_\pm$. This implies that the image of both $B_\pm$ in $M/\mathrm{S}^3 \times 1$ are totally geodesic. Since these strata are two dimensional and M/S^3 is four dimensional, both strata cannot be totally geodesic according to Petrunin's analogue [Pe] of Frankel's theorem for Alexandrov spaces. This finishes our claim. $\square$

We now use the classification of 7-dimensional compact simply connected primitive cohomogeneity one manifolds in [Ho]. Although the use of this classification is not necessary, it simplifies the argument and brings out its main points. Surprisingly there are, in addition to numerous linear actions on $\mathbb{S}^7$, only 6 (group) primitive families in the classification, 5 of them with $G = \mathrm{S}^3 \times \mathrm{S}^3$ and we apply Lemma 3.2 to exclude from them the ones that do not admit an invariant metric with positive curvature.

Example 1. The simplest primitive group diagram on a 7-manifold with $G = \mathrm{S}^3 \times \mathrm{S}^3$ is given by

$$\{e\} \subset \{\Delta\,\mathrm{S}^3, (e^{ip\theta}, e^{iq\theta})\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

for any p, q. A modification of this example is

$$\mathbb{Z}_2 = \{(1,-1)\} \subset \{\Delta\,\mathrm{S}^3 \cdot H, (e^{ip\theta}, e^{iq\theta})\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

with p even and q odd. Notice that the second family is a two fold branched cover of the first.

In the first case the normal weight is $k = 1$ and thus Lemma 3.2 (a) implies that there are no positively curved invariant metrics. In the second case $k = 2$ and it follows that $|p - q| = 1$ or $|p + q| = 1$, which, up to automorphisms of $\mathrm{S}^3 \times \mathrm{S}^3$, is the Eschenburg space E_p. Notice though that in both cases we also need to exclude the possibility that one of p or q is 0, which is not covered by Lemma 3.2. For this special case, one uses the product Lemma [GWZ, Lemma 2.6], which we will not discuss here.

Example 2. The second family has $H = \mathbb{Z}_4$:

$$\langle (i,i) \rangle \subset \{(e^{ip_-\theta}, e^{iq_-\theta}) \cdot H, (e^{jp_+\theta}, e^{jq_+\theta}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

where $p_-, q_- \equiv 1 \mod 4$, and p_+, q_+ are arbitrary. This family is excluded altogether. Indeed, if p_+ and q_+ are both odd, the normal weight at B_+ is $k = 2$, which implies that $|p_+ \pm q_+| = 1$, which is clearly impossible. If one is even, the other odd, $k = 1$ which is excluded by Lemma 3.2. If $p_+ = 0$ or $q_+ = 0$, the action is not group primitive.

Example 3. The third family has $H = \mathbb{Z}_2 \oplus \mathbb{Z}_4$:

$$\{(\pm 1, \pm 1), (\pm i, \pm i)\} \subset \{(e^{ip_-\theta}, e^{iq_-\theta}) \cdot H, (e^{jp_+\theta}, e^{jq_+\theta}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

where p_-, q_- is odd, p_+ even and q_+ odd. On the left, B_- has normal weight $k = 4$ and thus $(p_-, q_-) = (\pm 1, \pm 1)$ or $|p_- + q_-| = 2$ or $|p_- - q_-| = 2$. On the right, B_+ has normal weight $k = 2$ and thus $|p_+ + q_+| = 1$ or $|p_+ - q_+| = 1$. Notice also that we can assume that all integers are positive by conjugating all groups by i or j in one of the components, and observing that $p_+ = 0$ is not group primitive. Together with Lemma 3.2 (c), this leaves only the possibilities $\{(p_-, q_-), (p_+, q_+)\} = \{(1,3), (2,1)\}$ or $\{(p_-, q_-), (p_+, q_+)\} = \{(1,1), (p_+, q_+)\}$ with $|q_+ - p_+| = 1$. The first case is the exceptional manifold R. In the second case we can also assume that $q_+ > p_+$ by interchanging the two S^3 factors if necessary, and hence $(p_+, q_+) = (p, p+1)$ with $p > 0$. This gives us the family Q_k.

Example 4. The last family has $H = \Delta Q$:

$$\Delta Q \subset \{(e^{ip_-\theta}, e^{iq_-\theta}) \cdot H, (e^{jp_+\theta}, e^{jq_+\theta}) \cdot H\} \subset \mathrm{S}^3 \times \mathrm{S}^3,$$

where $p_\pm, q_\pm \equiv 1 \mod 4$. Now the weights on both normal spaces are 4 and hence $|q_\pm + p_\pm| = 2$ or $(p_\pm, q_\pm) = (1,1)$. Combining with Lemma 3.2 (c) yields only two possibilities. Either $\{(p_-, q_-), (p_+, q_+)\} = \{(1,-3), (-3,1)\}$ or $\{(1,1), (p_+, q_+)\}$ with $q_+ + p_+ = 2$. The first case is the Berger space B^7 and in the second case we can arrange that $\{(p_-, q_-), (p_+, q_+)\} = \{(1,1), (1+2k, 1-2k)\}$ with $k \geq 0$. The case $k = 0$ is excluded since it would not be group primitive. Thus we obtain the family P_k.

Remark. There is only one further family of compact simply connected 7-dimensional primitive cohomogeneity one manifolds, given by the action of $\mathrm{S}^1 \times \mathrm{S}^3 \times \mathrm{S}^3$ on the Kervaire sphere. For this action it was shown in [BH] that it cannot admit an invariant metric with positive curvature unless it is a linear action on a sphere. In [GVWZ] it was shown that in most cases it does not even admit an invariant metric with non-negative curvature. If we also allow non-primitive 7-dimensional cohomogeneity one manifolds, one finds 9 further families in [Ho]. He also shows that the only cohomogeneity one manifold in dimension 7 or below (primitive or not) where it is not yet known if it admits an invariant metric with non-negative curvature, are the two families in Example 1.

We also mention that in dimension 7, one finds a classification of positively curved cohomogeneity one manifold in [PV1, PV2] in the case where the group is not locally isomorphic to $\mathrm{S}^3 \times \mathrm{S}^3$ and that a classification in dimensions 6 and below was obtained in [Se].

4. Candidates and Hitchin metrics

The two families of cohomogeneity one manifolds P_k and Q_k have another remarkable and unexpected property. They carry a natural metric on them which is 3-Sasakian and can be regarded as an orbifold principal bundle over $\mathbb{S}^4$ or $\mathbb{CP}^2$, equipped with a self dual Einstein metric.

Before we discuss these metric, we give a description of our candidates using the language of self duality. If we consider an oriented four manifold M^4 equipped with a metric, we can use the Hodge star operator $\star\colon \Lambda^2 T^* \to \Lambda^2 T^*$ to define self dual and anti self dual 2-forms, i.e., forms ω with $\star\omega = \pm\omega$. They each form a 3-dimensional vector bundle $\Lambda^2_\pm T^*(M)$ over M with a fiber metric induced by the metric on M. Thus their principal frame bundles are two natural SO(3) principal bundles associated to the tangent bundle of M. The star operator depends on the conformal class of the metric but the isomorphism type of the vector bundles only depend on the given orientation. The same construction can be carried out for oriented orbifolds equipped with an orbifold metric since such a metric is locally the quotient of a smooth Riemannian metric under a finite group of isometries. The principal bundles are then orbifold bundles. In particular SO(3) will in general only act almost freely (i.e., all isotropy groups are finite) and the total space may only be an orbifold.

We use this construction now for the following orbifold structure O_k on $\mathbb{S}^4$. Consider the cohomogeneity one action of SO(3) on $\mathbb{S}^4$ described in Section 2. The two singular orbits are Veronese embeddings of $\mathbb{RP}^2$ into $\mathbb{S}^4$. We define an orbifold O_k by requiring that it be smooth along the regular orbits and one singular orbit, but along the second singular orbit it is smooth in the orbit direction and has an angle $2\pi/k$ normal to it. Since the normal space is two dimensional, this orbifold is still homeomorphic to $\mathbb{S}^4$. When k is even, we have the SO(3) equivariant two fold branched cover $\mathbb{CP}^2 \to \mathbb{S}^4$ mentioned in Section 2, and O_{2k} pulls back to an orbifold structure on $\mathbb{CP}^2$ with angle $2\pi/k$ normal to the real points $\mathbb{RP}^2 \subset \mathbb{CP}^2$. If we equip the orbifold with an orientation and with an orbifold metric invariant under the SO(3) action, we can define the SO(3) principal bundle H_k of the vector bundle of self dual 2-forms on O_k. The orientation we choose is adapted to the cohomogeneity one action as follows. Recall that we have a natural basis in the orbit direction corresponding to the action fields $X_1^* = E_{12}^*$, $X_2^* = E_{13}^*$, $X_3^* = E_{23}^*$ and we choose the orientation defined by $\gamma', X_1^*, X_2^*, X_3^*$ where γ is the normal geodesic chosen in Section 2. Along the singular orbits X_1^* respectively X_2^* vanishes and should be replaced by the derivative of the Jacobi field induced by their action fields. The isometric cohomogeneity one action of $\mathrm{SO}(3)$ on O_k clearly lifts to an action on the bundle of self dual 2-forms and thus onto the principal bundle H_k. It commutes with the principal bundle action of $\mathrm{SO}(3)$ and together they form an $\mathrm{SO}(3)\times\mathrm{SO}(3)$ cohomogeneity one action on H_k. We now show:

THEOREM 4.1. *The total space H_k of the* $\mathrm{SO}(3)$ *principal orbifold bundle of self dual 2-forms on O_k is smooth and the cohomogeneity one manifolds P_k and Q_k are equivariantly diffeomorphic to the (2-fold) universal covers of H_{2k-1} and H_{2k} respectively.*

Proof. Recall that for the $\mathrm{SO}(3)$ cohomogeneity one action on $\mathbb{S}^4$ in Section 2, the isotropy groups are given by $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \subset \{\mathrm{O}(2), \mathrm{O}'(2)\} \subset \mathrm{SO}(3)$ where the two singular isotropy groups are embedded in two different blocks. Since the metric on O_k is smooth near B_-, it follows that we still have $K^- \cong \mathrm{O}(2)$, which we can assume is embedded in the upper block, and hence $H \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ embedded as the set of diagonal matrices in $\mathrm{SO}(3)$. The normal angle along B_+ is $2\pi/k$ and a neighborhood of B_+ can be described as follows. The homomorphism $\phi_k\colon \mathrm{SO}(2) \to \mathrm{SO}(2)$, $\phi_k(A) = A^k$ gives rise to a homomorphism $j_k\colon K_0^+ \simeq \mathrm{SO}(2) \to \mathrm{SO}(3)$, where $A \in K_0^+$ goes to $\phi_k(A)$ followed by an embedding into the lower block of $\mathrm{SO}(3)$. We can now define $K^+ = \mathrm{O}(2)$ for k odd, $K^+ = \mathrm{O}(2) \times \mathbb{Z}_2$ for k even and extend the homomorphism to $j_k\colon K^+ \to \mathrm{SO}(3)$ such that $\mathrm{diag}(1,-1) \in \mathrm{O}(2)$ goes to $\mathrm{diag}(-1,1,-1)$ and the non-trivial element in $\mathbb{Z}_2$ goes to $\mathrm{diag}(1,-1,-1)$ when k is even. A neighborhood of the singular orbit on the right is then given by $D(B_+) = \mathrm{SO}(3) \times_{K^+} D^2_+$ where K^+ acts on $\mathrm{SO}(3)$ via j_k, K_0^+ acts on D^2_+ via ϕ_2, $\mathrm{diag}(1,-1) \in \mathrm{O}(2)$ acts as a reflection, and $\mathbb{Z}_2$ acts trivially. Indeed, we then have $\mathrm{SO}(3) \times_{K^+} D^2_+ = \mathrm{SO}(3) \times_{(K^+/\ker j_k)} (D^2_+/\ker j_k)$ with singular orbit $\mathrm{SO}(3)/\mathrm{O}(2)$ and normal disk $D^2_+/\ker j_k = D^2_+/\mathbb{Z}_k$. Furthermore, $\partial D(B_+) = \mathrm{SO}(3) \times_{K^+} \mathbb{S}^1_+ = \mathrm{SO}(3)/H$.

The vector bundle of self dual two forms can be viewed as follows: Let P be the $\mathrm{SO}(4)$ principal bundle of oriented orthonormal frames in the orbifold tangent bundle of O_k. This frame bundle is a smooth manifold since the finite isometric orbifold groups act freely on frames. $\mathrm{SO}(4)$ has two normal subgroups $\mathrm{SU}(2)_-$ and $\mathrm{SU}(2)_+$, given by left and right multiplication of unit quaternions, with $\mathrm{SO}(4)/\mathrm{SU}(2)_\pm \simeq \mathrm{SO}(3)$. The $\mathrm{SO}(3)$ principal bundles $P/\mathrm{SU}(2)_+$ and $P/\mathrm{SU}(2)_-$ are then the principal bundles for the vector bundle of self dual and the vector bundle of anti self dual 2 forms. This is due to the fact that the splitting $\Lambda^2 V \cong \Lambda^2_- V \oplus \Lambda^2_+ V$ for an oriented four dimensional vector space corresponds to the splitting of Lie algebra ideals $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ under the isomorphism $\Lambda^2 V \cong \mathfrak{so}(4)$. Alternatively, we can first project under the two fold cover $\mathrm{SO}(4) \to \mathrm{SO}(3)\,\mathrm{SO}(3)$ and then divide by one of the $\mathrm{SO}(3)$ factors. The action of $\mathrm{SO}(4)$ on P is only almost free since $P/\mathrm{SO}(4) = O_k$ but we will show that both $\mathrm{SU}(2)_+$ and $\mathrm{SU}(2)_-$ act freely on P, or equivalently, each $\mathrm{SO}(3)$ factor in $\mathrm{SO}(3)\,\mathrm{SO}(3)$ acts freely on $P/\{-\mathrm{Id}\}$. This then implies that $P/\mathrm{SU}(2)_+ = H_k$ is indeed a smooth manifold.

The description of the disc bundle $D(B_+)$ gives rise to a description of the corresponding $\mathrm{SO}(4)$ frame bundle $\mathrm{SO}(3) \times_{K^+} \mathrm{SO}(4)$ where the action of K^+ on $\mathrm{SO}(3)$ is given by j_k as above, and the action of K_0^+ on $\mathrm{SO}(4)$ is given via $\mathrm{SO}(2) \subset \mathrm{SO}(4)\colon A \in \mathrm{SO}(2) \to (\phi_k(A), \phi_2(A))$ acting on the splitting $T_+ \oplus T_+^\perp$ into tangent space and normal space of the singular orbit. Similarly for the left hand side where $k = 1$. On the left hand side the X_1^*

direction collapses, T_- is oriented by X_2^*, X_3^* and $T_-^\perp$ by $\gamma'(0), X_1^*$. On the right hand side the X_2^* direction collapses, T_+ is oriented by X_3^*, X_1^* and $T_+^\perp$ by $\gamma'(L), X_2^*$. Furthermore, $\mathrm{SO}(2) \subset \mathrm{O}(2)$ has negative weights on $T_\pm$, where we have endowed the isotropy groups on the left and on the right with orientations induced by X_1 and X_2 respectively. Indeed, $[E_{12}, E_{13}] = -E_{23}$ on the left and $[E_{13}, E_{23}] = -E_{12}$ on the right. On $T_-^\perp$, the weight is positive, and on $T_+^\perp$ negative. Hence $K_0^\pm \subset \mathrm{SO}(3)\,\mathrm{SO}(4)$ sits inside the natural maximal torus in $\mathrm{SO}(3)\,\mathrm{SO}(4)$ with slopes $(1, -1, 2)$ on the left, and $(k, -k, -2)$ on the right. To make this precise metrically, we can choose as a metric on H_k, the natural connection metric induced by the Levi Cevita connection on O_k. A parallel frame is then a geodesic in this metric. By equivariance under H, the unit vectors $X_i^*/|X_i^*|$ form such a parallel frame.

Under the homomorphism $\mathrm{SO}(4) \to \mathrm{SO}(3)\,\mathrm{SO}(3)$ and the natural maximal tori in $\mathrm{SO}(4)$ and in $\mathrm{SO}(3)\,\mathrm{SO}(3)$, a slope (p, q) circle goes into one with slope $(p+q, -p+q)$. Hence the slopes of $K_0^\pm$ in $\mathrm{SO}(3)\,\mathrm{SO}(3)\,\mathrm{SO}(3)$ are $(1, 1, 3)$ on the left, and $(k, -(k+2), k-2)$ on the right. This also implies that the second and third $\mathrm{SO}(3)$ factor each act freely on $P/\{-\mathrm{Id}\}$. Here we have used the fact that we already know that $\mathrm{SO}(4)$, and thus each $\mathrm{SO}(3)$, acts freely on the regular part and hence freeness only needs to be checked in K_0^+. Notice also that for k even, all slopes in K_0^+ should be divided by 2 to make the circle description effective. If we divide by the third $\mathrm{SO}(3)$ to obtain H_k, the slopes are $(1, 1)$ on the left and $(k, -(k+2))$ on the right. Finally, notice that the principle isotropy group of the $\mathrm{SO}(3)\,\mathrm{SO}(3)$ action on H_k is again $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ since this is true for the $\mathrm{SO}(3)$ action on O_k and $\mathrm{SO}(4)$ acts freely on the regular points in P. This determines the group diagram. For $k = 2m-1$, it is the group diagram of the two fold subcover of P_m obtained by dividing $G = \mathrm{S}^3 \times \mathrm{S}^3$ by its center. For $k = 2m$, this is the group picture of the two fold subcover of Q_m obtained by adding a component to all 3 isotropy groups (generated e.g. by (j, j)). This finishes our proof. ☐

Remarks. (a) The proof also shows that the $\mathrm{SO}(3)$ principal bundles $P/\,\mathrm{SU}(2)_-$ corresponding to the vector bundle of anti-self dual two forms is smooth and has slopes $(1, 3)$ on the left and $(k, k-2)$ on the right. Note that in the case of $k = 3$ one obtains the slopes for the exceptional manifold B^7 and in the case of $k = 4$ the ones for R (up to 2-fold covers).

(b) In the case of $k = 2\ell$, we can regard O_k as an orbifold metric O_ℓ on $\mathbb{CP}^2$. In this case it follows that the $\mathrm{SO}(3)$ principal bundle of the bundle of self dual two forms is Q_ℓ itself.

We now explain the relationship to the Hitchin metrics. Recall that a metric on M is called 3-Sasakian if $G = \mathrm{SU}(2)$ or $G = \mathrm{SO}(3)$ acts isometrically and almost freely with totally geodesic orbits of curvature 1. Moreover, for U tangent to the $\mathrm{SU}(2)$ orbits and X perpendicular, $X \wedge U$ is required to be an eigenvector of the curvature operator $\hat{R}$ with eigenvalue 1, in particular the sectional curvatures $\sec(X, U)$ are equal to 1. In the case we are interested in, where the dimension of M is 7, the quotient $B = M^7/G$ is 4-dimensional and its induced metric is self-dual Einstein with positive scalar curvature, although it is in general only an orbifold metric. Recall that a metric is called self dual if the curvature operator satisfies $\hat{R} \circ \star = \star \circ \hat{R}$. Conversely, given a self-dual Einstein orbifold metric on B^4 with positive scalar curvature, the $\mathrm{SO}(3)$ principal orbifold bundle of self dual 2-forms on

B^4 has a 3-Sasakian orbifold metric given by the naturally defined Levi Cevita connection metric. See [BG] for a survey on this subject.

Recall that $\mathbb{S}^4$ and $\mathbb{CP}^2$, according to Hitchin, are the only smooth self dual Einstein 4-manifolds. The 3-Sasakian metrics they give rise to are the metric on $\mathbb{S}^7(1)$ in the first case, and in the second case the metric on the Wallach space $W^7_{(2)}$ described in Section 2. However, in the more general context of orbifolds, Hitchin constructed in [Hi1] a sequence of self dual Einstein orbifolds O_k homeomorphic to $\mathbb{S}^4$, one for each integer $k > 0$. The metric is invariant under the cohomogeneity one action by SO(3) from Section 2 and has an orbifold singularity as in the orbifold O_k discussed earlier. The cases of $k = 1, 2$ correspond to the smooth standard metrics on $\mathbb{S}^4$ and on $\mathbb{CP}^2$ respectively. Hence the Hitchin metrics give rise to 3-Sasakian orbifold metrics on the seven dimensional orbifold H^7_k. Here one needs to check that the orientation we chose above agrees with the orientation in [Hi1]. As we saw in Theorem 4.1, H_k is actually smooth and the 3-Sasakian metric, as a quotient of the smooth connection metric on the principal frame bundle, is also smooth. Thus our candidates P_k and Q_k all admit a smooth 3-Sasakian metric. In the context of 3-Sasakian geometry, the examples P_k are particularly interesting since they are two connected, and so far, the only known 2-connected example in dimension 7, was $\mathbb{S}^7$.

It was shown by O.Dearricott in [De] (see also [CDR]) that a 3-Sasakian metric, scaled down in direction of the principal $G = \mathrm{SO}(3)$ or $\mathrm{SU}(2)$ orbits, has positive sectional curvature if and only if the self dual Einstein orbifold base has positive curvature. It is therefore interesting to examine the curvature properties of the Hitchin metrics, which we will now discuss shortly. The metric is described by the 3 functions $T_i(t) = |X_i^*(\gamma(t))|^2$ along a normal geodesic γ, since invariance under the isotropy group implies that these vectors are orthogonal. It turns out that in order to solve the ODE along γ given by the condition that the metric is self dual Einstein, it is convenient to change the arc length parameter from t to r. The metric is thus described by

$$g_{\gamma(r)} = f(r)dr^2 + T_1(r)d\theta_1^2 + T_2(r)d\theta_2^2 + T_3(r)d\theta_3^2,$$

where $d\theta_i$ is dual to X_i. In order to solve the ODE, Hitchin uses complex algebraic geometry on the twistor space of O_k. For general k, the solutions are explicit only in principal and it is thus a tour de force to prove the required smoothness properties of the metric. For small values of k though, one finds explicit solutions in [Hi1] and [Hi2]:

Example 1. The first non-smooth example is the Hitchin metric with normal angle $2\pi/3$. Here the functions are algebraic:

$$\begin{aligned}
T_1 &= \frac{80r^2(r^6 - 2r^5 - 5r^4 - 15r^3 - 20r^2 + 13r + 4)}{(3r^3 + 7r^2 + r + 1)^2(3r^3 - 13r^2 + r + 1)},\\
T_2 &= \frac{5r(3r-1)(r-\beta)(r+\beta+2)(2 - 3r - r^2 + r^3 + 5\beta r)^2}{(3r^3 + 7r^2 + r + 1)^2(r^2 + r - 1)(r^2 + r + 4)},\\
T_3 &= \frac{5r(3r-1)(r+\beta)(r-\beta+2)(2 - 3r - r^2 + r^3 - 5\beta r)^2}{(3r^3 + 7r^2 + r + 1)^2(r^2 + r - 1)(r^2 + r + 4)},\\
f &= \frac{5(3r-1)(r^2 + r + 4)(r+1)^2}{(r + r^2 - 1)/(3r^3 + 7r^2 + r + 1)^2},
\end{aligned}$$

with $\beta = \sqrt{\frac{r+r^2-1}{r}}$ and $\frac{\sqrt{5}-1}{2} \le r \le 1$.

Example 2. The simplest example of a non-smooth Hitchin metric has normal angle $2\pi/4$, where the functions are given by:

$$T_1 = \frac{(1-r^2)^2}{(1+r+r^2)(r+2)(2r+1)}\ ,\ T_2 = \frac{1+r+r^2}{(r+2)(2r+1)^2}\ ,\ T_3 = \frac{r(1+r+r^2)}{(r+2)^2(2r+1)},$$
$$f = \frac{1+r+r^2}{r(r+2)^2(2r+1)^2},$$

with $1 \le r < \infty$.

Example 3. Finally, we have the Hitchin metric with normal angle $2\pi/6$:

$$\begin{aligned}
T_1 &= \frac{(3r^2+2r+1)(r^2+2r-1)^2(r^2-2r+3)(r^2+1)}{(3r^2-2r+1)(r^2-2r-1)^2(r^2+2r+3)^2},\\
T_2 &= \frac{(3r^2-2r+1)(r^2-2r+3)(r+1)^3(r-1)}{(3r^2+2r+1)(r^2+2r+3)^2(r^2-2r-1)},\\
T_3 &= \frac{-4(3r^2-2r+1)(3r^2+2r+1)r}{(r^2+2r+3)^2(r^2-2r-1)(r^2-2r+3)},\\
f &= \frac{(r+1)(r^2-2r+3)(3r^2+2r+1)(3r^2-2r+1)}{r(1-r)(r^2-2r-1)^2(r^2+1)(r^2+2r+3)^2},
\end{aligned}$$

with $\sqrt{2}-1 \le r \le 1$.

Although one can in principle use the methods in [Hi1] to determine the functions for larger values of k, they quickly become even more complicated. The above 3 cases are sufficient though to understand the behavior in general. If the functions are given in arc length parameter, one has $\sec(\gamma', X_i^*) = -f_i''/f_i$, where $f_i = \sqrt{T_i}$, and hence positive curvature is equivalent to the concavity of f_i. In Figures 8-10 we therefore have drawn a graph of the length functions f_i in arc length parameter, together with a graph of $\sec(\gamma', X_i^*)$. The pictures are similar, but notice the difference in scale. One sees that the non-smooth singular orbit must occur at $t = L$ since it is necessarily totally geodesic, which implies that the non-collapsing functions have 0 derivative. The pictures show that the function f_1, which vanishes at the smooth singular orbit, is concave, whereas the other two are slightly convex near the smooth singular orbit.

For each of the 3 elements $g \in H$, the fixed point set of g is a 2-sphere, since this is clearly true for the linear action on $\mathbb{S}^4$ corresponding to $k = 1$. They are isometric to each other via an element of the Weyl group. Since the circle that commutes with g acts by isometries on the 2-sphere, it is rotationally symmetric with an orbifold point at one of the poles. It has positive curvature, except in a small region two thirds toward this pole. Figure 11 shows the length of the action field induced by the circle action on this 2-sphere (which is equal to $f_i/2$) in the case of $k = 3$ and $k = 6$. Notice that it extends from 0 to $3L$. These 2-spheres can also be isometrically embedded as surfaces of revolution in 3-space, which we exhibit in Figure 12.

One can show that, as a consequence of being self dual Einstein, $\sec(X_i^*, X_j^*) = \sec(\gamma', X_k^*)$, when i, j, k are distinct, and that if all 3 are positive, the curvature of any 2-plane is indeed positive also. Thus the Hitchin metric has positive curvature wherever the above orbifold 2-sphere has positive curvature. By Dearricott's theorem, this implies that the induced 3-Sasakian metric on our candidates, scaled down in direction of the principal SO(3) orbits, has positive sectional curvature on half of the manifold.

Thus this metric does not yet give the desired metrics of positive curvature on P_k and Q_k. It is also tempting to think that, as in the case of the known actions in Section 2, simple trigonometric expressions for the 9 functions describing a metric on P_k, Q_k or R might yield a metric with positive curvature on our candidates. But this does not seem to be the case either, since already the smoothness conditions and simple necessary convexity properties require trigonometric functions that are quite complicated.

It is intriguing that the (non-compact) space of 2-monopoles studied by Atiyah and Hitchin in [AH] has surprisingly similar properties to the above metric. It carries a self dual Einstein orbifold metric which in this case is Ricci flat, i.e., is Hyperkähler. It is invariant under SO(3) with principal orbits $\mathrm{SO}(3)/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and a singular orbit $\mathbb{RP}^2$ with normal angle $2\pi/k$. Of the 3 functions describing the metric, one is concave as well, and the other two are not. Thus one of the fixed point sets of elements in H is a (non-compact) surface of revolution with positive curvature.

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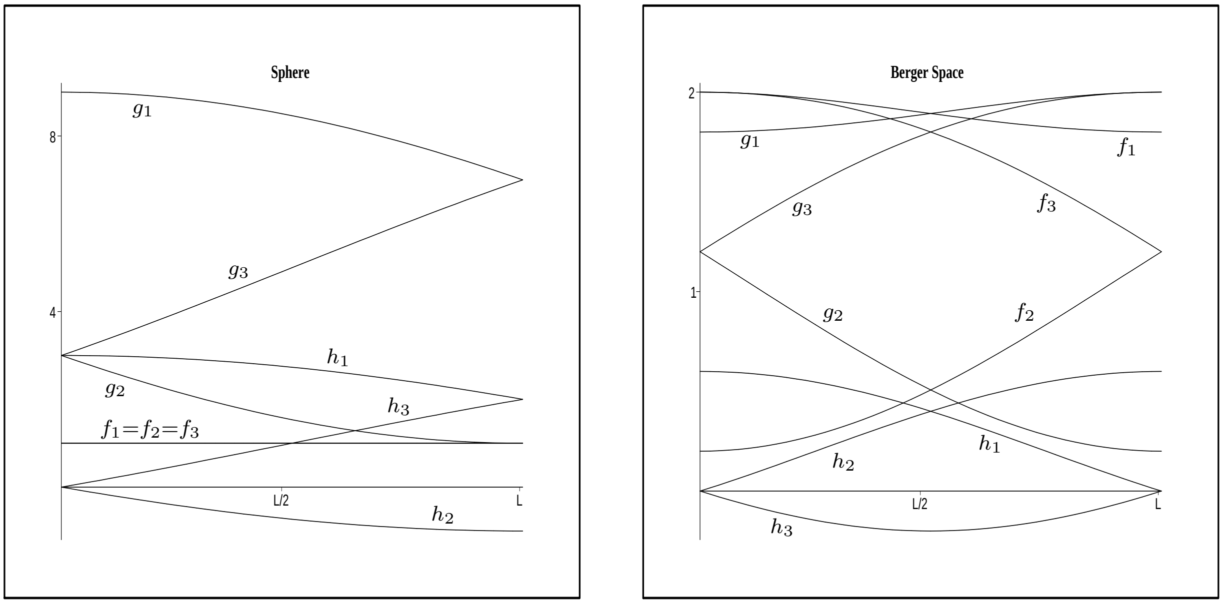

FIGURE 1. All 9 functions on $[0, L]$.

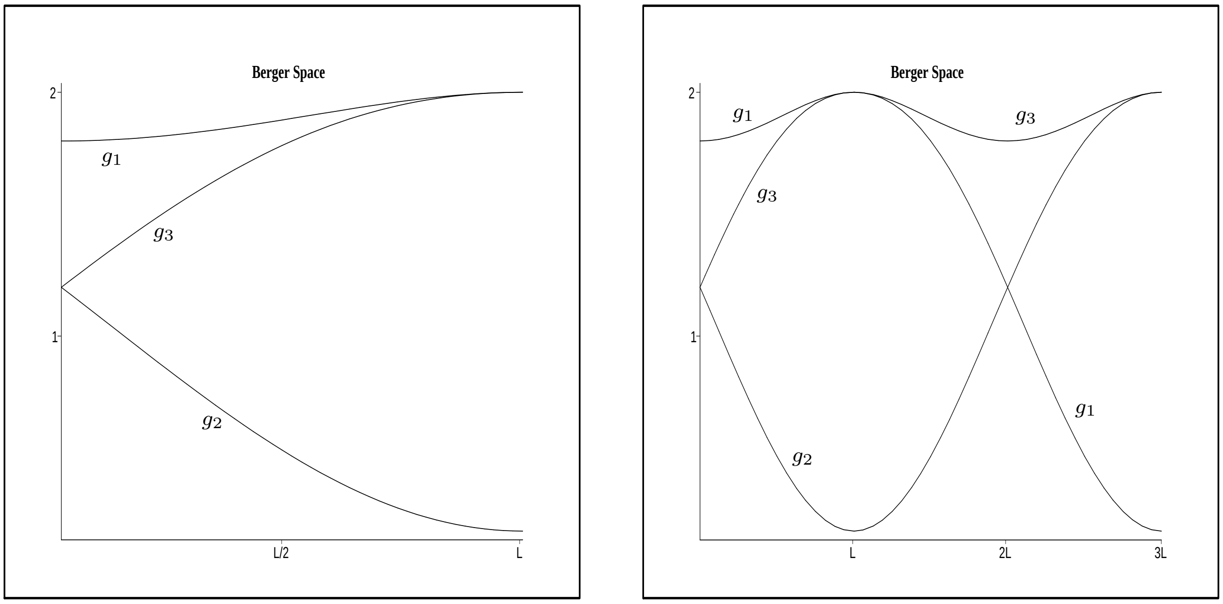

FIGURE 2. The g functions on $[0, L]$ and $[0, 3L]$.

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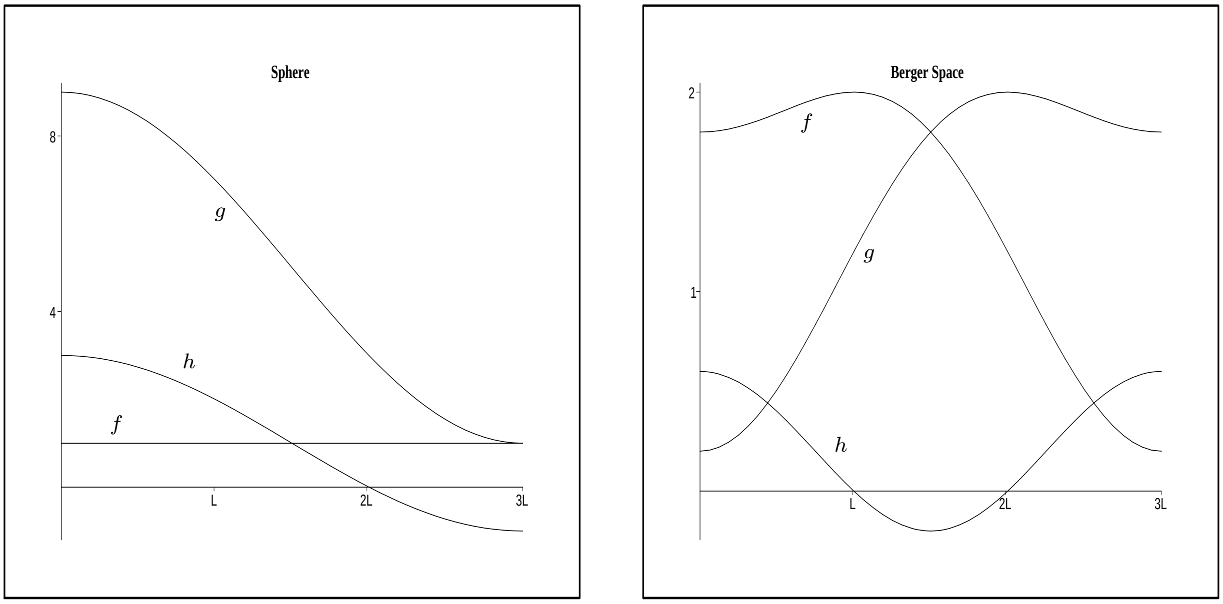

FIGURE 3. All functions on $[0, 3L]$.

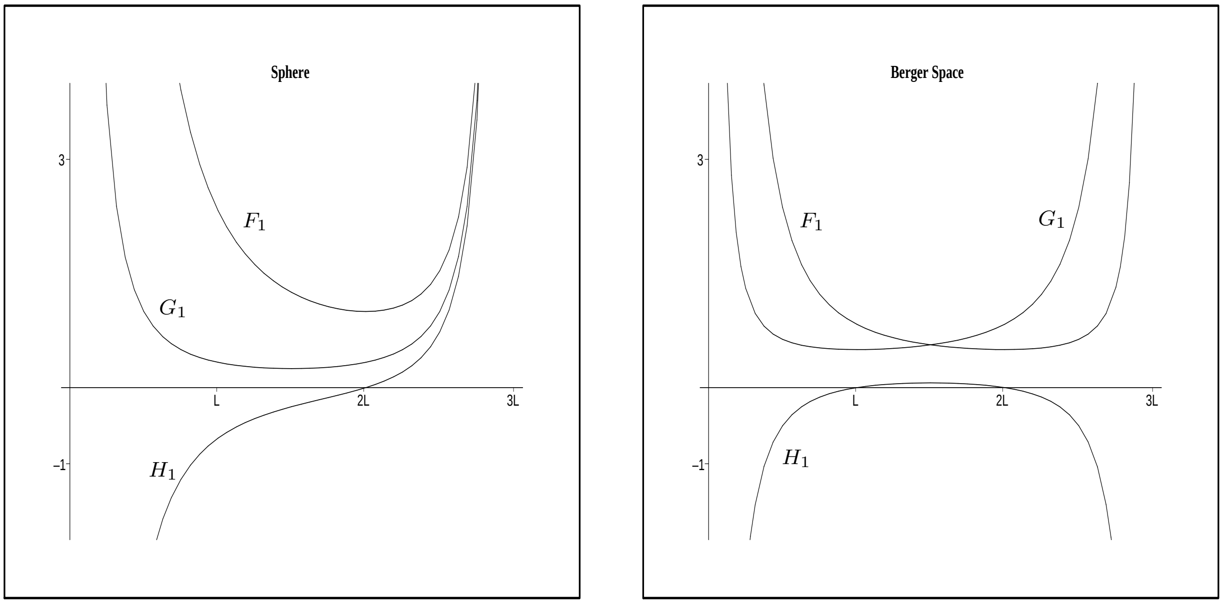

FIGURE 4. The inverse functions on $[0, 3L]$.

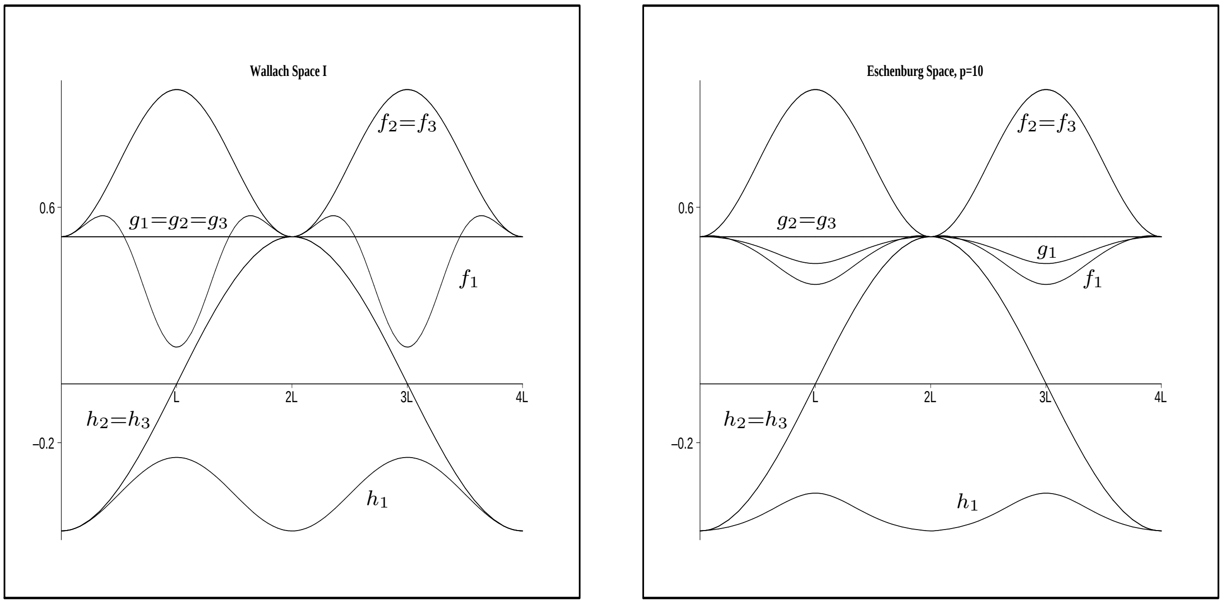

FIGURE 5. Wallach space $W^7_{(1)}$ and Eschenburg space E_{10} on $[0, 4L]$.

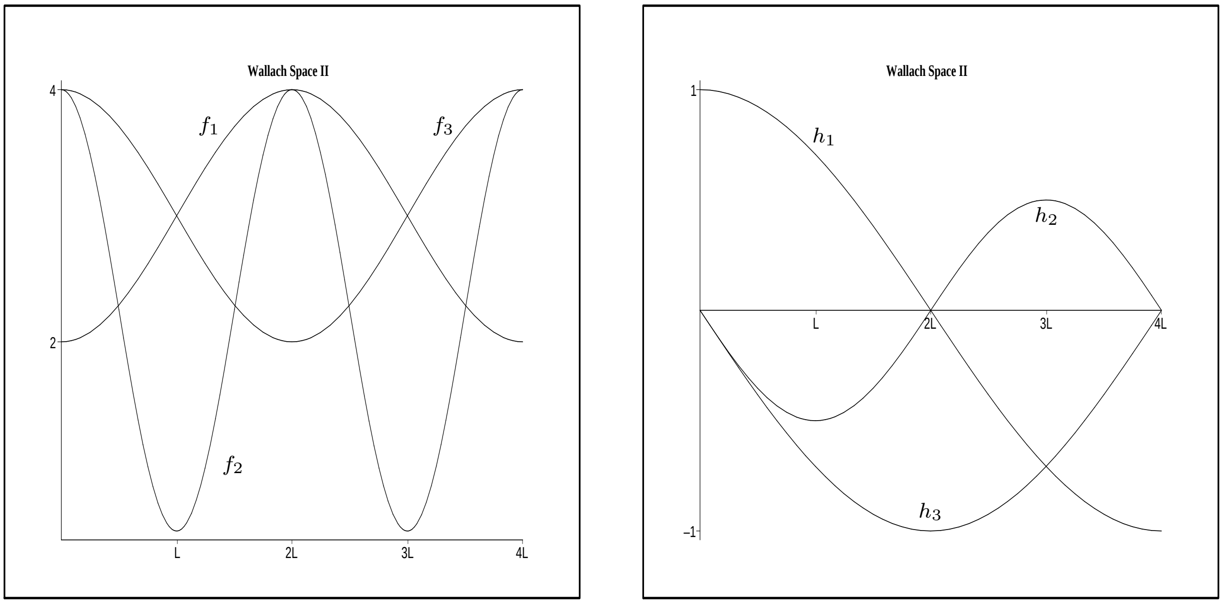

FIGURE 6. Wallach space $W^7_{(2)}$ on $[0, 4L]$.

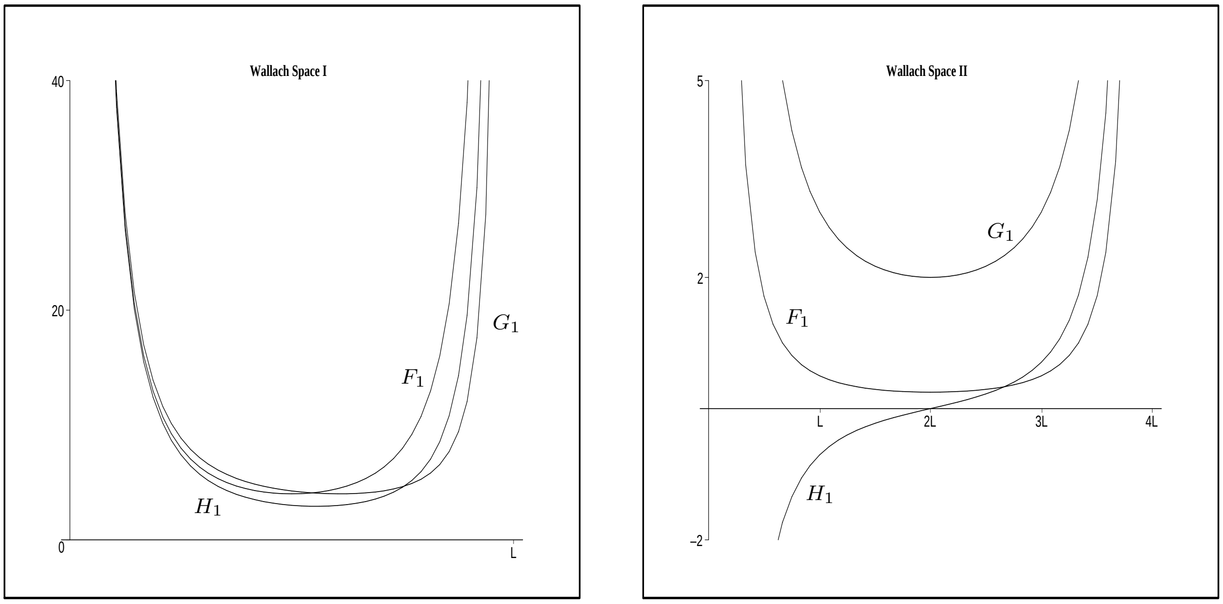

FIGURE 7. Inverse functions for $W^7_{(1)}$ and $W^7_{(2)}$.

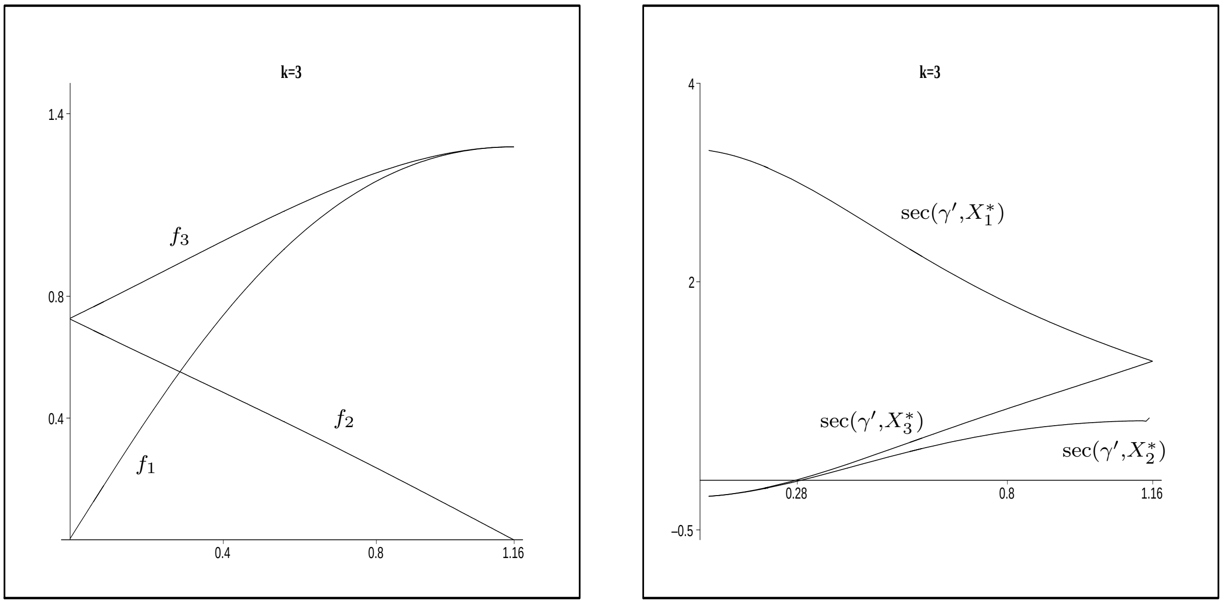

FIGURE 8. Hitchin metric with Normal angle $2\pi/3$.

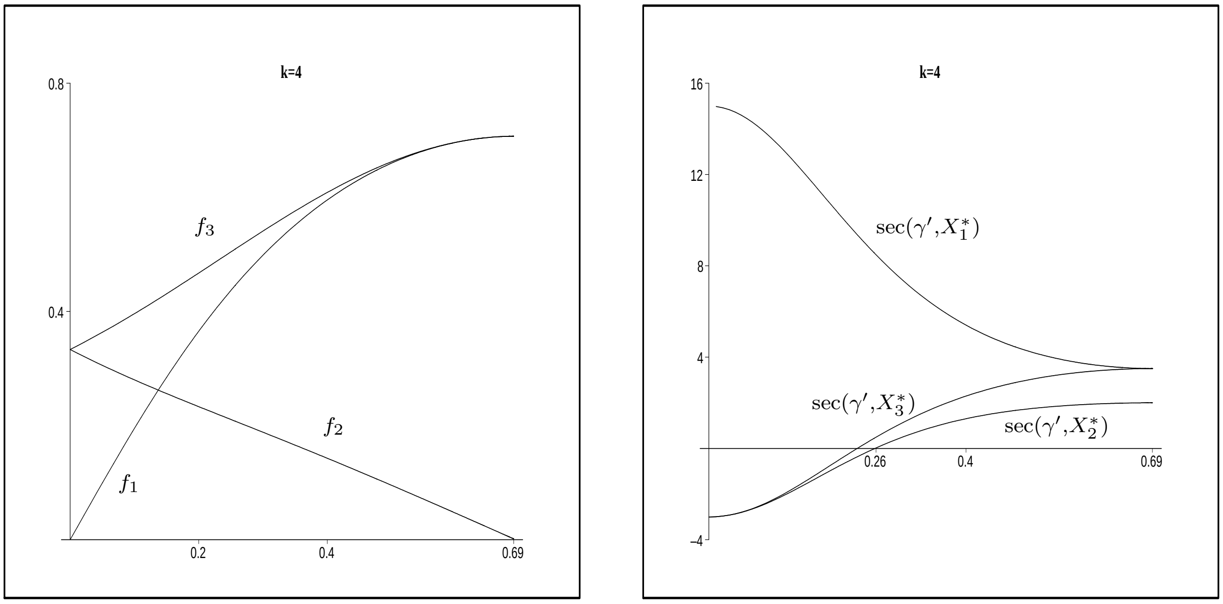

FIGURE 9. Hitchin metric with Normal angle $2\pi/4$.

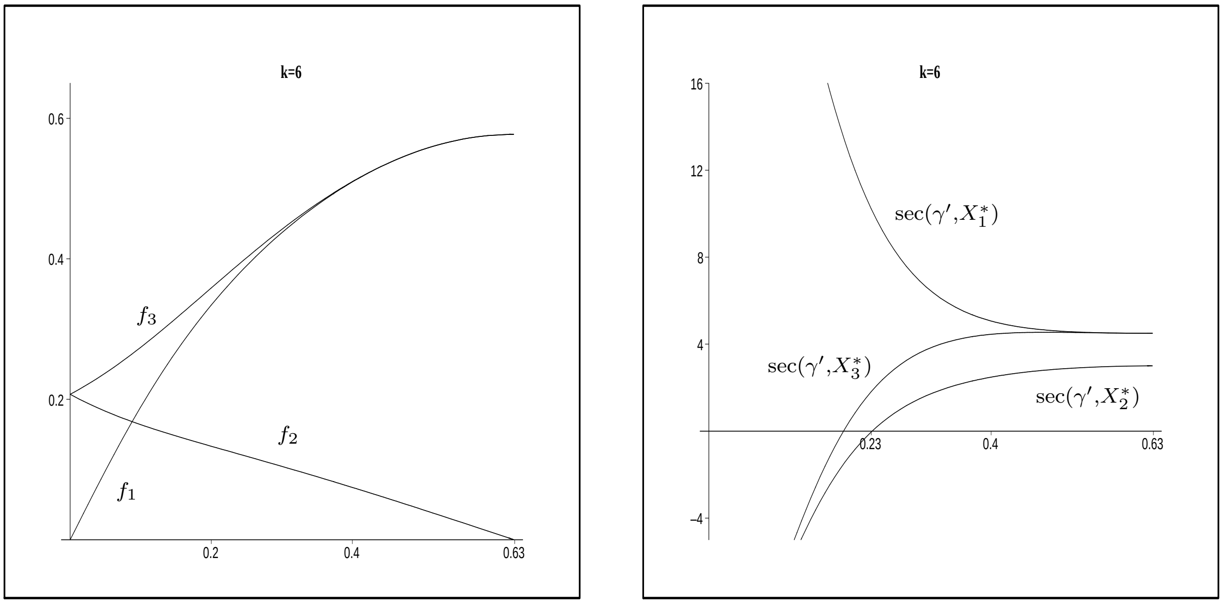

FIGURE 10. Hitchin metric with Normal angle $2\pi/6$.

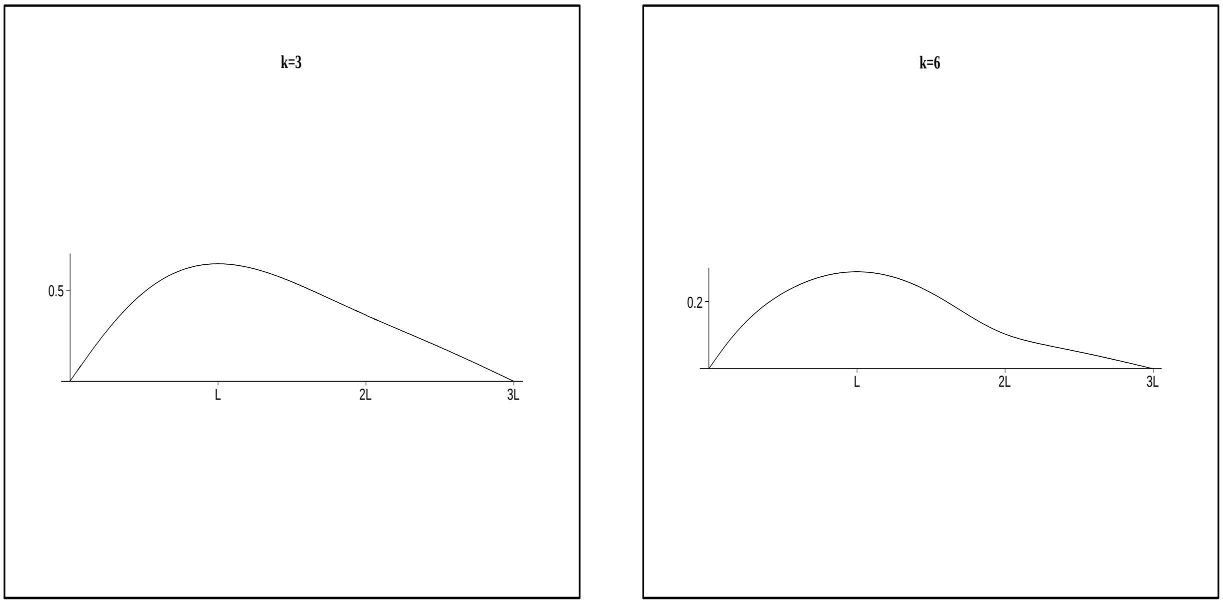

Figure 11. Hitchin metrics on $[0, 3L]$.

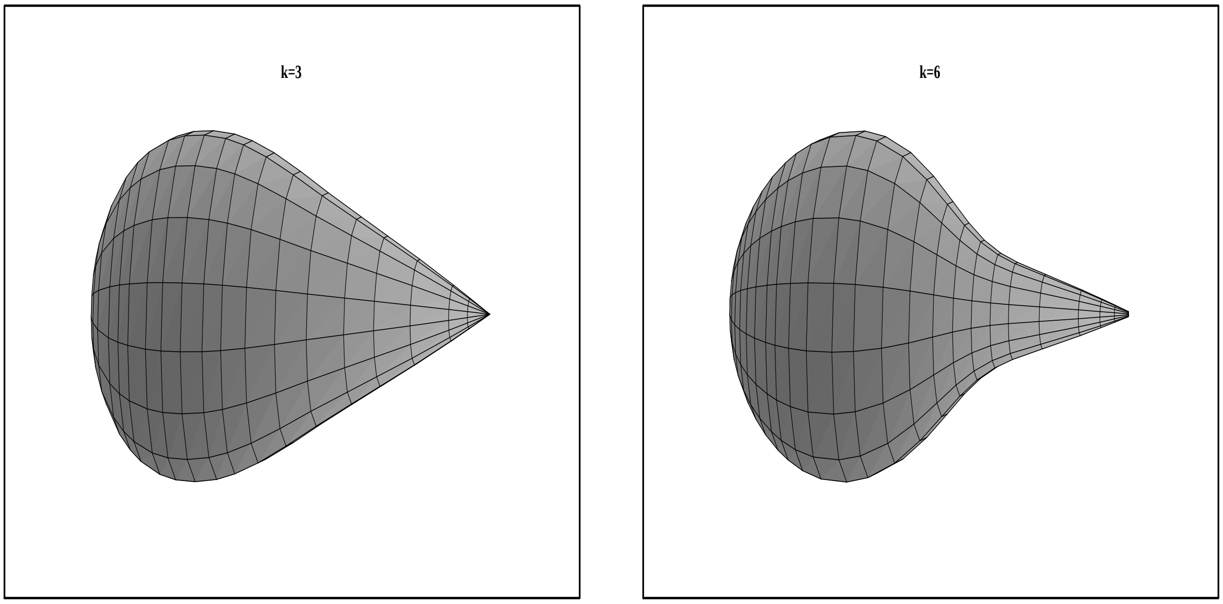

Figure 12. Fixed point orbifold 2-spheres in Hitchin metrics.