ON SOME POSSIBLE FORMULATION OF DIFFERENTIAL INVARIANTS FOR 4–MANIFOLDS

by

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Introduction

There are two objectives in this article. Firstly, using techniques recently introduced by Donaldson we define a differential invariant for those manifolds homeomorphic to $S^2 \times S^2$. Then we determine this invariant for the standard model. Secondly we discuss, and in some cases explicitely describe, the differences between moduli spaces of stable 2–bundles with $c_1 = 0$ over a simply–connected Kähler surface as the Kähler metric varies. These two discussions look quite unrelated but are in fact resulted from the consideration of extending the Γ–invariant and the polynomial invariants to formulate further differential invariants for smooth 4–manifolds with $b_2^+ = 1$. The precise relation between these two discussions will become clear in the course of our explanation followed.

To begin with, let X be a smooth compact simply–connected oriented 4–manifold with $b_2^+(X) = 1$. For integers $k > 0$, denote $\mathscr{C}_X^k$ the set of connected components of the positive cone in $H^2(X;\mathbb{R})$ after dividing by the system of walls $\bigcup_{1 \le \ell \le k} W_\ell$, where

$$W_\ell = \bigcup \{ \langle e \rangle^\perp \subset H^2(X;\mathbb{R}) \mid e \cdot e = -\ell \, ; \, e \in H^2(X;\mathbb{Z}) \} \ ,$$

and write simply $\mathscr{C}_X$ for $\mathscr{C}_X^1$. In § 1 we give a brief review on the definition of the invariant

$$\Gamma_X : \mathscr{C}_X \longrightarrow H^2(X;\mathbb{Z})$$

introduced in [D3] using Yang–Mills moduli spaces associated to an SU(2)–bundle $P \longrightarrow X$ with $c_2(P) = 1$. Working with $c_2(P) = k > 1$, we explain in § 2 it is still possible to define assignments

$$\Gamma_X^k : \mathscr{C}_X^k \longrightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$$

in the same spirit. As we shall see however, at present only in the case when $k = 2$ and X is homeomorphic to $S^2 \times S^2$ do we have a complete definition of a differential invariant for X. In this situation the cohomology group

$$H^2(X,\mathbb{R}) \simeq \{a_1h_1 + a_2h_2 \,|\, a_1,a_2 \in \mathbb{R}\}$$

is spanned by two (integral) generators h_1,h_2 over $\mathbb{R}$ while $\mathscr{C}_X^2$ is a set consisting of regions

$$C_+ = \{a_1 > a_2 > 0\}\ ,\ C_- = \{a_2 > a_1 > 0\}$$

together with $-C_+, -C_-$ as elements.

Theorem 1

(a) For any smooth manifold X homeomorphic to $S^2 \times S^2$ the polynomials $\Gamma_X^2(C_+)$, $\Gamma_X^2(C_-)$ satisfy a universal relation

$$\Gamma_X^2(C_+) = \Gamma_X^2(C_-) + (h_1 - h_2)^5 \ .$$

(b) For the standard model $S^2 \times S^2$, we have

$$\Gamma^2_{S^2 \times S^2}(C_+) = h_1^5 - 5(h_1^4 h_2) + 10(h_1^3 h_2^2) \quad \text{and}$$

$$\Gamma^2_{S^2 \times S^2}(C_-) = h_2^5 - 5(h_1 h_2^4) + 10(h_1^2 h_2^3)$$

where $(h_1^{\ell_1} h_2^{\ell_2})$ denotes the symmetrization of $h_1^{\ell_1} h_2^{\ell_2}$ in $(H^2(X;\mathbb{Z}))^{\otimes 5}$ for positive integers ℓ_1, ℓ_2.

The proof of theorem 1(a) will be postponed to § 6 where we consider the problem in some more general situation. We show theorem 1(b) in § 3. This is as far our first discussion goes but the determination of $\Gamma^2_{S^2 \times S^2}$ links it to the second. We realize $S^2 \times S^2$ as a smooth quadric surface $Q \subset \mathbb{P}_3(\mathbb{C})$ and then make use of certain facts on the moduli spaces $M_k^s(\omega)$ of ω–stable 2–bundles E with $(c_1(E), c_2(E)) = (0,k)$ over a simply–connected Kähler surface Y. To be more precise, let $\mathring{\Omega}_Y \subset \Omega_Y$ be the Kähler cone of Y and $\mathscr{C}_Y^k$ be the set of connected components of $\mathring{\Omega}_Y \setminus \bigcup_{1 \le \ell \le k} \mathring{W}_\ell$ where

$$\mathring{W}_\ell = \bigcup \{ \langle e \rangle^\perp \subset H^2(Y;\mathbb{R}) \mid e \cdot e = -\ell ;\ e \in H^{1,1}(Y;\mathbb{Z}) \} \ .$$

Theorem 2

(a) Supposing ω_{-1}, ω_1 are two Kähler forms on Y lying in a common components $\mathfrak{C} \in \mathscr{C}_Y^k$ of the divided Kähler cone $\mathfrak{K}_Y$, we have

$$M_k^s(\omega_{-1}) = M_k^s(\omega_1) \ .$$

(In other words, stability condition is uniformly defined on each such component.)

(b) Let $\{\omega_t \mid t \in [-1,1]\}$ be a path of Kähler forms on Y meeting only a single wall $\langle e\rangle^\perp$ of the system $\bigcup_{1 \le \ell \le k} \mathfrak{W}_\ell$. Assuming

$$\omega_{-1} \cdot e < 0 = \omega_0 \cdot e < \omega_1 \cdot e \quad \text{and} \quad e \cdot e = -k \ ,$$

we have that

$$M_k^s(\omega_{-1}) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^2) \setminus \{0\}) \quad \text{and}$$

$$M_k^s(\omega_1) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^{-2}) \setminus \{0\})$$

where L denotes the holomorphic line bundle over Y determined by e .

This theorem will be proved in § 4. Due to an observation of Donaldson, the latter part of theorem 2 can further be understood in terms of Yang–Mills theory by comparing the moment maps associated to:

(i) the S^1–action on the standard finite dimensional model of the anti–self–dual (ASD) equation around the reduction $L \oplus L^{-1}$, and

(ii) the gauge group action on the space of connections for Kähler surfaces modelling the ASD equation.

Following this idea, we explain in § 5 how to obtain an approximation of the ASD equation around a reduction over a simply–connected Kähler surface. More precisely we show in the standard finite dimensional model around a reduction A preserving $L \oplus L^{-1}$, the ASD equation is in essence a map

$$\hat{\phi} : H^1_A \longrightarrow \mathbb{R}$$

on a small neighbourhood of $0 \in H^1_A$ which is approximated by a (Morse) function

$$\hat{\phi}_0(v) = 2\left\{ \sum_{\alpha=1}^{h^1(L^2)} |Z_v^\alpha|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W_v^\beta|^2 \right\}$$

in the sense that

$$\hat{\phi}(v) = \hat{\phi}_0(v) + 0(|v|^3)$$

for $v \in H^1_A$ with $|v|$ small. In the above expression Z_v^α , W_v^β are certain complex coordinates for $H^{0,1}_{\bar\partial_A}(L^2) \oplus H^{0,1}_{\bar\partial_A}(L^{-2}) \simeq H^1_A$. Using such an approximation we interpret theorem 2 from an analytical point of view in § 6. This completes an outline of the paper.

It might worth pointing out that quite unexpectedly the systems of walls appeared in these two discussions are comparable. As a consequence, any calculation of Γ_Y^k over a simply–connected Kähler surface Y requires only identical moduli spaces of stable 2–bundles for each element in $\tilde{\mathscr{C}}_Y^k$. Furthermore, any change of these moduli spaces signals the difference of the polynomials Γ^k_Y . As one will see, this phenomenon culminates in the calculation of $\Gamma^2_{S^2\times S^2}$ in § 3.

I should emphasis the polynomials $\Gamma^2_{S^2\times S^2}(C_+)$, $\Gamma^2_{S^2\times S^2}(C_-)$ as described in theorem 1(b) are not Donaldson polynomials and quite on the contrary they reflect the construction of such polynomials depends upon the metrics as $b^+_2(S^2 \times S^2) = 1$. Moreover, in contrast to a result of [FMM], these two polynomials are not polynomials on the intersection form and the canonical class of a quadric surface Q realizing $S^2 \times S^2$.

Recently other differential invariants for certain smooth 4–manifolds have been obtained in [Kot], [OV] using SO(3)–bundle and our discussion here lies in a different stream. Other useful information relevant to our work can also be found in [FM1], [FM2].

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§ 1. A brief review of the Γ–invariant

In this section we give a brief review of the Γ–invariant to facilitate our future discussions. Assume always X is a smooth compact simply–connected oriented 4–manifold with $b_2^+(X) = 1$, the rank of the positive part associated to the intersection form on $H^2(X;\mathbb{Z})$. Recall first the invariant Γ_X takes the form of a mapping

$$\Gamma_X : \mathscr{C}_X \longrightarrow H^2(X;\mathbb{Z})$$

assigning to each "chamber" C, or an element of $\mathscr{C}_X$, a cohomology class $\Gamma_X(C)$ in $H^2(X;\mathbb{Z})$. To describe $\Gamma_X(C)$ it requires certain knowledge of 4–dimensional Yang–Mills theory on X and in this respect we introduce the following preliminary material on more general ground according to what we need later.

Let P be a SU(2)–bundle over X. Topologically such bundles are classified by $c_2(P) = k$. Let $\mathscr{A}$ be the (affine) space of connections on P. The gauge group $\mathscr{G} = \mathrm{Aut}\, P$ acts on $\mathscr{A}$ and we denote the quotient space $\mathscr{A}/\mathscr{G}$ by $\mathscr{B}_X$. Let $\mathscr{B}_X^*$ be the dense open subset of $\mathscr{B}_X$ consisting of equivalence classes of irreducible connections on P. For a smooth oriented real surface Σ in X, we define $\mathscr{B}_\Sigma$, $\mathscr{B}_\Sigma^*$ in the obvious way. Given a Riemannian metric m on X, we write

$$M_k(m) = \{A \in \mathscr{A} \mid *_m F(A) = -F(A)\}/\mathscr{G}$$

for the moduli space of anti–self–dual (ASD) connections on P relative to the metric m. Here F(A) denotes the curvature field associated to a connection A on P. Note that ASD connection on P exists only if $k = c_2(P) \geq 0$. In the case when $k = 0$, the moduli space is a single point $[\theta]$ carried by the trivial connection θ on X. For $k \geq 1$, a moduli space $M_k(m)$ relative to a generic metric m on X lives inside $\mathscr{B}_X^*$ and is a smooth oriented manifold of even dimension $2d$ where $d = 4k-3$. In general moduli spaces of this sort need not be compact.

Over a generic moduli space $M_k(m)$, one can construct certain complex line bundles which play a crucial role in our approach of defining differential invariants for X. To obtain these bundles, observe first that for a smooth oriented real surface Σ in X there defines a line bundle

$$\mathscr{L}_\Sigma = \Lambda^{\max}(\ker \not{\partial}_\Sigma)^* \otimes \Lambda^{\max}(\operatorname{coker} \not{\partial}_\Sigma)$$

over $\mathscr{B}_\Sigma^*$, where $\not{\partial}_\Sigma$ is the Dirac operator coupled to the restricted connection $A|_\Sigma$. This line bundle extends over degree zero reductions on Σ and it is possible to choose smooth sections on it not to vanish at the point $[\theta_\Sigma]$ in $\mathscr{B}_\Sigma$ carrying the trivial connection θ_Σ on Σ. Furthermore, for suitably chosen surfaces Σ, there are transversal sections of the bundle $\mathscr{L}_\Sigma$ $(= r_\Sigma^* \mathscr{L}_\Sigma)$ over $M_k(m)$ vanishing on codimension 2 submanifolds $V_\Sigma \cap M_k(m)$ of $M_k(m)$. Working with d surfaces $\Sigma_1,\ldots,\Sigma_d$ rather than one, we obtain transversal intersections $V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_d} \cap M_k(m)$ consisting of isolated points in $M_k(m)$. As one can see from the construction, the algebraic sums associated to these intersections do not have any invariant meaning in general but depend upon

(i) the choice of the metric m on X, and

(ii) the sections of the bundles $\mathscr{L}_\Sigma$ in the case when $M_k(m)$ fails to be compact.

Overcoming these difficulties leads to definitions of invariants for X and in this spirit the invariant Γ_X was defined.

To obtain Γ_X, one consider the case that $k = 1$. As $M_1(m)$ is generically a 2–dimensional manifold, we study intersection numbers associated to $V_\Sigma \cap M_1(m)$. If $M_1(m)$ is compact, so that $[M_1(m)]$ carries a homology class in $H_2(\mathscr{B}_X^*; \mathbb{Z})$, these intersection numbers are given by evaluations

$$<c_1(\mathscr{L}_\Sigma),\ [M_1(m)]> \in \mathbb{Z}\ .$$

In this way we obtain an assignment

$$\Sigma \longmapsto <c_1(\mathscr{L}_\Sigma),\ [M_1(m)]>$$

which lifts to homology level and defines an element in $\mathrm{Hom}(H_2(X,\mathbb{Z});\mathbb{Z}) = H^2(X;\mathbb{Z})$. In the non–compact case however $M_1(m)$ does not carry homology in $\mathscr{B}_X^*$ but it is still possible to define a homology class, say, $e_m \in H_2(\mathscr{B}_X^*;\mathbb{Z})$ using an argument in algebraic topology applied to certain finite dimensional model which describes the ends of $M_1(m)$ (cf. [D3]). Now we can assign to Σ the integer $<c_1(\mathscr{L}_\Sigma),e_m>$ instead and obtain whereby an element in $H^2(X;\mathbb{Z})$ as before. (Note that this pairing does not in general represent the intersection number of $V_\Sigma \cap M_1(m)$ any more.) However cohomology classes so obtained depend on the metric m and the orientation of $M_1(m)$ in some ways that we are going to explain.

As $b_2^+(X) = 1$, there is an L^2–normalized self–dual harmonic 2–form ω_m on X which is unique up to a sign. A choice of ω_m specifies a standard orientation of $M_1(m)$ and we write $M_1(\omega_m)$ for $M_1(m)$ with the assigned orientation. This process gives $M_1(-\omega_m)$ the opposite orientation compared with $M_1(\omega_m)$. Now if m_{-1}, m_1 are two generic metrics on X which can be joined by a (generic) path $\{m_t \mid t \in [-1,1]\}$ in such a way that $M_1(\omega_{m_t})$ contains no reduction for all t, then we can prove by a cobordism argument that

$$e_{m_{-1}} = e_{m_1} \in H^2(\mathscr{B}_X^*;\mathbb{Z})$$

in a usual way. The associated elements obtained in $H^2(X;\mathbb{Z})$ are therefore identical. By Hodge theory, this is the case provided the cohomology classes $[\omega_{m_t}]$ lie in a common chamber $C \in \mathscr{C}_X$ of the divided positive cone Ω_X. We have thus an assignment

$$\begin{aligned} \mathscr{C}_X &\longmapsto H^2(X;\mathbb{Z}) \\ C &\longmapsto \{[\Sigma] \longmapsto \langle c_1(\mathscr{L}_\Sigma), e_m \rangle\} \end{aligned}$$

choosing $[\omega_m] \in C$ and this is the idea of defining $\Gamma_X(C)$. If there is a universal bundle $\mathbb{P} \longrightarrow \mathscr{B}_X^* \times X$, as is always the case when $c_2(P) = 1$, the characteristic class $c_1(\mathscr{L}_\Sigma)$ can alternatively be realized by $c_2(\mathbb{P})/[\Sigma] \in H^2(\mathscr{B}_X^*;\mathbb{Z})$ and so one can describe more elegantly

$$\Gamma_X(C) = c_2(\mathbb{P})/e_m \in H^2(X;\mathbb{Z}) \quad ,$$

the version introduced in [D3].

For two chambers $C_1, C_2 \in \mathscr{C}_X$ the difference between $\Gamma_X(C_1)$ and $\Gamma_X(C_2)$, caused by reductions, has further been determined in [D3]. This enables us to described completely the invariant Γ_X for the manifold X once $\Gamma_X(C)$ is determined for just one single chamber C. Meanwhile, this discloses also the fact that the difference term is universal and is not in the nature of the differentiable structure of X . As we shall see in the next section, the determination of such difference terms is at present the main difficulty of generalizing the invariant Γ_X in our approach.

§ 2. A differential invariant of $S^2 \times S^2$

Following previous discussion it is conceivable that by using moduli spaces $M_k(m)$ with $k > 1$ it might still be possible to define some other differential invariants for X. In fact the case $k = 1$ is in some sense special since apart from which one can always arrange transversal intersections $V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_d} \cap M_k(m)$ to be compact assuming $b_2^+(X) = 1$ (cf. [D5] lemma (3.1)). Indeed, this is the case should one work with sufficiently general metrics m on X so that all moduli spaces $M_1(m),\ldots,M_{k-1}(m)$ in addition to $M_k(m)$ are smooth manifolds of formal dimension containing no U(1)–reduction. For this purpose one is to assume $[\omega_m]$ lies in some chamber $C \in \mathscr{C}_X^k$ in order that it does not meet the system of walls $\bigcup_{1\leq\ell\leq k} W_\ell$. In such cases, we can define a symmetric multi–linear map

$$q_{k,X}(\omega_m) : \underbrace{H_2(X;\mathbb{Z}) \times \ldots \times H_2(X;\mathbb{Z})}_{d \text{ times}} \longrightarrow \mathbb{Z}$$

using assignments

$$([\Sigma_1],\ldots,[\Sigma_d]) \longmapsto \text{the algebraic sum of a transversal intersection } V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_d} \cap M_k(\omega_m)\ .$$

Should we write $\mu([\Sigma])$ for $c_1(\mathscr{L}_\Sigma) \in H^2(\mathscr{B}_X^*;\mathbb{Z})$, these intersection numbers are given by the natural pairings

$$<\mu([\Sigma_1]) \cup \ldots \cup \mu([\Sigma_d])\ ,\ [M_k(\omega_m)]>$$

and if we consider

$$q_{k,X}(\omega_m) = <\mu^d, [M_k(\omega_m)]>$$

an element in $\mathrm{Sym}^d(H^2(X;\mathbb{Z}))$ this construction gives an assignment

(2.1)
$$\begin{aligned} \Gamma^k_X : \mathscr{C}^k_X &\longrightarrow \mathrm{Sym}^d(H^2(X;\mathbb{Z})) \\ C &\longmapsto q_{k,X}(\omega_m) \end{aligned}$$

assuming $[\omega_m] \in C$. This discussion lays out a framework for new differential invariants for X but in regard to the problem of comparing $\Gamma^k_X(C_{-1})$, $\Gamma^k_X(C_1)$ for two chambers $C_{-1}, C_1 \in \mathscr{C}^k_X$ we find only the following particular situation is known for the moment.

(2.2) <u>Lemma</u>. Suppose $\{\omega_t \mid t \in [-1,1]\}$ is a smooth path of self–dual harmonic forms on X meeting only a single wall $<e>^{\perp}$ of the system $\bigcup_{1\leq\ell\leq k} W_\ell$. Assuming

(i) $e \cdot e = -k$ and

(ii) $\omega_{-t} \cdot e < 0 = \omega_0 \cdot e < \omega_t \cdot e$ for $t > 0$,

we have then

$$\Gamma^k_X(C_{-1}) = \Gamma^k_X(C_1) + (-1)^k e^d$$

where $[\omega_{-1}] \in C_{-1}$ and $[\omega_1] \in C_1$.

This lemma, to be proved in § 6, does not give comparison formulas for Γ_X^k in the cases when

$$(2.3) \qquad e\cdot e = -1,\ldots,-k+1$$

and therefore does little help in making Γ_X^k a differential invariant of X in general. We can however avoid this difficulty in the special case when the intersection form of X is even so that (2.3) has no lattice solution if we work with $k = 2$; one never solves

$$e\cdot e = -1 \quad \text{since} \quad e\cdot e \equiv 0 \pmod 2 \ .$$

In such situation the intersection form of X is bound to be a copy of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and so, by a theorem of Freedman, X must be homeomorphic to $S^2 \times S^2$ (cf. [D2]). For such manifolds X one finds $H^2(X;\mathbb{Z})$ is freely generated by some h_1,h_2 and it is easy to check

$$e = h_2-h_1 \quad \text{and} \quad -e = h_1-h_2$$

are the only lattice solutions to the equation $e\cdot e = -2$. In this situation the system of walls

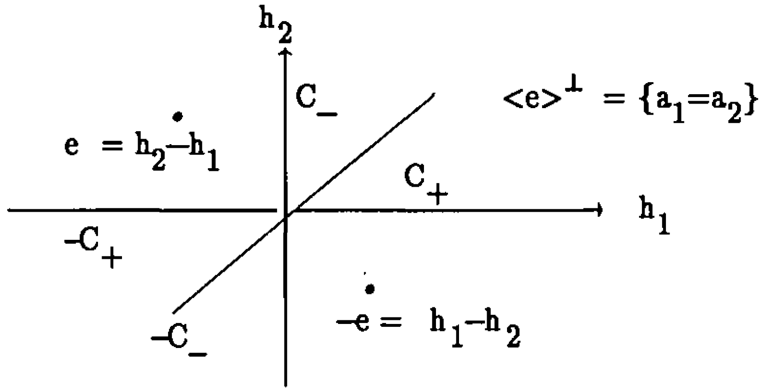
$$\bigcup_{1\leq \ell \leq 2} W_\ell = \{\langle h_2-h_1\rangle^\perp \subset H^2(X;\mathbb{R})\}$$

divides the positive cone

$$\Omega_X = \{a_1h_1 + a_2h_2 \in H^2(X;\mathbb{R}) \mid a_1a_2 > 0\}$$

into four chambers $C_+, C_-, -C_+, -C_-$ as shown below.

Diagram



Now we can state a complete definition of a differential invariant Γ^2_X for this kind of manifolds. More precisely, for manifolds X homeomorphic to $S^2 \times S^2$, the assignment

$$(2.4) \qquad \Gamma^2_X : \mathscr{C}^2_X \longrightarrow \mathrm{Sym}^5(H^2(X;\mathbb{Z}))$$

has the following properties:

(2.5) (i) $\Gamma^2_X(-C) = -\Gamma^2_X(C)$

(ii) $\Gamma^2_X(C_+) = \Gamma^2_X(C_-) + (h_1-h_2)^5$

(iii) If $f : X_1 \longrightarrow X_2$ is an orientation preserving diffeomorphism between two such manifolds, then $\Gamma^2_{X_1}(f^*(C)) = f^*\Gamma^2_{X_2}(C)$.

Note that (2.5) (ii) is a consequence of lemma (2.2) should one put $C_1 = C_+$, $C_{-1} = C_-$ and $e = h_2-h_1$ to get the formula

$$\Gamma^2_X(C_{-1}) = \Gamma^2_X(C_+) + (h_2-h_1)^5 \ .$$

We shall determine the invariant $\Gamma^2_{S^2\times S^2}$ for the standard $S^2 \times S^2$ in the next section and postpone the proof of lemma (2.2) to § 6 combining with some other discussion.

§ 3. The invariant for the standard model

In this section we determine the invariant $\Gamma^2_{S^2\times S^2}$ for the standard $S^2 \times S^2$. For simplicity we write

$$q_+ = \Gamma^2_{S^2\times S^2}(C_+) \quad\text{and}\quad q_- = \Gamma^2_{S^2\times S^2}(C_-) \ .$$

Clearly, the knowledge of q_+ and q_- determines $\Gamma^2_{S^2\times S^2}$ completely. To find q_+ and q_- we are to use some arguments in algebraic geometry. It is a well–known fact that $S^2 \times S^2$ can be realized as a complex quadric surface $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$ in the complex projective 3–space $\mathbb{P}_3$ and all the ample line bundles H_{r_1,r_2} on Q are of the form $\mathcal{O}(r_1,r_2) = pr_1^*\mathcal{O}_{\mathbb{P}_1}(r_1) \otimes pr_2^*\,\mathcal{O}_{\mathbb{P}_1}(r_2)$ where r_1,r_2 are strictly positive integers and pr_i denotes the projection map from $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$ to the i–th factor for $i = 1,2$. For each ample line bundle H_{r_1,r_2}, let M_{r_1,r_2} be the moduli space of H_{r_1,r_2}–stable 2–bundles E over Q with $\Lambda^2E \simeq \mathcal{O}_Q$ and $c_2(E) = 2$. (The definition of stability can be found in § 4.) The moduli spaces M_{r_1,r_2} are smooth and if $r_1 \neq r_2$ they are naturally identified with Yang–Mills moduli spaces $M_2(m)$ for compatible Kähler metrics m on Q by a theorem of Uhlenbeck and Yau. It follows from the general theory to determine q_+ and q_- it suffices to pick two moduli spaces M_{r_1,r_2}, one for $r_1 > r_2$ and one for $r_1 < r_2$. (The case $r_1 = r_2$ is special.) As we shall see however, the moduli spaces themselves are in fact divided into three kinds, according to the comparison between r_1 and r_2. This can be summarized as follws.

(3.1) Proposition. Associated to a quadric surface there are three spaces M_Δ, M_+, M_-

such that

$$M_{r_1,r_2} = \begin{cases} M_+ & \text{if} \quad r_1 > r_2 \\ M_\Delta & \text{if} \quad r_1 = r_2 \\ M_- & \text{if} \quad r_1 < r_2 \end{cases} .$$

In addition, we have

$$M_+ = M_\Delta \coprod \mathbb{P}_2^+ , \quad M_- = M_\Delta \coprod \mathbb{P}_2^-$$

where $\mathbb{P}_2^+, \mathbb{P}_2^-$ are two (distinct) copies of the complex projective plane parametrizing respectively non–trivial extensions of the following exact sequences:

$$0 \longrightarrow \mathcal{O}(1,-1) \longrightarrow E \longrightarrow \mathcal{O}(-1,1) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow E \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0 \ .$$

Extending the work of [SC], this proposition was proved in [M] by a direct argument using explicitely the description that $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$. We shall show however in the next section that this proposition can be obtained on general ground and is in fact a consequence of theorem 2 stated in the introduction. For this reason we omit the proof of (3.1) here and proceed to determine the polynomials q_+ and q_-.

<u>Remark</u>. Complete descriptions of the spaces M_+, M_Δ, M_- has been found in [B] using monads, giving moreover an example of a non–Hausdorff moduli space of simple bundles $\mathbb{P}_\Delta^+ \coprod M_\Delta \coprod \mathbb{P}_2^-$ over a quadric surface Q. Compare also [SC]. We shall however not make any use of these descriptions in the calculations of $\Gamma^2_{S^2 \times S^2}$, despite the determina-

tion of q_+, q_- is most explicit.

Let $L_i = pr_i^{-1}(\cdot) \simeq \mathbb{P}_1$ be the fibres of the projection map pr_i on $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$ for $i = 1,2$. To obtain q_+, q_- it suffices to establish the following table of evaluations for μ^5 .

(3.2) Table

Number of		$<\mu^5,[M_+]>$	$<\mu^5,[M_-]>$
L_1–lines	L_2–lines		
0	5	1	0
1	4	–1	0
2	3	1	0
3	2	0	1
4	1	0	–1
5	0	0	1

We shall only check the column for $<\mu^5,[M_+]>$ as the evaluation for $<\mu^5,[M_-]>$ is similar. Note that $q_+ = <\mu^5,[M_+]>$ in $\mathrm{Sym}^5(H^2(X;\mathbb{Z}))$.

Our calculation for q_+ hinges on the fact that a line L_i on the quadric Q is a copy of $\mathbb{P}_1$ and so one can adapt an argument in [D1] to show that the zero sets $V_{L_i} \cap M_+$ used to define q_+ can be taken to have the following concrete form:

$$\{[E] \in M_+ \mid E|_{L_i} \text{ is not trivial}\} \quad (3.3)$$

It is well–known that holomorphic 2–bundles on a projective line $\mathbb{P}_1 \simeq L_i$ always split and so we have

$$E|_{L_i} \simeq \mathcal{O}_{L_i}(a) \oplus \mathcal{O}_{L_i}(-a)$$

for some integer $a \geq 0$. The condition $E|_{L_i}$ is not trivial in (3.3) means $a \neq 0$ in the splitting of $E|_{L_i}$. In this case we say L_i is a jumping line of the bundle E.

To see the validity of (3.3) it is no loss to work with L_1–lines on Q. Then the determinant bundle $\mathscr{L}_{L_1}$ over $\mathscr{B}^*_{L_1}$ is given by

$$\mathscr{L}_{L_1} = (\Lambda^{\max}(\ker \not{\partial}_{L_1})^*) \otimes \Lambda^{\max}(\operatorname{coker} \not{\partial}_{L_1}) \ .$$

This line bundle has an alternative interpretation in complex geometry as we are going to explain. The Dirac operator when coupled with a connection on the restricted bundle $E|_{L_1}$ identifies with the twisted Cauchy–Riemann operator on the bundle $E|_{L_1} \otimes K^{1/2}_{L_1} \simeq E(0,-1)|_{L_1}$:

$$\bar{\partial}_{E(0,-1)|_{L_1}} : \Omega^0(E(0,-1)|_{L_1}) \longrightarrow \Omega^{0,1}(E(0,-1)|_{L_1}) \ .$$

Here we write $E(0,-1)$ for $E \otimes \mathcal{O}(0,-1)$ and use the fact $K^{1/2}_{L_1} \simeq \mathcal{O}_{L_1}(-1)$ for the projective line $L_1 \simeq \mathbb{P}_1$. The operator $\bar{\partial}_{E(0,-1)|_{L_1}}$ is an isomorphism if it has no kernel and cokernel, or equivalently

$$H^0(E(0,-1)|_{L_1}) = H^1(E(0,-1)|_{L_1}) = 0 \ .$$

This is the case precisely when the bundle $E(0,-1)$ has trivial splitting type over the line L_1. In this setting one can regard $\mathscr{L}_{L_1}$ the bundle over $\mathscr{B}^*_{L_1}$ coming from the assignment

$$E \longrightarrow \Lambda^{\max}(H^0(E(0,-1)|_{L_1})^* \otimes \Lambda^{\max} H^1(E(0,-1)|_{L_1}) \ .$$

As this bundle has a holomorphic sections s which vanishes precisely when $\bar{\partial}_{E(0,-1)|_{L_1}}$ is not an isomorphism (cf. [Q]), we conclude $V_{L_1} \cap M_+$ can be represented by the zero set $\{s = 0\} = V^J_{L_1} \subset M_+$ containing elements in M_+ precisely as described in (3.3). Of course one is to check the transversality for the zero sets $V^J_{L_1}$ and this will be shown in due course.

Suggested by this discussion, it is natural to investigate the splitting behaviour of elements $[E] \in M_+$ when restricted to a line L on the quadric Q. We first observe its splitting type is rather confined.

(3.4) <u>Lemma</u>. For a stable 2–bundle E over Q with $c_1(E) = 0$, $c_2(E) = 2$, we have either

$$E|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L \qquad \text{(trivial) or}$$
$$E|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \qquad \text{(jumping)}$$

for all lines L on Q .

<u>Proof</u>. The argument is a direct consequence of the Riemann–Roch formular. Associated to each L_1–line there is an exact sequence

$$0 \longrightarrow E(-1,0) \longrightarrow E \longrightarrow E|_{L_1} \longrightarrow 0 \ . \tag{3.5}$$

The stability of E gives $h^0(E) = 0$ and therefore the corresponding long exact sequence of (3.5) reads

$$0 \longrightarrow H^0(E|_{L_1}) \longrightarrow H^1(E(-1,0)) \longrightarrow H^1(E) \longrightarrow \dots \ .$$

One checks readily by the Riemann–Roch formula

$$\chi(E(r_1,r_2)) = 2(r_1+1)(r_2+1)-2$$

that $h^1(E) = 0$ and $h^1(E(-1,0)) = 2$. It follows then

$$h^0(E|_{L_1}) = h^0(\mathcal{O}_{L_1}(a) \oplus \mathcal{O}_{L_1}(-a)) = 2$$

which can possibly happen only when $a = 0,1$. The argument for L_2–lines is similar and this proves the lemma.

Now we come to count the number of jumping lines a stable bundle $E \longrightarrow Q$ can possibly have. We denote for instance $H_{r_1 \geq r_2}$ the ample line bundle H_{r_1,r_2} on Q if

$r_1 \geq r_2 > 0$.

(3.6) Lemma. An $H_{r_1 \geq r_2}$–stable bundle E can have at most two jumping lines in the line system $L_1 = pr_1^{-1}(\cdot)$. Similarly, an $H_{r_1 \leq r_2}$–stable bundle E can have at most two jumping lines in the line system $L_2 = pr_2^{-1}(\cdot)$.

As the moduli space M_0 is contained in M_+ and M_- by Theorem 2, the following corollary is immediate.

(3.7) Corollary. A bundle $E \longrightarrow Q$ can have at most two jumping lines in each line system of Q if $[E] \in M_0$.

To prove lemma (3.6), we show first for $[E] \in M_+$ the splitting type $E|_{L_1}$ is generically trivial. Suppose not, one finds by lemma (3.4)

$$E|_{L_1} \simeq \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_1}(-1)$$

holds uniformly for all L_1 and consequently that

$$E(0,-1)|_{L_1} \simeq \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_1}(-2)$$

on all such lines. Thus $(pr_1)_* E(0,-1)$ defines a line bundle, say, $\mathcal{O}_{\mathbb{P}_1}(\ell)$ over the base curve $\mathbb{P}_1$. It follows then

$$pr_1^*((pr_1)_* E(0,-1)) \simeq \mathcal{O}(\ell,0)$$

defines a line subbundle of $E(0,-1)$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{O}(\ell,0) \xrightarrow{\text{ev}} E(0,-1) \longrightarrow \mathcal{O}(-\ell,-2) \longrightarrow 0$$

via the natural evaluation map ev. As $c_2(E(0,-1)) = 2$, one finds

$$\mathcal{O}(\ell,0) \cdot \mathcal{O}(-\ell,-2) = -2\ell = 2$$

which gives that $\ell = -1$. We conclude therefore E comes from an extension

$$(3.8) \qquad 0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow E \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0 \ .$$

This however contradicts the $H_{r_1 \geq r_2}$–stability of E since

$$H_{r_1 \geq r_2} \cdot \mathcal{O}(-1,1) = r_1 - r_2 \geq 0$$

(cf. Definition (4.1)). Thus for those $[E] \in M_+$ the restrictions $E|_{L_1}$ is generically trivial.

To determine the number of L_1–jumping lines an $H_{r_1 \geq r_2}$–stable bundle can possibly have, it is easiest to consider E as a <u>family</u> of holomorphic bundles over a projective line $\mathbb{P}_1 \simeq L_2$. In this interpretation, the number of L_1–jumping lines for E is exactly the number of elements in the zero set $V^J_{L_1} \cap L_2$ where $V^J_{L_1}$ denotes the zero set of the canonical section of the determinant bundle $\mathcal{L}_{L_1}$ for this family. As $V^J_{L_1} \cap L_2$ represents the zero set of a (non–trivial) section of $\mathscr{L}_{L_1} \longrightarrow L_2$ with

$$c_1(\mathscr{L}_{L_1}) = c_2(E)/[L_1] = 2h_1$$

(cf. [D2]), we conclude $V^J_{L_1} \cap L_2$ contains at most two points and therefore $E|_{L_1}$ is non–trivial for at most two L_1–lines. The argument for $H_{r_1 \leq r_2}$–stable bundle E is similar and this proves the lemma.

One infers easily from this lemma that in table (3.2)

$$q_+(L_1^5) = q_+(L_1^4 L_2) = q_+(L_1^3 L_2^2) = 0 \tag{3.9}$$

by using zero sets $V^J_{L_1(z_i)}$ on M_+ associated to three (distinct) lines $L_1(z_i)$, $i = 1,2,3$. Indeed, in these situations the number of L_1–lines we are working with is no less than three and so (3.9) follows if one can show

$$V^J_{L_1(z_1)} \cap V^J_{L_1(z_2)} \cap V^J_{L_1(z_3)} \cap M_+ = \phi \ .$$

This is however a trivial consequence of lemma (3.6) as no $H_{r_1 \geq r_2}$–stable bundle E can "jump" on three distinct L_1–lines.

To find remaining evaluations for q_+ we apply the same argument to three (distinct) L_2–lines. This time we get a non–empty (set) intersection

$$(3.10) \qquad \left\{ \bigcup_{i=1}^{3} V^J_{L_2(w_i)} \right\} \cap M_+$$

$$= \left\{ \bigcup_{i=1}^{3} V^J_{L_2(w_i)} \cap M_0 \right\} \coprod \left\{ \bigcap_{i=1}^{3} V^J_{L_2(w_i)} \cap \mathbb{P}_2^+ \right\} \qquad \text{(by proposition (3.1))}$$

$$= \bigcap_{i=1}^{3} V^J_{L_2(w_i)} \cap \mathbb{P}_2^+ \qquad \text{(by lemma (3.6))}$$

$$= \mathbb{P}_2^+ .$$

Here the final equality follows from the fact that $\mathbb{P}_2^+$ parametrized non–trivial extensions of

$$0 \longrightarrow \mathcal{O}(1,-1) \longrightarrow E \longrightarrow \mathcal{O}(-1,1) \longrightarrow 0$$

and one checks readily

$$E|_{L_2} \simeq \mathcal{O}_{L_2}(1) \oplus \mathcal{O}_{L_2}(-1) \quad \text{for all } L_2\text{–lines}$$

using the fact $E|_{L_2}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_{L_2}(1) \longrightarrow E|_{L_2} \longrightarrow \mathcal{O}_{L_2}(-1) \longrightarrow 0$$

which always splits as $H^1(\mathcal{O}_{L_2}(2)) = 0$. We shall see in a moment the intersection (3.10) is transversal in general. Assuming this, we can proceed to determine q_+ in the remaining cases by studying the universal bundle $\mathcal{E}_+$ over the product space $\mathbb{P}_2^+ \times Q$:

$$(3.11) \qquad 0 \longrightarrow P^*_{\mathbb{P}_2^+}\mathcal{O}_{\mathbb{P}_2^+}(1) \otimes P_Q^*\mathcal{O}(1,-1) \longrightarrow \mathscr{E}_+ \longrightarrow P_Q^*\mathcal{O}(-1,1) \longrightarrow 0$$

where $P_{\mathbb{P}_2^+}$ and P_Q are the obvious projection maps (cf. [R] lemma 2.3).

To show $q_+(L_1^2 L_2^3) = 1$, we consider the intersection between $\mathbb{P}_2^+$ and two (more) zero sets $V^J_{L_1(z_1)}$, $i = 1,2$. These zero sets on $\mathbb{P}_2^+$ represent the determinant bundle $\mathscr{L}_{L_1}$ with

$$c_1(\mathscr{L}_{L_1}) = c_2(\mathscr{E}_+)/[L_1] = h_+ \ ,$$

where h_+ denotes the standard generator of $H^2(\mathbb{P}_2^+;\mathbb{Z})$. It follows up to a sign $q_+(L_1^2 L_2^3)$ is given by

$$<c_2(\mathscr{E}_+)/_{[L_1(z_1)]} \cup c_2(\mathscr{E}_+)/_{[L_1(z_2)]}, [\mathbb{P}_2^+]> = h_+ \cdot h_+ = 1 \ .$$

As the algebraic sum associated to an intersection of five zero sets on M_+ defined by holomorphic sections must be non–negative, we conclude $\mathbb{P}_2^+$ has its usual complex orientation and therefore $q_+(L_1^2 L_2^3) = 1$, as stated in (3.2). Similarly, using

$$c_1(\mathscr{L}_{L_2}) = c_2(\mathscr{E}_+)/_{[L_2]} = -h_+$$

and one derives

$$q_+(L_1L_2^4) = h_+ \cdot (-h_+) = -1 \ ,$$
$$q_+(L_2^5) = (-h_+) \cdot (-h_+) = 1$$

as wished.

To show the intersection $\left\{\bigcap_{i=1}^{3} V^J_{L_2(w_i)}\right\} \cap M_+ \simeq \mathbb{P}_2^+$ is transversal in general, we first describe a local defining function f for the zero set $V^J_{L_2}$ near a point $[E] \in M_+$ coming from a non–trivial extension

$$0 \longrightarrow \mathcal{O}(1,-1) \longrightarrow E \longrightarrow \mathcal{O}(-1,1) \longrightarrow 0 \ .$$

Consider the following extension

$$0 \longrightarrow \mathcal{O}_{L_2}(-1) \longrightarrow F \longrightarrow \mathcal{O}_{L_2}(1) \longrightarrow 0 \ .$$

It is well–known that an element $t \in H^1(\mathcal{O}_{L_2}(-2)) \simeq \mathbb{C}$ determines a bundle F_t in the extension which has non–trivial splitting type precisely when $t = 0$. Moreover, this family of bundles $\{F_t\}$ constitute a versal deformation of the bundle $\mathcal{O}_{L_2}(1) \oplus \mathcal{O}_{L_2}(-1)$ over $L_2 \simeq \mathbb{P}_1$ having the property that every deformation family $\{E_u\}$ over a parameter space U of this bundle $\mathcal{O}_{L_2}(1) \oplus \mathcal{O}_{L_2}(-1)$ can be realized as the pullback of $\{F_t\}$ via a map

$$f : U \longrightarrow H^1(\mathcal{O}_{L_2}(-2)) \simeq \mathbb{C} \ .$$

If U is a small neighbourhood of [E] in M_+, then by restricting bundles parametrized by U to L_2 we obtain in this way local defining functions f for the jumping divisor $V^J_{L_2}$ near $[E] \in M_+$. Regarding E as a deformation family of $E|_{L_2}$, one finds the differential of this map is given by the restriction map

$$r_{L_2} : H^1(\mathrm{End}\, E) \longrightarrow H^1((\mathrm{End}\, E)|_{L_2}) \simeq H^1(\mathcal{O}_{L_2}(-2)) \ .$$

Then it follows from the following commutative diagram that this restriction map is surjective and we conclude therefore zeros sets $V^J_{L_2} \cap M_+$ are always transversal.

(3.12) <u>Diagram</u>.

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^1(E(1,-1)) & \longrightarrow & H^1(\mathrm{End}\ E) & \longrightarrow & H^1(E(-1,1)) & \longrightarrow 0 \\
 & \downarrow & & \downarrow r_{L_2} & & \downarrow r_{L_2} & \\
 & 0 & \longrightarrow & H^1((\mathrm{End}\ E)|_{L_2}) & \longrightarrow & H^1(E(-1,1)|_{L_2}) & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
\end{array}$$

Here we have used the fact $h^0(E(1,-1)) = 0$ which follows from $h^0(E(-1,1)) = 1$ and the simplicity of E.

Now we are to show the local defining functions f_i so obtained for the zero sets $V^J_{L_2(w_i)}$ constitute a map

$$(df_1, df_2, df_3) : H^1(\mathrm{End}\ E) \longrightarrow \mathbb{C}^3$$

of rank 3, or equivalently, the intersection

$$\bigcap_{i=1}^{3} \left\{ \ker r_{L_2(w_i)} \right\} \subset H^1(\text{End } E)$$

is 2–dimensional, provided the points $w_i \in L_1$ are general. Let $H^1(\text{End } E) \simeq \mathbb{C}^5$ be spanned by vectors $(v_1,v_2;v_3,v_4,v_5) \in \mathbb{C}^5$ and we assume without loss of generality that the image of $H^1(E(1,-1)) \simeq \mathbb{C}^2$ in $H^1(\text{End } E)$ is spanned by v_1,v_2 while the vectors v_3,v_4,v_5 are lifted from $H^1(E(-1,1)) \simeq \mathbb{C}^3$. By naturality, the space $\text{Ker } r_{L_2}$ in $H^1(E(-1,1))$ is isomorphic to a lifting of $\text{Ker } r_{L_2}$ in $H^1(E(-1,1))$ while the space $\{\text{Ker } r_{L_2} \subset H^1(E(-1,1))\}$ can be described in the following way. Consider the commutative diagram below in where

$$H^1(\mathcal{O}(-2,2)) \simeq H^0(\mathcal{O}_{L_1}(2)) \otimes H^1(\mathcal{O}_{L_2}(-2)) \quad \text{and}$$
$$H^1(\mathcal{O}(-2,2)|_{L_2}) \simeq H^1(\mathcal{O}_{L_2}(-2)) \ .$$

(3.13) Diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(E(-1,1)) & \xrightarrow{\sim} & H^1(\mathcal{O}(-2,2)) & \longrightarrow & 0 \\ & & \downarrow r_{L_2} & & \downarrow r_{L_2} & & \\ 0 & \longrightarrow & H^1(E(-1,1)|_{L_2}) & \xrightarrow{\sim} & H^1(\mathcal{O}(-2,2)|_{L_2}) & \longrightarrow & 0 \ . \end{array}$$

Fix an isomorphism $H^1(\mathcal{O}_{L_2}(-2)) \simeq \mathbb{C}$ and we may assume $H^1(\mathcal{O}(-2,2)) \simeq H^0(\mathcal{O}_{L_1}(2)) \simeq \mathbb{C}^3$ is spanned by homogeneous polynomials a_0^2, a_0a_1, a_1^2 where $[a_0, a_1]$ denotes homogeneous coordinates of L_1. Now if $L_2 = pr_2^{-1}(w)$ for some $w = [a_0(w), a_1(w)] \in L_1$, then the kernel of r_{L_2} in

$$H^1(E(-1,1)) \simeq \{v_1a_0^2 + v_2a_0a_1 + v_3a_1^2 \in H^0(\mathcal{O}_{L_1}(2)) \mid v_i \in \mathbb{C}\,,\ i = 1,2,3\}$$

is given by

$$\{(v_3,v_4,v_5) \in \mathbb{C}^3 \mid v_3(a_0(w))^2 + v_4(a_0(w))\cdot(a_1(w)) + v_5(a_1(w))^2 = 0\}$$

and we conclude the intersection of three such planes is a point in $\mathbb{C}^3$, provided the points $w_1,w_2,w_3 \in L_1$ are in general positions. It follows then the defining functions of the zero sets $V^J_{L_2(w_i)}$ are of maximal rank and the intersection $\left\{\bigcap_{i=1}^{3} V^J_{L_2(w_i)}\right\} \cap M_+$ is therefore transversal in general.

Now we wish to explain why q_+ , q_- are not polynomials of the intersection form

$$q_Q = h_1h_2 + h_2h_1$$

and the canonical class

$$k_Q = -2h_1 - 2h_2$$

on a quadric surface $Q \simeq S^2 \times S^2$. Supposing on the contrary q_+ , say, admits such an expression, the coefficient a_0 of k_Q^5 would then be detected by the evaluation $q_+(L_1^5)$ or $q_+(L_2^5)$ as the intersection form is zero in either case. A contradiction is immediate since we have $a_0 \neq 0$ by $q_+(L_2^5) = 1$ while $q_+(L_1^5) = 0$ gives $a_0 = 0$.

Obviously the failure of q_+, q_- admitting such expressions lies in the fact that the construction of these polynomials depends upon the choice of metrics on Q . However we can get around this dependence just by averaging, or taking the sum of q_+ and q_-. Thus, as Q is a complete intersection, we can apply [FMM] theorem 5 to conclude $q_+ + q_-$ is a polynomial on q_Q and k_Q. Indeed, one can find by a direct calculation

$$q_+ + q_- = -\frac{1}{32}\, k_Q^5 + \frac{5}{8}\,(k_Q^3\, q_Q) - \frac{15}{4}\,(k_Q\, q_Q^2)$$

where the brackets ( ) denote symmetrizations of k_Q and q_Q.

In a future article we shall explain how to obtain certain polynomials on the blow–up of the complex projective plane at one point but now we move on to discuss how to prove proposition (3.1) on general ground.

§ 4. The stability condition on a Kähler surface

Here we give the proof of theorem 2, spelt out in (4.4) and (4.6) below. Suppose always that Y is a simply–connected Kähler surface and E is a holomorphic 2–bundle on Y with $\Lambda^2 E \simeq \mathcal{O}_Y$ and $c_2(E) = k > 0$. For a given Kähler form ω on Y we define

$$\deg_\omega F = \int_Y c_1(\det F) \wedge \omega$$

for each holomorphic bundle $F \longrightarrow Y$. Note that the degree $\deg_\omega F$ of F depends upon the Kähler form ω.

(4.1) Definition. A 2–bundle $F \longrightarrow Y$ is ω–stable if for every non–trivial holomorphic bundle map $\varphi : \mathcal{L} \longrightarrow F$ from a holomorphic line bundle $\mathcal{L}$ into F we have

$$\deg_\omega \mathcal{L} < \tfrac{1}{2} \deg_\omega F \ .$$

For simplicity we write $\omega \cdot \mathcal{L}$ in place of $\deg_\omega \mathcal{L}$ if $\mathcal{L}$ is a line bundle over Y.

(4.2) Remark. Regarding φ an element of $H^0(\mathcal{L}^{-1} \otimes E)$, we can actually require in the definition (4.1) that the inequality holds only for those non–zero $\varphi \in H^0(\mathcal{L}^{-1} \otimes E)$ with isolated zeros on Y. Indeed, if $\varphi \in H^0(\mathcal{L}^{-1} \otimes E)$ vanishes along an effective divisor $D \geq 0$, we may find a non–zero bundle map $\tilde{\varphi}$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{L} \otimes D \xrightarrow{\tilde{\varphi}} E \longrightarrow \mathcal{L}^{-1} \otimes D^{-1} \otimes I \longrightarrow 0$$

where $\tilde{\varphi} \in H^0(\mathscr{L}^{-1} \otimes D^{-1} \otimes E)$ has isolated zeros defining an ideal sheaf I. Now the requirement

$$\deg_\omega(\mathscr{L} \otimes D) < \tfrac{1}{2} \deg_\omega F$$

for such situations certainly implies the inequality for $\mathscr{L}$ in (4.1) as $\deg_\omega D \geq 0$.

It follows from this remark to test the ω–stability of a 2–bundle $E \longrightarrow Y$ with $c_1(E) = 0$ it suffices to check the inequality $\omega \cdot \mathscr{L} < 0$ holds for all possible exact sequences

(4.3) $$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

induced from non–zero elements φ in $H^0(\mathscr{L}^{-1} \otimes E)$ with isolated zero. This is our main tool of studying how the moduli space $M_k^s(\omega)$ of ω–stable 2–bundles $E \longrightarrow Y$ with $c_1(E) = 0$, $c_2(E) = k > 0$ changes as the Kähler form ω varies. Compare the following lemma with [F] remark (2.2).

(4.4) Lemma. Let ω_{-1}, ω_1 be two Kähler forms on Y lying in a common connected component of the Kähler cone $\Omega_Y \subset H^2(Y;\mathbb{R})$ after dividing by the system of walls

$$\{<e>^\perp \subset H^2(Y;\mathbb{R}) \,|\, -1 \leq e \cdot e \leq -k \,;\ \ e \in H^{1,1}(Y;\mathbb{Z})\} \ .$$

Then we have

$$M_k^s(\omega_{-1}) = M_k^s(\omega_1) \ .$$

Proof. Note first the assumption ω_{-1}, ω_1 lie in a common connected component of the divided Kähler cone $\mathfrak{K}_Y$ is equivalent to that

$$\text{sign}(\omega_1 \cdot e) = \text{sign}(\omega_{-1} \cdot e) \neq 0 \tag{4.5}$$

for all lattice points $e \in H^{1,1}(Y;\mathbb{Z})$ with $e \cdot e = -1,\dots,-k$. Now suppose on the contrary $M_k^s(\omega_{-1}) \neq M_k^s(\omega_1)$ and so we may assume without loss in such cases there is an element $[E] \in M_k^s(\omega_{-1}) \backslash M_k^s(\omega_1)$. Then, as E is not ω_1–stable, we can find an ω_1–destabilizing line bundle $\mathscr{L}$ fitting into an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

as in (4.3) with $\omega_1 \cdot \mathscr{L} \geq 0$. Note that $\omega_{-1} \cdot \mathscr{L} < 0$ in this situation as E is ω_{-1}–stable. Our aim here is to check that

$$\mathscr{L} \cdot \mathscr{L} = -1,\dots,-k \ .$$

Granted this one infers from (4.5)

$$\text{sign}(\omega_1 \cdot \mathscr{L}) = \text{sign}(\omega_{-1} \cdot \mathscr{L}) < 0$$

which however contradicts $\omega_1 \cdot \mathscr{L} \geq 0$ and the lemma will then follow.

To check $\mathscr{L} \cdot \mathscr{L} = -1,\dots,-k$, one observes first

$$0 \leq c_2(E \otimes \mathscr{L}^{-1}) = c_2(E) + \mathscr{L} \cdot \mathscr{L}$$

which gives $\mathscr{L} \cdot \mathscr{L} \geq -k$ as $c_2(E) = k$. To show on the other hand $\mathscr{L} \cdot \mathscr{L} < 0$ we apply the Hodge index theorem to a Kähler form ω_0 on Y attaining $\omega_0 \cdot \mathscr{L} = 0$. Such ω_0 exists somewhere in the path

$$\{(1-t)\omega_{-1} + t\omega_1 \mid t \in [0,1]\}$$

of Kähler forms as $\omega_1 \cdot \mathscr{L} \geq 0 > \omega_{-1} \cdot \mathscr{L}$. This proves the lemma.

Remark. There might have interest to know using similar argument one can prove the semi–stability condition and the stability condition are actually equivalent on each chamber $\mathfrak{C} \in \tilde{\mathscr{C}}_Y^k$. (Following the present context, a 2–bundle $E \longrightarrow Y$ is ω–semi–stable if it satisfies weaker requirements that $\omega \cdot \mathscr{L} \leq 0$ in definition (4.1).) Thus, in some sense, "essential" ω–semi–stable bundles can possibly occur only when the polarization ω lives in $\langle e\rangle^{\perp}$ for some lattice $e \in H^{1,1}(Y;\mathbb{Z})$ with $-1 \leq e\cdot e \leq -k$. We find proposition (3.1) fits well in this discussion. Relative to the ample line bundle $\mathcal{O}(1,1)$ over a quadric Q there are semi–stable 2–bundles, the non–trivial extensions parametrized by the two copies of projective plane $\mathbb{P}_2^+$, $\mathbb{P}_2^-$. These bundles however become stable in other polarizations provided only that they are semi–stable.

Now we wish to compare $M_k^s(\omega_{-1})$ with $M_k^s(\omega_1)$ when ω_{-1}, ω_1 lie in two different chambers $\mathfrak{C}_{-1}$, $\mathfrak{C}_1$ in the divided Kähler cone $\mathfrak{N}_Y$. For convenience, we write $\{\omega_t \mid t \in [-1,1]\}$, or simply $\{\omega_t\}$, to denote the path of Kähler forms on Y joining ω_{-1}, ω_1 in a usual way. Assume also that $\{\omega_t\}$ meets the system of walls

$$\{\langle e\rangle^{\perp} \subset H^2(Y;\mathbb{R}) \mid e \in H^{1,1}(Y;\mathbb{Z}),\ e\cdot e = -1,\dots,-k\}$$

only at $t = 0$. We consider then the following situation first as it is easiest to describe.

Denote t_- for an element $t \in [-1,0)$ in the following argument. Let $L = L_e$ be the line bundle determined by $e \in H^{1,1}(Y;\mathbb{Z})$.

(4.6) Lemma. Suppose the path $\{\omega_t\}$ of Kähler form meets only a single wall $<L>^{\perp}$ at $t = 0$ where $L \cdot L = -k$. Assuming $\omega_{-1} \cdot L < 0 = \omega_0 \cdot L < \omega_1 \cdot L$, we have

$$M_k^s(\omega_{-1}) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^2)\backslash\{0\}) \quad \text{and} \tag{4.7}$$

$$M_k^s(\omega_1) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^{-2})\backslash\{0\}) \ . \tag{4.8}$$

Proof. We shall only prove (4.7) as the argument for (4.8) is completely similar. It is not difficult to see that

$$M_k^s(\omega_0) \subset M_k^s(\omega_{-1}) \ .$$

Indeed, if on the contrary there is an element $[E] \in M_k^s(\omega_0)\backslash M_k^s(\omega_{-1})$, then we can find as before an exact sequence

$$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

having the properties that

(i) $\omega_{-1} \cdot \mathscr{L} \geq 0 > \omega_0 \cdot \mathscr{L}$ and

(ii) $-1 \leq \mathscr{L} \cdot \mathscr{L} \leq -k$.

Since ω_{t_-}, ω_{-1} lie in a common chamber, one inters from (ii) that

(4.9) $$\operatorname{sign}(\omega_{t_-} \cdot \mathscr{L}) = \operatorname{sign}(\omega_{-1} \cdot \mathscr{L})$$

which moreover has to be (strictly) positive as $\omega_{-1} \cdot \mathscr{L} > 0$ in this situation by (i). It follows then

(4.10) $$\omega_{t_-} \cdot \mathscr{L} > 0 > \omega_0 \cdot \mathscr{L} .$$

Now set $t_- \longrightarrow 0$ and one sees immediately a contradiction in (4.10) as $\omega_{t_-} \cdot \mathscr{L} \longrightarrow \omega_0 \cdot \mathscr{L}$ when t_- approaches zero.

To identify $M_k^s(\omega_{-1}) \backslash M_k^s(\omega_0)$, one argues similarly that every element $[E] \in M_k(\omega_{-1}) \backslash M_k(\omega_0)$ can be obtained by an extension

(4.11) $$0 \longrightarrow \mathscr{L} \longrightarrow E \longrightarrow \mathscr{L}^{-1} \otimes I \longrightarrow 0$$

with $\omega_0 \cdot \mathscr{L} \geq 0 > \omega_{-1} \cdot \mathscr{L}$ and that $-1 \leq \mathscr{L} \cdot \mathscr{L} \leq -k$. Again, using (4.9) we have

$$\omega_0 \cdot \mathscr{L} \geq 0 > \omega_{t_-} \cdot \mathscr{L}$$

and from which one infers $\omega_0 \cdot \mathscr{L} = 0$ by setting $t_- \longrightarrow 0$. Assuming $\{\omega_t\}$ meets only the wall $\langle L \rangle^\perp$, we conclude $\mathscr{L} = L^{\pm 1}$. Furthermore, the assumption $\omega_{-1} \cdot L < 0$ determines that $\mathscr{L} = L$ as $\mathscr{L} \cdot \omega_{-1} < 0$ by the ω_{-1}–stability of E . In the case when $L \cdot L = \mathscr{L} \cdot \mathscr{L} = -k$, we have $|I| = 0$ and therefore (4.11) reads

(4.12) $$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \longrightarrow 0 .$$

Now assertion (4.7) follows should one prove that non–trivial extensions of (4.12) define ω_{-1}–stable bundles. To see this is the case, consider the following potential destabilizing model for E , where $\mathscr{L}_1 \longrightarrow Y$ is a line bundle with $\omega_{-1} \cdot \mathscr{L}_1 \geq 0$.

(4.13) Diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{L}_1 & & & & \\
 & & & & \downarrow{\scriptstyle \beta} & & & & \\
0 & \longrightarrow & L & \longrightarrow & E & \xrightarrow{\alpha} & L^{-1} & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \mathscr{L}_1^{-1} \otimes I_1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

We are to prove the map β has to be zero if the bundle E does not split.

The case when the composition map $\alpha \circ \beta$ is identically zero never causes problem since then the map β factors through L and from this one infers

$$\omega_{-1} \cdot \mathscr{L}_1 \leq \omega_{-1} \cdot L < 0 \ ,$$

a contradiction to the assumption that $\omega_{-1} \cdot \mathscr{L}_1 \geq 0$. In the case when $\alpha \circ \beta \not\equiv 0$, we have either

(i) $\alpha \circ \beta$ vanishes somewhere, or

(ii) $\alpha \circ \beta$ is nowhere vanishing.

In the former case it is easy to observe

$$\omega_0 \cdot (\mathscr{L}_1^{-1} \otimes L^{-1}) = \omega_0 \cdot \mathscr{L}_1^{-1} > 0$$

as $\omega_0 \cdot L = 0$. It follows then

$$\omega_0 \cdot \mathscr{L}_1 < 0 \leq \omega_{-1} \cdot \mathscr{L}_1 \qquad (4.14)$$

and one deduces as before that $-1 \leq \mathscr{L} \cdot \mathscr{L} \leq -k$. Thus, as ω_{t_-} and ω_{-1} lie in a common chamber, we have

$$\mathrm{sign}(\omega_{t_-} \cdot \mathscr{L}_1) = \mathrm{sign}(\omega_{-1} \cdot \mathscr{L}_1) > 0$$

and by setting $t_- \longrightarrow 0$ we obtain $\omega_0 \cdot \mathscr{L}_1 \geq 0$. This however contradicts (4.14) and we may then exclude the possibility of (i).

Now if $\alpha o \beta$ is nowhere vanishing, one finds $\mathscr{L} = L^{-1}$ and so to prove $\beta \in H^0(E \otimes L)$ is zero in this situation it suffices to check non–trivial extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \longrightarrow 0$$

give simple bundles, i.e. $h^0(E^* \otimes E) = 1$. Indeed, as $h^0(E \otimes L^{-1}) > 0$, the simplicity of E ensures $h^0(E \otimes L) = 0$ and hence that $\beta \in H^0(E \otimes L)$ is zero.

The simplicity of E follows conveniently from a lemma of Oda which asserts as a special case that 2–bundles $E \longrightarrow Y$ obtained from non–trivial extensions of two line bundles, say L and L^{-1} in our situation, are simple if $h^0(L^{\pm 2}) = 0$ (cf. [O]). This is certainly the case here as $\omega_0 \cdot L^{\pm 2} = 0$ and ω_0 is a Kähler form on Y. Thus we exclude the possibility of (ii) and complete the proof of this lemma.

(4.15) Remark. By applying these two lemmas (4.4) and (4.6) to a quadric surface $Q \subset \mathbb{P}_3$ discussed in the last section, one deduces readily proposition (3.1) which describes how the moduli space $M_2^s(\omega)$ changes as the Kähler form ω on Q varies.

These arguments applies equally well to the cases when $L \cdot L = -1, \dots, -k+1$. In such situations, one proves as before

$$M_k^s(\omega_0) \subset M_k^s(\omega_{-1}), \quad M_k^s(\omega_0) \subset M_k^s(\omega_1)$$

and that $M_k^s(\omega_{-1}) \backslash M_k^s(\omega_0)$, $M_k^s(\omega_1) \backslash M_k^s(\omega_0)$ contain locally free extensions

$$0 \longrightarrow L \longrightarrow E \longrightarrow L^{-1} \otimes I \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow L^{-1} \longrightarrow E \longrightarrow L \otimes I \longrightarrow 0$$

respectively. It is not clear to me though what the description of $M_k(\omega_{-1}) \backslash M_k(\omega_0)$ or $M_k(\omega_1) \backslash M_k(\omega_0)$ in general would be.

In the next two sections we shall explain how the two copies of projective space

$$\mathbb{P}(H^1(L^2) \backslash \{0\}) \quad \text{and} \quad \mathbb{P}(H^1(L^{-2}) \backslash \{0\})$$

in lemma (4.6) can be accounted for in terms of Yang–Mills theory.

§ 5. Reducible anti–self–dual connections on Kähler surfaces

The goal of this section is to derive an approximation of the ASD equation around a U(1)–reduction on a simply–connected Kähler surface. To serve some other purpose, we recall first the following general facts about a reducible ASD connection A on a smooth compact simply–connected oriented 4–manifold X. We work with a fixed metric m_0 on X throughout this section.

Suppose A is a reducible ASD connection on an SU(2)–bundle $P \longrightarrow X$ preserving a splitting $L \oplus L^{-1}$ for some line bundle $L \longrightarrow X$, where $c_2(P) = -L \cdot L = k > 0$. It is a well–known fact that a neighbourhood of $[A] \in M_k(m_0)$ can be modelled as an S^1–quotient $\phi^{-1}(0)/S^1$ for some finite dimensional equivariant map

$$\phi : H^1_A \longrightarrow H^2_A \tag{5.1}$$

defined on a small neighbourhood of $\underline{0} \in H^1_A$. Here we write H^i_A, $i = 0,1,2$, for the cohomology groups of the Atiyah–Hitchen–Singer deformation complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\, P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\, P) \xrightarrow{d_A^+} \Omega^2_+(\mathrm{ad}\, P) \longrightarrow 0$$

associated to the ASD connection A. More precisely in (5.3) we find a smooth map

$$v \in H^1_A \longrightarrow \tilde{v} \in \ker d_A^* \subset \Omega^1(\mathrm{ad}\, P)$$

modelled on suitable Hilbert spaces so that for $|v| << 1$ the map $\tilde{v}$ solves

$$\phi(v) = F_+(A + \tilde{v}) \in H^2_A$$

and in the same time satisfies the estimate that

$$|\tilde{v} - v| \le \text{const}\,|v|^2 \tag{5.2}$$

(cf. [D2]). It is clear from (5.5) and (5.6) the map ϕ satisfies

$$\phi(0) = 0\ , \quad d\phi(0) = 0$$

and so it is of interest to identify the second order approximation of the map ϕ about $\underline{0} \in H^1_A$. This can be achieved on a simply–connected Kähler surface.

In order to explain this, we pass the above discussion to a simply–connected Kähler surface Y as was discussed in § 4. So assume the metric m_0 is Kähler and $L \longrightarrow Y$ denotes a holomorphic line bundle satisfying

$$L \cdot L = -k\ , \quad \omega_0 \cdot L = 0\ ,$$

where ω_0 is the Kähler form on Y associated to m_0 . Note first in this situation the map ϕ takes a particular simple form:

$$\phi : \left\{ v \in H^1_{A_0} \,\middle|\, |v| << 1 \right\} \longrightarrow \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix} . \tag{5.3}$$

The reason for this is as follows. Working over complex manifolds one can associate to the connection A a twisted Cauchy–Riemann operator

$$\bar{\partial}_A : \Omega^{0,0}(\text{ad } P) \longrightarrow \Omega^{0,1}(\text{ad } P)$$

and define whereby Dolbeault cohomology groups $H^{0,i}_{\bar{\partial}_A}$, $i = 0,1,2$ in a natural way. As Y is Kähler, there are natural isomorphisms

$$H^1_A \simeq H^{0,1}_{\bar{\partial}_A} \quad \text{and} \tag{5.4}$$

$$H^2_A \simeq H^{0,2}_{\bar{\partial}_A} \oplus H^0_A \tag{5.5}$$

relating cohomology groups of these two kinds (cf. [Kob] p. 248). Here in our discussion, one interprets

$$H^0_A \simeq \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix}$$

in (5.5). Now, as Y is simply–connected, the reduction [A] presents an isolated singularity in $M_k(m_0)$ and consequently each connection in

$$\{A_0 + \tilde{v} \mid 0 < |v| << 1\}$$

determines an ω_0–stable bundle over Y in a usual way. Since these bundles are holomorphic, one finds for $|v| << 1$

$$F_+^{(0,2)}(A + \tilde{v}) = 0$$

and hence that

$$\phi(v) = F_+(A + \tilde{v}) \in \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix} ,$$

the only part in H_A^2 containing type (1,1) forms by (5.5).

Using the simple description of ϕ we can define a dual map

$$\hat{\phi} : H_A^1 \longrightarrow \mathbb{R}$$

$$v \longmapsto -\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} , \phi(v)\right] \wedge \omega_0$$

on $\{|v| << 1\}$ having the property that

$$\phi(v) = \frac{\hat{\phi}(v)}{4\ \mathrm{vol}\ (Y)} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix} \in \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix} .$$

Clearly then $\hat{\phi}^{-1}(0)/S^1$ provides an alternative model for a small neighbourhood of $[A] \in M_k^s(m_0)$. Similar to the map ϕ, one finds $\hat{\phi}$ satisfies $\hat{\phi}(0) = d\hat{\phi}(0) = 0$ and so we study the second order approximation of $\hat{\phi}$ about $\underline{0} \in H_A^1$. Let

$$\hat{\phi}_0(v) = -\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} , F_+(A + v)\right] \wedge \omega_0 \tag{5.6}$$

where $v \in H^1_A$ with $|v| << 1$.

(5.7) Lemma. On small neighbourhoods $\{|v| << 1\}$ of $0 \in H^1_A$, the function $\hat{\phi}$ is approximated by $\hat{\phi}_0$ in the sense that

$$\hat{\phi}(v) = \hat{\phi}_0(v) + 0(|v|^3) \ . \tag{5.8}$$

Proof. Assuming $p = \tilde{v} - v$ one finds

$$F_+(A + \tilde{v}) = F_+(A + v) + d_A^+ p + (v \wedge p + p \wedge v + p \wedge p)_+$$

and hence that

$$-\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, F_+(A + \tilde{v})\right] \wedge \omega_0$$

$$= -\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, F_+(A + v)\right] \wedge \omega_0 - \int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, d_A^+ p\right] \wedge \omega_0 + 0(|v|^3)$$

as $|p| \leq \mathrm{const}|v|^2$ by (5.2). Now (5.8) follows should one notice

$$-\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, d_A^+ p\right] \wedge \omega_0 = \int_Y \mathrm{Tr}\left[d_A \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \wedge p\right] \wedge \omega_0 = 0$$

since $d_A \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = 0$ relative to the splitting $L \oplus L^{-1}$. This completes the proof.

Now we identify $\hat{\phi}_0$. For this purpose observe first relative to the splitting $L \oplus L^{-1}$

the Dolbeault cohomology group $H^{0,1}_{\bar\partial_A}$ is naturally a direct sum of Hermitian vector spaces

$$H^{0,1}_{\bar\partial_A} \simeq H^{0,1}_{\bar\partial_A}(L^2) \oplus H^{0,1}_{\bar\partial_A}(L^{-2}) \ .$$

By taking two sets of unitary basis, say,

$$\{\varphi_\alpha \mid 1 \le \alpha \le h^1(L^2)\} \quad \text{and} \quad \{\psi_\beta \mid 1 \le \beta \le h^1(L^{-2})\} \ ,$$

for $H^{0,1}_{\bar\partial_A}(L^2)$ and $H^{0,1}_{\bar\partial_A}(L^{-2})$ respectively, one finds

$$H^{0,1}_{\bar\partial_A} \simeq \left\{ \sum_{\alpha=1}^{h^1(L^2)} Z_\alpha \begin{bmatrix} 0 & \varphi_\alpha \\ 0 & 0 \end{bmatrix} + \sum_{\beta=1}^{h^1(L^{-2})} W_\beta \begin{bmatrix} 0 & 0 \\ \psi_\beta & 0 \end{bmatrix} \;\middle|\; Z_\alpha, W_\beta \in \mathbb{C} \right\} .$$

Furthermore, via the isomorphism $H^1_A \simeq H^{0,1}_{\bar\partial_A}$, we obtain in turn a (real) basis for H^1_A :

$$a_\alpha = \begin{bmatrix} 0 & \varphi_\alpha \\ -\bar\varphi_\alpha & 0 \end{bmatrix} , \ Ia_\alpha = \begin{bmatrix} 0 & i\varphi_\alpha \\ -\overline{i\varphi_\alpha} & 0 \end{bmatrix} , \ \alpha = 1,\dots,h^1(L^2) \ ;$$

$$b_\beta = \begin{bmatrix} 0 & -\bar\psi_\alpha \\ \psi_\beta & 0 \end{bmatrix} , \ Ib_\beta = \begin{bmatrix} 0 & -\overline{i\psi_\beta} \\ i\psi_\beta & 0 \end{bmatrix} , \beta = 1,\dots,h^1(L^{-2}) \ .$$

In these notations, it is not difficult to check every vector $v \in H^1_A$ can be uniquely as a combination

$$v = \begin{bmatrix} 0 & \Sigma Z_v^\alpha \varphi_\alpha \\ -\Sigma \overline{Z_v^\alpha}\,\overline{\varphi_\alpha} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\Sigma \overline{W_v^\beta}\,\overline{\psi_\beta} \\ \Sigma W_v^\beta \psi_\beta & 0 \end{bmatrix}$$

$$= \Sigma(\mathrm{Re}\, Z_v^\alpha a_\alpha + \mathrm{Im}\, Z_v^\alpha\, Ia_\alpha) + \Sigma(\mathrm{Re}\, W_v^\beta b_\beta + \mathrm{Im}\, W_v^\beta\, Ib_\beta)$$

for some complex scalars Z_v^α, W_v^β. Now we can describe the approximation $\hat{\phi}_0$ explicitely as follows. Assume $\mathrm{vol}\, Y = 1$.

(5.9) Lemma. For a vector $v \in H_A^1$ with $|v|$ small, we have

$$\hat{\phi}_0(v) = 2\left\{ \sum_{\alpha=1}^{h^1(L^2)} |Z_v^\alpha|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W_v^\beta|^2 \right\} .$$

Proof. We show $\hat{\phi}_0$ satisfies the system of differential equations

$$(5.10)\qquad \begin{aligned} \left.\frac{\partial\hat{\phi}_0}{\partial a_\alpha}\right|_v &= 4\,\mathrm{Re}\, Z_v^\alpha , \quad \left.\frac{\partial\hat{\phi}_0}{\partial Ia_\alpha}\right|_v = 4\,\mathrm{Im}\, Z_v^\alpha , \quad \alpha = 1,\dots,h^1(L^2) \ ; \\ \left.\frac{\partial\hat{\phi}_0}{\partial b_\beta}\right|_v &= -4\,\mathrm{Re}\, W_v^\beta , \quad \left.\frac{\partial\hat{\phi}_0}{\partial Ib_\beta}\right|_v = -4\,\mathrm{Im}\, W_v^\beta , \quad \beta = 1,\dots,h^1(L^{-2}) \ . \end{aligned}$$

Then, as $\hat{\phi}_0(0) = 0$, it follows easily that

$$\hat{\phi}_0(v) = 2\sum_\alpha \left\{ \left[\mathrm{Re}\, Z_v^\alpha\right]^2 + \left[\mathrm{Im}\, Z_v^\alpha\right]^2 \right\} - 2\sum_\beta \left\{ \left[\mathrm{Re}\, W_v^\beta\right]^2 + \left[\mathrm{Im}\, W_v^\beta\right]^2 \right\}$$

$$= 2\left\{\sum_{\alpha} |Z_v^\alpha|^2 - \sum_{\beta} |W_v^\beta|^2\right\} ,$$

as wished. To show (5.10) we check only

$$\left.\frac{\partial \hat{\phi}_0}{\partial a_1}\right|_v = 4 \text{ Re } Z_v^1 \tag{5.11}$$

as the argument for other cases are similar. It is however just a routine matter of showing this:

$$d\hat{\phi}_0\Big|_v(a_1) = -\int_Y \text{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix} , d^+_{A+v}\begin{bmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{bmatrix}\right] \wedge \omega_0$$

$$= \int_Y \text{Tr}\left[\left[v, \begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix}\right] \wedge \begin{bmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{bmatrix}\right] \wedge \omega_0$$

$$= -2\int_Y \text{Tr}\left[\begin{bmatrix} 0 & i\, Z_v^1 \varphi_1 \\ i\, \overline{Z_v^1}\, \bar{\varphi}_1 & 0\end{bmatrix} \wedge \begin{bmatrix} 0 & \varphi_1 \\ -\bar{\varphi}_1 & 0\end{bmatrix}\right] \wedge \omega_0$$

$$= 2\int_Y \left[Z_v^1 + \overline{Z_v^1}\right] i\, \varphi_1 \wedge \bar{\varphi}_1 \wedge \omega_0$$

$$= 4 \text{ Re } Z_v^1 .$$

(5.12) Remark. The reason behind this argument as mentioned in the introduction is that the moment map (cf. [AB])

$$\mathcal{A} \longrightarrow \Omega^4(\mathrm{ad}\, P)$$

$$A \longmapsto F_+(A) \wedge \omega_0$$

for the gauge group action on $\mathcal{A}$ when restricted to the finite dimensional model

$$\phi : H^1_A \longrightarrow \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix}$$

for a reducible ASD connection A is in essence the one associated to the S^1–action

$$(z^1,\dots,z^{h^1(L^2)};\ w^1,\dots,w^{h^1(L^{-2})})$$

$$\longmapsto (e^{2i\theta}z^1,\dots,e^{2i\theta}z^{h^1(L^2)};\ e^{-2i\theta}w^1,\dots,e^{-2i\theta}w^1,\dots,e^{-2i\theta}w^{h^1(L^{-2})})$$

on $\mathbb{C}^{h^1(L^2)+h^1(L^{-2})}$. From this point of view, lemma (5.9) is not surprising at all since the moment map in the latter situation has explicitely been known to be

$$\mathbb{C}^{h^1(L^2)+h^1(L^{-2})} \longrightarrow i\,\mathbb{R}$$

$$(z^1,\dots,z^{h^1(L^2)};\ w^1,\dots,w^{h^1(L^{-2})}) \longmapsto \frac{i}{2}\left\{ \sum_{\alpha=1}^{h^1(L^2)} |Z^\alpha|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W^\beta|^2 \right\} .$$

This observation is due to Donaldson.

Now combining (5.8) and (5.9) we obtain the following description

$$\hat{\phi}(v) = 2\left\{\sum_{\alpha}^{h^1(L^2)} |Z_v^{\alpha}|^2 - \sum_{\beta}^{h^1(L^{-2})} |W_v^{\beta}|^2\right\} + 0(|v|^3) \tag{5.13}$$

which will be useful in the next section to explain theorem 2. Using (5.13) one can also deduce, in the case when both $h^1(L^2)$ and $h^1(L^{-2})$ are strictly positive, the link of the reduction $[A] \in M_k^s(m_0)$ is a quotient

$$\left[S^{2h^1(L^2)-1} \times S^{2h^1(L^{-2})-1}\right] / S^1$$

where S^1 acts diagonally on the spheres $S^{2h^1(L^2)-1}$ and $S^{2h^1(L^{-2})-1}$. This point however is not required in the present discussion.

§ 6. Two effects of changing metrics

We concern ourselves with two problems of similar outset here but for the moment just focus on the first, the one that is directly related to the discussion of the last section.

We have derived an approximation $\hat{\phi}_0$ for the finite dimensional model

$$\phi : H^1_A \longrightarrow \mathbb{R} \cdot \begin{bmatrix} i\omega_0 & 0 \\ 0 & -i\omega_0 \end{bmatrix}$$

of the ASD equation around a reduction $[A] \in M^s_k(m_0)$ on a simply–connected Kähler surface Y. In order to understand theorem 2(b) we extend this model to a small path of Kähler metrics $\{m_t \mid t \in [-1,1]\}$ on Y. With little additional effort we obtain as before an equivariant map

$$\Phi : H^1_A \times [-1,1] \longrightarrow \mathbb{R} \cdot \begin{bmatrix} i\omega_t & 0 \\ 0 & -i\omega_t \end{bmatrix} \tag{6.1}$$

on $\{|v| << 1 ; v \in H^1_A\} \subset H^1_A \times [-1,1]$ which solves

$$\Phi(v,t) = F_{+,m_t}(A + \tilde{v}_t) \in \mathbb{R} \cdot \begin{bmatrix} i\omega_t & 0 \\ 0 & -i\omega_t \end{bmatrix}$$

for some $\tilde{v}_t = \tilde{v}_t(v,t) \in \ker d^*_A$ in $\Omega^1(\mathrm{ad}\, P)$. Now we put

$$\hat{\Phi}(v,t) = -\int_Y \mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}, \Phi(v,t)\right] \wedge \omega_t \tag{6.2}$$

and assume $\mathrm{vol}\, Y \equiv 1$ for all metrics m_t . One finds then

$$\Phi = \frac{1}{4}\hat{\Phi} \cdot \begin{bmatrix} i\omega_t & 0 \\ 0 & -i\omega_t \end{bmatrix}$$

and so $\hat{\Phi}^{-1}(0)/S^1$ models $\bigcup_t \left\{ M_k^s(\omega_t) \,|\, t \in [-1,1] \right\}$ about the reduction class $[A]$. This time we are interested in an approximation of $\hat{\Phi}$ in the hope of finding a model for $\hat{\Phi}^{-1}(0)/S^1$ up to diffeomorphism. Such an approximation for $\hat{\Phi}$ can be found if the path $\{m_t\}$ of Kähler metrics passes through the wall $\langle L\rangle^{\perp} \subset H^2(Y;\mathbb{R})$ transversally in the sense that $\partial\hat{\Phi}/\partial t \neq 0$ at $(\underline{0},0)$.

(6.3) Lemma. If $\partial\hat{\Phi}/\partial t \neq 0$ at $(\underline{0},0)$, then by a small isotopy $\hat{\Phi}^{-1}(0)/S^1$ is diffeomorphic to the S^1–quotient of

$$\sum_{\alpha=1}^{h^1(L^2)} |Z_v^{\alpha}|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W_v^{\beta}|^2 = \frac{t}{2} \cdot \frac{\partial\hat{\Phi}}{\partial t}(\underline{0},0)$$

defined on sufficiently small neighbourhoods $\{(v,t) \in H_A^1 \times [-1,1] \,\big|\, |v| + |t| << 1\}$ of $(\underline{0},0)$ in $H_A^1 \times [-1,1]$.

The proof of this lemma, to be omitted here, is a standard argument for the isotopy theorem in differential topology incorporate with lemma (5.9). Now we determine the sign of $\partial\hat{\Phi}/\partial t$ at $(\underline{0},0)$ as it is also required in our discussion. The method we use is in parallel with [D3] proposition (2.12).

(6.4) Lemma. Assuming $L \cdot \omega_{-t} < 0 = L \cdot \omega_0 < L \cdot \omega_t$ for small $t > 0$, we have

$$\frac{\partial \hat{\Phi}}{\partial t}(0,0) < 0 \ .$$

Proof. We check the (smooth) function $\hat{\Phi}(\underline{0},t)$ of t is strictly decreasing at $t = 0$. Relative to the splitting $L \oplus L^{-1}$ preserving by the reduction A, one finds for small t

$$\hat{\Phi}(0,t) = \int_Y -\mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, F_{+,m_t}(A)\right] \wedge \omega_t$$

$$= \int_Y -\mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, F(A)\right] \wedge \omega_t$$

$$= \int_Y -\mathrm{Tr}\left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} -2\pi i\, c_1(L) & 0 \\ 0 & 2\pi i\, c_1(L) \end{bmatrix}\right] \wedge \omega_t$$

$$= -4\pi L \cdot \omega_t$$

$$= \begin{cases} > 0 & \text{if } t < 0 \\ = 0 & \text{if } t = 0 \\ < 0 & \text{if } t > 0 \end{cases} .$$

This shows the lemma.

For simplicity we assume $\partial \hat{\Phi} / \partial t = -2$ at $(\underline{0},0)$ so that $\hat{\Phi}^{-1}(0)/S^1$ is modelled on the S^1–quotient of

$$\text{(6.5)} \qquad \sum_{\alpha=1}^{h^1(L^2)} |Z_v^\alpha|^2 - \sum_{\beta=1}^{h^1(L^{-2})} |W_v^\beta|^2 = -t \ .$$

Now we are in a position to explain how (6.5) derived from Yang–Mills theory can lead to a better understanding of the descriptions

$$\text{(6.6)} \qquad M_k^s(\omega_t) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^2)\setminus\{0\}) \quad \text{for} \quad t < 0 \ ,$$

$$\text{(6.7)} \qquad M_k^s(\omega_t) = M_k^s(\omega_0) \coprod \mathbb{P}(H^1(L^{-2})\setminus\{0\} \quad \text{for} \quad t > 0$$

stated in theorem 2(b). In the case when $t < 0$, one deduces readily from (6.5) that $M_k^s(\omega_t)$ contains a copy of

$$\text{(6.8)} \qquad \left\{ \sum_{\alpha=1}^{h^1(L^2)} |Z_v^\alpha|^2 = -t \ ; \ w_v^1 = \dots = w_v^{h^1(L^{-2})} = 0 \right\} / S^1 \simeq \mathbb{P}_{h^1(L^2)-1}$$

which accounts for the projective space $\mathbb{P}(H^1(L^2)\setminus\{0\})$ in (6.6). Furthermore, as $t < 0$ increases to zero, this copy of projective space (6.8) degenerates into a single point $\underline{0} \in H_A^1$ corresponding to the reduction $[A]$ in the Yang–Mills moduli space

$$M_k(m_0) = M_k^s(\omega_0) \coprod \{[A]\} \ .$$

As t passes through zero and become positive, there emerges another copy of projective space

$$\text{(6.9)} \qquad \left\{ Z_v^1 = \dots = Z_v^{h^1(L^2)} = 0 \ ; \ \sum_{\beta=1}^{h^1(L^{-2})} |W_v^\beta|^2 = t \right\} / S^1 \simeq \mathbb{P}_{h^1(L^{-2})-1}$$

in $M_k^s(\omega_t)$ accounting for $\mathbb{P}(H^1(L^{-2})\backslash\{0\})$ in (6.7). Apart from these, one can make use of (6.5) to find small diffeomorphic deleted neighbourhoods for

$$\begin{array}{rll} \mathbb{P}(H^1(L^2)\backslash\{0\}) \subset M_k(m_t) \simeq M_k^s(\omega_t) & \text{if} & t < 0\ , \\ [A] \in M_k(m_0) & \text{if} & t = 0\ , \text{and} \\ \mathbb{P}(H^1(L^{-2})\backslash\{0\}) \subset M_k(m_t) \simeq M_k^s(\omega_t) & \text{if} & t > 0\ . \end{array}$$

All these fit well in the description of (6.6), (6.7) and this concludes the analytical interpretation of theorem 2(b) that we want to discuss as the first problem of this section.

Now we come to second problem, the proof of

$$\Gamma_X^k(C_{-1}) = \Gamma_X^k(C_1) + (-1)^k e^d \tag{6.10}$$

in lemma (2.2). Our argument is analogous to [D3] and so we shall be brief at some points. Working with manifolds X with $b_2^+(X) = 1$, we can assume in the proof of (6.10) that the ASD equation about a reduction A reduces to a finite dimensional map

$$\Phi : H_A^1 \times [-1,1] \longrightarrow \mathbb{R} \cdot \begin{bmatrix} i\omega_t & 0 \\ 0 & -i\omega_t \end{bmatrix}$$

as in (6.1) for a small path of generic metric $\{m_t \mid t \in [-1,1]\}$ on X. This time $\{\omega_t\}$ denotes a smooth path of L_2–normalized self–dual harmonic forms on X relative to $\{m_t\}$ which crosses the wall $\langle e\rangle^{\perp} \subset H^2(X;\mathbb{R})$ transversally. Define as before a dual map

$$\hat{\Phi} : H_A^1 \times [-1,1] \longrightarrow \mathbb{R}$$

for Φ by (6.2) so that $\hat{\Phi}^{-1}(0)/S^1$ describes

$$\mathcal{M} = \bigcup_t \{M_k(\omega_t) \mid t \in [-1,1]\}$$

about the reduction $[A]$. Using this model we can get rid of the singular point $([A],0)$ in $\mathcal{M}$ by cutting off a small r–neighbourhood

$$N_r = \{|v| < r\,;\, v \in H_A^1\}/S^1 \subset \hat{\Phi}^{-1}(0)/S^1 \ .$$

This process introduces a boundary component diffeomorphic to the complex projective space

$$\{v \in H_A^1 \,\big|\, |v| = r\}/S^1 \simeq \mathbb{P}_d$$

in $\mathcal{B}_X^*$. We write $\mathbb{P}_d$ with its usual complex orientation inherited from $H_A^1 \simeq \mathbb{C}^{d+1}$ understood. Now the standard cobordism argument applies to compact transversal intersections

$$V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_d} \cap (\mathcal{M} \setminus N_r)$$

and enables us to determine (6.10) up to a sign:

$$\text{(6.11)} \qquad <\mu^d, M_k(\omega_{-1})> = <\mu^d, M_k(\omega_1)> \pm <\mu^d, [\mathbb{P}_d]> \ .$$

Indeed, as the line bundle $\mathcal{L}_\Sigma \longrightarrow \mathbb{P}_d$ identifies with $\mathcal{O}_{\mathbb{P}_d}(-\Sigma \cdot e)$ by lemma (2.28) of [D2], one infers

$$\langle \mu^d, [\mathbb{P}_d] \rangle = -e^d \tag{6.12}$$

as $d = 4k-3$ must be odd. It follows then

$$\Gamma_X^k(C_-) = \Gamma_X^k(C_+) \pm e^d \tag{6.13}$$

in $\mathrm{Sym}^d(H^2(X;\mathbb{Z}))$ by (6.11). The complete determination for (6.13) therefore depends on the orientation of $\mathbb{P}_d$ that should be used in the calculation of (6.11).

For this purpose we choose an orthogonal basis

$$\left\{ v^\alpha = \begin{bmatrix} 0 & h_\alpha \\ -\bar{h}_\alpha & 0 \end{bmatrix} \; ; \quad Iv^\alpha = \begin{bmatrix} 0 & ih_\alpha \\ -\overline{ih_\alpha} & 0 \end{bmatrix} \right\}_{\alpha=1}^{d+1}$$

for $H_A^1 \simeq \mathbb{C}^{d+1}$ compatible with the complex structure $I = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and write for a vector $v \in H_A^1$

$$v = \sum_\alpha \begin{bmatrix} 0 & y_v^\alpha h_\alpha \\ -\overline{y_v^\alpha}\, \overline{h_\alpha} & 0 \end{bmatrix}$$

$$= \sum_\alpha \left\{ \mathrm{Re}\, y_v^\alpha \cdot v^\alpha + \mathrm{Im}\, y_v^\alpha \cdot Iv^\alpha \right\}$$

where $y_v^1, \ldots, y_v^{d+1}$ are complex scalars. To determine the sign for e^d in (6.13) it suffices to consider a particular case that

$$\hat{\Phi}(v,t) = \sum_{\alpha=1}^{d+1} |y_v^\alpha|^2 - t\,.$$

In this situation $M_k(\omega_1)$ contains a copy of $\mathbb{P}_d \simeq \{v \in H_A^1 \,\big|\, |v| = 1\}/S^1$ in $\mathscr{B}_X^*$ as an addition contribution in homology compared with $M_k(\omega_{-1})$. We are going to check for this particular case the standard orientation of $\mathbb{P}_d$, as a component of $M_k(\omega_1)$, differs from the usual complex orientation by the sign of $(-1)^k$. Granted this, (6.11) reads particularly

$$\begin{aligned} <\mu^d, M_k(\omega_{-1})> &= <\mu^d, M_k(\omega_1)> - (-1)^k <\mu^d, [\mathbb{P}_d]> \\ &= <\mu^d, M_k(\omega_1)> + (-1)^k e^d \end{aligned}$$

by (6.12). Lemma (2.2) will then follow.

The standard orientation of the component $\mathbb{P}_d \subset M_k(\omega_1)$ is built in the framework developed in [D4] and can be derived from

(i) the "comparison formular" (cf. [D4] proposition (3.25)) and

(ii) the "cancellation rule" (cf. [D4] p. 422).

To see this, we recall first some basic conventions introduced in [D4]. Associated to each $A \in \mathscr{A}$ there defines a differential operator

$$\mathscr{D}_A = -d_A^* \oplus d_A^+ : \Omega^1(\mathrm{ad}\, P) \longrightarrow (\Omega^0 \oplus \Omega_+^2)(\mathrm{ad}\, P)$$

and one obtains whereby a bundle $\Lambda \longrightarrow \mathscr{B}_X$ induced from the assignment

$$A \longmapsto \Lambda^{\max}(\ker \mathscr{D}_A) \otimes \Lambda^{\max}(\operatorname{coker} \mathscr{D}_A)^* \ .$$

This bundle is trivial. Using the homology orientation

$$-1 \wedge \omega_1 \in \Lambda^{\max}(H^1(X)) \otimes \Lambda^{\max}(H^0(X) \oplus H^2_+(X))$$

we can fix a standard orientation of $\Lambda \longrightarrow \mathscr{B}_x$ which restricts and defines the standard orientation of $M_k(\omega_1)$. Around the reduction A preserving the splitting

$$L^{-1} \oplus L \sim (\mathbb{C} \oplus L^2) \otimes L^{-1} \ ,$$

one can apply the comparison formular and infers that the standard orientation of

$$\Lambda_A \sim \Lambda_{A|\mathbb{R}} \cdot \Lambda_{A|_{L^2}}$$

over the reduction A is given by

$$(-1)^{(-L)\cdot(-L)}\{\text{usual orientation of } H^1_A\} \otimes \{-1 \wedge \omega_1\} \ . \tag{6.14}$$

Notice that $(-L) \cdot (-L) = k$ in this discussion. Now to deduce the standard orientation of $\mathbb{P}_d \subset M_k(\omega_1)$ we make use of the cancellation rule to (6.14) in the following manner.

First we assume for simplicity $\tilde{v}_t = v$ so that for each $t \in [-1,1]$ the zero set of

$$\Phi(v,t) = F_{+,m_t}(A+\tilde{v}_t) = \frac{1}{4}\left[\sum_\alpha |y^\alpha_v|^2 - t\right] \cdot \begin{bmatrix} i\omega_t & 0 \\ 0 & -i\omega_t \end{bmatrix}$$

in H_A^1 contains ASD connections relative to the metric m_t. In particular, the copy of $\mathbb{P}_d$ in $M_k(\omega_1)$ is exactly given by the S^1–quotient of

$$S^{2d+1} = \{v \in H_A^1 \,\big|\, |v| = 1\} .$$

Also we assume cohomology groups H_A^1 are identical for different metrics m_t. (Removing this assumption amounts to introduce small error terms wherever necessary in the discussion but one sees this point is negligible in the argument.) For a vector $v \in S^{2d+1}$, it is easy to see that $T_v H_A^1 \simeq H_A^1$ is naturally oriented by

$$v \wedge Iv \wedge \{\text{the usual orientation of } T_{[v]}\mathbb{P}_d\} . \tag{6.15}$$

We shall check that for the operator $\mathscr{D}_{A+v} = -d_{A+v}^* \oplus d_{A+v}^+$ we have

$$\mathscr{D}_{A+v}(Iv) = \left[\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, 0\right] , \tag{6.16}$$

$$\mathscr{D}_{A+v}(v) = \left[0, \begin{pmatrix} i\omega_1 & 0 \\ 0 & -i\omega_1 \end{pmatrix}\right] \quad \text{and} \tag{6.17}$$

$$\mathscr{D}_{A+v} \equiv \underline{0} \quad \text{on} \quad \langle Iv\rangle^{\perp} \cap \langle v\rangle^{\perp} \tag{6.18}$$

inside $H_A^0 \oplus H_A^2$. Then the cancellation rule applies to this situation and gives

$$\begin{aligned}
&\text{the standard orientation of } T_{[v]}\mathbb{P}_d \\
&= (-1)^d \{\text{the usual orientation of } T_{[v]}H_A^1\} \otimes \{-1 \wedge \omega_1\} \text{ by (6.14)} \\
&= (-1)^d \{v \wedge Iv \wedge (\text{the usual orientation of } T_{[v]}\mathbb{P}_d\} \otimes \{-1 \wedge \omega_1\} \text{ by (6.15)} \\
&= (-1)^d \{Iv \wedge v \wedge (\text{the usual orientation of } T_{[v]}\mathbb{P}_d\} \otimes \{\mathscr{D}_{A+v}(Iv) \wedge \mathscr{D}_{A+v}(v)\}
\end{aligned}$$

$$= (-1)^d \text{ the usual orientation of } T_{[v]}\mathbb{P}_d$$

by eliminating Iv with $\mathscr{D}_{A+v}(Iv)$ and v with $\mathscr{D}_{A+v}(v)$. In this way we determine the sign of e^d in (6.10).

To establish (6.16)–(6.18) one simply observes for any vector $u \in H^1_A$ we have

$$\left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, d^*_{A+v}u \right\rangle_{L^2} = \left\langle d_{A+v}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, u \right\rangle_{L^2}$$
$$= -2 \langle Iv,u \rangle_{L^2}$$

and that

$$\left\langle d^+_{A+v}(u), \begin{bmatrix} i\omega_1 & 0 \\ 0 & -i\omega_1 \end{bmatrix} \right\rangle_{L^2} = \lim_{t\to 0} \int_X -\mathrm{Tr}\left[F_{+,m_1}(A+v+tu), \begin{bmatrix} i\omega_1 & 0 \\ 0 & -i\omega_1 \end{bmatrix}\right]$$

$$= \frac{\partial\hat{\phi}}{\partial v}\bigg|_{(v,1)}(u,0)$$

$$= 2\langle v,u \rangle_{L^2}$$

This finishes the proof of lemma (2.2).

As a final remark we wish to point out in the case when $k = 1$ this argument shows the standard orientation of the projective space $\mathbb{P}_d \simeq \mathbb{P}_1$ in such situation is opposite to the usual complex orientation, a fact that has been known in [D3]. Working with Yang–Mills moduli spaces with $k = 2$ as in the definition of Γ^2_X for manifolds X homeomorphic to $S^2 \times S^2$, the projective space $\mathbb{P}_d \simeq \mathbb{P}_5$ we come across by contrast has its usual complex orientation agreed with the assigned standard orientation. Such comparisons all depend on the parity of the integer k, the second Chern class of the SU(2)–bundle $P \longrightarrow X$ in question.

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