ON THE ORTHOGONAL GROUP OF UNIMODULAR QUADRATIC FORMS: ORBITS OF ℓ-TUPLES

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ABSTRACT. Generalizing work of [W] and [A1], [A2], we define a *complete set* of isometric invariants for an l-tuple of pairwise orthogonal, linearly independent, primitive, elements and prove that it characterizes the orbits of such pairs under the action of the group of isometries, when a certain indefiniteness condition holds for the lattice.

1. Introduction

In ([W]) C. T. C. Wall proved that the group of isometries $O(L)$ of a unimodular, integral lattice $(L,\cdot)$, whose rank $r(L)$ and signature $\sigma(L)$ satisfy $r(L)-|\sigma(L)|\geq 4$, acts transitively on elements of the lattice of the same square, type and divisibility (type refers to characteristic or ordinary: $a\in L$ is characteristic if $a\cdot x\equiv x\cdot x \bmod 2,\ \ \forall x\in L$. It is otherwise called ordinary). In [A1], [A2] we isolated a complete set of isometric invariants which characterize the orbit of a pair of linearly independent, mutually orthogonal primitive, ordinary elements provided that $r(L)-|\sigma(L)|\geq 6$. In this paper we formulate a *complete set* of isometric invariants of an arbitrary number of linearly independent, mutually orthogonal, primitive elements $\alpha_i,\ \ i=1,..,l$ with $\alpha_i\cdot\alpha_j=0$ when $i\neq j$ and prove the transitivity of the action of $O(L)$ on all such sets of elements with the same *complete set* of isometric invariants. In addition to the squares of the elements we formulate the invariants into what we call the **modular invariants** which are defined for every lattice (even or odd) which arise essentially from torsion information and consist of various torsion groups together with certain elements in these groups, and to the **type** of the l-tuple which is a generalization of the type of a single element. It was somewhat of a surprise to me that briefly speaking a l-tuple can be **characteristic** (cf. the definiton of Section 4) even though none of its members are! The type invariant need not be predicted by the modular invariants.

2. Some Basic Notation

We recall some notation and terminology from [A1]. Let R_s be the symmetric, bilinear, unimodular, integral lattice with basis $x_i, y_i,\ i=1,...,s$ and form given by: $x_i\cdot y_i=1\ \ \forall i$ and all others zero (i.e. orthogonal direct sum of s hyperbolic pieces). Let $O(R_s)$ be the orthogonal group. The generators of $O(R_2)$ are explicitly

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listed in [W] and we list them here for convenience.

	a^n	b^n	c	d	d'	e
$x_1 \rightarrow$	$x_1 + nx_2$	x_1	x_2	$-x_1$	x_1	x_1
$y_1 \rightarrow$	y_1	$y_1 - ny_2$	y_2	$-y_1$	y_1	y_1
$x_2 \rightarrow$	x_2	$x_2 + nx_1$	x_1	x_2	$-x_2$	y_2
$y_2 \rightarrow$	$y_2 - ny_1$	y_2	y_1	y_2	$-y_2$	x_2

When in R_s we mean isometry a between the i^{th} and j^{th} summands we will write $ij - a$. When we apply Wall's result regarding transitivity on a single element in the i^{th} and j^{th} summands we will write $ij -$ [W]. When dealing with an odd lattice L of zero signature and rank $r(L) = r$ we will view it as $R_r \oplus (1) \oplus (-1)$, with basis $x_i, y_i,\ \ i = 1, \dots, r,$ and u, v such that $u^2 = 1 = -v^2$. An odd lattice L of signature $\sigma(L) = \sigma \geq 1$ and rank $r(L) = 2r + \sigma$ will be thought of as $R_r \oplus_\sigma (+1)$, with basis $x_i, y_i,\ \ i = 1, \dots, r,\ \ h_j,\ \ j = 1, \dots, \sigma,\ h_j^2 = 1$. We need to introduce some isometries of $R_r \oplus_\sigma (+1)$, $R_r \oplus (1) \oplus (-1)$, which we will use later. Some notation will be necessary. Let $\overline{m} = (m_1, ..., m_\sigma)$. Then $\overline{m}_i$, (resp. $\overline{m}_{ij}$) will be vectors with 1 in the i-th (resp. i-th and j-th) slots and zeros elsewhere. The isometries then are $\chi_{i,\overline{m}}$ given by:

$$h_j \rightarrow h_j + m_j x_i, \quad x_i \rightarrow x_i$$
$$y_i \rightarrow y_i - \tfrac{1}{2}(\sum_j m_j^2) x_i - \sum_j m_j h_j$$

In the same way we have isometries $\chi_{i,(m_1,m_2)}$ of $R_r \oplus (1) \oplus (-1)$, given by:

$$u \rightarrow u + m_1 x_i, \quad v \rightarrow v + m_2 x_i, \quad x_i \rightarrow x_i$$
$$y_i \rightarrow y_i + \tfrac{1}{2}(-m_1^2 + m_2^2) x_i - m_1 u + m_2 v$$

Finally here is some more notation we will be using throughout the paper. We write $I_{k+1} = (1, ..., k+1)$, by $\overline{X}_i(m)$ we will denote an element of the form: $-x_i + my_i$, Y(m) will denote the element $(2m-1)u + (2m+1)v$, for an element α of the lattice $\alpha|_m$ will denote the part of the element which lives in the span of $x_i, y_i,\ \ 1 \leq i \leq m$.

3. The Modular Invariants

First notice that the torsion of $L/ \prec \alpha_1, ..., \alpha_l \succ$ is the same as that of $\oplus_l Z/Im(\tau)$, where $\tau : L \rightarrow \oplus_l Z$ is given by $\tau(x) = (x \cdot \alpha_1, ..., x \cdot \alpha_l)$ for $x \in L$. Now we begin decomposing L inductively in such a way that it will lead us to the sought after definition of the *modular invariants*. Before we begin we fix the ordered set of elements $(\alpha_1, ..., \alpha_l)$ as well as their squares $\alpha_i^2 = \alpha_i \cdot \alpha_i$ i=1,...,l. We will write $L_i^\perp$ for $\prec \alpha_1 \succ^\perp \cap \dots \cap \prec \alpha_i \succ^\perp$. Then $\alpha_i \in L_{i-1}^\perp$, $\forall i = 1, \dots, l$, where $L_0^\perp = L$. Let d_{I_i} be the unique positive integer such that $x \cot \alpha_i \equiv 0 \bmod d_{I_i}$, $\forall x \imath L_{i-1}^\perp$. Call this integer the **modularity of α_i in $L_{i-1}^\perp$**. Notice that unimodularity of L implies $d_{I_1} = 1$. Now let $\alpha'_{I_i} \in L_{i-1}^\perp$ be such that $\alpha'_{I_i} \cdot \alpha_i = d_{I_i}$ and call it a **modular dual to α_i**. It is well defined modulo $L_i^\perp$. Obviously for all $i, n \in \{1, \dots, l\}$:

$$L_{i-1}^\perp = \prec \alpha'_{I_i} \succ \oplus L_I^\perp, \quad L = \prec \alpha'_{I_1} \succ \oplus \cdots \oplus \prec \alpha'_{I_n} \succ \oplus L_n^\perp$$

Now we are ready to define $u_{I_i s} \in Z/d_{I_s}$ for $s = 2, \dots, l$ and $i = 1, \dots, l-1$ as follows. For all $i = 1, \dots, l$ we can take $\alpha'_{I_i} \cdot \alpha_s$, $s = i, \dots l$. Recall $\alpha'_{I_i} \cdot \alpha_i = d_{I_i}$. Then since α'_{I_i} is well defined modulo $L_i^\perp$, $\alpha'_{I_i} \cdot \alpha_{i+1}$ is well defined modulo $d_{I_{i+1}}$.

So we get an element $[u_{I_i(i+1)}] \in Z/d_{I_{i+1}}$, where here and from now on $u_{I_i(s)}$ will be a canonical representative. Procceeding inductively suppose that we have defined $[u_{I_is}] \in Z/d_{I_s}$ forall $s = i+1, \dots, n-1$. Then look at the set of all α'_{I_i} such that $\alpha'_{I_i} \cdot \alpha_s = u_{I_is}$. Any such element in well defined modulo $L^{\perp}_{n-1}$ and hence $\alpha'_{I_i} \cdot \alpha_n$ is well defined modulo d_{I_n} thus giving rise to $[u_{I_in}] \in Z/d_{I_n}$. So we can now give:

Definition 3.1. *For a l-tuple of mutually orthogonal, linearly independent, primitive elements it's modular invariants are $u_{I_ij} \in Z/d_{I_j}$, $2 \le j \le l$, $1 \le i \le j-1$.*

4. The Type Invariant

In the case of an even lattice the **modular invariants** together with the squares of the elements will turn out to determine the orbit. In the case of an odd lattice there is a further invariant which is related to the type of the elements. For an l-tuple of mutually perpendicular linearly independent primitive (but not necessarily ordinary) elements α_i, $i = 1, \dots, l$ there is the issue of the type of the lattices: $L^{\perp}_k = \cap_{s=1}^k \prec \alpha_s \succ^{\perp}$ (i.e. whether they are even or odd). If they are odd for all $k < \xi_o$ and the one for ξ_0 turns even then $L^{\perp}_k$'s are even for all $k > \xi_0$. This number $\xi_0 \in \{0, 1, \dots, l\}$ (defined to be zero if $L^{\perp}_l$ is odd) is of course an invariant of the l-tuple and as it turns out, need not be predicted by the modular invariants we listed so far. Furthermore suppose that we have $k = \xi_0 \in \{1, \dots, l\}$. Suppose that α'_{I_s} are such that $\alpha'_{I_s} \cdot \alpha_{\xi_0} = u_{I_s\xi_0}$ for all $s = 1, \dots, \xi_0 - 1$. Then these are well defined modulo $L^{\perp}_{\xi_0}$. Since the latter is even the parities ζ_s of the squares of α'_{I_s}, $s = 1, \dots, \xi_0 - 1$ are invariants. So we are ready to state:

Definition 4.1. *For an l-tuple of mutually orthogonal, linearly independent, primitive elements in an odd lattice L define its type to be an element:*

$$(\xi_0; \zeta_1, \dots, \zeta_{l-1}) \in (Z/(l+1)) \oplus_{l-1} (Z/2).$$

It will be called ordinary if $(\xi_0, \zeta_1, \dots, \zeta_{l-1}) = (0; 0, \dots, 0)$ and characteristic of type $(\xi_0, \zeta_1, \dots, \zeta_{l-1})$ if $\xi_0 \neq 0$.

Remark 4.2. *Notice that $L^{\perp}_1$ is even iff α_1 is characteristic. So in a way this invariant generalizes the type of one element to type of an l-tuple. This is what made* [A2] *necessary. Although we were treating the case of a pair of ordinary elements in* [A1] *the pair itself may turn out to be "characteristic" in the sense of our definition above.*

5. Statement and Outline Of Proof

Theorem 5.1. *Let $(L, \cdot)$ be an integral, unimodular, lattice whose signature $\sigma(L)$ and rank $r(L)$ satisfy: $r(l) - |\sigma(L)| \ge l + 1$. The Orthogonal Group $O(L)$ of L acts transitively on all ordered l-tuples of pairwise orthogonal, linearly independent elements $(\alpha_1, ..., \alpha_l)$ in a lattice provided they have the same squares, divisibilities, type and modular invariants.*

That the divisisibilities are invariants is obvious so assume that all elements are primitive. One then fixes the squares and tries, after fixing bases and a suitable set of generating elements for $O(L)$, to find isometries which take the l-tuple to a form in which the coordinates of all elements are determined by the invariants. The basic strategy is to use the coordinates of a nice basis for the various $L^{\perp}_i$ as the guides to which isometries to use. We devide the proof in various parts spread over several sections as follows:

Section 6: Proof when L is even
Section 7: Proof when L is odd and $\sigma(\mathrm{L}) = 0$
  Subsection 7.1: α_1 is characteristic
  Subsection 7.2: α_1 is ordinary and VPS $= 1$
  Subsection 7.3: α_1 is ordinary and there is no PS
  Subsection 7.4: α_1 is ordinary, α_1^2 is even and there is PS
Section 8: Proof when L is odd and $\sigma(\mathrm{L}) \neq 0$
  Subsection 8.1: α_1 is characteristic
  Subsection 8.2: α_1 isordinary

The proof in the case of an even lattice is simple and elegant. The complications in the odd lattice case arise basically due to the fact that there are three possibilities for a primitive element in such a lattice: Ordinary (of even or odd square: see VPS) or characteristic.

6. Proof when L is even

We will assume that our elements are primitive. The canonical ordered l-tuple with invariants as in the statement of the theorem is given by :

$$\begin{aligned} \alpha_1 &= -x_1 + n_1 y_1 \\ \alpha_l &= \sum_{s=1}^{l-1}[r_{sl}x_s + B_{ls}y_s] + d_{I_l}(-x_l + n_l y_l), \end{aligned} \tag{1}$$

where $r_{I_i s} \equiv -u_{I_i s} \bmod d_{I_s}$ and $0 \leq u_{I_i s} \leq d_{I_s} - 1$ and the B_{ij}'s are inductively expressed as functions of the modular invariants via $\alpha_i \cdot \alpha_j = 0$. Notice also that the n_i's are determined by the invariants as well. So now we have to prove that there is an isometry which carries an l-tuple as in the statement to a canonical l-tuple. Before we begin we need to find a basis for $L_l^\perp$. It is easy to see that such a basis consists of

$$e_{ls}^\perp = x_s + n_s y_s + \sum_{t=s+1}^{l} P_{st} y_t, \quad s = 1, \dots, l, \quad x_i, y_i \quad i \geq l+1 \tag{2}$$

where the P_{ji}'s are computed in terms of the invariants via $e_{lj}^\perp \cdot \alpha_i = 0$. We will procceed by induction using [A1] as the base case. So suppose that we have found an isometry ϕ which takes the first k elements to the desired canonical form. Then $\phi(\alpha_{k+1})$ looks like:

$$\sum_{s=1}^{k}[R_{I_s(k+1)}x_s + V_{(k+1)s}y_s] + D_1(-x_{k+1} + N y_{k+1})$$

where $R_{I_i(k+1)} \equiv r_{I_i(k+1)} \bmod d_{I_{k+1}}$ and

$$V_{(k+1)i} = \frac{1}{d_{I_i}} \sum_{s=1}^{i-1}(r_{I_s i}V_{(k+1)s} + R_{I_s(k+1)}B_{is}) + R_{I_i(k+1)}n_i$$

Using (2) and the difinition of $d_{I_{k+1}}$ we can easily see that:

$$d_{I_{k+1}} = \gcd(e_{k1}^\perp \cdot \alpha_{k+1}, \dots, e_{kk}^\perp \cdot \alpha_{k+1}, D_1) \tag{3}$$

Now procceed by isometries. Let β be the part of α_{k+1} which lives in the first k blocks.

For all i=1,...,k, apply $(i)(k+2)-a^{n_i}\circ e\circ b^{-1}$ followed by $(i+j)(k+2)-a^{P_{(i)(i+j)}}$ for all j=1,...,k-i-1. The effect of these isometries is that at the i-th step while the first k elements stay fixed the last element gets sent to an element of the form $\beta+D_{i+1}(-x_{k+1}+N_iy_{k+1})+G_ix_{k+2}$ with $D_{i+1}=\gcd(D_i,(e_{li}^{\perp}\cdot\alpha_l)$ and some integer N_i. Then apply a suitable $(k+1)(k+2)-[W]$ to keep $\alpha_1,\dots,\alpha_k$ fixed and send the last element to $\beta+D_{i+1}(-x_{k+1}+N_{i+1}y_{k+1})$. Clearly then $D_{k+1}=d_{I_{k+1}}$. In the end making use of 3 we have $D_{k+1}=d_{I_{k+1}}$ and we found an isometry which fixes $\alpha_1,\dots,\alpha_k$ and sends α_{k+1} to an element of the form $\beta+d_{I_{k+1}}(-x_{k+1}+My_{k+1})$ Now let $R_{I_ik+1}=d_{I_{k+1}}Q_i+r_{I_ik+1}$. For all i=1,...,k apply: First $(k+1)(k+2)-a^{-1}$ and then $(i)(k+2)-a^{Q_in_i}\circ e\circ b^{-Q_i}$ followed by: $(i+j)(k+2)-a^{Q_iP_{(i)(i+j)}}$ for j=1,...,k-i-1 and a suitable $(k+1)(k+2)-[W]$ to send α_{k+1} at the end of the proccess to its desired form while keeping the first k elements fixed.

7. L is odd and $\sigma(L)=0$.

We need to define some quantities which are determined by the invariants and which will be usefull devices for the write-up of the proof. For any ordinary l-tuple like the ones we consider we can define certain *parity quantities* π_i. What will be important about these quantities is the first index for which they become (if ever) odd. They are defined by:

$$\pi_i=\{\frac{1}{d_{I_i}}[\alpha_i+\sum_{s=1}^{i-1}u_{I_si}e_{(i-1)s}^{\perp}]\}^2$$

If for a tuple π_s is even $\forall s\le i-1$ and π_i is even we say that a parity switch occurs otherwise that it doesn't and i will be called the *value of the parity switch* and we will write it VPS.

7.1. α_1 is characteristic.

The canonical l-tuple is of the form:

$$\begin{aligned}&\alpha_1=(2n_1-1)u+(2n_1+1)v,\\ &\text{and for } i=2,\dots,l,\ \alpha_i=\rho_{1i}((2n_1+1)u+(2n_1-1)v)+\\ &+\sum_{s=1}^{i-2}(-u_{I_{s+1}i}x_s+B_{is}y_s)+d_{I_i}(\overline{X}_{i-1}(n_i))\end{aligned}\tag{4}$$

where the ρ_{1i}'s, which satisfy $0\le\rho_{1i}\le d_{I_i}-1$, are inductively determined via:

$$\begin{aligned}&\rho_{1i}(4n_1-1)-u_{1i}\equiv\lambda_i \bmod d_{I_i},\quad \lambda_2=0 \text{ and for } 3\le i\le l,\\ &\lambda_i=\sum_{s=2}^{i-1}\frac{\rho_{1s}(4n_1-1)-u_{1s}-\lambda_s}{d_{I_s}}u_{I_si}\end{aligned}\tag{5}$$

whereas the B_{is}'s are determined by the invariants via the equations $\alpha_i\cdot\alpha_{s+1}=0$. A generating set for $L_l^{\perp}$ is easily seen to be given by:

$$\begin{aligned}&e_{l1}^{\perp}=(2n_1+1)u+(2n_1-1)v+\sum_{s=1}^{l-1}P_{l1s}y_s,\\ &e_{lj}^{\perp}=x_{j-1}+n_jy_{j-1}+\sum_{s=j+1}^{l-1}P_{ljs}y_s,\quad j=2,\dots,l\\ &x_i,y_i,\quad i\ge l\end{aligned}\tag{6}$$

where P_{l1s} is determined by $\alpha_{s+1} \cdot e_{l1}^{\perp} = 0$ and for $j \geq 2$, P_{ljs} is determined by $\alpha_{s+1} \cdot e_{lj}^{\perp} = 0$. We now begin the proof by induction. That α_1 is taken to an element of the form $(2n_1+1)u + (2n_1-1)v$ is due to C. T. C. Wall. For $l = 2$ then after we have applied the isometry that takes α_1 to its canonical form α_2 can be assumed to look like: $\alpha_2 = x_{1i}((2n_1+1)u + (2n_1-1)v) + D_{12}(-x_1 + N_2 y_1)$. Now one can easily see that: $d_{I_2} = \gcd(8n_1 x_{1i}, D_{12})$. Applying $\chi_{2,(2n_1+1,2n_2-1)}$ followed by $12 - [W]$ we get to send α_2 to an element of the form: $\alpha_2 = x_{1i}((2n_1+1)u + (2n_1-1)v) + d_{I_2}(-x_1 + N_2 y_1)$ for some N_2. It is easy to see that x_{12} is determined $\mod d_{I_2}$ by $x_{12}(4n_1 - 1) \equiv u_{12} \mod d_{12}$ as in (5). Now letting $x_{1i} = d_{I_2} q_2 + \rho_{1i}$ we apply $(12) - a^{-1}$ followed by $\chi_{2,(q_2(2n_1+1),-q_2(2n_2-1))}$ and finish with $(12) - [W]$ to get the canonical form. This finishes the case $l = 2$.

Now assume the canonical form for l. We wish to prove it for $l + 1$. After applying the isometries which take the first l elements to their canonical form as in (4), α_{l+1} looks like:

$$x_{1(l+1)}((2n_1+1)u+(2n_1-1)v)+\sum_{s=1}^{l-1}(-U_{I_{s+1}(l+1)}x_s + V_{(l+1)s}y_s)+D_{I_{(l+1)}}(\overline{X}_l(N_{l+1})),$$

where $x_{1(l+1)}$ is determined by the invariants via an equation analogous to (5):

$$x_{1(l+1)}(4n_1-1) \equiv \Lambda_{l+1} \mod d_{I_{l+1}}, \text{ with } \Lambda_{l+1} = \sum_{s=2}^{l} \frac{\rho_{1s}(4n_1-1) - u_{1s} - \lambda_s}{d_{I_s}} U_{I_s(l+1)}$$

the $U_{I_{s+1}(l+1)} \equiv u_{I_{s+1}(l+1)} \mod d_{I_{l+1}}$ and the $V_{(l+1)s}$'s are determined via the equations $\alpha_{l+1} \cdot \alpha_{s+1} = 0$. Now using (6) and the definition of $d_{I_{l+1}}$ we see that

$$d_{I_{l+1}} = \gcd(e_{l1}^{\perp} \cdot \alpha_{l+1}, \dots, e_{ll}^{\perp} \cdot \alpha_{l+1}, D_{I_{(l+1)}})$$

Now apply isometries as follows:

Apply $\chi_{l+1,(2n_1+1,-(2n_1-1))}$ followed by $s(l+1) - a^{P_{l1s}}$ for $s = 1, \dots, l-1$ and finish with a suitable $l(l+1) - [W]$ so that in the l^{th} block we have something of the form:

$$\gcd(e_{l1}^{\perp} \cdot \alpha_{l+1}, D_{I_{(l+1)}})(\overline{X}_l(N_{l+1}))$$

for some N_{l+1}. Then for all $j = 2, \dots, l$ apply: $(j-1)(l+1) = a^{n_j} \circ e \circ b^{-1}$ followed by $s(l+1) - a^{P_{ljs}}$ for $s = j+1, \dots, l-1$ and for every j finish with a suitable $l(l+1) - [W]$ so that at the end of the j^{th} series of isometries in the l^{th} block we have an element of the form:

$$\gcd(e_{l1}^{\perp} \cdot \alpha_{l+1}, \dots, e_{lj}^{\perp} \cdot \alpha_{l+1}, D_{I_{(l+1)}})(\overline{X}_l(N_{l+1}))$$

for some N_{l+1}. In the end of this sequence of isometries we have kept the first l elements fixed while we have sent α_{l+1} to an element of the form:

$$x_{1(l+1)}((2n_1+1)u+(2n_1-1)v)+\sum_{s=1}^{l-1}(-U_{I_{s+1}(l+1)}x_s + V_{(l+1)s}y_s)+d_{I_{(l+1)}}(\overline{X}_l(N_{l+1})),$$

Now before continuing with the final sequence of isometries set:

$$x_{1(l+1)} = q_1 d_{I_{l+1}} + \rho_{1(l+1)}, \quad U_{I_{s+1}(l+1)} = q_{s+1} d_{I_{l+1}} + u_{I_{s+1}(l+1)}$$

Apply $l(l+1) - e \circ a^{-1}$ and then $\chi_{l+1,q_1(2n_1+1,-(2n_1-1))}$ followed by $s(l+1) - a^{q_1 P_{l1s}}$ for $s = 1, \dots, l-1$ and finish with a suitable $l(l+1) - [W]$ so that in the l^{th} block we have something of the form $d_{I_{l+1}}(\overline{X}_l(N_{l+1}))$ for some N_{l+1}. Then for all $j = 2, \dots, l$ apply: $l(l+1) - a^{-1}$ followed by $(j-1)(l+1) = a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by $s(l+1) - a^{q_j P_{ljs}}$ for $s = j+1, \dots, l-1$ and for every j finish with a suitable $l(l+1) - [W]$ so that at the end of the j^{th} series of isometries in the l^{th} block we have an element of the form $d_{I_{l+1}}(\overline{X}_l(N_{l+1}))$ for some N_{l+1}. In the end of this sequence of isometries we have kept the first l elements fixed while we have sent α_{l+1} to its desired canonical form.

7.2. α_1 **is ordinary and** $VPS = 1$. First we state and prove the canonical form of a $k-1$-tuple for which $L_{k-1}^{\perp}$ is odd . The canonical form in this case is:

$$\begin{aligned} &\alpha_1 = n_1 u + (n_1+1)v \quad \forall i \leq k-1, \ \ \alpha_i = \\ &-\rho_{1i}((n_1+1)u + n_1 v) \quad +\sum_{s=1}^{i-2}(-u_{I_{s+1}i}x_s + B_{is}y_s) \quad +d_{I_i}(\overline{X}_{i-1}(n_i)), \end{aligned} \tag{7}$$

where $\rho_{1j} = \sum_{s=1}^{j-1}\delta_s u_{I_s j} + \delta_j d_{I_j}$, $\delta_s \in \{0,1\}$, $\delta_1 = 1$ and $\alpha_i \cdot \alpha_{j+1} = 0$ inductively determine the various B_{ij}'s in terms of the invariants. For any $i \leq k-1$, $L_i{}^{\perp}$ is generated by:

$$\begin{aligned} &e_{i1}^{\perp} = (n_1+1)u + n_1 v + \sum_{s=1}^{i-1} P_{i1s}y_s \\ &e_{ij}^{\perp} = x_{j-1} + n_j y_{j-1} + \sum_{s=j}^{i-1} P_{ijs}y_s, j = 2, \dots, i \\ &x_j, y_j, \ \ j \geq i \end{aligned} \tag{8}$$

where the various P_{ijs}'s are easily seen to be determined inductively by the invariants by writing down the successive equation which express their perpendicularity to the α_t's, $t = 1, \dots, k-1$.

It is now obvious that the modular invariants determine such a tuple. The δ_j's can be seen to be determined by them as well by an inductive argument.

procceed by induction. That there are isometries which take α_1 to its desired form is a result of C. T. C. Wall. When we have taken α_1 to that form, α_2 is in general of the form:

$$\alpha_2 = -x_{12}((n_1+1)u + n_1 v) + D_{12}(-x_1 + N_2 y_1).$$

It can easily be seen that $d_{I_2} = \gcd(x_{12}(2n_1+1), D_{12})$. Applying $\chi_{2,2(n_1+1,-n_1)}$ followed $12 - [W]$ we get one of:

$$\alpha_2 = -x_{12}((n_1+1)u + n_1 v) + d_{12}(-x_1 + N_2 y_1)$$

$$\alpha_2 = -x_{12}((n_1+1)u + n_1 v) + 2d_{12}(-x_1 + N_2 y_1)$$

for some N_2 according as $\prec \alpha_1 \succ^{\perp} \cap \prec \alpha_2 \succ^{\perp}$ is odd or even (technically according as D_{12} isn't or is divisible by $2d_{12}$). Since we are assuming that $\cap_{i=1}^{k-1} \prec \alpha_i \succ^{\perp}$ is odd the latter is not possible. Now assume the former and after setting $x_{12} = d_{I_2}2q_2 + \rho_{12}$ with $\rho_{12} = u_{12}$ or $d_{12} + u_{12}$ apply: $12 - a^{-1}$ followed by: $\chi_{2,2q_2(n_1+1,-n_1)}$ and finish with a suitable $12 - [W]$ to sent, while keeping α_1 fixed, α_2 to an element of the form:

$$\alpha_2 = -\rho_{12}((n_1+1)u + n_1 v) + d_{12}(-x_1 + N_2 y_1).$$

This finishes the case of two elements ($k = 3$).

Now suppose that we have proved that for m elements our canonical form is given by (7) for $m = k-1$. We wish to prove it for $m+1$ elements. After we applied isometries which take the first m elements to the canonical form as in (7), α_{m+1} is of the form:

$$-x_{1(m+1)}((n_1+1)u+n_1v)+\sum_{s=1}^{m-1}(-U_{I_{s+1}(m+1)}x_s+V_{(m+1)s}y_s)+D_{m+1}(\overline{X}_m(N_{m+1}))$$

By (8) and the definitions $d_{I_{m+1}} = \gcd(e_{m1}^{\perp}\cdot\alpha_{m+1},\dots,e_{mm}^{\perp}\cdot\alpha_{m+1},D_{m+1})$. Now apply isometries:

Apply $\chi_{m+1,2(n_1+1,-n_1)}$ followed by $(s(m+1)) - a^{P_{m1s}}$ for $s=1,\dots,m-1$ and finish with a suitable $(m,m+1)-[W]$. (Suitable as always means so that in the m^{th} block there is a multiple of an element of the form $\overline{X}_m(N)$ and the $(m+1)^{th}$ block is empty. Then for $j=2,\dots,m-1$ apply: $(j-1)(m+1) - a^{n_j}\circ e\circ b^{-1}$ followed by $(s(m+1)) - a^{P_{mjs}}$ for $s=j,\dots,m-1$ and for every j finish with a suitable $m(m+1)-[W]$ as above. At the end of this we have kept the first m elements fixed while α_{m+1} is taken to:

$$-x_{1(m+1)}((n_1+1)u+n_1v)+\sum_{s=1}^{m-1}(-U_{I_{s+1}(m+1)}x_s+V_{(m+1)s}y_s)+d_{I_{m+1}}(\overline{X}_m(N_{m+1}))$$

$$-x_{1(m+1)}((n_1+1)u+n_1v)+\sum_{s=1}^{m-1}(-U_{I_{s+1}(m+1)}x_s+V_{(m+1)s}y_s)+2d_{I_{m+1}}(\overline{X}_m(N_{m+1}))$$

according as $2d_{I_{m+1}}$ doesn't or does divide $\gcd(e_{m2}^{\perp}\cdot\alpha_{m+1},\dots,e_{mm}^{\perp}\cdot\alpha_{m+1})$. But the latter case leads to an $(m+1)$-tuple with $L_{m+1}^{\perp}$ even and so it doesn't occur here. So assume the former and before procceeding by isometries let:

$$x_{1(m+1)} = d_{I_{m+1}}2q_1+\rho_{1(m+1)},\, U_{I_{s+1}(m+1)} = d_{I_{m+1}}q_{s+1}+u_{I_{s+1}(m+1)} \tag{9}$$

Apply $m(m+1) - e\circ a^{-1}$ followed by $\chi_{m+1,-2q_1(n_1+1,-n_1)}$ followed by $(s(m+1)) - a^{-2q_1P_{m1s}}$ for $s=1,\dots,m-1$ and finish with a suitable $(m,m+1)-[W]$. (Suitable as always means so that in the m^{th} block there is a multiple of an element of the form $\overline{X}_m(N)$ and the $(m+1)^{th}$ block is empty.) Then for $j=2,\dots,m-1$ apply $m(m+1)-a$ followed by $(j-1)(m+1) - a^{q_{j+1}n_j}\circ e\circ b^{-q_{j+1}}$ followed by $(s(m+1)) - a^{q_{j+1}P_{mjs}}$ for $s=j,\dots,m-1$ and for every j finish with a suitable $m(m+1)-[W]$ as above. At the end of this we have kept the first m elements fixed while α_{m+1} is taken to its desired canonical form.

Now suppose that $L_{k-1}^{\perp}$ is odd and $L_k^{\perp}$ is even. Then the canonical form is:

$$\begin{aligned}&\alpha_1,\dots,\alpha_{k-1}\ \text{ as in (7) and }\ \alpha_k = -\rho_{1k}((n_1+1)u+n_1v)\\ &+\sum_{s=1}^{k-2}(-r_{I_{s+1}k}x_s+B_{ks}y_s)+2d_{I_k}(\overline{X}_{k-1}(n_k))\end{aligned} \tag{10}$$

$$\frac{e_{(k-1)1}^{\perp}\cdot\alpha_k}{d_{I_i}} \equiv 1 \bmod 2,\quad \frac{e_{(k-1)j}^{\perp}\cdot\alpha_k}{d_{I_i}} \equiv 0 \bmod 2,\quad j\ge 2$$

$$\rho_{1k} = u_{1k}+\sum_{s=2}^{k-1}\delta_s r_{I_sk}+\delta_k d_{I_k},\quad r_{I_{s+1}k} = u_{I_{s+1}k}+\delta_{sk}d_{I_k},\quad \delta_{sk}\in\{0,1\} \tag{11}$$

and the B_{ks}'s are inductively determined by the equations: $\alpha_k\cdot\alpha_{s+1}=0$. The δ_k, δ_{sk}'s are new features not predicted by the modular invariants. They are predicted by the type of the k-tuple as follows First of all notice that $\alpha'_{I_k} = e_{(k-1)1}^{\perp} + (\frac{e_{(k-1)1}^{\perp}\cdot\alpha_k/d_{I_k}-1}{2})y_{k-1}$ is of odd square. Then notice that the α'_1 which achieves: $\alpha_1' \cdot \alpha_k = u_{1k}$ is given by $\alpha_1' = (-u-v) + \sum_{s=1}^{k-2} \delta_{s+1} y_s + \delta_k \alpha_{I_k}'$, and the parity of its square is determined by the value of δ_k (even when zero and otherwise odd). Similarly the α_{I_i}' such that $\alpha_{I_i}' \cdot \alpha_k = u_{I_i k}$, $i = 2, \ldots, k-1$ is given by $\alpha_{I_i}' = -y_{i-1} + \delta_{(i-1)k} \alpha_{I_k}'$, $i = 2, \ldots, k-1$, and the parity of its square is determined by the value of $\delta_{(i-1)k}$ (even when zero and otherwise odd). The space $L_k^{\perp}$ is generated by:

$$\begin{aligned} &e_{k1}^{\perp} = 2e_{(k-1)1}^{\perp} + P_{k1(k-1)} y_{k-1} \\ &e_{kj}^{\perp} = e_{(k-1)j}^{\perp} + P_{kj(k-1)} y_{k-1}, j = 2, \ldots, k \\ &x_j, y_j, \quad j \geq k \end{aligned} \tag{12}$$

where : $P_{k1(k-1)} = (\frac{e_{(k-1)1}^{\perp} \cdot \alpha_k / d_{I_k} - 1}{2})$, and $P_{kj(k-1)} = \frac{e_{(k-1)j}^{\perp} \cdot \alpha_k}{2d_{I_k}}$.

Here we apply the same isometries as in the previous case (the ones before (9) as they are and they will lead to the second of the options mentioned there and then by dividing by $2d_{I_{m+1}}$ rather than $d_{I_{m+1}}$ in the equations (9) we get almost to the claimed canonical form (10) except that α_k gets sent to:

$$-x((n_1+1)u + n_1 v) + \sum_{s=1}^{k-2} (-r_{I_{s+1}k} x_s + B_{ks} y_s) + 2d_{I_k}(\overline{X}_{k-1}(n_k))$$

with $x = \rho_{1k} = u_{1k} + \sum_{s=2}^{k-1} \delta_s r_{I_s k} + \delta d_{I_k}$ and $\delta \in \{0,1,2,3\}$ Now there remains to find isometries which keep the first elements fixed while they send the k^{th} with $\delta = 0$ (resp. $\delta = 1$) to the k^{th} with $\delta = 1$ (resp. $\delta = 3$) provided that the invariants are all the same. The following series of isometries does just that. By the first equation of (11) we can write:

$$2\lambda = \frac{e_{(k-1)1}^{\perp} \cdot \alpha_k}{d_{I_k}} + 1.$$

Define for $s = 1, \ldots, k-2$, $\psi_s = S_{x_s} \circ [s-e] \circ \chi_{s,(\frac{-e_{s1}^{\perp} \cdot \alpha_{s+1}}{d_{I_{s+1}}}, \frac{e_{s1}^{\perp} \cdot \alpha_{s+1}}{d_{I_{s+1}}})}$ and let $\psi = \psi_{k-2} \circ \ldots \circ \psi_1$ Now define

$$\phi_\mu = [(k-1)k - b^\mu] \circ [(k-1) - e] \circ [(k-1)k - b^{n_1\mu}] \circ [(k-1)k - b^\lambda] \circ [(k) - e] \circ$$

$$\circ \chi_{(k-1),(\lambda,-\lambda)} \circ [(k-1) - e] \circ S_{x_{k-1}} \circ \phi \circ \chi_{k,(n,-n)} \circ [(k-1)k - e] \circ \chi_{k,(-1,1)}$$

and

$$\phi = [(k-1) - e] \circ S_u \circ [(k-1)k - a] \circ [(k-1) - e] \circ \chi_{k,(-1,1)}$$

The desired isometry is then given by: $\phi_1^{-1} \circ \phi \circ \phi_0$. This finishes the case.

Now we write down the canonical form of an l-tuple with $l \geq k$.

$$\begin{aligned} &\alpha_1, \ldots, \alpha_k \text{ as in (10) and } \forall i \geq k+1, \\ &\alpha_i = -\rho_{1i}((n_1+1)u + n_1 v) + \sum_{s=1}^{k-2} (-r_{I_{s+1}i} x_s + B_{is} y_s) + \\ &-r_{I_k i} x_{k-1} + B_{i(k-1)} y_{k-1} + \sum_{s=k}^{i-1} (-u_{I_{s+1}i} x_s + B_{is} y_s) + d_{I_i}(\overline{X}_{i-1}(n_i)) \end{aligned} \tag{13}$$

where the k quantities ρ_{1i}, $r_{I_{s+1}i}$, $s = 1, \dots, k-1$ are solutions to the system mod d_{I_i} of k linear equations in k unknowns:

$$\begin{aligned} &\alpha_i \cdot \alpha_1' = u_{1i}, \quad \alpha_i \cdot \alpha_{I_j}' = u_{I_j i}, \quad j = 2, \dots, k-1, \\ &\alpha_i \cdot (-\alpha_{I_k}' - y_{k-1}) = u_{I_k i} \end{aligned} \tag{14}$$

whose coefficients are determined by the invariants of the tuple of the k first elements and the modular invariants $u_{I_s i}, s = 1, \dots, k$ of α_i. All but ρ_{1i} are canonical representatives and $\rho_{1i} = r_{1i} + \delta_i d_{I_i}$, with $\delta \in \{0,1\}$ easily seen to be predicted by the invariants and r_{1i} a canonical representative of the solution to the first of (14). The space $L_l^{\perp}$ is generated by:

$$\begin{aligned} &e_{l1}^{\perp} = 2e_{(k-1)1}^{\perp} + P_{l1(k-1)}y_{k-1} + \sum_{s=k}^{l-1} P_{l1s}y_s \\ &e_{lj}^{\perp} = e_{(k-1)j}^{\perp} + P_{lj(k-1)}y_{k-1} + \sum_{s=k}^{l-1} P_{ljs}y_s, j = 2, \dots, l \\ &x_j, y_j, \quad j \geq l \end{aligned} \tag{15}$$

We prove this canonical form by induction on $l - k$. If $l - k = 0$ then we are done since it is the previous case. Suppose we have proved it for $l - 1$ and we wish to prove it for l. After applying an isometry which brings the first $l - 1$ elements to their canonical form as in (13) α_l looks like:

$$\begin{aligned} \alpha_l = &-x_{1l}((n_1+1)u + n_1 v) + \sum_{s=1}^{k-2}(-R_{I_{s+1}l}x_s + V_{ls}y_s) + \\ &-R_{I_k l}x_{k-1} + V_{l(k-1)}y_{k-1} + \sum_{s=k}^{l-1}(-U_{I_{s+1}l}x_s + V_{ls}y_s) + D_l(\overline{X}_{l-1}(N_l)) \end{aligned} \tag{16}$$

Then by definition and (15), $d_{I_l} = \gcd(e_{(l-1)1}^{\perp} \cdot \alpha_l, \dots, e_{(l-1)(l-1)}^{\perp} \cdot \alpha_l, D_l)$.

Apply $\chi_{l,2(n_1+1,-n_1)}$ followed by $(s(l)) - a^{2P_{(l-1)1s}}$, $s = 1, \dots, k-2$, and then $(s(l)) - a^{P_{l1s}}$, $s = k-1, \dots, l-1$ and finish with $(l-1)l - [W]$ so that the $(l)^{th}$ block is empty and the $(l-1)^{th}$ block of the same form as before but now with divisibility $\gcd(e_{(l-1)1}^{\perp} \cdot \alpha_l, D_l)$. Then for all $j = 2, \dots, l$ apply: $(j-1)(l) - a^{n_j} \circ e \circ b^{-1}$ followed by $(s(l)) - a^{P_{(k-1)js}}$, $s = j, \dots, k-2$ followed by $(s(l)) - a^{P_{ljs}}$, $s = k-1, \dots, l-1$ and finish for every j with a suitable $((l-1)l) - [W]$ as above so that at the end of these isometries the first l - 1-elements remain fixed while α_l is sent to:

$$\begin{aligned} \alpha_l = &-x_{1l}((n_1+1)u + n_1 v) + \sum_{s=1}^{k-2}(-R_{I_{s+1}l}x_s + V_{ls}y_s) + \\ &-R_{I_k l}x_{k-1} + V_{l(k-1)}y_{k-1} + \sum_{s=k}^{l-1}(-U_{I_{s+1}l}x_s + V_{ls}y_s) + d_{I_l}(\overline{X}_{l-1}(N_l)) \end{aligned} \tag{17}$$

Now set $x_{1l} = \rho 1l + 2q_1 d_{I_l}$, $R_{I_{s+1}l} = r_{I_{s+1}l} + q_s d_{I_l}$, and $U_{I_{s+1}l} = u_{I_{s+1}l} + q_s d_{I_l}$ and procceed by isometries.

Apply $(l-1)l - e \circ a^{-1}$, then $\chi_{l,-2q_1(n_1+1,-n_1)}$ followed by $(s(l)) - a^{2P_{(l-1)1s}}$, $s = 1, \dots, k-2$, and then $(s(l)) - a^{P_{l1s}}$, $s = k-1, \dots, l-1$ and finish with $(l-1)l - [W]$ so that the $(l)^{th}$ block is empty and the $(l-1)^{th}$ block of the same form as before. Then for all $j = 2, \dots, l$ apply $((l-1)l) - a^{-1}$, then $(j-1)(l) - a^{n_j} \circ e \circ b^{-1}$ followed by $(s(l)) - a^{P_{(k-1)js}}$, $s = j, \dots, k-2$ followed by $(s(l)) - a^{P_{ljs}}$, $s = k-1, \dots, l-1$

and finish for every j with a suitable $((l-1)l)-[W]$ as above so that at the end of these isometries the first $l-1$-elements remain fixed while α_l is sent to its desired canonical form.

7.3. **α_1^2 is even and there is no parity switch.** The canonical l-tuple is of the form:

$$\begin{aligned}
&\alpha_1 = -x_1 + n_1 y_1, \quad \text{and for } 2 \le \mathrm{i} \le \mathrm{k}-1 \\
&\alpha_i = -u_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{i-1}(-u_{I_s i} x_s + B_{is} y_s) + d_{I_i}(\overline{X}_i(n_i)), \\
&\alpha_k = -r_{1k}(x_1 + n_1 y_1) + \sum_{s=2}^{k-1}(-r_{I_s k} x_s + B_{ks} y_s) + d_{I_k}(Y(n_k)), \\
&\text{and for } \mathrm{k}+1 \le \mathrm{i} \le \mathrm{l}, \\
&\alpha_i = -u_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{k-1}(-u_{I_s i} x_s + B_{is} y_s) + \sum_{s=k}^{i-2}(-u_{I_s i} x_s + B_{is} y_s) \\
&+[B + (2n_k+1) r_{I_k i}]u + [B + (2n_k - 1) r_{I_k i}]v + d_{I_i}(\overline{X}_{i-1}(n_i)),
\end{aligned} \tag{18}$$

where the B_{ij}'s are inductively expressed in terms of the invariants by the equations $\alpha_i \cdot \alpha_j = 0$, $B = \frac{1}{2d_{I_k}} \alpha_i|_{k-1} \cdot \alpha_k$ and $r_{I_k i} \equiv u_{I_k i} + (1-2n_k)B \bmod d_{I_i}$. Now we need to compute $\cap_{i=1}^{k-1}\alpha_i^{\perp}$, $\cap_{i=1}^{k}\alpha_i^{\perp}$, and $\cap_{i=1}^{l}\alpha_i^{\perp}$. For all $i \le k-1$, $\cap_{s=1}^{i}\alpha_s^{\perp}$ is easily seen to be generated by:

$$\begin{aligned}
&e_{ij}^{\perp} = x_j + n_j y_j + \sum_{s=j+1}^{i} P_{ijs} y_s, \quad j = 1, \dots i \\
&x_i, y_i, \quad i \ge i+1
\end{aligned} \tag{19}$$

Notice that $e_{(k-1)j}^{\perp} \cdot \alpha_k \equiv 0 \bmod 2d_{I_k}$ for all j. This is due to the fact that we assume $L_k^{\perp}$ to be even. Now we need to also compute a generating set for $L_k^{\perp}$. It is easily seen to be given by:

$$\begin{aligned}
e_{kj}^{\perp} &= e_{(k-1)j}^{\perp} - P_{kjk}(u+v), \quad j = 1, \dots k-1 \\
e_{kk}^{\perp} &= (2n_k+1)u + (2n_k - 1)v, \quad x_i, y_i, \quad i \ge k,
\end{aligned} \tag{20}$$

where $\frac{P_{kjk} = e_{(k-1)j}^{\perp} \cdot \alpha_k}{2d_{I_k}}$. Finally it can be seen that $L_l^{\perp}$ is given by:

$$\begin{aligned}
&e_{lj}^{\perp} = e_{(k)j}^{\perp} + \sum_{s=k}^{l-1} P_{ljs} y_s, \quad j = 1, \dots, k-1 \\
&e_{lj}^{\perp} = (x_{j-1} + n_j y_{j-1}) + \sum_{s=j}^{l-1} P_{ljs} y_s \quad j = k+1, \dots, l \\
&x_i, y_i, \quad i \ge l
\end{aligned} \tag{21}$$

We give the proof by induction. First we prove the canonical form before $\mathrm{L}_k^{\perp}$ becomes even. We do the case of two elements first. That there is an isometry which takes α_1 to its desired canonical form is a result of C. T. C. Wall's. With α_1 in canonical form α_2 might look like one of the following:

$$\alpha_2 = -x(-x_1 + n_1 y_1) + D_2 \overline{X}, \tag{22}$$

where $\overline{X}$ is of the form of one of $-x_2 + N_2 y_2$ or $Y(N_2)$. Apply $(12) - a^{n_1} \circ e \circ b_{-1}$ in any case. Applying $[W]$ in the direct sum of the 2^{nd} hyperbolic form with the span of u and v we sent α_2 in the first case to an element of the form:

$$\alpha_2 = -x(-x_1 + n_1 y_1) + d_{I_2}(-x_2 + N_2 y_2)$$

and in the second to the same as above if $x2n_1/d_{I_2}$ is odd and to an element of the form:

$$\alpha_2 = -x(-x_1 + n_1 y_1) + d_{I_2} Y(N_2)$$

if $x2n_1/d_{I_2}$ is even. The latter case doesn't occur because in that case $L_2^{\perp}$ is even. So now assume α_2 is of the form (22). Applying a suitable $[W]$ in the span of x_2, y_2, u, v we sent α_2 to an element of the form:

$$\alpha_2 = -x(-x_1 + n_1 y_1) + d_{I_2}(-x_2 + (M_k + 1)u + M_k y)$$

Then set $x = d_{I_2} q_1 + u_{12}$ and simply apply $a^{q_1 n_1} \circ e \circ b^{-q_1}$ and a suitable $[W]$ in the span of x_2, y_2, u, v to sent α_2 to its desired canonical form while α_1 is kept fixed. Now suppose that we have proved the canonical form in this case for $m-1$ elements with $L_{m-1}^{\perp}$ and we wish to prove it for m. With the first $m-1$ elements in canonical position α_m can be assumed to be of the form

$$-U_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-U_{I_s m} x_s + B_{ms} y_s) + D_m(\overline{X}) \tag{23}$$

where $\overline{X}$ is of the form $-x_m + N_m y_m$ or $Y(N_m)$. In the first case apply the following isometries: For all $j = 1, \ldots, m-1$ apply $jm - a^{n_j} \circ e \circ b^{-1}$ followed by $sm - a^{P_{(m-1)js}}$ for $s = j+1, \ldots, m-1$ and finish with a suitable $[W]$ in the span of x_m, y_m, u, v so that the span of u, v is empty and the part of α_l in the m^{th} block is of the same form as at the biggining. At the end of these isometries α_m is taken to an element of the form:

$$-U_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-U_{I_s m} x_s + B_{ms} y_s) + d_{I_m}(-x_m + N_m y_m) \tag{24}$$

In the second case we procceed with the same isometries as above except when we come to the point of applying "a suitable $[W]$ in the span of x_m, y_m, u, v". There a suitable $[W]$ exists in the j^{th} step, to keep the part of α_l in the span of x_m, y_m, u, v of the same form, only as long as $e_{(m-1)j}^{\perp} \cdot \alpha_m \equiv 0 \bmod 2d_{I_m}$. The first time that the latter is not satisfied a suitable $[W]$ in the span of x_m, y_m, u, v transforms the part of α_m there into a multiple of an element of the form $-x_m + N_m y_m$. If that happens then from that point on we continue with exactly the same isometries as before. So depending on whether $e_{(m-1)j}^{\perp} \cdot \alpha_m \equiv 0 \bmod 2d_{I_m}$ for all j or not we get to keep the first $m-1$ elements fixed while α_m is taken to, respectively one of:

$$-U_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-U_{I_s m} x_s + B_{ms} y_s) + d_{I_m}(Y(N_m)) \tag{25}$$

or (24). Since we are assuming here that $L_m^{\perp}$ is odd it has to be the latter. So now assume that the first $m-1$ elements are in canonical form and α_m looks like (24). Apply an isometry in the span of x_m, y_m, u, v which sends $d_{I_m}(Y(N_m))$ to an element of the form $d_{I_m}[x_m + (M_m + 1)u + M_m v]$ and then after setting $U_{I_s m} = d_{I_m} q_s + u_{I_s m}$ apply for all $j = 1, \ldots, m-1$ $jm - a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by $sm - a^{q_j P_{(m-1)js}}$ for $s = j+1, \ldots, m-1$ and finish for every j with a suitable $[W]$ in the span of x_m, y_m, u, v so that the part of α_m in the span of x_m, y_m, u, v is of the same form as when we begun the procces for $j-1$. For $j = m-1$ apply a different $[W]$ so that in the span of x_m, y_m, u, v we get something of the form $d_{I_m}(-x_m + n_m y_m)$. At the end of these isometries we keep the first $m-1$ elements fixed while α_m is sent to an element of the desired form. Now assume that we have k elements, that $L_{k-1}^{\perp}$ is odd while $L_k^{\perp}$ is even and that the first $k-1$ elements are in canonical form as in (18). Then from the discussion above we can find isometries so that while the first $k-1$ elements are kept in canonical form, α_k is of the same form as in (25):

$$-R_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-R_{I_s m}x_s + B_{ms}y_s) + d_{I_m}(Y(N_m)) \tag{26}$$

Now set $R_{I_s m} = d_{I_m}2q_s + r_{I_s m}$ for all s. Then apply $\chi_{k,(-1,1)}$ and then apply for all $j = 1,\dots,m-1$ $jm - a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by $sm - a^{q_j P_{(m-1)js}}$ for $s = j+1,\dots,m-1$ and finish for every j with a suitable $[W]$ in the span of x_m, y_m, u, v so that the part of α_m in the span of x_m, y_m, u, v is of the same form as when we begun the procces for $j-1$. For $j = m-1$ apply a different $[W]$ so that in the span of x_m, y_m, u, v we get something of the form $d_{I_m}(Y(n_m))$. At the end of these isometries we keep the first $m-1$ elements fixed while α_m is sent to an element of the desired form. The final step of this case is to prove the canonical form for $l \geq k$. We do this by induction on $l-k$. Assume the canonical form for $l-1$ elements. With the first $l-1$ elements then in canonical form as in (18) α_l looks like:

$$\begin{aligned}\alpha_l = &-U_{1l}(x_1 + n_1 y_1) + \sum_{s=2}^{k-1}(-U_{I_s l}x_s + V_{is}y_s) + \sum_{s=k}^{l-2}(-U_{I_s l}x_s + V_{ls}y_s)\\ &+[V + (2n_k+1)R_{I_k l}]u + [V + (2n_k - 1)R_{I_k l}]v + D_l(\overline{X}_{l-1}(n_l))\end{aligned}$$

By definition and (21): $d_{I_l} = \gcd(e_{(l-1)1}^{\perp}\cdot\alpha_l, \dots, e_{(l-1)(l-1)}^{\perp}\cdot\alpha_l, D_l)$. Now begin with isometries.

For $j = 1,\dots,k$ apply $(jl) - a^{n_j} \circ e \circ b^{-1}$ followed by: $(sl) - a^{P_{kjs}}$ for $s = j+1,\dots,k$, then $\chi_{l,(P_{kjk},-P_{kjk})}$ followed by $(sl) - a^{P_{ljs}}$ for $s = k,\dots,l-1$. Finish for each j with a suitable $(l-1)l - [W]$ so that the part of α_l in the $l-1$, l blocks is of the same form as at the beggining. Then apply $\chi_{l,(2n_k+1,-(2n_k-1))}$ followed by $(sl) - a^{P_{lks}}$, for $s = k,\dots,l-1$. Finally for $j = k+1,\dots,l$ apply $((j-1)l) - a^{n_j} \circ e \circ b^{-1}$, followed by $(sl) - a^{P_{ljs}}$ for $s = j,\dots,l-1$ and finish for each j with a suitable $(l-1)l - [W]$. At the end of these isometries we have achieved to keep the first $l-1$ elements fixed while α_l is sent to:

$$\begin{aligned}\alpha_l = &-U_{1l}(x_1 + n_1 y_1) + \sum_{s=2}^{k-1}(-U_{I_s l}x_s + V_{is}y_s) + \sum_{s=k}^{l-2}(-U_{I_s l}x_s + V_{ls}y_s)\\ &+[B + (2n_k+1)R_{I_k l}]u + [B + (2n_k - 1)R_{I_k l}]v + d_{I_l}(\overline{X}_{l-1}(n_l))\end{aligned}$$

Now set $U_{I_s l} = d_{I_l}q_s + u_{I_s l}$, $R_{I_k l} = d_{I_l}q_k + r_{I_k l}$ and $V = d_{I_l}q + B$ and procced by isometries: For $j = 1,\dots,k-1$, $(l-1)l - a^{-1}$, then $(jl) - a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by: $(sl) - a^{q_j P_{kjs}}$ for $s = j+1,\dots,k$, then $\chi_{l,(q_j P_{kjk},-q_j P_{kjk})}$ followed by $(sl) - a^{q_j P_{ljs}}$ for $s = k,\dots,l-1$. Finish for each j with a suitable $(l-1)l - [W]$ so that the part of α_l in the $l-1$, l blocks is of the same form as at the beggining. Then apply $(l-1)l - e \circ a^{-1}$ $\chi_{l,q_k(2n_k+1,-(2n_k-1))}$ followed by $(sl) - a^{q_k P_{lks}}$, for $s = k,\dots,l-1$. Finally after $(l-1)l - a^{-1}$ apply for $j = k+1,\dots,l$, $((j-1)l) - a^{q_{j-1}n_j} \circ e \circ b^{-1}$, followed by $(sl) - a^{q_{j-1}P_{ljs}}$ for $s = j,\dots,l-1$ and finish for each j with a suitable $(l-1)l - [W]$. At the end of these isometries we have achieved to keep the first

$l-1$ elements fixed while α_l is sent to its desired canonical form. This finishes this case.

7.4. **α_1 is ordinary, α_1^2 is even and there is a parity switch.** We first state the canonical forms and the corresponding perpendicular subspaces. The canonical form of a $m-1$-tuple with $L_{m-1}^{\perp}$ odd and no parity switch is given by:

$$\begin{aligned} &\alpha_1 = -x_1 + n_1 y_1, \quad \text{and for } 2 \leq \mathrm{i} \leq \mathrm{m}-1 \\ &\alpha_i = -u_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{i-1}(-u_{I_s i} x_s + B_{is} y_s) + d_{I_i}(\overline{X}_i(n_i)), \end{aligned} \tag{27}$$

where the B_{ij}'s are inductively expressed in terms of the invariants by the equations $\alpha_i \cdot \alpha_j = 0$, and for all $i \leq m-1$, $L_{m-1}^{\perp}$ is easily seen to be generated by:

$$\begin{aligned} &e_{ij}^{\perp} = x_j + n_j y_j + \sum_{s=j+1}^{i} P_{ijs} y_s, \quad j = 1, \dots i \\ &x_j, y_j, \quad j \geq i+1 \end{aligned} \tag{28}$$

The canonical of an $k-1$ tuple with $VPS = m \leq k-1$ and $L_{k-1}^{\perp}$ odd is given by:

$$\begin{aligned} &\alpha_1, \dots, \alpha_{m-1} \text{ as in (27) } \alpha_m = \\ &-r_{1m}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-r_{I_s m} x_s + B_{ms} y_s) + d_{I_m}(n_m u + (n_m + 1) v), \\ &\text{and for } \mathrm{m}+1 \leq \mathrm{i} \leq \mathrm{k}-1 \quad \alpha_i = \\ &-u_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1} +(-u_{I_s i} x_s + B_{is} y_s) + \\ &+(B_{im} - r_{I_m i}(n_m + 1))u + (B_{im} - r_{I_m i} n_m) v + \\ &+\sum_{s=m}^{i-2}(-u_{I_{s+1} i} x_s + B_{is} y_s) + d_{I_i}(\overline{X}_{i-1}(n_i)), \end{aligned} \tag{29}$$

and $L_i^{\perp}$ for $m \leq i \leq k-1$ is easily seen to be generated by:

$$\begin{aligned} &e_{ij}^{\perp} = e_{(m-1)j}^{\perp} + P_{im}(u+v) + \sum_{s=m}^{i-1} P_{ijs} y_s, \quad j \leq m-1 \\ &e_{im}^{\perp} = (n_m + 1) u + n_m v + \sum_{s=m}^{i-1} P_{im} \\ &e_{ij}^{\perp} = x_{j-1} + n_j y_{j-1} + \sum_{s=j}^{i-1} P_{ijs} y_s, \quad m+1 \leq j \leq i \\ &x_j, y_j, \quad j \geq i \end{aligned} \tag{30}$$

The canonical form of an l-tuple with $VPS = m$ and change of parity of $L_i^\perp$ at $i = k$ is given by:

$$
\begin{aligned}
&\alpha_1, \dots, \alpha_{k-1} \text{ as in (29) } \alpha_k = \\
&-r_{1k}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-r_{I_s k}x_s + B_{ks}y_s) + \\
&+(B_{km} - r_{I_m k}(n_m + 1))u + (B_{km} - r_{I_m k}n_m)v + \\
&+\sum_{s=m}^{k-2}(-r_{I_{s+1}k}x_s + B_{ks}y_s) + 2d_{I_k}(-x_{k-1} + n_k y_{k-1}), \\
&\text{and for } k+1 \le i \le l \ \alpha_i = \\
&-u_{1i}(x_1 + n_1 y_1) + \sum_{s=2}^{m-1}(-u_{I_s i}x_s + B_{is}y_s) + \\
&+(B_{im} - r_{I_m i}(n_m + 1))u + (B_{im} - r_{I_m i}n_m)v + \\
&+\sum_{s=m}^{k-2}(-u_{I_{s+1}i}x_s + B_{is}y_s) + (-r_{I_k i}x_{k-1} + B_{i(k-1)}y_{k-1}) + \\
&+\sum_{s=k}^{i-2}(-u_{I_{s+1}i}x_s + B_{is}y_s) + d_{I_l}(-x_{l-1} + n_l y_{l-1})
\end{aligned}
\tag{31}
$$

Notice that $e_{(k-1)j}^\perp \cdot \alpha_i / d_{I_i} \equiv 1 \bmod 2$ for $j = m$ and otherwise to 0. This is, as is not hard to see, due to the fact that $L_k^\perp$ is even. Taking these into account we can now compute $L_i^\perp$:

$$
\begin{aligned}
&e_{ij}^\perp = e_{(k-1)j}^\perp + \sum_{s=k-1}^{i-1} P_{ijs}y_s, \quad j \le k-1 \text{ and } j \ne m \\
&e_{im}^\perp = 2e_{(k-1)m}^\perp + \sum_{s=k-1}^{i-1} P_{ims}y_s \\
&e_{ij}^\perp = x_{j-1} + n_j y_{j-1} + \sum_{s=j}^{i-1} P_{ijs}y_s, \quad k \le j \le i \\
&x_j, y_j, \quad j \ge k
\end{aligned}
\tag{32}
$$

In all these canonical forms the various quantities are functions of the invariants as one can check. The proof of these canonical forms runs in the same way as the ones we did before.

8. L is odd and $\sigma(L) \neq 0$

8.1. **α_1 is characteristic.** We can by [W] assume that α_1 is of the form: $2(-x_1 + n_1 y_1) + \sum_{s=1}^{\sigma} h_s$. $L_1^\perp$ is easily seen to be generated by:

$$
\begin{aligned}
&e_{11}^\perp = x_1 + n_1 y_1, \quad x_s, y_s, \quad s \ge 2 \\
&h_{11} = y_1 + 2h_1, \quad h_{1s} = -h_1 + h_s, \quad s \ge 2
\end{aligned}
\tag{33}
$$

Now with α_1 in as above there is an isometry which takes α_2 to an element of the form:

$$
r_{12}x_1 + B_{21}y_1 + d_{I_2}(-x_2 + n_2 y_2) + (2w_2 - (\sigma - 1)u_{12})h_1 + u_{12}\sum_{s \ge 2} h_s
$$

where $r_{12} = -2u_{12}$, $B_{21} = (r_{12}n_1 + w_2)$ and w_2 is a canonical representative of a solution to the equation $2x \equiv \sigma u_{12} \bmod d_{I_2}$. Notice that the latter equation might have up to two solutions (one if d_{I_2} or σ is odd and two if d_{I_2} and σ are even given in this case by $(\sigma/2)u_1 2 + id_{I_2}$, $i \in \{0,1\}$. But it is easy to see that in this case i is determined by the invariants and hence proving that we can bring α_2 to the aforementioned form finishes this case for two elements: With α_1 in its canonical form α_2 looks like:

$$R_{12}x_1 + V_{21}y_1 + D_2(-x_2 + N_2y_2) + (2W_2 - \sum_{s\geq 2} C_s)h_1 + \sum_{s\geq 2} C_s h_s,$$

with $V_{21} = (R_{12}n_1 + W_2)$. Now by definition and (33) we have:

$$d_{I_2} = \gcd(e_{11}^{\perp} \cdot \alpha_2, h_{11} \cdot \alpha_2, \ldots, h_{1\sigma} \cdot \alpha_2, D_2) \tag{34}$$

Now we are ready to apply isometries which bring α_2 to its desired canonical form while they keep α_1 fixed. First apply $13 - a^{n_1} \circ e \circ b^{-1}$ follwed by a suitable $23 - [W]$. Then apply $\chi_{3,(2\overline{m}_1)}$ followed by $13 - a$ and a suitable $23 - [W]$ and finally for all $s \geq 2$ apply: $\chi_{3,-\overline{m}_1+\overline{m}_s}$ followed for each s by a suitable $23 - [W]$. At the end of these isometries while α_1 is kept fixed α_2 is taken to an element of the form:

$$R_{12}x_1 + V_{21}y_1 + d_{I_2}(-x_2 + N_2y_2) + (2W_2 - \sum_{s\geq 2} C_s)h_1 + \sum_{s\geq 2} C_s h_s$$

$$\begin{array}{l} R_{12} \equiv -2u_{12} \bmod d_{I_2}, \quad R_{12} = -2u_{12} + d_{I_2}q_1, \\ 2W_2 \equiv \sigma u_{12} \bmod d_{I_2}, \quad \mathrm{w}_2 \text{ a canonical representative,} \\ \text{and } \mathrm{W}_2 = \mathrm{w}_2 + \mathrm{d}_{\mathrm{I}_2}\mathrm{q}, \text{ and for all s} \\ C_s \equiv u_{12} \bmod d_{I_2}, \quad C_s = u_{12} + d_{I_2}Q_s \end{array} \tag{35}$$

Now continue by isometries. First apply $23 - a^{-1}$ followed by $13 - a^{q_1 n_1} \circ e \circ b^{-q_1}$ followed by a suitable $23 - [W]$. Then apply $23 - e \circ a^{-1}$, $\chi_{3,(2q\overline{m}_1)}$ followed by $13 - a^q$ and a suitable $23 - [W]$ and finally for all $s \geq 2$ apply $23 - e \circ a^{-1}$ followed by $\chi_{3,-Q_s\overline{m}_1+Q_s\overline{m}_s}$ followed for each s by a suitable $23 - [W]$. At the end of these isometries while α_1 is kept fixed α_2 is taken to its desired canonical form. Now we procceed to state the canonical form of an l-tuple in this case and prove it by induction on l. The canonical form of an l-tuple is:

$$\begin{array}{l} 2(-x_1 + n_1y_1) + \sum_{s=1}^{\sigma} h_s, \text{ and for } \mathrm{i} = 2, \ldots, \mathrm{l} \\ \alpha_i = r_{1i}x_1 + B_{i1}y_1 + \sum_{s=2}^{i-1}[-u_{I_s i}x_s + B_{is}y_s] + \\ +d_{I_i}(-x_i + n_iy_i) + (2w_i - (\sigma - 1)u_{1i})h_1 + u_{1i}\sum_{s\geq 2} h_s, \end{array} \tag{36}$$

where $B_{i1} = (r_{1i}n_1 + w_i)$, $r_{1i} = -2u_{1i}$ and w_i is a canonical representative of a solution to the equation and $2x \equiv \sigma u_{1i} \bmod d_{I_i}$. Notice that the latter equation might have up to two solutions. But it is easy to see that the solution is determined by the invariants and hence proving that we can bring α_2 to the aforementioned form finishes this case for two elements. The various other B_{ij}'s are determined in terms of the invariants by the equations $\alpha_i \cdot \alpha_j = 0$. Now we need to compute $L_l^{\perp}$.

It is easily seen to be generated by:

$$e_{ij}^{\perp} = x_j + n_j y_j + \sum_{s=j+1}^{i} P_{ijs} y_s, \quad j = 1, \dots, i, \quad x_s, y_s, \quad s \geq i+1$$
$$h_{i1} = \sum_{s=2}^{i} P'_{i1s} y_s + y_1 + 2h_1, \quad h_{ij} = \sum_{s=2}^{i} P'_{ijs} y_s - h_1 + h_j, \quad j \geq 2 \tag{37}$$

Assume that we have found isometries which bring the first $l-1$ elements to their desired canonical form as in (36). Then α_l is of the form:

$$R_{1l}x_1 + V_{l1}y_1 + \sum_{s=2}^{l-1}[-U_{I_s l}x_s + V_{ls}y_s] + D_l(-x_l + N_l y_l) + (2W_l - \sum_{s\geq 2} C_{ls})h_1 + \sum_{s\geq 2} C_{ls}h_s$$

where the various V_{ij}'s are computed by equations similar to the ones above for B_{ij}'s. Now by definition and (37) we see that d_{I_l} is given by:

$$\gcd\,(e^{\perp}_{(l-1)1} \cdot \alpha_l, \dots, e^{\perp}_{(l-1)(l-1)} \cdot \alpha_l, h_{(l-1)1} \cdot \alpha_l, \dots, h_{(l-1)\sigma} \cdot \alpha_l, D_l) \tag{38}$$

Now we apply the first group of isometries. For $j = 1, \dots, l-1$ apply $j(l+1) - a^{n_j} \circ e \circ b^{-1}$ followed by $s(l+1) - a^{P_{(l-1)js}}$ for all $s = j+1, \dots, l-1$ and finish for each j with a suitable $l(l+1) - [W]$. Then apply $\chi_{l+1, 2\overline{m}_1}$ followed by $s(l+1) - a^{P'_{(l-1)1s}}$ for all $s = 2, \dots, l-1$ and finish with a suitable $l(l+1) - [W]$. Finally for every $j = 2, \dots, \sigma$ apply $\chi_{(l+1), -\overline{m}_1 + \overline{m}_j}$ followed by $s(l+1) - a^{P'_{(l-1)js}}$ for all $s = 2, \dots, l-1$ and finish for each j with $l(l+1) - [W]$. Now α_l is of the form:

$$R_{1l}x_1 + V_{l1}y_1 + \sum_{s=2}^{l-1}[-U_{I_s l}x_s + V_{ls}y_s] + d_{I_l}(-x_l + N_l y_l) + (2W_l - \sum_{s\geq 2} C_{ls})h_1 + \sum_{s\geq 2} C_{ls}h_s \; .$$

$$\begin{aligned}
&R_{1l} \equiv -2u_{1l} \bmod d_{I_l}, \quad R_{1l} = -2u_{1l} + d_{I_l}q_1, \\
&U_{I_s l} \equiv u_{I_s l} \bmod d_{I_l}, \quad U_{I_s l} = u_{I_s l} + q_s d_{I_l} \\
&2W_l \equiv \sigma u_{1l} \bmod d_{I_l}, \quad \mathrm{w_l} \text{ a canonical representative,} \\
&\text{and } \mathrm{W_l = w_l + d_{I_l} q}, \text{ and for all s} \\
&C_{ls} \equiv u_{1l} \bmod d_{I_l}, \quad C_{ls} = u_{1l} + d_{I_l} Q_s
\end{aligned} \tag{39}$$

Now procceed with the second group of isometries to reduce the various quantities defined only $\bmod\, d_{I_l}$ to their canonical representatives.

For $j = 1, \dots, l-1$ apply $l(l+1) - a^{-1}$ then $j(l+1) - a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by $s(l+1) - a^{q_j P_{(l-1)js}}$ for all $s = j+1, \dots, l-1$ and finish for each j with a suitable $l(l+1) - [W]$. Then apply $l(l+1) - e \circ a^{-1}$, $\chi_{l+1, 2q\overline{m}_1}$ followed by $s(l+1) - a^{qP'_{(l-1)1s}}$ for all $s = 2, \dots, l-1$ and finish with a suitable $l(l+1) - [W]$. Finally for every $j = 2, \dots, \sigma$ apply $l(l+1) - e \circ a^{-1}$, $\chi_{(l+1), -Q_j\overline{m}_1 + Q_j\overline{m}_j}$ followed by $s(l+1) - a^{Q_j P'_{(l-1)js}}$ for all $s = 2, \dots, l-1$ and finish for each j with $l(l+1) - [W]$. This brings α_l to its desired canonical form and finishes this case.

8.2. α_1 **is ordinary.** Now we procceed to settle the final case. First we deal with the case when $L_l^\perp$ is odd. The canonical form is:

$$\begin{aligned} \alpha_1 &= -x_1 + n_1 y_1 && +\Delta_1 h_1, \\ \alpha_i &= \sum_{s=1}^{i-1}(-u_{I_s i}x_s + B_{is}y_s) + d_{I_i}(-x_i + n_i y_i) && +\Delta_i h_1, \quad i = 2, \dots, l \end{aligned} \tag{40}$$

where $\Delta_i = \sum_{s=1}^{i-1} \delta_s u_{I_s i} + \delta_i d_{I_i}$ for $i = 1, \dots, l$ and the $\delta_i \in \{0,1\}$ are easily seen to be determined by the invariants. The various B_{ij}'s are determined by the equations $\alpha_i \cdot \alpha_j = 0$. We now need to determine a basis for $L_i^\perp$. Such a basis can be seen to consist of:

$$\begin{aligned} e_{ij}^\perp &= x_j + n_j y_j + \sum_{s=j+1}^{l} P_{ijs} y_s, \quad j = 1, \dots, i, \quad x_s, y_s, \quad s \geq i+1 \\ h_{i1} &= -\sum_{s=1}^{l} \delta_s y_s + h_1, \quad h_s, \quad 2 \leq s \leq \sigma + 1 \end{aligned} \tag{41}$$

Now suppose that the first $k-1$ elements are in canonical form as in (40). Then α_k is of the form:

$$\sum_{s=1}^{k-1}(-U_{I_s(k)}x_s + V_{ks}y_s) + D_k(\overline{Z}) + \Delta h_1$$

where, $U_{I_s k} \equiv u_{I_i(k+1)} \bmod d_{I_k}$, the $V_{(k)j}$'s are determined similarly to the B_{ij}'s above and $\overline{Z}$ is of the form $(1+\delta)(-x_k + N_k y_k) + \delta \sum_{s=2}^{\sigma} h_s$, $\delta \in \{0,1\}$. Let us now first examine the case $\delta = 0$. By definition and (41) d_{I_k} is equal to:

$$\gcd(e_{(k-1)1}^\perp \cdot \alpha_k, \dots, e_{(k-1)(k-1)}^\perp \cdot \alpha_k, h_{(k-1)1} \cdot \alpha_k, D_k).$$

For $j = 1, \dots, k$ apply: $[j,(k+1)] - a^{n_j} \circ e \circ b^{-1}$ followed by $[s,(k+1)] - a^{P_{(k-1)js}}$ for every $s = j+1, \dots, k-1$ and finish for each j with an appropriate [(k),(k+1)]-[W]. Then apply $\chi_{(k+1),\overline{m}_1}$ followed by $[s,(k+1)] - a^{-\delta_s}$ for $s = 1, \dots, k-1$, and finish with $[(k+1),(k+2)] -$ [W]. At the end of these isometries the first $k-1$ elements remain fixed while α_k goes to:

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + d_{I_k}(-x_k + N_k y_k) + \Delta h_1$$

Now set $U_{I_i(k+1)} = d_{I_{k+1}} q_{i(k+1)} + u_{I_i(k+1)}$, $\Delta = \Delta_k + 2qd_{I_k}$ and apply for $j = 1, \dots, k$, $k(k+1) - a^{-1}$, $[j,(k+1)] - a^{q_j n_j} \circ e \circ b^{-q_j}$ followed by $[s,(k+1)] - a^{q_j P_{(k-1)js}}$ for every $s = j+1, \dots, k-1$ and finish for each j with an appropriate [(k),(k+1)]-[W]. Then apply $k(k+1) - e \circ a^{-1}$ $\chi_{(k+1),2q\overline{m}_1}$ followed by $[s,(k+1)] - a^{-2q\delta_s}$ for $s = 1, \dots, k-1$, and finish with a suitable $[(k+1),(k+2)] -$ [W]. At the end of these isometries the first $k-1$ elements remain fixed while α_k goes to its desired cononical form. This finishes the case $\delta = 0$. We now take up the case $\delta = 1$. We separate two cases. By definition and (41), $d_{I_k} = \gcd(e_{(k-1)1}^\perp \cdot \alpha_k, \dots, e_{(k-1)(k-1)}^\perp \cdot \alpha_k, h_{(k-1)1} \cdot \alpha_k, D_k)$, if $\sigma \geq 2$ and to $\gcd(e_{(k-1)1}^\perp \cdot \alpha_k, \dots, e_{(k-1)(k-1)}^\perp \cdot \alpha_k, h_{(k-1)1} \cdot \alpha_k, 2D_k)$, if $\sigma = 1$. In the first case working basically the same way as in the first group of isometries in the case $\delta = 0$

we send, while keeping the first $k-1$ elements fixed α_k to:

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + d_{I_k}[2(-x_k + N_k y_k) + \sum_{s=2}^{\sigma} h_s] + \Delta h_1$$

if $[e^{\perp}_{(k-1)j} \cdot \alpha_k]/d_{I_k}$ is even for all j (in which case $[h_{(k-1)1} \cdot \alpha_k]/d_{I_k}$ is odd) and D_k/d_{I_k} is odd and otherwise to:

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + d_{I_k}(-x_k + N_k y_k) + \Delta h_1.$$

Of course the case must be the second because we are assuming that $L^{\perp}_k$ is odd. If $\sigma = 1$ then working the same way we get to send α_k to:

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + 2d_{I_k}(-x_k + N_k y_k) + \Delta h_1$$

if $[e^{\perp}_{(k-1)j} \cdot \alpha_k]/d_{I_k}$ is even for all j (in which case $[h_{(k-1)1} \cdot \alpha_k]/d_{I_k}$ is odd) and $d_{I_k}|D_k$ and otherwise to:

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + d_{I_k}(-x_k + N_k y_k) + \Delta h_1.$$

Once again for the same reasons the second happens. In either case now the rest of the proccess in identical to the one for $\delta = 0$. Now we deal with case when $L^{\perp}_i$ is odd for all $i \leq k$ and $L^{\perp}_k$ is even. From the discussion in the previous paragraph we see that we can assume the first $k-1$ elements in canonical position as in (40) and α_k of the form

$$\sum_{s=1}^{k-1}(-U_{I_s k}x_s + V_{ks}y_s) + d_{I_k}[2(-x_k + N_k y_k) + \sum_{s=2}^{\sigma} h_s] + \Delta h_1$$

with $[e^{\perp}_{(k-1)j} \cdot \alpha_k]/d_{I_k}$ even, for all j and $[h_{(k-1)1} \cdot \alpha_k]/d_{I_k}$ odd. Now we can procced by isometries to determine the canonical form of α_k. Applying the same isometries as in the second group of the case $\delta = 0$ (devide by $2d_{I_k}$ instead of by d_{I_k}) we can send α_k to an element of the form:

$$\sum_{s=1}^{k-1}(-r_{I_s k}x_s + V_{ks}y_s) + d_{I_k}[2(-x_k + N_k y_k) + \sum_{s=2}^{\sigma} h_s] + \Delta h_1$$

where $r_{I_s k} = u_{I_s k}$ or $u_{I_s k} + d_{I_k}$ (What the case is will be distinguished by the coordinates of the invariant characteristic vector of the tuple) and Δ is now equal to

$$\Delta = \sum_{s=1}^{k-1} \delta_s r_{I_s k} + \delta' d_{I_k}$$

with $\delta' \in \{0,1,2,3\}$. In fact $\delta' \in \{1,3\}$ because $[h_{(k-1)1} \cdot \alpha_k]/d_{I_k}$ is odd. If $\delta' = 3$ we can apply isometries to change it into 1 as follows: Apply, $k(k+1) - e \circ a^{-1}$ and for $j = 1, \ldots, k-1$, $j(k+1) - a^{\delta_j}$ followed by $k(k+1) - e$. Then apply $\chi_{k+1,-2\overline{m}_1}$ and finish by a sign change of h_1. This sends α_k to:

$$\sum_{s=1}^{k-1}(-r_{I_s k}x_s + V_{ks}y_s) + d_{I_k}[2(-x_k + n_k y_k) + \sum_{s=2}^{\sigma} h_s] + \Delta_k h_1 \tag{42}$$

where $\Delta_k = \sum_{s=1}^{k-1} \delta_s r_{I_s k} + d_{I_k}$ and achieves the canonical form in this case. We now compute $L_k^\perp$. It is easily seen to be generated by elements of the form:

$$\begin{aligned} &e_{kj}^\perp = e_{(k-1)j}^\perp + P_{kjk} y_k, \quad j = 1, \dots, k-1, \\ &x_k + n_k y_k, \quad x_s, y_s, \quad s \geq k+1 \\ &h_{k1} = 2h_{(k-1)1} + Q_{k1} y_k \quad h_{ks} = h_s + h_{(k-1)1}, \quad 2 \leq s \leq \sigma + 1 \end{aligned} \tag{43}$$

Now we finally need to find and prove the canonical form in the case $k \leq l$. The general canonical form looks like:

$$\begin{aligned} &\alpha_1, \dots, \alpha_k, \quad \text{as in } (40), (42) \\ &\alpha_i = \sum_{1 \leq s \leq i-1}^{s \neq k} (-u_{I_s i} x_s + B_{is} y_s) + (-2u_{I_{k-1} k} x_k + B_{ik} y_k) \\ &+ d_{I_i}(-x_i + n_i y_i) + \Delta_i h_1, \quad i = 2, \dots, l \end{aligned} \tag{44}$$

Where the Δ_i's are determined by the invariants in the same manner as the previous ones. (may or maynot have a summand of d_{I_i} together with their canonical representatives but what the case is, is determined by the invariants. The lattice $L_l^\perp$ is generated by:

$$\begin{aligned} &e_{lj}^\perp = e_{kj}^\perp + \sum_{s=k+1}^{l} P_{kjs} y_k, \quad j = 1, \dots, k-1, \\ &e_{lj}^\perp = x_j + n_j y_j + \sum_{s=j+1}^{l}, \quad j \geq k, \quad x_s, y_s, \quad s \geq l+1 \\ &h_{l1} = h_{k1} + \sum_{s=k+1}^{l} Q_{l1s} \quad h_{kj} = h_{kj} + \sum_{s=k+1}^{l} Q_{ljs}, \quad 2 \leq j \leq \sigma + 1 \end{aligned} \tag{45}$$

To finish this now one works as always by induction. Using $L_l^\perp$ define d_{I_l} and then use the vectors of $L_l^\perp$ as a guide to reduce the various quantities to their canonical representatives. This finishes the proof.

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