ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 4.

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Dedicated to the memory of
Professor N.M.Korobov.

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§2.0. Foreword.

Nikolaj Mikhajlovich Korobov was not only an outstandig mathematician, but also a good man, a gentleman. Among his pupils are
V.A. Bykovskij, Mark Levin, D.A. Mit'kin.

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§4.1. Pass from the operator δ to the operator $\nu^{-1}\delta$.

Let λ be a variable. We denote by $T_{n,\lambda}$ the diagonal $n\times n$-matrix, i-th diagonal element of which is equal to λ^{i-1} for $i=1, ..., n$. We return now to the equation (32) of the Part 2, i.e. to the equality

$$\nu^{3+2l}X_{l,k}(z;\nu-1) = A_l^*(z;\nu)X_{l,k}(z;\nu), \tag{1}$$

where $l=0$, 1, 2 and $k\in\mathfrak{K}_l$, $|z|>1, \nu\in\mathbb{Z}$, and $X_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, i-th of which is equal to $\delta^{i-1}f_{l,k}^{\vee}(z,\nu)$ for $i=1$, ..., $4+2l$. Let $Y_{l,k}(z;\nu)$ be the columnn with $4+2l$ elements, i-th of which is equal to $(\nu^{-1}\delta)^{i-1}f_{l,k}^{\vee}(z,\nu)$ for $i=1$, ..., $4+2l$, where $\nu\neq 0$. Clearly,

$$X_{l,k}(z;\nu) = T_{4+2l,\nu}Y_{l,k}(z;\nu). \tag{2}$$

Let

(3) $$A_l^{\sim}(z;\nu) = \nu^{-3-2l}(T_{4+2l,\nu})^{-1}A_l^*(z;\nu)T_{4+2l,\nu}.$$

Then equation (1) transforms ourself into equation

(4) $$T_{4+2l,1-\nu^{-1}}Y_{l,k}(z;\nu-1) = A_l^{\sim}(z;\nu)Y_{l,k}(z;\nu).$$

In view of (5), (6), (17), (18), (19) in the Part 3,

(5) $$Q_l^*(z;\nu) = Q_l^*(\nu) := \sum_{s=0}^{6+4l}\nu^s R_l^*(s),$$

(6) $$W_l^*(z;\nu) = W_l^*(\nu) := \sum_{s=0}^{6+4l}\nu^s V_l^*(s),$$

(7) $$S_l^*(z;\nu) = S_l^*(\nu) := \sum_{s=0}^{6+4l}\nu^s H_l^*(s)).$$

(8) $$A_l^*(z;\nu) = S_l^*(\nu) + zW_l^*(\nu).$$

Let

(9) $$S_l^{\sim}(\nu) = \nu^{-3-2l}(T_{4+2l,\nu})^{-1}S_l^*(\nu)T_{4+2l,\nu},$$

(10) $$V_l^{\sim}(s,\nu) = \nu^{-3-2l}(T_{4+2l,\nu})^{-1}V_l^*(s)T_{4+2l,\nu},$$

where $s = 0$, ..., $6+4l$,

(11) $$W_l^{\sim}(\nu) = \nu^{-3-2l}(T_{4+2l,\nu})^{-1}W_l^*(\nu)T_{4+2l,\nu} =$$
$$\sum_{s=0}^{6+4l}\nu^s V_l^{\sim}(s),$$

Then, in view of (8),

(12)
$$A_l^{\sim}(z;\nu) = S_l^{\sim}(\nu) + zW_l^{\sim}(\nu)\nu^{-3-2l}(T_{4+2l,\nu})^{-1}A_l^*(z,\nu)T_{4+2l,\nu},$$

(13) $$A_l^*(z;\nu) = \nu^{3+2l}(T_{4+2l,\nu})A_l^{\sim}(z,\nu)T_{4+2l,\nu}^{-1},$$

and it follows from the equalities

(14) $$A_l^*(z;\nu)A_l^*(z,-\nu) =$$
$$-\nu^{6+4l}E_{4+2l}),\ T_{4+2l,\nu}^{-1}T_{4+2l,-\nu}) = T_{4+2l,-1},$$

that

$$A_l^{\sim}(z,\nu)T_{4+2l,-1}A_l^{\sim}(z,-\nu)=T_{4+2l,-1} \tag{15}$$

for $l = 0, 1, 2$. Therefore the equality (15) is equivalent to the following system of equalities (16) – (18)

$$S_l^{\sim}(\nu)T_{4+2l,-1}S_l^{\sim}(-\nu)=T_{4+2l,-1} \tag{16}$$

$$W_l^{\sim}(\nu)T_{4+2l,-1}W_l^{\sim}(-\nu)=0E_{4+2l} \tag{17}$$

$$S_l^{\sim}(\nu)T_{4+2l,-1}W_l^{\sim}(-\nu)=-W_l^{\sim}(\nu)T_{4+2l,-1}S_l^{\sim}(-\nu) \tag{18}$$

for $l = 0, 1, 2$. Clearly, the equality (16) is equivalent for $l = 0, 1, 2$ to the system (20) – (21) in the Part 3, the equality (17) is equivalent for $l = 0, 1, 2$ to the system (15) in the Part 3 (see section 4.5 here) and the equality (18) is equivalent for $l = 0, 1, 2$ to the system (22) in the Part 3.

Clearly,

$$(T_{4+2l,\nu})^{-1}(N_{4+2l})^kT_{4+2l,\nu}=\nu^k(N_{4+2l})^k.$$

Therefore, in view of (47) in the Part 3,

$$(T_{4+2l,\nu})^{-1}H_l^*(s)T_{4+2l,\nu}=\nu^k(N_{4+2l})^k=h_l^*(s)\nu^{3+2l-s}(N_{4+2l})^{3+2l-s},$$

and, in view of (7), (9),

$$S_l^{\sim}(\nu):=\sum_{s=0}^{6+4l}h_l^*(s)(N_{4+2l})^{3+2l-s}=\sum_{s=0}^{3+2l}h_l^*(s)(N_{4+2l})^{3+2l-s} \tag{19}$$

doesn't depend from ν; more precicely

$$S_0^{\sim}(\nu)=S_0^{\sim}:=\begin{pmatrix}1&-4&8&-12\\0&1&-4&8\\0&0&1&-4\\0&0&0&1\end{pmatrix} \tag{20}$$

$$S_1^{\sim}(\nu)=S_1^{\sim}:=\begin{pmatrix}-1&6&-18&38&-66&102\\0&-1&6&-18&38&-66\\0&0&-1&6&-18&38\\0&0&0&-1&6&-18\\0&0&0&0&-1&6\\0&0&0&0&0&-1\end{pmatrix}, \tag{21}$$

(22)

$$S_2^{\sim}(\nu)=S_2^{\sim}:=\begin{pmatrix}1&-8&32&-88&192&-360&608&-952\\0&1&-8&32&-88&192&-360&608\\0&0&1&-8&32&-88&192&-360\\0&0&0&1&-8&32&-88&192\\0&0&0&0&1&-8&32&-88\\0&0&0&0&0&1&-8&32\\0&0&0&0&0&0&1&-8\\0&0&0&0&0&0&0&1\end{pmatrix}=$$

$$\begin{pmatrix} K_2 & L_2 \\ 0E_4 & K_2 \end{pmatrix},$$

where

$$K_2 = \begin{pmatrix} 1 & -8 & 32 & -88 \\ 0 & 1 & -8 & 32 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}, L_2 = \begin{pmatrix} 192 & -360 & 608 & -952 \\ -88 & 192 & -360 & 608 \\ 32 & -88 & 192 & -360 \\ -8 & 32 & -88 & 192 \end{pmatrix}.$$

In view of (10)

$$V_l^{\sim}(s,\nu) = \sum_{k=0}^{6+4l} \nu^{-k} V_l^{\sim\vee}(s,k) = \tag{23}$$

$$\sum_{\kappa=-3-2l}^{3+2l} \nu^{-3-2l-\kappa} V_l^{\sim\vee}(s, 3+2l+\kappa)$$

with $V_l^{\sim\vee}(k) \in Mat_{4+2l}(\mathbb{Q})$, where $l = 0,\ 1,\ 2,\ k = 0,\ ...,6+4l$. Therefore, in view of (11),

$$W_l^{\sim}(\nu) = \sum_{s=0}^{6+4l} \left(\sum_{k=0}^{6+4l} \nu^{s-k} V_l^{\sim\vee}(k) \right) = \tag{24}$$

$$\sum_{i=-6+4l}^{6+4l} \nu^{-i} V_l^{\sim *}(i),$$

where

$$V_l^{\sim *}(i) = \sum_{k=i}^{6+4l} V_l^{\sim\vee}(k-i,k) \tag{25}$$

for $i = 0,\ ...,\ 6+4l$,

$$V_l^{\sim *}(-i) = \sum_{k=0}^{6+4l-i} V_l^{\sim\vee}(k+i,k) \tag{26}$$

for $i = 1,\ ...,\ 6+4l$.

As in Part 3 of this work, if $n \in \mathbb{N}$, then E_n denotes the $n \times n$ unit matrix, $e_{n,i,k}$ for for $i = 1,\ ...,\ n,\ k = 1,\ ...,\ n$ denotes the $n \times n$- matrix, which has 1 on the intersection its i-th row and k-th column, and all the other its elements are equal to 0,

$$N_n = \sum_{k=0}^{n-1} e_{n,k,k+1},\ N_n^{\wedge} = \sum_{k=0}^{n-1} e_{n,k+1,k},$$

by $\mathfrak{R}_{\mathfrak{n}}^{\vee}$ and $\mathfrak{R}_{\mathfrak{n}}^{\wedge}$ are denoted the subrings in $Mat_n(\mathbb{C})$ consisting respectively from all the upper and respectively lower triangle matrices in $Mat_n(\mathbb{C})$. In view of (23) and (10),

$$V_l^{\sim\vee}(s, 3+2l+\kappa) = \sum_{\substack{\{\eta,\theta\}\subset[1,4+2l]\cap\mathbb{Z} \\ \eta-\theta=\kappa}} v_{l,\eta,\theta}^{*}(s) e_{4+2l,\eta,\theta},$$

In view of (58), (59) in the Part 3 (see also section 4.5 below)

(27) $$V_l^*(3+2l+\kappa)\in\mathfrak{R}_{4+2l}(N^\wedge)^\kappa\cap(N^\wedge)^\kappa\mathfrak{R}_{4+2l}$$

for $l=0,\ 1,\ 2,\kappa=0,\ ..,\ ,\ 3+2l.$

§4.2. The matrix $W_0^\sim(\nu)$.

In view of (10) and resuls of the part 3,

$$V_0^\sim(0,\nu)=\nu^{-3}\begin{pmatrix}0&0&12\nu^2&12\nu^3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\sum_{s=0}^{6}\nu^{-s},V_0^{\sim\vee}(0,s),$$

where

$$V_0^{\sim\vee}(0,0)=12e_{4,1,4},V_0^{\sim\vee}(0,1)=12e_{4,1,3},$$

$V_0^{\sim\vee}(0,s)=0E_4$ for $s=2,\ ...,\ 6,$

$$V_0^\sim(1,\nu)=(\nu)^{-3}\begin{pmatrix}0&-24\nu&-8\nu^2&0\\0&0&-8\nu&-8\nu^2\\0&0&0&0\\0&0&0&0\end{pmatrix}=\sum_{s=0}^{6}\nu^{-s}V_0^{\sim\vee}(1,s),$$

where

$$V_0^{\sim\vee}(1,1)=-8e_{4,1,3}-8e_{4,2,4},$$
$$V_0^{\sim\vee}(1,2)=-24e_{4,1,2}-8e_{4,2,3},$$

$V_0^{\sim\vee}(1,s)=0E_4$ for $s=0,\ 3,\ ...,\ 6,$

$$V_0^\sim(2,\nu)=(\nu)^{-3}\begin{pmatrix}12&-20\nu&0&0\\0&16&4\nu&0\\0&0&4&4\nu\\0&0&0&0\end{pmatrix}=\sum_{s=0}^{6}\nu^{-s}V_0^{\sim\vee}(2,s),$$

where

$$V_0^{\sim\vee}(2,2)=-20e_{4,1,2}+4e_{4,2,3}+4e_{4,3,4},$$
$$V_0^{\sim\vee}(2,3)=12e_{4,1,1}+16e_{4,2,2}+4e_{4,3,3},$$

$V_0^{\sim\vee}(2,s)=0E_4$ for $s=0,\ 1,\ 4,\ 5,\ 6,$

$$V_0^\sim(3,\nu)=(\nu)^{-3}\begin{pmatrix}16&0&0&0\\-8\nu^{-1}&16&0&0\\0&-8\nu^{-1}&0&0\\0&0&0&0\end{pmatrix}=\sum_{s=0}^{6}\nu^{-s}V_0^{\sim\vee}(3,s),$$

where

$$V_0^{\sim\vee}(3,3)=16e_{4,1,1}+16e_{4,2,2},$$
$$V_0^{\sim\vee}(3,4)=-8e_{4,2,1}-8e_{4,3,2},$$
$$V_0^{\sim\vee}(3,s)=0E_4 \text{ for } s=0,\ 1,\ 2,\ 5,\ 6$$

$$V_0^{\sim}(4,\nu) = (\nu)^{-3}\begin{pmatrix} 0 & 0 & 0 & 0\\ -12\nu^{-1} & 0 & 0 & 0\\ 4\nu^{-2} & -12\nu^{-1} & 0 & 0\\ 0 & 0 & -4\nu^{-1} & 0\end{pmatrix} = \sum_{s=0}^{6}\nu^{-s}V_0^{\sim\vee}(4,s),$$

where

$$V_0^{\sim\vee}(4,4) = -12e_{4,2,1} - 12e_{4,3,2} - 4e_{4,4,3},$$
$$V_0^{\sim\vee}(4,5) = 4e_{4,3,1}$$
$$V_0^{\sim\vee}(4,s) = 0E_4 \ \text{ for } \ s = 0,\ 1,\ 2,\ 3,\ 6,$$

$$V_0^{\sim}(5,\nu) = (\nu)^{-3}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 8\nu^{-2} & 0 & 0 & 0\\ 0 & 8\nu^{-2} & 0 & 0\end{pmatrix} = \sum_{s=0}^{6}\nu^{-s}V_0^{\sim\vee}(5,s),$$

where

$$V_0^{\sim\vee}(5,5) = 8e_{4,3,1} + 8e_{4,4,2},\ V_0^{\sim\vee}(5,s) = 0E_4 \text{ for } s = 0,\ 1,\ 2,\ 3,\ 4,\ 6,$$

$$V_0^{\sim}(6,\nu) = \nu^{-3}\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ -4\nu^{-3} & 0 & 0 & 0\end{pmatrix} = \sum_{s=0}^{6}\nu^{-s},V_0^{\sim\vee}(6,s),$$

where

$$V_0^{\sim\vee}(6,6) = -4e_{4,4,1},\ V_0^{\sim\vee}(6,s) = 0E_4 \text{ for } s = 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

We see that $V_0^{\sim\vee}(r,s) = 0E_4$, if $r \in [1,6]\cap\mathbb{Z}$ and $s \in [0, r-1]\cap\mathbb{Z}$. Therefore, in view in view of (26), $V_2^{\sim *}(-i) = 0E_4$ for $i = 0$, ..., 6.

In view of (25),

$$V_0^{\sim *}(0) = V_0^{\sim\vee}(0,0) + V_0^{\sim\vee}(1,1) +$$
$$V_0^{\sim\vee}(2,2) + V_0^{\sim\vee}(3,3) + V_0^{\sim\vee}(4,4) +$$
$$V_0^{\sim\vee}(5,5) + V_0^{\sim\vee}(6,6) =$$
$$12e_{4,1,4} - 8e_{4,1,3} - 8e_{4,2,4} +$$
$$(-20)e_{4,1,2} + 4e_{4,2,3} + 4e_{4,3,4} +$$
$$16e_{4,1,1} + 16e_{4,2,2} - 12e_{4,2,1} - 12e_{4,3,2} - 4e_{4,4,3} +$$
$$8e_{4,3,1} + 8e_{4,4,2} + (-4)e_{4,4,1} =$$
$$\begin{pmatrix} 16 & -20 & -8 & 12\\ -12 & 16 & 4 & -8\\ 8 & -12 & 0 & 4\\ -4 & 8 & -4 & 0\end{pmatrix} = 4\begin{pmatrix} 4 & -5 & -2 & 3\\ -3 & 4 & 1 & -2\\ 2 & -3 & 0 & 1\\ -1 & 2 & -1 & 0\end{pmatrix},$$

$$V_0^{\sim *}(1) = V_0^{\sim\vee}(0,1) + V_0^{\sim\vee}(1,2) +$$
$$V_0^{\sim\vee}(2,3) + V_0^{\sim\vee}(3,4) + V_0^{\sim\vee}(4,5) +$$
$$V_0^{\sim\vee}(5,6) =$$
$$12e_{4,1,3} - 24e_{4,1,2} - 8e_{4,2,3} + 12e_{4,1,1} + 16e_{4,2,2} +$$

$$4e_{4,3,3}-8e_{4,2,1}-8e_{4,3,2}+4e_{4,3,1}=$$
$$\begin{pmatrix}12&-24&12&0\\-8&16&-8&0\\4&-8&4&0\\0&0&0&0\end{pmatrix}=4\begin{pmatrix}3&-6&3&0\\-2&4&-2&0\\1&-2&1&0\\0&0&0&0\end{pmatrix},$$

$$V_0^{\sim *}(2)=V_0^{\sim\vee}(0,2)+V_0^{\sim\vee}(1,3)+$$
$$V_0^{\sim\vee}(2,4)+V_0^{\sim\vee}(3,5)+V_0^{\sim\vee}(4,6)=0E_4,$$

$$V_0^{\sim *}(3)=V_0^{\sim\vee}(0,3)+V_0^{\sim\vee}(1,4)+$$
$$V_0^{\sim\vee}(2,5)+V_0^{\sim\vee}(3,6)=0E_4.$$

$$V_0^{\sim *}(4)=V_0^{\sim\vee}(0,4)+V_0^{\sim\vee}(1,5)+$$
$$V_0^{\sim\vee}(2,6)=0E_4,$$

$$V_0^{\sim *}(5)=V_0^{\sim\vee}(0,5)+V_0^{\sim\vee}(1,6)=0E_4$$
$$V_0^{\sim *}(6)+V_0^{\sim\vee}(0,6)=0E_4.$$

Easy calculations show that

(28)
$$V_0^{\sim *}(0)T_{4,-1}V_0^{\sim *}(1)=$$
$$16\begin{pmatrix}4&5&-2&-3\\-3&-4&1&2\\2&3&0&-1\\-1&-2&-1&0\end{pmatrix}\begin{pmatrix}3&-6&3&0\\-2&4&-2&0\\1&-2&1&0\\0&0&0&0\end{pmatrix}=0E_4,$$

(29)
$$V_0^{\sim *}(1)T_{4,-1}V_0^{\sim *}(0)=$$
$$16\begin{pmatrix}3&6&3&0\\-2&-4&-2&0\\1&2&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}4&-5&-2&3\\-3&4&1&-2\\2&-3&0&1\\-1&2&-1&0\end{pmatrix}=0E_4,$$

(30)
$$V_0^{\sim *}(0)T_{4,-1}V_0^{\sim *}(0)=$$
$$16\begin{pmatrix}4&5&-2&-3\\-3&-4&1&2\\2&3&0&-1\\-1&-2&-1&0\end{pmatrix}\begin{pmatrix}4&-5&-2&3\\-3&4&1&-2\\2&-3&0&1\\-1&2&-1&0\end{pmatrix}=0E_4,$$

(31)
$$V_0^{\sim *}(1)T_{4,-1}V_0^{\sim *}(1)=$$
$$16\begin{pmatrix}3&6&3&0\\-2&-4&-2&0\\1&2&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}3&-6&3&0\\-2&4&-2&0\\1&-2&1&0\\0&0&0&0\end{pmatrix}=0E_4,$$

and, in view of (26),

(32)
$$S_0^{\sim}T_{4,-1}S_0^{\sim}T_{4,-1} = \begin{pmatrix}1 & -4 & 8 & -12\\ 0 & 1 & -4 & 8\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 4 & 8 & 12\\ 0 & 1 & 4 & 8\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & 1\end{pmatrix} = E_4,$$

(33)
$$S_0^{\sim}T_{4,-1}V_0^{\sim}(0) + V_0^{\sim}(0)T_{4,-1}S_0^{\sim} =$$

$$4\begin{pmatrix}1 & 4 & 8 & 12\\ 0 & -1 & -4 & -8\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}4 & -5 & -2 & 3\\ -3 & 4 & 1 & -2\\ 2 & -3 & 0 & 1\\ -1 & 2 & -1 & 0\end{pmatrix}+$$

$$4\begin{pmatrix}4 & 5 & -2 & -3\\ -3 & -4 & 1 & 2\\ 2 & 3 & 0 & -1\\ -1 & -2 & -1 & 0\end{pmatrix}\begin{pmatrix}1 & -4 & 8 & -12\\ 0 & 1 & -4 & 8\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 1\end{pmatrix}=$$

$$4\begin{pmatrix}-4 & 11 & -10 & 3\\ 3 & -8 & 7 & -2\\ -2 & 5 & 1 & 4\\ 1 & -2 & 1 & 0\end{pmatrix}+4\begin{pmatrix}4 & -11 & 10 & -3\\ -3 & 8 & -7 & 2\\ 2 & -5 & 4 & -1\\ -1 & 2 & -1 & 0\end{pmatrix}=0E_4,$$

(34)
$$S_0^{\sim}T_{4,-1}V_0^{\sim}(1) - V_0^{\sim}(1)T_{4,-1}S_0^{\sim} =$$

$$4\begin{pmatrix}1 & 4 & 8 & 12\\ 0 & -1 & -4 & -8\\ 0 & 0 & 1 & 4\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}3 & -6 & 3 & 0\\ -2 & 4 & -2 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}+$$

$$4\begin{pmatrix}3 & 6 & 3 & 0\\ -2 & -4 & -2 & 0\\ 1 & 2 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & -4 & 8 & -12\\ 0 & 1 & -4 & 8\\ 0 & 0 & 1 & -4\\ 0 & 0 & 0 & 1\end{pmatrix}=$$

$$4\begin{pmatrix}3 & -61 & 3 & 0\\ -2 & 4 & -2 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}-4\begin{pmatrix}3 & -61 & 3 & 0\\ -2 & 4 & -2 & 0\\ 1 & -2 & 1 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}=0E_4.$$

The equality (17) for $l = 0$ follows from (28) – (31). The equality (16) in the case $l = 0$ follows from (20), (32). The equality (18) in the case $l = 0$ follows from (20), (33) – (34); moreover these equalities substantinate the test, which was made for the eaquality $A_0(z,\nu)A_0(z,-\nu) = -\nu^6E_4$ in the Part 3.

So, we had made full test of the equality

$$A_0(z,\nu)A_0(z,-\nu) = -\nu^6E_4.$$

§4.3. Continuation of the test of the equality $$-\nu^{6+4l}E_{4+2l}=A_l^*(z;\nu)A_l^*(z;-\nu)$$ in the case $l=2$.

Further we have (see the Part 3):

$$t^{\vee}_{2,4,5}(10)=t^{\vee}_{2,4,8}(10)=32(6)h_2^*(7)+$$
$$(11)8h_2^*(6)+32h_2^*(5)+8h_2^*(4))=$$
$$32(6-22+32-22)=-192,$$
$$t^{\vee}_{2,4,6}(10)=t^{\vee}_{2,4,7}(10)=3t^{\vee}_{2,4,5}(10)=-576$$
$$t^{\wedge}_{2,4,5}(10)=32(6)h_2^*(7)+8(96)h_2^*(6)+$$
$$(32)(36)h_2^*(5)+8(96)h_2^*(4)+192h_2^*(3)=$$
$$192(h_2^*(7)+4h_2^*(6)+6h_2^*(5)+4h_2^*(4)+h_2^*(3))-$$
$$192t^{\wedge\wedge}_{2,1,5}(7)=192$$
$$t^{\wedge}_{2,4,6}(10)=32(18)h_2^*(7)+8(203)h_2^*(6)+$$
$$(32)(23)h_2^*(5)+8(-222)h_2^*(4)-(32)67h_2^*(3)-680h_2^*(2)-$$
$$88t^{**}_{2,1,6}(6)=576h_2^*(7)+(8)192h_2^*(6)+$$
$$+(192)2h_2^*(5)-(576)4h_2^*(4)-(32)26(3)(192)+(768)360=$$
$$576(1+352-832+480)+192(64-64)=576,$$
$$t^{\wedge}_{2,4,7}(10)=32(18)h_2^*(7)+8(118)h_2^*(6)-$$
$$(32)(36)h_2^*(5)+8(-316)h_2^*(4)+(32)4h_2^*(3)+(8)246h_2^*(2)+$$
$$(8)(104)h_2^*(1)-8(104)t^{**}_{2,1,7}(5)=$$
$$32(18)h_2^*(7)+8(118)h_2^*(6)-32(62)h_2^*(5)-(8)732h_2^*(4)+$$
$$8(-608)h_2^*(3)+8(-170)h_2^*(2))=t^{\wedge}_{2,4,6}(10)-$$
$$8(85)h_2^*(6)-32(85)h_2^*(5)-8(510)h_2^*(4)-32(85)h_2^*(3)-$$
$$8(85)h_2^*(2)+(8)85t^{**}_{2,1,6}(6)=t^{\wedge}_{2,4,6}(10),$$
$$t^{\wedge}_{2,4,8}(10)=32(6)h_2^*(7)+8(11)h_2^*(6)-$$
$$(32)(23)h_2^*(5)+8(-43)h_2^*(4)+(32)(32)h_2^*(3)+(8)61h_2^*(2)-$$
$$(32)19h_2^*(1)-(8)45h_2^*(0)+(8)45t^{**}_{2,1,8}(4)=$$
$$192+8(11)h_2^*(6)-(32)(23)h_2^*(5)+16h_2^*(4)+(32)77h_2^*(3)+$$
$$8(331)h_2^*(2)+32(26)h_2^*(1)=192+$$
$$64(-33-(16)23+(77)96-(331)45+13(608)=192+$$
$$64(-33-368+33(224)-(33)450-45+7904)=192+$$
$$64(-452-33(226)+7910)=192+64(-(70)113+7910)=192,$$
$$t^{*}_{2,4,5}(10)=t^{\vee}_{2,4,5}(10)+t^{\wedge}_{2,4,5}(10)=-192+192=0,$$

$$t^*_{2,4,6}(10) = t^\vee_{2,4,6}(10) + t^\wedge_{2,4,6}(10) = -576 + 576 = 0,$$
$$t^*_{2,4,7}(10) = t^\vee_{2,4,7}(10) + t^\wedge_{2,4,7}(10) = -576 + 576 = 0,$$
$$t^*_{2,4,8}(10) = t^\vee_{2,4,8}(10) + t^\wedge_{2,4,8}(10) = -192 + 192 = 0,$$
$$t^*_{2,5,6}(10) = t^\wedge_{2,5,6}(10) = 8(11)h^*_2(6)+$$
$$(32)(11)h^*_2(5) + 8(66)h^*_2(4) + (8)44h^*_2(3) + 88h^*_2(2) =$$
$$88t^{**}_{2,1,7}(5) = 0,$$
$$t^*_{2,5,7}(10) = t^\wedge_{2,5,7}(10) = 8(33)h^*_2(6)+$$
$$(32)(23)h^*_2(5) + 8(38)h^*_2(4) - 32(27)h^*_2(3) - 8(127)h^*_2(2)-$$
$$-8(40)h^*_2(1) + 320t^{**}_{2,1,7}(5) = 8(33)h^*_2(6)+$$
$$(32)(33)h^*_2(5) + 8(198)h^*_2(4) + 32(33)h^*_2(3) + (8)33h^*_2(4) =$$
$$264t^{**}_{2,1,6}(6) = 0,$$

$$t^*_{2,5,8}(10) = t^\wedge_{2,5,8}(10) = 8(33)h^*_2(6)+$$
$$(32)(13)h^*_2(5) + 8(-71)h^*_2(4) - 32(36)h^*_2(3) + 8(19)h^*_2(2)+$$
$$32(31)h^*_2(1) + 8(51)h^*_2(0) - 8(51)t^{**}_{2,1,8}(4) =$$
$$8(33)h^*_2(6) + (32)(13)h^*_2(5) + 8(-122)h^*_2(4) + 32(-87)h^*_2(3)+$$
$$8(-287)h^*_2(2) - 8(80)h^*_2(1) + (8)80t^{**}_{2,1,7}(5) =$$
$$8(33)h^*_2(6) + (32)33(13)h^*_2(5) + 8(198)h^*_2(4) + 32(33)h^*_2(3)+$$
$$8(33)h^*_2(2) = 264t^{**}_{2,1,6}(6) = 0,$$

$$t^*_{2,6,7}(10) = t^\wedge_{2,6,7}(10) =$$
$$32h^*_2(5) + 32(4)h^*_2(4) + 32(6)h^*_2(3) + 32(4)h^*_2(2)+$$
$$32h^*_2(1) = 32t^{**}_{2,1,7}(5) = 0,$$

$$t^*_{2,6,8}(10) = t^\wedge_{2,6,8}(10) =$$
$$32(3)h^*_2(5) + (8)33h^*_2(4) + 32(3)h^*_2(3) - 8(42)h^*_2(2)+$$
$$32(-12)h^*_2(1) + 8(-15)h^*_2(0) + 8(15)t^{**}_{2,1,8}(4) =$$
$$32(3)h^*_2(5) + (8)48h^*_2(4) + 32(18)h^*_2(3) + 8(48)h^*_2(2)+$$
$$32(3)h^*_2(1) = 96t^{**}_{2,1,7}(5) = 0,$$

$$t^*_{2,7,8}(10) = t^\wedge_{2,7,8}(10) =$$
$$8h^*_2(4) + 32h^*_2(3) + 48h^*_2(2)+$$
$$32h^*_2(1) + 8h^*_2(0) = 8t^{**}_{2,1,8}(4) = 0.$$

In view of (91), (94), (86), (89) in the Part 3,

$$t^*_{2,1,k}(11) = 0, \text{ for } k = 5,\ 6,\ 7,\ 8$$

$$t^*_{2,2,k}(11) = 0,\ \text{for}\ k = 1,\ 6,\ 7,\ 8$$
$$t^*_{2,3,k}(11) = 0,\ \text{for}\ k = 1,\ 2,\ 7, 8$$
$$t^*_{2,4,k}(11) = 0,\ \text{for}\ k = 1,\ 2,\ 3,\ 8,$$
$$t^*_{2,5,k}(11) = 0,\ \text{for}\ k = 1,\ 2,\ 3,\ 4,$$
$$t^*_{2,6,k}(11) = 0,\ \text{for}\ k = 1,\ 2,\ 3,\ 4,\ 5,$$
$$t^*_{2,7,k}(11) = 0,\ \text{for}\ k = 1,\ 2,\ 3,\ 4,\ 5,\ 6,$$
$$t^{\vee}_{2,i,k}(10) = 0$$

for $i = 6,\ ...,\ 8; k = 1,\ ...,\ 8,$

$$t^{\vee}_{2,1,1}(11) = (8)(119)h^*_2(7) + 32(19)h^*_2(6)+$$
$$8(45)h^*_2(5) + 32(6)h^*_2(4) + 88h^*_2(3) + 32h^*_2(2)+$$
$$8h^*_2(1) - 8t^{**}_{2,1,7}(5) = (8)(119)h^*_2(7)+$$
$$32(19)h^*_2(6) + 8(44)h^*_2(5) + 32(5)h^*_2(4)+$$
$$8(5)h^*_2(3) - (8)5t^{\wedge\wedge}_{2,1,5}(7) =$$
$$(8)(119)h^*_2(7) + (32)14h^*_2(6) + (8)14h^*_2(5) = 952,$$
$$t^{\wedge}_{2,1,1}(11) = -(8)(119)h^*_2(7) = -952,$$
$$t^*_{2,1,1}(11) = t^{\vee}_{2,1,1}(11) + t^{\wedge}_{2,1,1}(11) = 952 - 952 = 0,$$
$$t^{\vee}_{2,1,2}(11) = (8)(-1243)h^*_2(7) + 32(-202)h^*_2(6)+$$
$$8(-489)h^*_2(5) + 32(-67)h^*_2(4)+$$
$$8(-127)h^*_2(3) + 32(-12)h^*_2(2)+$$
$$8(-13)h^*_2(1) - 32h^*_2(0) + 32t^{**}_{2,1,8}(4) =$$

$$(8)(-1243)h^*_2(7) + 32(-202)h^*_2(6)+$$
$$8(-489)h^*_2(5) + 32(-66)h^*_2(4)+$$
$$8(-111)h^*_2(3) + 32(-6)h^*_2(2) + (8)3h^*_2(1)-$$
$$(8)3t^{**}_{2,1,7}(5) =$$

$$(8)(-1243)h^*_2(7) + 32(-202)h^*_2(6)+$$
$$8(-492)h^*_2(5) + 32(-69)h^*_2(4)+$$
$$8(-129)h^*_2(3) + 32(-9)h^*_2(2)+$$
$$(8)36t^{**}_{2,1,6}(6) =$$

$$(8)(-1243)h^*_2(7) + 32(-193)h^*_2(6)+$$
$$8(-348)h^*_2(5) + 32(-15)h^*_2(4)+$$
$$8(15)h^*_2(3) - (8)15t^{\wedge\wedge}_{2,1,5}(7) =$$

$$(8)(-1243)h_2^*(7)+32(-208)h_2^*(6)+$$
$$8(-438)h_2^*(5)+32(-30)h_2^*(4),$$

$$t_{2,1,2}^{\wedge}(11)=(8)(1243)h_2^*(7)+32(100)h_2^*(6),$$

$$t_{2,1,2}^{*}(11)=t_{2,1,2}^{\vee}(11)+t_{2,1,2}^{\wedge}(11)=$$
$$32(-108)h_2^*(6)+$$
$$8(-438)h_2^*(5)+32(-30)h_2^*(4)=$$
$$96((-36)(-8)-8(146)+(10)88)=(96)8(36-146+110)=0,$$

$$t_{2,1,3}^{\vee}(11)=(8)(-113)h_2^*(7)+32(-11)h_2^*(6)+$$
$$8(-3)h_2^*(5)+32(4)h_2^*(4)+$$
$$8(19)h_2^*(3)+32(3)h_2^*(2)+$$
$$8h_2^*(1)-(8)8h_2^*(0)+8(8)2,1,8^{**}(4)=$$

$$(8)(-113)h_2^*(7)+32(-11)h_2^*(6)+$$
$$8(-3)h_2^*(5)+32(6)h_2^*(4)+$$
$$8(51)h_2^*(3)+32(15)h_2^*(2)+$$
$$(8)(33)h_2^*(1)=(8)8h_2^*(5)+32(6)h_2^*(4)+$$
$$8(51)h_2^*(3)+32(15)h_2^*(2)+$$
$$(8)33h_2^*(1),$$

$$t_{2,1,3}^{\wedge}(11)=(8)(113)h_2^*(7)+32(-255)h_2^*(6)+$$
$$8(-455)h_2^*(5)=(8)(113)h_2^*(7)+8(-200)h_2^*(5)=$$

$$t_{2,1,3}^{*}(11)=t_{2,1,3}^{\vee}(11)+t_{2,1,3}^{\wedge}(11)=$$
$$24(-64)h_2^*(5)+(24)8h_2^*(4)+(24)17h_2^*(3)+(24)20h_2^*(2)+$$
$$(24)11h_2^*(1)-(24)11t_{2,1,7}^{**}(5)=$$
$$24(-75)h_2^*(5)+24(-36)h_2^*(4)+24(-49)h_2^*(3)+$$
$$24(-24)h_2^*(2)=576(-100+132-(49)8+360)=$$
$$576(-100+140-400+360)=0,$$

$$t_{2,1,4}^{\wedge}(11)=(8)(-545)h_2^*(7)+32(-91)h_2^*(6)+$$
$$8(249)h_2^*(5)+32(44)h_2^*(4),$$

$$t_{2,1,4}^{\vee}(11)=(8)(545)h_2^*(7)+32(91)h_2^*(6)+$$

$$8(227)h_2^*(5) + 32(32)h_2^*(4)+$$
$$8(61)h_2^*(3) + 32(5)h_2^*(2)+$$
$$-8h_2^*(1) - (8)8h_2^*(0)$$

$$t_{2,1,4}^*(11) = 32(119)h_2^*(5) + 32(76)h_2^*(4)+$$
$$8(61)h_2^*(3) + 32(5)h_2^*(2)+$$
$$-8h_2^*(1) - (8)8h_2^*(0) =$$
$$48(119)h_2^*(5) + 32(76)h_2^*(4)+$$
$$8(61)h_2^*(3) + 32(5)h_2^*(2) - 8h_2^*(1) + 8t_{2,1,7}h^{**}(5) =$$
$$(8)715h_2^*(5) + (32)77h_2^*(4) + 8(67)h_2^*(3) + (32)6h_2^*(2) =$$
$$(8)32(715 - 847 + (67)6 - 270) = (8)(32)(-132 + 402 - 270) = 0.$$

$$t_{2,2,2}^{\wedge}(11) = -(8)(304)h_2^*(7) - 32(19)h_2^*(5),$$

$$t_{2,2,2}^{\vee}(11) = (8)(304)h_2^*(7) + 32(45)h_2^*(6)+$$
$$8(96)h_2^*(5) + 32(11)h_2^*(4)+$$
$$8(16)h_2^*(3) + 32h_2^*(2),$$

$$t_{2,2,2}^*(11) = 32(26)h_2^*(6)+$$
$$8(96)h_2^*(5) + 32(11)h_2^*(4)+$$
$$8(16)h_2^*(3) + 32h_2^*(2) - 32t_{2,1,6}^{**}(6) =$$
$$32(25)h_2^*(6) + (32)4(5)h_2^*(5) + (32)5h_2^*(4) =$$
$$160(-40 + 128 - 88) = 0,$$

$$t_{2,2,3}^{\wedge}(11) = (8)(642)h_2^*(7) + 32(202)h_2^*(6)+$$
$$8(259)h_2^*(5) = (8)(642)h_2^*(7) + (8)32(259 - 202)$$

$$t_{2,2,3}^{\vee}(11) = (8)(-642)h_2^*(7) + 32(-99)h_2^*(6)+$$
$$8(-222)h_2^*(5) + 32(-27)h_2^*(4)+$$
$$8(-42)h_2^*(3) + 32(-3)h_2^*(2) + 8(-6)h_2^*(1),$$

$$t_{2,2,3}^*(11) = (156)8h_2^*(5)+$$
$$8(-222)h_2^*(5) + 32(-27)h_2^*(4)+$$
$$8(-42)h_2^*(3) + 32(-3)h_2^*(2) + 8(-6)h_2^*(1)+$$
$$8(6)t_{2,1,7}^{**}(5) = 8(-60)h_2^*(5) + 32(-21)h_2^*(4)+$$
$$8(-6)h_2^*(3) + 32(3)h_2^*(2) = (8)96(-20 + 77 - 12 - 45) = 0,$$

$$t^{\wedge}_{2,2,4}(11) = (8)(996)h^*_2(7) + 32(11)h^*_2(6)+$$
$$8(-682)h^*_2(5) + 32(-75)h^*_2(4),$$

$$t^{\vee}_{2,2,4}(11) = (8)(-996)h^*_2(7) + 32(-148)h^*_2(6)+$$
$$8(-316)h^*_2(5) + 32(-36)h^*_2(4)+$$
$$8(-52)h^*_2(3) + 32(-4)h^*_2(2) + 8(-12)h^*_2(1),$$

$$t^{*}_{2,2,4}(11) = 32(-137)h^*_2(6)+$$
$$8(-998)h^*_2(5) + 32(-219)h^*_2(4)+$$
$$8(-52)h^*_2(3) + 32(-4)h^*_2(2) + 8(-12)h^*_2(1) =$$
$$8(32)(137 - 998 + 1221 - 312 + 180 - 228) = (8)32(1538 - 1538) = 0,$$

$$t^{\vee}_{2,2,5}(11) = (8)(-169)h^*_2(7) + 32(-23)h^*_2(6)+$$
$$8(-43)h^*_2(5) + 32(-4)h^*_2(4)+$$
$$8(-5)h^*_2(3) + 32(-1)h^*_2(2) + 8(-7)h^*_2(1),$$

$$t^{\wedge}_{2,2,4}(11) = (8)(169)h^*_2(7) + 32(-91)h^*_2(6)+$$
$$8(-227)h^*_2(5) + 32(44)h^*_2(4) + 8(119)h^*_2(3),$$

$$t^{*}_{2,2,5}(11) = 32(-114)h^*_2(6)+$$
$$8(-270)h^*_2(5) + 32(40)h^*_2(4)+$$
$$8(114)h^*_2(3) + 32(-1)h^*_2(2) + 8(-7)h^*_2(1) =$$
$$(32)8(114 - 270 - 440 + 684 + 45 - 133) = 128(843 - 843) = 0,$$
$$t^{\wedge}_{2,3,3}(11) = (8)(-270)h^*_2(7) + 32(-45)h^*_2(6)+$$
$$8(-45)h^*_2(5),$$

$$t^{\vee}_{2,3,3}(11) = (8)(270)h^*_2(7) + 32(36)h^*_2(6)+$$
$$8(66)h^*_2(5) + 32(6)h^*_2(4)+$$
$$(48)h^*_2(3),$$

$$t^{*}_{2,3,3}(11) = 32(-9)h^*_2(6)+$$
$$8(21)h^*_2(5) + 32(6)h^*_2(4)+$$
$$(48)h^*_2(3) = (32)8(9 + 21 - 66 + 36) = 0$$

$$t^{\wedge}_{2,3,4}(11) = (8)(-186)h^*_2(7) + 32(99)h^*_2(6)+$$
$$8(489)h^*_2(5) + (32)(39)h^*_2(4),$$

$$t^{\vee}_{2,3,4}(11) = (8)(186)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(38)h^*_2(5) + 32(3)h^*_2(4)+$$
$$(16)h^*_2(3) - 32h^*_2(2),$$

$$t^{*}_{2,3,4}(11) = 32(122)h^*_2(6)+$$
$$8(527)h^*_2(5) + 32(42)h^*_2(4)+$$
$$(16)h^*_2(3) - 32h^*_2(2)+$$
$$= (32)8(-122 + 527 - 462 + 12 + 45) = 0,$$

$$t^{\wedge}_{2,3,5}(11) = (8)(253)h^*_2(7) + 32(148)h^*_2(6)+$$
$$8(3)h^*_2(5) + (32)(-107)h^*_2(4) + 8(-185)h^*_2(3),$$

$$t^{\vee}_{2,3,5}(11) = (8)(-253)h^*_2(7) + 32(-36)h^*_2(6)+$$
$$8(-71)h^*_2(5) + 32(-7)h^*_2(4)+$$
$$8(-9)h^*_2(3) - 64h^*_2(2),$$

$$t^{*}_{2,3,5}(11) = 32(112)h^*_2(6)+$$
$$8(-68)h^*_2(5) + 32(-114)h^*_2(4)+$$
$$8(-194)h^*_2(3) - 64h^*_2(2)+$$
$$= (32)8(-112 - 68 + 1254 - 1164 + 90) = 256(180 - 180) = 0,$$
$$t^{\wedge}_{2,3,6}(11) = (8)(169)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(-227)h^*_2(5) + (32)(-32)h^*_2(4) + 8(119)h^*_2(3)+$$
$$(32)19h^*_2(2),$$

$$t^{\vee}_{2,3,6}(11) = (8)(-169)h^*_2(7) + 32(-23)h^*_2(6)+$$
$$8(-43)h^*_2(5) + 32(-4)h^*_2(4)+$$
$$8(-5)h^*_2(3) - 32h^*_2(2),$$

$$t^{*}_{2,3,6}(11) = 8(-270)h^*_2(5) + 32(-36)h^*_2(4)+$$
$$8(114)h^*_2(3) + 32(18)h^*_2(2) =$$
$$(32)8(-270 + 396 + 684 - 810) = 256(1080 - 1080) = 0,$$
$$t^{\wedge}_{2,4,4}(11) = (8)(-96)h^*_2(7) + 32(-36)h^*_2(6)+$$
$$8(-96)h^*_2(5) + (32)(-6)h^*_2(4),$$

$$t^{\vee}_{2,4,4}(11) = (8)(96)h^*_2(7) + 32(11)h^*_2(6)+$$
$$8(16)h^*_2(5) + 32h^*_2(4),$$

$$t^*_{2,4,4}(11) = 32(-25)h^*_2(6)+$$
$$8(-80)h^*_2(5) + 32(-5)h^*_2(4) =$$
$$-160t^{**}_{2,4,4}(11),$$

where

$$t^{**}_{2,4,4}(11) = 5h^*_2(6) + 4h^*_2(5) + h^*_2(4) = -40 + 128 - 88 = 0,$$

$$t^{\wedge}_{2,4,5}(11) = (8)(-203)h^*_2(7) + 32(-23)h^*_2(6)+$$
$$8(222)h^*_2(5) + (32)(67)h^*_2(4) + (8)85h^*_2(3)$$

$$t^{\vee}_{2,4,5}(11) = (8)(203)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(33)h^*_2(5) + 32(2)h^*_2(4) - 8h^*_2(3),$$

$$t^*_{2,4,5}(11) = (8)(203)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(255)h^*_2(5) + 32(69)h^*_2(4) + (8)84h^*_2(3) =$$
$$32(8)(255 - 759 + 504) = 0,$$

$$t^{\wedge}_{2,4,6}(11) = (8)(-118)h^*_2(7) + 32(36)h^*_2(6)+$$
$$8(316)h^*_2(5) + (32)(-4)h^*_2(4) + (8)(-246)h^*_2(3) - (32)26h^*_2(2),$$

$$t^{\vee}_{2,4,6}(11) = (8)(118)h^*_2(7) + 32(13)h^*_2(6)+$$
$$8(18)h^*_2(5) + 32h^*_2(4) - 8(2)h^*_2(3),$$

$$t^*_{2,4,6}(11) = 32(49)h^*_2(6) + 8(334)h^*_2(5)+$$
$$(32)(-3)h^*_2(4) + (8)(-248)h^*_2(3) - (32)26h^*_2(2) =$$
$$(8)32(-49 + 334 + 33 - 1488 + 1170) = 256(318 - 318) = 0,$$

$$t^{\wedge}_{2,4,7}(11) = (8)(-11)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(43)h^*_2(5) + (32)(-32)h^*_2(4) + (8)(-61)h^*_2(3)+$$
$$(32)19h^*_2(2) + (8)45h^*_2(1) =$$
$$(8)(-11)h^*_2(7) + 32(23)h^*_2(6)+$$
$$8(43)h^*_2(5) + (32)(-32)h^*_2(4) + (8)(-61)h^*_2(3),$$

$$t^{\vee}_{2,4,7}(11) = (8)(11)h^*_2(7) + 32h^*_2(6)+$$
$$8h^*_2(5) + 32(0)h^*_2(4) - 8h^*_2(3),$$

$$t^*_{2,4,7}(11) = 32(24)h^*_2(6) + 8(44)h^*_2(5)+$$
$$(32)(-32)h^*_2(4) + (8)(-62)h^*_2(3) = 256(-24 + 44 + 352 - 372) = 0,$$

$$t^{\wedge}_{2,5,5}(11) = (8)(-11)h^*_2(7) + 32(-11)h^*_2(6)+$$
$$8(-66)h^*_2(5) + (32)(-11)h^*_2(4) + (8)(-11)h^*_2(3) =$$
$$(8)(-11)h^*_2(7) + 88t^{\wedge\wedge}_{2,1,5}(7) = (8)(-11)h^*_2(7)$$
$$t^{\vee}_{2,5,5}(11) = (8)(11)h^*_2(7) + 32h^*_2(6)+$$
$$8h^*_2(5) = (8)(11)h^*_2(7),\ t^*_{2,5,5}(11) = -88 + 88 = 0.$$

§4.4. The matrix $W^{\sim}_2(z;\nu)$.

In view of (10) and resuls of the part 3,

$$V^{\sim}_2(0,\nu) = \nu^{-7}8(119)\begin{pmatrix} 0 & 0 & 0 & 0 & \nu^4 & 3\nu^5 & 3\nu^6 & \nu^7\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14}\nu^{-s}V^{\sim\vee}_2(0,s),$$

where

$$V^{\sim\vee}_2(0,0) = (8)119e_{8,1,8},$$
$$V^{\sim\vee}_2(0,1) = (8)357e_{8,1,7},$$
$$V^{\sim\vee}_2(0,2) = (8)357e_{8,1,6},$$
$$V^{\sim\vee}_2(0,3) = (8)119e_{8,1,5},$$

$V^{\sim\vee}_2(0,s) = 0E_8$ for $s = 4, ..., 14$,

$$V^{\sim}_2(1,\nu) =$$

$$\nu^{-7}32\begin{pmatrix} 0 & 0 & 0 & -119\nu^3 & -257\nu^4 & -157\nu^5 & -19\nu^6 & 0\\ 0 & 0 & 0 & 0 & -19\nu^3 & -57\nu^4 & -57\nu^5 & -19\nu^6\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14}\nu^{-s}V^{\sim\vee}_2(1,s),$$

where

$$V^{\sim\vee}_2(1,1) = (32)(-19)e_{8,1,7} + (32)(-19)e_{8,2,8},$$

$$V_2^{\sim\vee}(1,2)=(32)(-157)e_{8,1,6}+(32)(-57)e_{8,2,7},$$
$$V_2^{\sim\vee}(1,3)=(32)(-257)e_{8,1,5}+(32)(-57)e_{8,2,6},$$
$$V_2^{\sim\vee}(1,4)=(32)(-119)e_{8,1,4}+(32)(-19)e_{8,2,5},$$
$V_2^{\sim\vee}(1,s)=0E_8$ for $s=0,5,\ ...,14,$

$$V_2^{\sim}(2,\nu)=$$
$$\nu^{-7}8\begin{pmatrix}0&0&714\nu^2&542\nu^3&-603\nu^4&-431\nu^5&0&0\\0&0&0&304\nu^2&653\nu^3&394\nu^4&45\nu^5&0\\0&0&0&0&45\nu^2&135\nu^3&135\nu^4&45\nu^5\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}=$$
$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(2,s),$$

where
$$V_2^{\sim\vee}(2,2)=(8)(-431)e_{8,1,6}+(8)45e_{8,2,7}+(8)45e_{8,3,8},$$
$$V_2^{\sim\vee}(2,3)=(8)(-603)e_{8,1,5}+(8)394e_{8,2,6}+(8)135e_{8,3,7},$$
$$V_2^{\sim\vee}(2,4)=(8)542e_{8,1,4}+(8)653e_{8,2,5}+(8)135e_{8,3,6},$$
$$V_2^{\sim\vee}(2,5)=(8)714e_{8,1,3}+(8)304e_{8,2,4}+(8)45e_{8,3,5},$$
$V_2^{\sim\vee}(2,s)=0E_8$ for $s=0,\ 1,\ 6,\ ...,14,$

$$V_2^{\sim}(3,\nu)=\nu^{-7}32\times$$
$$\begin{pmatrix}0&-119\nu&243\nu^2&388\nu^3&70\nu^4&0&0&0\\0&0&-114\nu&-83\nu^2&101\nu^3&70\nu^4&0&0\\0&0&0&-45\nu&-96\nu^2&-57\nu^3&-6\nu^4&0\\0&0&0&0&-6\nu&-18\nu^2&-18\nu^3&-6\nu^4\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}=$$
$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(3,s),$$

where
$$V_2^{\sim\vee}(3,3)=(32)70e_{8,1,5}+$$
$$(32)70e_{8,2,6}+(32)(-6)e_{8,3,7}+(32)(-6)e_{8,4,8},$$
$$V_2^{\sim\vee}(3,4)=(32)388(e_{8,1,4}+$$
$$(32)101e_{8,2,5}+(32)(-57)e_{8,3,6}+(32)(-18)e_{8,4,7},$$
$$V_2^{\sim\vee}(3,5)=(32)243e_{8,1,3}+$$

$$(32)(-83)e_{8,2,4}+(32)(-96)e_{8,3,5})+(32)(-18)e_{8,4,6},$$

$$V_2^{\sim\vee}(3,6)=(32)(-119)e_{8,1,2}+$$

$$(32)(-114)e_{8,2,3}+(32)(-45)e_{8,3,4}+(32)(-6)e_{8,4,5},$$

$V_2^{\sim\vee}(3,s)=0E_8$ for $s=0,\ 1,\ 2,\ 7,\ ...,14,$

$$V_2^{\sim}(4,\nu)=\nu^{-7}8\times$$

$$\begin{pmatrix} 119 & -1243\nu & -113\nu^2 & 545\nu^3 & 0 & 0 & 0 & 0\\ 0 & 304 & -642\nu & -996\nu^2 & -169\nu^3 & 0 & 0 & 0\\ 0 & 0 & 270 & 186\nu & -253\nu^2 & -169\nu^3 & 0 & 0\\ 0 & 0 & 0 & 96 & 203\nu & 118\nu^2 & 11\nu^3 & 0\\ 0 & 0 & 0 & 0 & 11 & 33\nu & 33\nu^2 & 11\nu^3\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}=$$

$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(4,s),$$

where

$$V_2^{\sim\vee}(4,4)=(8)545e_{8,1,4}+(8)(-169)e_{8,2,5}+$$

$$(8)(-169)e_{8,3,6}+(8)(11)e_{8,4,7}+(8)(11)e_{8,5,8},$$

$$V_2^{\sim\vee}(4,5)=(8)(-113)e_{8,1,3}+(8)(-996)e_{8,2,4}+$$

$$(8)(-253)e_{8,3,5}+(8)(118)e_{8,4,6}+(8)(33)e_{8,5,7},$$

$$V_2^{\sim\vee}(4,6)=(8)(-1243)e_{8,1,2}+(8)(-642)e_{8,2,3}+$$

$$(8)(186)e_{8,3,4}+(8)(203)e_{8,4,5}+(8)(33)e_{8,5,6},$$

$$V_2^{\sim\vee}(4,7)=(8)(119)e_{8,1,1}+(8)(304)e_{8,2,2}+$$

$$(8)(270)e_{8,3,3}+(8)(96)e_{8,4,4}+(8)(11)e_{8,5,5},$$

$V_2^{\sim\vee}(4,s)=0E_8$ for $s=0,\ ...,\ 3,\ 8,\ ...,14,$

$$V_2^{\sim}(5,\nu)=\nu^{-7}\times$$

$$32\begin{pmatrix} 100 & -255\nu & -91\nu^2 & 0 & 0 & 0 & 0 & 0\\ -19\nu^{-1} & 202 & 11\nu & -91\nu^2 & 0 & 0 & 0 & 0\\ 0 & -45\nu^{-1} & 99 & 148\nu & 23\nu^2 & 0 & 0 & 0\\ 0 & 0 & -36\nu^{-1} & -23 & 36\nu & 23\nu^2 & 0 & 0\\ 0 & 0 & 0 & -11\nu^{-1} & -23 & -13\nu & -1\nu^2 & 0\\ 0 & 0 & 0 & 0 & -1\nu^{-1} & -3 & -3\nu & -1\nu^2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}=$$

$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(5,s),$$

where

$$V_2^{\sim\vee}(5,5)=(32)(-91)e_{8,1,3}+(32)(-91)e_{8,2,4}+$$

$$(32)(23)e_{8,3,5} + (32)(23)e_{8,4,6}+$$
$$(32)(-1)e_{8,5,7} + (32)(-1)e_{8,6,8},$$
$$V_2^{\sim\vee}(5,6) = (32)(-255)(e_{8,1,2} + (32)11e_{8,2,3}+$$
$$(32)(148)e_{8,3,4} + (32)(36)e_{8,4,5}+$$
$$(32)(-13)e_{8,5,6} + (32)(-3)e_{8,6,7},$$
$$V_2^{\sim\vee}(5,7) = (32)100e_{8,1,1} + (32)(202)e_{8,2,2}+$$
$$(32)(99)e_{8,3,3} + (32)(-23)e_{8,4,4}+$$
$$(32)(-23)e_{8,5,5} + (32)(-3)e_{8,6,6},$$
$$V_2^{\sim\vee}(5,8) = (32)(-19)e_{8,2,1} + (32)(-45)e_{8,3,2}+$$
$$(32)(-36)e_{8,4,3} + (32)(-11)e_{8,5,4} + (32)(-1)e_{8,6,5},$$
$V_2^{\sim\vee}(5,s) = 0E_8$ for $s = 0, ..., 4, 9, ..., 14,$

$$V_2^{\sim}(6,\nu) = \nu^{-7}\times$$
$$8\begin{pmatrix} 455 & -249\nu & 0 & 0 & 0 & 0 & 0 & 0 \\ -259\nu^{-1} & 682 & 227\nu & 0 & 0 & 0 & 0 & 0 \\ 45\nu^{-2} & -489\nu^{-1} & -3 & 227\nu & 0 & 0 & 0 & 0 \\ 0 & 96\nu^{-2} & -222\nu^{-1} & -316 & -43\nu & 0 & 0 & 0 \\ 0 & 0 & 66\nu^{-2} & 38\nu^{-1} & -71 & -43\nu & 0 & 0 \\ 0 & 0 & 0 & 16\nu^{-2} & 33\nu^{-1} & 18 & 1\nu & 0 \\ 0 & 0 & 0 & 0 & 1\nu^{-2} & 3\nu^{-1} & 3 & 1\nu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$
$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(6,s),$$

where

$$V_2^{\sim\vee}(6,6) = (8)(-249)e_{8,1,2} + (8)(227)e_{8,2,3}+$$
$$(8)(227)e_{8,3,4} + (8)(-43)e_{8,4,5}+$$
$$(8)(-43)e_{8,5,6} + (8)e_{8,6,7}) + (8)e_{8,7,8},$$
$$V_2^{\sim\vee}(6,7) = (8)(455)(e_{8,1,1} + (8)682e_{8,2,2}+$$
$$(8)(-3)e_{8,3,3} + (8)(-316)e_{8,4,4}+$$
$$(8)(-71)e_{8,5,5} + (8)(18)e_{8,6,6} + (8)3e_{8,7,7},$$
$$V_2^{\sim\vee}(6,8) = (8){-259}(e_{8,2,1} + (8)(-489)e_{8,3,2}+$$
$$(8)(-222)e_{8,4,3} + (8)(38)e_{8,5,4}) + (8)(33)e_{8,6,5}+$$
$$(8)(3)e_{8,7,6},$$
$$V_2^{\sim\vee}(6,9) = (8)(45)e_{8,3,1} + (8)(96)e_{8,4,2}+$$
$$(8)(66)e_{8,5,3} + (8)(16)e_{8,6,4} + (8)e_{8,7,5},$$
$V_2^{\sim\vee}(6,s) = 0E_8$ for $s = 0, ..., 5, 10, ..., 14,$

$$V_2^{\sim}(7,\nu) = \nu^{-7}32\times$$

$$\begin{pmatrix} 44 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -75\nu^{-1} & 44 & 0 & 0 & 0 & 0 & 0 & 0 \\ 39\nu^{-2} & -107\nu^{-1} & -32 & 0 & 0 & 0 & 0 & 0 \\ -6\nu^{-3} & 67\nu^{-2} & -4\nu^{-1} & -32 & 0 & 0 & 0 & 0 \\ 0 & -11\nu^{-3} & 27\nu^{-2} & 36\nu^{-1} & 4 & 0 & 0 & 0 \\ 0 & 0 & -6\nu^{-3} & -3\nu^{-2} & 7\nu^{-1} & 4 & 0 & 0 \\ 0 & 0 & 0 & -\nu^{-3} & -2\nu^{-2} & -\nu^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(7,s),$$

where

$$V_2^{\sim\vee}(7,7) = (32)44)e_{8,1,1} + (32)(44)e_{8,2,2} +$$
$$(32)(-32)e_{8,3,3} + (32)(-32)e_{8,4,4} +$$
$$(32)(4)e_{8,5,5} + (32)(4)e_{8,6,6},$$
$$V_2^{\sim\vee}(7,8) = (32)(-75)(e_{8,2,1} + (32)(-107)e_{8,3,2} +$$
$$(32)(-4)e_{8,4,3} + (32)(36)e_{8,5,4} +$$
$$(32)(7)e_{8,6,5} + (32)(-1)e_{8,7,6},$$
$$V_2^{\sim\vee}(7,9) = (32)39e_{8,3,1} + (32)(67)e_{8,4,2} +$$
$$(32)(27)e_{8,5,3} + (32)(-3)e_{8,6,4} +$$
$$(32)(-2)e_{8,7,5},$$
$$V_2^{\sim\vee}(7,10) = (32)(-6)e_{8,4,1} + (32)(-11)e_{8,5,2} +$$
$$(32)(-6)e_{8,6,3} + (32)(-1)e_{8,7,4},$$

$V_2^{\sim\vee}(7,s) = 0E_8$ for $s = 0$, ..., 6, 11, ..., 14,

$$V_2^{\sim}(8,\nu) = \nu^{-7}\times$$

$$8\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -119\nu^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 185\nu^{-2} & -119\nu^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -85\nu^{-3} & 246\nu^{-2} & 61\nu^{-1} & 0 & 0 & 0 & 0 & 0 \\ 11\nu^{-4} & -127\nu^{-3} & 19\nu^{-2} & 61\nu^{-1} & 0 & 0 & 0 & 0 \\ 0 & 16\nu^{-4} & -42\nu^{-3} & -52\nu^{-2} & -5\nu^{-1} & 0 & 0 & 0 \\ 0 & 0 & 6\nu^{-4} & 2\nu^{-3} & -9\nu^{-2} & -5\nu^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\nu^{-3} & -2\nu^{-2} & -\nu^{-1} & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(8,s),$$

where

$$V_2^{\sim\vee}(8,8) = (8)(-119)e_{8,2,1} + (8)(-119)e_{8,3,2} +$$
$$(8)(61)e_{8,4,3} + (8)(61)e_{8,5,4} +$$
$$(8)(-5)e_{8,6,5} + (8)(-5)e_{8,7,6} + (8)(-1)e_{8,8,7},$$
$$V_2^{\sim\vee}(8,9) = (8)(185)e_{8,3,1} + (8)246e_{8,4,2} +$$

$$(8)(19)e_{8,5,3} + (8)(-52)e_{8,6,4}+$$
$$(8)(-9)e_{8,7,5} + (8)(-2)e_{8,8,6},$$
$$V_2^{\sim\vee}(8,10) = (8)(-85)e_{8,4,1} + (8)(-127)e_{8,5,2}+$$
$$(8)(-42)e_{8,6,3} + (8)(2)e_{8,7,4} + (8)(-1)e_{8,8,5},$$
$$V_2^{\sim\vee}(8,11) = (8)(11)e_{8,5,1} + (8)(16)e_{8,6,2}+$$
$$(8)(6)e_{8,7,3},$$

$V_2^{\sim\vee}(8,s) = 0E_8$ for $s = 0,\ ...,\ 7,\ 12,\ ...,\ 14,$

$$V_2^{\sim}(9,\nu) = \nu^{-7}\times$$
$$32\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 19\nu^{-2} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -26\nu^{-3} & 19\nu^{-2} & 0 & 0 & 0 & 0 & 0 & 0\\ 10\nu^{-4} & -31\nu^{-3} & -5\nu^{-2} & 0 & 0 & 0 & 0 & 0\\ -1\nu^{-5} & 12\nu^{-4} & -3\nu^{-3} & -5\nu^{-2} & 0 & 0 & 0 & 0\\ 0 & -1\nu^{-5} & 3\nu^{-4} & 4\nu^{-3} & 1\nu^{-2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\nu^{-4} & 2\nu^{-3} & 1\nu^{-2} & 0 & 0 \end{pmatrix} =$$
$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(9,s),$$

where

$$V_2^{\sim\vee}(9,9) = (32)(19)e_{8,3,1} + (32)(19)e_{8,4,2}+$$
$$(32)(-5)e_{8,5,3} + (32)(-5)e_{8,6,4}+$$
$$32e_{8,7,5} + 32e_{8,8,6},$$
$$V_2^{\sim\vee}(9,10) = (32)(-26)(e_{8,4,1} + (32)(-31)e_{8,5,2}+$$
$$(32)(-3)e_{8,6,3} + (32)(4)e_{8,7,4} + (32)(2)e_{8,8,5},$$
$$V_2^{\sim\vee}(9,11) = (32)10e_{8,5,1} + (32)(12)e_{8,6,2}+$$
$$(32)(3)e_{8,7,3} + 32e_{8,8,4},$$
$$V_2^{\sim\vee}(9,12) = (32)(-1)e_{8,6,1} + (32)(-1)e_{8,7,2},$$

$V_2^{\sim\vee}(9,s) = 0E_8$ for $s = 0,\ ...,\ 8,\ 13,\ 14,$

$$V_2^{\sim}(10,\nu) = \nu^{-7}\times$$
$$8\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -45\nu^{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 51\nu^{-4} & -45\nu^{-3} & 0 & 0 & 0 & 0 & 0 & 0\\ -15\nu^{-5} & 50\nu^{-4} & -1\nu^{-3} & 0 & 0 & 00 & 0 & 0\\ 1\nu^{-6} & -13\nu^{-5} & 1\nu^{-4} & -1\nu^{-3}0 & 0 & 0 & 0 & \\ 0 & 0 & -6\nu^{-5} & -12\nu^{-4} & -7\nu^{-3} & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(10, s),$$

where

$$V_2^{\sim\vee}(10,10) = 8(-45)e_{8,4,1} + 8(-45)e_{8,5,2} +$$
$$8(-1)e_{8,6,3} + 8(-1)e_{8,7,4} + 8(-7)e_{8,8,5},$$
$$V_2^{\sim\vee}(10,11) = 8(51)e_{8,5,1} + (8)50e_{8,6,2} + 8e_{8,7,3} +$$
$$8(-12)e_{8,8,4},$$
$$V_2^{\sim\vee}(10,12) = 8(-15)e_{8,6,1} + 8(-13)e_{8,7,2} + 8(-6)e_{8,8,3},$$
$$V_2^{\sim\vee}(10,13) = 8e_{8,7,1},$$

$V_2^{\sim\vee}(10, s) = 0E_8$ for $s = 0, ..., 9, 14,$

$$V_2^{\sim}(11,\nu) = \nu^{-7}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 192\nu^{-5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -160\nu^{-6} & 192\nu^{-5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 32\nu^{-7} & -96\nu^{-6} & 64\nu^{-5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 32\nu^{-7} & 64\nu^{-6} & 64\nu^{-5} & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(11, s),$$

where

$$V_2^{\sim\vee}(11,11) = (32)(6)e_{8,5,1} + (32)(6)e_{8,6,2} +$$
$$(32)(2)e_{8,7,3} + (32)(2)e_{8,8,4},$$
$$V_2^{\sim\vee}(11,12) = (32)(-5)(e_{8,6,1} + (32)(-3)e_{8,7,2} +$$
$$(32)(2)e_{8,8,3},$$
$$V_2^{\sim\vee}(11,13) = 32e_{8,7,1} + 32e_{8,8,2},$$

$V_2^{\sim\vee}(11, s) = 0E_8$ for $s = 0, ..., 10, 14,$

$$V_2^{*}(12) = \nu^{-7}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88\nu^{-5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40\nu^{-6} & -88\nu^{-5} & 0 & 0 & 0 & 0 & 0 & 0 \\ -8\nu^{-7} & -16\nu^{-6} & -56\nu^{-5} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\sum_{s=0}^{14} \nu^{-s} V_2^{\sim\vee}(12, s),$$

where

$$V_2^{\sim\vee}(12,12)=8(-11)e_{8,6,1}+8(-11)e_{8,7,2}+8(-7)e_{8,8,3},$$
$$V_2^{\sim\vee}(12,13)=8(5)e_{8,7,1}+(8)(-2)e_{8,8,2},$$
$$V_2^{\sim\vee}(12,14)=8(-1)e_{8,8,1},$$

$V_2^{\sim\vee}(12,s)=0E_8$ for $s=0$, ..., 11,

$$V_2^{\sim}(13)=\nu^{-7}\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\32\nu^{-6}&0&0&0&0&0&0&0\\0&32\nu^{-6}&0&0&0&0&0&0\end{pmatrix}=$$
$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(13,s),$$

where

$$V_2^{\sim\vee}(13,13)=32e_{8,7,1}+32e_{8,8,2},$$

$V_2^{\sim\vee}(13,s)=0E_8$ for $s=0$, ..., 12, 14,

$$V_2^{\sim}(14)=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\-8\nu^{-7}&0&0&0&0&0&0&0\end{pmatrix}=$$
$$\sum_{s=0}^{14}\nu^{-s}V_2^{\sim\vee}(14,s),$$

where

$$V_2^{\sim\vee}(14,14)=8(-1)e_{8,8,1},\ V_2^{\sim\vee}(14,s)=0E_8\ \text{ for } s=0,\ ...,\ 13.$$

We see that $V_2^{\sim\vee}(r,s)=0E_8$, if $r\in[1,14]\cap\mathbb{Z}$ and $s\in[0,r-1]\cap\mathbb{Z}$. Therefore, in view in view of (26), $V_2^{\sim *}(-i)=0$ for $i=0$, ..., 14.

In view of (25),

$$V_2^{\sim *}(0)=V_l^{\sim\vee}(0,0)+V_l^{\sim\vee}(1,1)+$$
$$V_2^{\sim\vee}(2,2)+V_l^{\sim\vee}(3,3)+V_l^{\sim\vee}(4,4)+$$
$$V_2^{\sim\vee}(5,5)+V_l^{\sim\vee}(6,6)+V_l^{\sim\vee}(7,7)+$$
$$V_2^{\sim\vee}(8,8)+V_l^{\sim\vee}(9,9)+V_l^{\sim\vee}(10,10)+$$
$$V_2^{\sim\vee}(11,11)+V_l^{\sim\vee}(12,12)+V_l^{\sim\vee}(13,13)+$$

$$V_2^{\sim\vee}(14,14) =$$

$$(8)119e_{8,1,8}+$$
$$(32)(-19)e_{8,1,7} + (32)(-19)e_{8,2,8}+$$
$$(8)(-431)e_{8,1,6} + (8)45e_{8,2,7} + (8)45e_{8,3,8}+$$
$$(32)70e_{8,1,5}+$$
$$(32)70e_{8,2,6} + (32)(-6)e_{8,3,7} + (32)(-6)e_{8,4,8}+$$
$$(8)545e_{8,1,4} + (8)(-169)e_{8,2,5}+$$
$$(8)(-169)e_{8,3,6} + (8)(11)e_{8,4,7} + (8)(11)e_{8,5,8}+$$
$$(32)(-91)e_{8,1,3} + (32)(-91)e_{8,2,4}+$$
$$(32)(23)e_{8,3,5} + (32)(23)e_{8,4,6}+$$
$$(32)(-1)e_{8,5,7} + (32)(-1)e_{8,6,8}+$$
$$(8)(-249)e_{8,1,2} + (8)(227)e_{8,2,3}+$$
$$(8)(227)e_{8,3,4} + (8)(-43)e_{8,4,5}+$$
$$(8)(-43)e_{8,5,6} + (8)e_{8,6,7}) + (8)e_{8,7,8}+$$
$$(32)(44)e_{8,1,1} + (32)(44)e_{8,2,2}+$$
$$(32)(-32)e_{8,3,3} + (32)(-32)e_{8,4,4}+$$
$$(32)(4)e_{8,5,5} + (32)(4)e_{8,6,6}+$$
$$(8)(-119)e_{8,2,1} + (8)(-119)e_{8,3,2}+$$
$$(8)(61)e_{8,4,3} + (8)(61)e_{8,5,4}+$$
$$(8)(-5)e_{8,6,5} + (8)(-5)e_{8,7,6} + (8)(-1)e_{8,8,7}+$$
$$(32)(19)e_{8,3,1} + (32)(19)e_{8,4,2}+$$
$$(32)(-5)e_{8,5,3}) + (32)(-5)e_{8,6,4}+$$
$$32e_{8,7,5}) + 32e_{8,8,6}+$$
$$8(-45)e_{8,4,1} + 8(-45)e_{8,5,2}+$$
$$8(-1)e_{8,6,3} + 8(-1)e_{8,7,4} + 8(-7)e_{8,8,5}+$$
$$(32)(6)e_{8,5,1} + (32)(6)e_{8,6,2})+$$
$$(32)(2)e_{8,7,3} + (32)(2)e_{8,8,4}+$$
$$8(-11)e_{8,6,1} + 8(-11)e_{8,7,2} + 8(-7)e_{8,8,3}+$$
$$32e_{8,7,1} + 32e_{8,8,2}+$$
$$8(-1)e_{8,8,1} =$$

$$(32)(44)e_{8,1,1} + (8)(-249)e_{8,1,2} + (32)(-91)(e_{8,1,3}+$$
$$(8)545e_{8,1,4} + (32)70e_{8,1,5} + (8)(-431)e_{8,1,6}+$$
$$(32)(-19)e_{8,1,7} + (8)119e_{8,1,8} + (32)(-32)e_{8,4,4}+$$

$$(8)(-119)e_{8,2,1}+(32)(44)e_{8,2,2}+(8)(227)e_{8,2,3}+$$
$$(32)(-91)e_{8,2,4}+(8)(-169)e_{8,2,5}+(32)70e_{8,2,6}+$$
$$(8)45e_{8,2,7}+(32)(-19)e_{8,2,8}+$$

$$(32)(19)e_{8,3,1}+(8)(-119)e_{8,3,2}+(32)(-32)e_{8,3,3}+$$
$$(8)(227)e_{8,3,4}+(32)(23)e_{8,3,5}+(8)(-169)e_{8,3,6}+$$
$$(32)(-6)e_{8,3,7}+(8)45e_{8,3,8}+$$

$$8(-45)e_{8,4,1}+(32)(19)e_{8,4,2})+(8)(61)e_{8,4,3}+(32)(-32)e_{8,4,4}+$$
$$(8)(-43)e_{8,4,5}+(32)(23)e_{8,4,6}+(8)(11)e_{8,4,7}+(32)(-6)e_{8,4,8}+$$

$$(32)(6)e_{8,5,1}+8(-45)e_{8,5,2}+(32)(-5)e_{8,5,3})+(8)(61)e_{8,5,4}+$$
$$(32)(4)e_{8,5,5}+(8)(-43)e_{8,5,6}+(32)(-1)e_{8,5,7}+(8)(11)e_{8,5,8}+$$

$$8(-11)e_{8,6,1}+(32)(6)e_{8,6,2}+8(-1)e_{8,6,3}+(32)(-5)e_{8,6,4}+$$
$$(8)(-5)e_{8,6,5}+(32)(4)e_{8,6,6}+(8)e_{8,6,7}+(32)(-1)e_{8,6,8}+$$

$$32e_{8,7,1}+8(-11)e_{8,7,2}+(32)(2)e_{8,7,3}+8(-1)e_{8,7,4}+$$
$$32e_{8,7,5}+(8)(-5)e_{8,7,6}+(8)e_{8,7,8}+$$

$$8(-1)e_{8,8,1}+32e_{8,8,2}+8(-7)e_{8,8,3}+(32)(2)e_{8,8,4}+$$
$$8(-7)e_{8,8,5}+32e_{8,8,6}+(8)(-1)e_{8,8,7}=$$

$$8\begin{pmatrix} 176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\ -119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\ 76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\ -45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\ 24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\ -11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\ 4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\ -1 & 4 & -7 & 8 & -7 & 4 & -1 & 0 \end{pmatrix}=$$

$$8\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix},$$

where

$$A_0=\begin{pmatrix} 176 & -249 & -364 & 545 \\ -119 & 176 & 227 & -364 \\ 76 & -119 & -128 & 227 \\ -45 & 76 & 61 & -128 \end{pmatrix}, B_0=\begin{pmatrix} 280 & -431 & -76 & 119 \\ -169 & 280 & 45 & -76 \\ 92 & -169 & -24 & 45 \\ -43 & 92 & 11 & -24 \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 24 & -45 & -20 & 61 \\ -11 & 24 & -1 & -20 \\ 4 & -11 & 8 & -1 \\ -1 & 4 & -7 & 8 \end{pmatrix}, D_0 = \begin{pmatrix} 16 & -43 & -4 & 11 \\ -5 & 16 & 1 & -4 \\ 4 & -5 & 0 & 1 \\ -7 & 4 & -1 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(1) = V_l^{\sim \vee}(0,1) + V_l^{\sim \vee}(1,2)+$$
$$V_2^{\sim \vee}(2,3) + V_l^{\sim \vee}(3,4) + V_l^{\sim \vee}(4,5)+$$
$$V_2^{\sim \vee}(5,6) + V_l^{\sim \vee}(6,7) + V_l^{\sim \vee}(7,8)+$$
$$V_2^{\sim \vee}(8,9) + V_l^{\sim \vee}(9,10) + V_l^{\sim \vee}(10,11)+$$
$$V_2^{\sim \vee}(11,12) + V_l^{\sim \vee}(12,13) + V_l^{\sim \vee}(13,14) =$$

$$(8)357e_{8,1,7}+$$
$$(32)(-157)e_{8,1,6} + (32)(-57)e_{8,2,7}+$$
$$(8)(-603)e_{8,1,5} + (8)394e_{8,2,6} + (8)135e_{8,3,7}+$$
$$(32)388e_{8,1,4}+$$
$$(32)101e_{8,2,5} + (32)(-57)e_{8,3,6} + (32)(-18)e_{8,4,7}+$$
$$(8)(-113)e_{8,1,3} + (8)(-996)e_{8,2,4}+$$
$$(8)(-253)e_{8,3,5} + (8)(118)e_{8,4,6} + (8)(33)e_{8,5,7}+$$
$$(32)(-255)e_{8,1,2} + (32)11e_{8,2,3}+$$
$$(32)(148)e_{8,3,4} + (32)(36)e_{8,4,5}+$$
$$(32)(-13)e_{8,5,6} + (32)(-3)e_{8,6,7}+$$
$$(8)(455)e_{8,1,1} + (8)682e_{8,2,2}+$$
$$(8)(-3)e_{8,3,3} + (8)(-316)e_{8,4,4}+$$
$$(8)(-71)e_{8,5,5} + (8)(18)e_{8,6,6} + (8)3e_{8,7,7}+$$
$$(32)(-75)(e_{8,2,1} + (32)(-107)e_{8,3,2}+$$
$$(32)(-4)e_{8,4,3} + (32)(36)e_{8,5,4}+$$
$$(32)(7)e_{8,6,5} + (32)(-1)e_{8,7,6}+$$
$$(8)(185)e_{8,3,1} + (8)246e_{8,4,2}+$$
$$(8)(19)e_{8,5,3} + (8)(-52)e_{8,6,4}+$$
$$(8)(-9)e_{8,7,5} + (8)(-2)e_{8,8,6}+$$
$$(32)(-26)(e_{8,4,1} + (32)(-31)e_{8,5,2}+$$
$$(32)(-3)e_{8,6,3} + (32)(4)e_{8,7,4} + (32)(2)e_{8,8,5}+$$
$$8(51)e_{8,5,1} + (8)50e_{8,6,2} + 8e_{8,7,3}+$$
$$8(-12)e_{8,8,4})+$$
$$(32)(-5)e_{8,6,1} + (32)(-3)e_{8,7,2})+$$
$$(32)(2)e_{8,8,3}+$$
$$8(5)e_{8,7,1} + (8)(-2)e_{8,8,2} =$$

$$(8)(455)e_{8,1,1}+(32)(-255)e_{8,1,2}+(8)(-113)e_{8,1,3}+$$
$$(32)388e_{8,1,4}+(8)(-603)e_{8,1,5}+(32)(-157)e_{8,1,6}+$$
$$(8)357e_{8,1,7}+$$

$$(32)(-75)e_{8,2,1}+(8)682e_{8,2,2}+(32)11e_{8,2,3}+(8)(-996)e_{8,2,4}+$$
$$(32)101e_{8,2,5}+(8)394e_{8,2,6}+(32)(-57)e_{8,2,7}+$$

$$(8)(185)e_{8,3,1}+(32)(-107)e_{8,3,2}+(8)(-3)e_{8,3,3}+$$
$$(32)(148)e_{8,3,4}+(8)(-253)e_{8,3,5}+(32)(-57)e_{8,3,6}+$$
$$(8)135e_{8,3,7}+$$

$$(32)(-26)e_{8,4,1}+(8)246e_{8,4,2}+(32)(-4)e_{8,4,3}+$$
$$(8)(-316)e_{8,4,4}+(32)(36)e_{8,4,5}+(8)(118)e_{8,4,6}+$$
$$(32)(-18)e_{8,4,7}+$$

$$8(51)e_{8,5,1}+(32)(-31)e_{8,5,2}+(8)(19)e_{8,5,3}+(32)(36)e_{8,5,4}+$$
$$(8)(-71)e_{8,5,5}+(32)(-13)e_{8,5,6}+(8)(33)e_{8,5,7}+$$

$$(32)(-5)e_{8,6,1}+(8)50e_{8,6,2}+(32)(-3)e_{8,6,3}+(8)(-52)e_{8,6,4}+$$
$$(32)(7)e_{8,6,5}+(8)(18)e_{8,6,6}+(32)(-3)e_{8,6,7}+$$

$$8(5)e_{8,7,1}+(32)(-3)e_{8,7,2}+8e_{8,7,3}+(32)(4)e_{8,7,4}+$$
$$(8)(-9)e_{8,7,5})+(32)(-1)e_{8,7,6})+(8)3e_{8,7,7})+$$

$$(8)(-2)e_{8,8,2}+(32)(2)e_{8,8,3}+8(-12)e_{8,8,4})+(32)(2)e_{8,8,5}+$$
$$(8)(-2)e_{8,8,6}=$$

$$8\begin{pmatrix}455&-1020&-113&1552&-603&-628&357&0\\-300&682&44&-996&404&394&-228&0\\185&-428&-3&592&-253&-228&135&0\\-104&246&-16&-316&144&118&-72&0\\51&-124&19&144&-71&-52&33&0\\-20&50&-12&-52&28&18&-12&0\\5&-12&1&16&-9&-4&3&0\\0&-2&8&-12&8&-2&0&0\end{pmatrix}=$$

$$8\begin{pmatrix}A_1&B_1\\C_1&D_1\end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 455 & -1020 & -113 & 1552 \\ -300 & 682 & 44 & -996 \\ 185 & -428 & -3 & 592 \\ -104 & 246 & -16 & -316 \end{pmatrix}, B_1 = \begin{pmatrix} -603 & -628 & 357 & 0 \\ 404 & 394 & -228 & 0 \\ -253 & -228 & 135 & 0 \\ 144 & 118 & -72 & 0 \end{pmatrix},$$

$$C_1 = \begin{pmatrix} 51 & -124 & 19 & 144 \\ -20 & 50 & -12 & -52 \\ 5 & -12 & 1 & 16 \\ 0 & -2 & 8 & -12 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -71 & -52 & 33 & 0 \\ 28 & 18 & -12 & 0 \\ -9 & -4 & 3 & 0 \\ 8 & -2 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(2) = V_l^{\sim\vee}(0,2) + V_l^{\sim\vee}(1,3) +$$
$$V_2^{\sim\vee}(2,4) + V_l^{\sim\vee}(3,5) + V_l^{\sim\vee}(4,6) +$$
$$V_2^{\sim\vee}(5,7) + V_l^{\sim\vee}(6,8) + V_l^{\sim\vee}(7,9) +$$
$$V_2^{\sim\vee}(8,10) + V_l^{\sim\vee}(9,11) + V_l^{\sim\vee}(10,12) +$$
$$V_2^{\sim\vee}(11,13) + V_l^{\sim\vee}(12,14) =$$

$$(8)357e_{8,1,6} +$$
$$(32)(-257)e_{8,1,5} + (32)(-57)e_{8,2,6} +$$
$$(8)542e_{8,1,4} + (8)653e_{8,2,5} + (8)135e_{8,3,6} +$$
$$(32)243e_{8,1,3} +$$
$$(32)(-83)e_{8,2,4} + (32)(-96)e_{8,3,5} + (32)(-18)e_{8,4,6} +$$
$$V_2^{\sim\vee}(4,6) = (8)(-1243)e_{8,1,2} + (8)(-642)e_{8,2,3} +$$
$$(8)(186)e_{8,3,4} + (8)(203)e_{8,4,5}) + (8)(33)e_{8,5,6} +$$
$$(32)100e_{8,1,1} + (32)(202)e_{8,2,2} +$$
$$(32)(99)e_{8,3,3} + (32)(-23)e_{8,4,4} +$$
$$(32)(-23)e_{8,5,5} + (32)(-3)e_{8,6,6} +$$
$$V_2^{\sim\vee}(6,8) = (8){-259}e_{8,2,1} + (8)(-489)e_{8,3,2} +$$
$$(8)(-222)e_{8,4,3} + (8)(38)e_{8,5,4} + (8)(33)e_{8,6,5} +$$
$$(8)(3)e_{8,7,6} + (32)39e_{8,3,1} + (32)(67)e_{8,4,2} +$$
$$(32)(27)e_{8,5,3} + (32)(-3)e_{8,6,4} + (32)(-2)e_{8,7,5} +$$
$$(8)(-85)e_{8,4,1} + (8)(-127)e_{8,5,2} +$$
$$(8)(-42)e_{8,6,3} + (8)(2)e_{8,7,4} + (8)(-1)e_{8,8,5} +$$
$$(32)10e_{8,5,1} + (32)(12)e_{8,6,2} +$$
$$(32)(3)e_{8,7,3} + 32e_{8,8,4} +$$

$$8(-15)e_{8,6,1} + 8(-13)e_{8,7,2} + 8(-6)e_{8,8,3}+$$
$$32e_{8,7,1} + 32e_{8,8,2} + 8(-1)e_{8,8,1} =$$

$$(32)100e_{8,1,1} + (8)(-1243)e_{8,1,2} + (32)243e_{8,1,3}+$$
$$(8)542e_{8,1,4} + (32)(-257)e_{8,1,5} + (8)357e_{8,1,6}+$$

$$(8){-259}e_{8,2,1} + (32)(202)e_{8,2,2} + (8)(-642)e_{8,2,3}+$$
$$(32)(-83)e_{8,2,4} + (8)653e_{8,2,5} + (32)(-57)e_{8,2,6}+$$

$$(32)39e_{8,3,1} + (8)(-489)e_{8,3,2} + (32)(99)e_{8,3,3}+$$
$$(8)(186)e_{8,3,4} + (32)(-96)e_{8,3,5} + (8)135e_{8,3,6}+$$

$$(8)(-85)e_{8,4,1} + (32)(67)e_{8,4,2} + (8)(-222)e_{8,4,3}+$$
$$(32)(-23)e_{8,4,4} + (8)(203)e_{8,4,5} + (32)(-18)e_{8,4,6}+$$

$$(32)10e_{8,5,1} + (8)(-127)e_{8,5,2} + (32)(27)e_{8,5,3}+$$
$$(8)(38)e_{8,5,4} + (32)(-23)e_{8,5,5} + (8)(33)e_{8,5,6}+$$

$$8(-15)e_{8,6,1} + (32)(12)e_{8,6,2} + (8)(-42)e_{8,6,3}+$$
$$(32)(-3)e_{8,6,4} + (8)(33)e_{8,6,5} + (32)(-3)e_{8,6,6}+$$

$$32e_{8,7,1} + 8(-13)e_{8,7,2} + (32)(3)e_{8,7,3} + (8)(2)e_{8,7,4}+$$
$$(32)(-2)e_{8,7,5} + (8)(3)e_{8,7,6}+$$

$$8(-1)e_{8,8,1} + 32e_{8,8,2} + 8(-6)e_{8,8,3} + 32e_{8,8,4}+$$
$$(8)(-1)e_{8,8,5} =$$

$$8\begin{pmatrix} 400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\ -259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\ 156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\ -85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\ 40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\ -15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\ 4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \end{pmatrix} =$$

$$8\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix},$$

where

$$A_2 = \begin{pmatrix} 400 & -1243 & 972 & 542 \\ -259 & 808 & -642 & -332 \\ 156 & -489 & 396 & 186 \\ -85 & 268 & -222 & -92 \end{pmatrix}, B_2 = \begin{pmatrix} -1028 & 357 & 0 & 0 \\ 653 & -228 & 0 & 0 \\ -384 & 135 & 0 & 0 \\ 203 & -72 & 0 & 0 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 40 & -127 & 108 & 38 \\ -15 & 48 & -42 & -12 \\ 4 & -13 & 12 & 2 \\ -1 & 4 & -6 & 4 \end{pmatrix},$$

$$D_2 = \begin{pmatrix} -92 & 33 & 0 & 0 \\ 33 & -12 & 0 & 0 \\ -8 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$V_2^{\sim *}(3) = V_l^{\sim\vee}(0,3) + V_l^{\sim\vee}(1,4)+$$
$$V_2^{\sim\vee}(2,5) + V_l^{\sim\vee}(3,6) + V_l^{\sim\vee}(4,7)+$$
$$V_2^{\sim\vee}(5,8) + V_l^{\sim\vee}(6,9) + V_l^{\sim\vee}(7,10)+$$
$$V_2^{\sim\vee}(8,11) + V_l^{\sim\vee}(9,12) + V_l^{\sim\vee}(10,13)+$$
$$V_2^{\sim\vee}(11,14) =$$
$$(8)119e_{8,1,5} + (32)(-119)e_{8,1,4} + (32)(-19)e_{8,2,5}+$$
$$(8)714e_{8,1,3} + (8)304e_{8,2,4} + (8)45e_{8,3,5} + (32)(-119)e_{8,1,2}+$$
$$(32)(-114)e_{8,2,3} + (32)(-45)e_{8,3,4} + (32)(-6)e_{8,4,5}+$$
$$(8)(119)e_{8,1,1} + (8)(304)e_{8,2,2}+$$
$$(8)(270)e_{8,3,3} + (8)(96)e_{8,4,4} + (8)(11)e_{8,5,5}+$$
$$(32)(-19)e_{8,2,1} + (32)(-45)e_{8,3,2}+$$
$$(32)(-36)e_{8,4,3} + (32)(-11)e_{8,5,4} + (32)(-1)e_{8,6,5}+$$
$$(8)(45)e_{8,3,1} + (8)(96)e_{8,4,2}+$$
$$(8)(66)e_{8,5,3} + (8)(16)e_{8,6,4} + (8)e_{8,7,5}+$$
$$(32)(-6)e_{8,4,1} + (32)(-11)e_{8,5,2}+$$
$$(32)(-6)e_{8,6,3} + (32)(-1)e_{8,7,4}+$$
$$(8)(11)e_{8,5,1} + (8)(16)e_{8,6,2} + (8)(6)e_{8,7,3}+$$
$$(32)(-1)e_{8,6,1} + (32)(-1)e_{8,7,2} + 8e_{8,7,1} =$$

$$(8)(119)e_{8,1,1} + (32)(-119)e_{8,1,2} + (8)714e_{8,1,3}+$$
$$(32)(-119)e_{8,1,4} + (8)119e_{8,1,5}+$$

$$(32)(-19)e_{8,2,1} + (8)(304)e_{8,2,2} + (32)(-114)e_{8,2,3}+$$
$$(8)304e_{8,2,4} + (32)(-19)e_{8,2,5}+$$

$$(8)(45)e_{8,3,1}+(32)(-45)e_{8,3,2}+(8)(270)e_{8,3,3}+$$
$$(32)(-45)e_{8,3,4}+(8)45e_{8,3,5}+$$

$$(32)(-6)e_{8,4,1}+(8)(96)e_{8,4,2}+(32)(-36)e_{8,4,3}+$$
$$(8)(96)e_{8,4,4}+(32)(-6)e_{8,4,5}+$$

$$(8)(11)e_{8,5,1}+(32)(-11)e_{8,5,2}+(8)(66)e_{8,5,3}+$$
$$(32)(-11)e_{8,5,4}+(8)(11)e_{8,5,5}+$$

$$(32)(-1)e_{8,6,1}+(8)(16)e_{8,6,2}+(32)(-6)e_{8,6,3}+$$
$$(8)(16)e_{8,6,4}+(32)(-1)e_{8,6,5}+$$

$$8e_{8,7,1}+(32)(-1)e_{8,7,2}+(8)(6)e_{8,7,3}+$$
$$(32)(-1)e_{8,7,4}+(8)e_{8,7,5}=$$

$$8\begin{pmatrix}119 & -476 & 714 & -476 & 119 & 0 & 0 & 0\\ -76 & 304 & -456 & 304 & -76 & 0 & 0 & 0\\ 45 & -180 & 270 & -180 & 45 & 0 & 0 & 0\\ -24 & 96 & -144 & 96 & -24 & 0 & 0 & 0\\ 11 & -44 & 66 & -44 & 11 & 0 & 0 & 0\\ -4 & 16 & -24 & 16 & -4 & 0 & 0 & 0\\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}=$$

$$8\begin{pmatrix}A_3 & B_3\\ C_3 & D_3\end{pmatrix},$$

where

$$A_3=\begin{pmatrix}119 & -476 & 714 & -476\\ -76 & 304 & -456 & 304\\ 45 & -180 & 270 & -180\\ -24 & 96 & -144 & 96\end{pmatrix}, B_3=\begin{pmatrix}119 & 0 & 0 & 0\\ -76 & 0 & 0 & 0\\ 45 & 0 & 0 & 0\\ -24 & 0 & 0 & 0\end{pmatrix},$$

$$C_3=\begin{pmatrix}11 & -44 & 66 & -44\\ -4 & 16 & -24 & 16\\ 1 & -4 & 6 & -4\\ 0 & 0 & 0 & 0\end{pmatrix},$$

$$D_3=\begin{pmatrix}11 & 0 & 0 & 0\\ -4 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

Since $V_2^{\sim\vee}(r,s)=0E_8$, if $r=0,\ldots,10$, $s\in[k+4,\,14]\cap\mathbb{Z}$, it follows that $V_2^{\sim*}(i)=0$ for $i=s-r\geq 4$.

In view of (22),

$$(K_2T_{4,-1})^2 = E_4, K_2T_{4,-1}K_2T_{4,-1}K_2T_{4,-1}L_2T_{4,-1} = \tag{35}$$

$$-L_2T_{4,-1}K_2T_{4,-1} = \begin{pmatrix} -192 & -1176 & -3872 & -9288 \\ 88 & 512 & 1640 & 3872 \\ -32 & -168 & -512 & -1176 \\ 8 & 32 & 88 & 192 \end{pmatrix},$$

$$(S_2^{\sim}(\nu)T_{8,-1})^2 = E_8 \tag{36}$$

and (16) holds for l=2.

Since

$$K_2T_{4,-1}A_0T_{4,-1} + L_2T_{4,-1}C_0T_{4,-1} = \tag{37}$$

$$-A_0T_{4,-1}K_2T_{4,-1} = \begin{pmatrix} -176 & -1159 & -3276 & -5153 \\ 119 & 776 & 2173 & 3388 \\ -76 & -489 & -1352 & -2083 \\ 45 & 284 & 771 & 1168 \end{pmatrix},$$

$$K_2T_{4,-1}B_0T_{4,-1} + L_2T_{4,-1}D_0T_{4,-1} = \tag{38}$$

$$-(A_0T_{4,-1}L_2T_{4,-1} + B_0TK2T) = \begin{pmatrix} -4872 & -2769 & -876 & -119 \\ 3177 & 1792 & 563 & 76 \\ -1932 & -1079 & -336 & -45 \\ 1067 & 588 & 181 & 24 \end{pmatrix},$$

$$K_2T_{4,-1}C_0T_{4,-1} = -C_2T_{4,-1}K_0T_{4,-1} = \tag{39}$$

$$\begin{pmatrix} -246 & -147 & -388 & -573 \\ 11 & 64 & 161 & 228 \\ -4 & -21 & -48 & -63 \\ 15 & 4 & 7 & 8 \end{pmatrix},$$

$$C_0T_{4,-1}L_2T_{4,-1} + D_0TK_2T = \tag{40}$$

$$-K_2T_{4,-1}D_0T_{4,-1} = \begin{pmatrix} 512 & 277 & 84 & 11 \\ -197 & -104 & -31 & -4 \\ 52 & 27 & 8 & 1 \\ -7 & -4 & -1 & 0 \end{pmatrix},$$

it follows that

$$S_2^{\sim}(\nu)T_{8,-1}V_2^{\sim *}(0) = -V_2^{\sim *}(0)T_{8,-1}S_2^{\sim}(\nu) \tag{41}$$

Since

$$K_2T_{4,-1}A_1T_{4,-1} + L_2T_{4,-1}C_1T_{4,-1} = \tag{42}$$

$$A_1T_{4,-1}K_2T_{4,-1}=\begin{pmatrix}455 & 2620 & 6287 & 8048\\ -300 & -1718 & -4100 & -5220\\ 185 & 1052 & 2493 & 3152\\ -104 & -586 & -1376 & -1724\end{pmatrix},$$

(43) $$K_2T_{4,-1}B_1T_{4,-1}+L_2T_{4,-1}D_1T_{4,-1}=$$

$$-(A_1T_{4,-1}L_2T_{4,-1}+B_1TK2T)=\begin{pmatrix}5797 & 2228 & 357 & 0\\ -3740 & -1430 & -228 & 0\\ 2243 & 852 & 135 & 0\\ -1216 & -458 & -72 & 0,\end{pmatrix},$$

(44) $$K_2T_{4,-1}C_1T_{4,-1}=C_2T_{4,-1}K_1T_{4,-1}=$$

$$\begin{pmatrix}51 & 284 & 659 & 816\\ -20 & -110 & -252 & -308\\ 5 & 28 & 65 & 80\\ 0 & -2 & -8 & -12\end{pmatrix},$$

(45) $$C_1T_{4,-1}L_2T_{4,-1}+D_1TK_2T=K_2T_{4,-1}D_1T_{4,-1}=$$

$$\begin{pmatrix}569 & 212 & 33 & 0\\ -212 & -78 & -12 & 0\\ 55 & 20 & 3 & 0\\ -8 & -2 & 0 & 0\end{pmatrix},$$

it follows that

(46) $$S_2^{\sim}(\nu)T_{8,-1}V_2^{\sim *}(1)=V_2^{\sim *}(1)T_{8,-1}S_2^{\sim}(\nu)$$

Since

(47) $$K_2T_{4,-1}A_2T_{4,-1}+L_2T_{4,-1}C_2T_{4,-1}=$$

$$-A_2T_{4,-1}K_2T_{4,-1}=\begin{pmatrix}-400 & -1957 & -3828 & -3742\\ 259 & 1264 & 2466 & 2404\\ -156 & -759 & -1476 & -1434\\ 85 & 412 & 798 & 772\end{pmatrix},$$

(48) $$K_2T_{4,-1}B_2T_{4,-1}+L_2T_{4,-1}D_2T_{4,-1}=$$

$$-(A_2T_{4,-1}L_2T_{4,-1}+B_2TK2T)=\begin{pmatrix}-1828 & -357 & 0 & 0\\ 1171 & 228 & 0 & 0\\ -696 & -135 & 0 & 0\\ 373 & 72 & 0 & 0\end{pmatrix},$$

(49) $$K_2T_{4,-1}C_2T_{4,-1}=-C_2T_{4,-1}K_2T_{4,-1}=$$

$$\begin{pmatrix} -40 & -193 & -372 & -358 \\ 15 & 72 & 138 & 132 \\ -4 & -19 & -36 & -34 \\ 1 & 4 & 6 & 4 \end{pmatrix},$$

(50) $$C_2T_{4,-1}L_2T_{4,-1} + D_2TK_2T =$$

$$-K_2T_{4,-1}D_2T_{4,-1} = \begin{pmatrix} 172 & 33 & 0 & 0 \\ -63 & -12 & 0 & 0 \\ 16 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

(51) $$S_2^{\sim}(\nu)T_{8,-1}V_2^{\sim *}(2) = -V_2^{\sim *}(2)T_{8,-1}S_2^{\sim}(\nu)$$

Since

(52) $$K_2T_{4,-1}A_3T_{4,-1} + L_2T_{4,-1}C_3T_{4,-1} =$$

$$A_3T_{4,-1}K_2T_{4,-1} = \begin{pmatrix} 119 & 476 & 714 & 476 \\ -76 & -3044 & -456 & -304 \\ 45 & 180 & 270 & 180 \\ -24 & -96 & -144 & -96 \end{pmatrix},$$

(53) $$K_2T_{4,-1}B_3T_{4,-1} + L_2T_{4,-1}D_3T_{4,-1} =$$

$$-(A_3T_{4,-1}L_2T_{4,-1} + B_3TK2T) = \begin{pmatrix} 119 & 0 & 0 & 0 \\ -76 & 0 & 0 & 0 \\ 45 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \end{pmatrix},$$

(54) $$K_2T_{4,-1}C_3T_{4,-1} = C_2T_{4,-1}K_2T_{4,-1} =$$

$$\begin{pmatrix} 11 & 44 & 66 & 44 \\ -4 & -16 & -24 & -16 \\ 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(55) $$C_3T_{4,-1}L_2T_{4,-1} + D_3TK_2T = K_2T_{4,-1}D_3T_{4,-1} =$$

$$\begin{pmatrix} 11 & 33 & 0 & 0 \\ -4 & -12 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0, \end{pmatrix},$$

it follows that

(56) $$S_2^{\sim}(\nu)T_{8,-1}V_2^{\sim *}(3) = V_2^{\sim *}(3)T_{8,-1}S_2^{\sim}(\nu)$$

In view of (41), (46), (51) and (56), the equality (18) holds for $l = 2$.

(57) $$A_0T_{4,-1}A_0T_{4,-1} = -B_0T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} -1794 & -1896 & 5806 & 7584 \\ 872 & 694 & -3488 & -4146 \\ -298 & 0 & 1886 & 1896 \\ 0 & -298 & -872 & -602 \end{pmatrix},$$

(58) $$A_0T_{4,-1}B_0T_{4,-1} = -B_0T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} -2854 & -5240 & 570 & 1280 \\ 1656 & 2866 & -320 & -694 \\ -846 & -1312 & 154 & 312 \\ 352 & 418 & -56 & -94 \end{pmatrix},$$

(59) $$C_0T_{4,-1}A_0T_{4,-1} = -D_0T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} 94 & 312 & 318 & 32 \\ -56 & -154 & -96 & 46 \\ -42 & -64 & 78 & 136 \\ 128 & 230 & -136 & -346 \end{pmatrix},$$

(60) $$C_0T_{4,-1}B_0T_{4,-1} = -D_0T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} -102 & -24 & 10 & 0 \\ 24 & -30 & 0 & 10 \\ -46 & -96 & 10 & 24 \\ 96 & 242 & -24 & -62 \end{pmatrix},$$

it follows that

(61) $$V_2^{\sim *}(0)T_{8,-1}V_2^{\sim *}(0) = 0E_8.$$

Since

(62) $$A_0T_{4,-1}A_1T_{4,-1} = -B_0T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} -5280 & -12020 & 880 & 18120 \\ 2794 & 6264 & -802 & -9968 \\ -1192 & -2580 & 664 & 4616 \\ 298 & 552 & -490 & -1520 \end{pmatrix},$$

(63) $$A_0T_{4,-1}B_1T_{4,-1} = -B_0T_{4,-1}D_1T_{4,-1} =$$

$$\begin{pmatrix} 8080 & -6260 & -3840 & 0 \\ -4362 & 3416 & 2082 & 0 \\ 1944 & -1556 & -936 & 0 \\ -570 & 488 & 282 & 0 \end{pmatrix},$$

(64)
$$C_0T_{4,-1}A_1T_{4,-1} = -D_0T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} 64 & 236 & 304 & 136 \\ -70 & -200 & -130 & 80 \\ -104 & -244 & -8 & 328 \\ 282 & 680 & 86 & -816 \end{pmatrix},$$

(65)
$$C_0T_{4,-1}B_1T_{4,-1} = -D_0T_{4,-1}D_1T_{4,-1} =$$

$$\begin{pmatrix} -16 & -20 & 0 & 0 \\ 70 & -40 & -30 & 0 \\ 152 & -116 & -72 & 0 \\ -394 & 296 & 186 & 0 \end{pmatrix},$$

it follows that

(66)
$$V_2^{\sim *}(0)T_{8,-1}V_2^{\sim *}(1) = 0E_8.$$

Since

(67)
$$A_0T_{4,-1}A_2T_{4,-1} = -B_0T_{4,-1}C_2T_{4,-1} =$$

$$\begin{pmatrix} -4550 & -14360 & -11940 & 4840 \\ 2456 & 7742 & 6408 & -2668 \\ -1094 & -3440 & -2820 & 1240 \\ 320 & 998 & 792 & -412 \end{pmatrix},$$

(68)
$$A_0T_{4,-1}B_2T_{4,-1} = -B_0T_{4,-1}D_2T_{4,-1} =$$

$$\begin{pmatrix} 10810 & 3840 & 0 & 0 \\ -5872 & -2082 & 0 & 0 \\ 2650 & 936 & 0 & 0 \\ -808 & -282 & 0 & 0 \end{pmatrix},$$

(69)
$$C_0T_{4,-1}A_2T_{4,-1} = -D_0T_{4,-1}C_2T_{4,-1} =$$

$$\begin{pmatrix} 10 & 40 & 60 & 40 \\ -40 & -130 & -120 & 20 \\ -86 & -272 & -228 & 88 \\ 224 & 710 & 600 & -220 \end{pmatrix},$$

(70)
$$C_0T_{4,-1}B_2T_{4,-1} = -D_0T_{4,-1}D_2T_{4,-1} =$$

$$\begin{pmatrix} 10 & 0 & 0 & 0 \\ 80 & 30 & 0 & 0 \\ 202 & 72 & 0 & 0 \\ -520 & -186 & 0 & 0 \end{pmatrix},$$

it follows that

(71) $$V_2^{\sim *}(0)T_{8,-1}V_2^{\sim *}(1) = 0E_8.$$

Since

(72) $$A_0T_{4,-1}A_3T_{4,-1} = -B_0T_{4,-1}C_3T_{4,-1} =$$

$$\begin{pmatrix} -1280 & -5120 & -7680 & -5120 \\ 694 & 2776 & 4164 & 2776 \\ -312 & -1248 & -1872 & -1248 \\ 94 & 376 & 564 & 376 \end{pmatrix},$$

(73) $$A_0T_{4,-1}B_3T_{4,-1} = -B_0T_{4,-1}D_3T_{4,-1} =$$

$$\begin{pmatrix} -1280 & 0 & 0 & 0 \\ -694 & 0 & 0 & 0 \\ -312 & 0 & 0 & 0 \\ 94 & 0 & 0 & 0 \end{pmatrix},$$

(74) $$C_0T_{4,-1}A_3T_{4,-1} = -D_0T_{4,-1}C_3T_{4,-1} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -10 & -40 & -60 & -40 \\ -24 & -96 & -144 & 96 \\ 62 & 248 & 372 & 248 \end{pmatrix},$$

(75) $$C_0T_{4,-1}B_3T_{4,-1} = -D_0T_{4,-1}D_3T_{4,-1} =$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ -24 & 0 & 0 & 0 \\ 62 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

(76) $$V_2^{\sim *}(0)T_{8,-1}V_3^{\sim *}(3) = 0E_8.$$

Since

(77) $$A_1T_{4,-1}A_0T_{4,-1} = -B_1T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} 19952 & 38280 & -14288 & -49700 \\ -13118 & -25128 & 9510 & 32752 \\ 8040 & 15372 & -5912 & -20128 \\ -4466 & -8520 & 3388 & 11216 \end{pmatrix},$$

(78)
$$A_1T_{4,-1}B_0T_{4,-1} = -B_1T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} 11360 & 34192 & -3040 & -8788 \\ -7522 & -22536 & 2010 & 5792 \\ 4648 & 13852 & -1240 & -3560 \\ -2606 & -7720 & 694 & 1984 \end{pmatrix},$$

(79)
$$C_1T_{4,-1}A_0T_{4,-1} = -D_1T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} 2144 & 4080 & -1632 & -5404 \\ -822 & -1560 & 638 & 2080 \\ 248 & 468 & -200 & -632 \\ -170 & -312 & 162 & 448 \end{pmatrix},$$

(80)
$$C_1T_{4,-1}B_0T_{4,-1} = -D_1T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} 1264 & 3720 & -336 & -956 \\ -490 & -1432 & 130 & 368 \\ 152 & 436 & -40 & -112 \\ -118 & -312 & 30 & 80 \end{pmatrix},$$

it follows that

(81)
$$V_2^{\sim *}(1)T_{8,-1}V_2^{\sim *}(0) = 0E_8.$$

Since

(82)
$$A_1T_{4,-1}A_1T_{4,-1} = -B_1T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} 41528 & 101888 & 18636 & -113776 \\ -27344 & -67060 & -12176 & 75016 \\ 16788 & 41152 & 7408 & -46128 \\ -9344 & -22892 & -4080 & 25720 \end{pmatrix},$$

(83)
$$A_1T_{4,-1}B_1T_{4,-1} = -B_1T_{4,-1}D_1T_{4,-1} =$$

$$\begin{pmatrix} -57184 & 41232 & 26364 & 0 \\ 37664 & -27188 & -17376 & 0 \\ -23132 & 16720 & 10680 & 0 \\ 12880 & -9324 & -5952 & 0 \end{pmatrix},$$

(84)
$$C_1T_{4,-1}A_1T_{4,-1} = -D_1T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} 4496 & 11008 & 1940 & -12400 \\ -1728 & -4228 & -736 & 4776 \\ 524 & 1280 & 216 & -1456 \\ -368 & -892 & -128 & 1048 \end{pmatrix},$$

$$C_1T_{4,-1}B_1T_{4,-1}=-D_1T_{4,-1}D_1T_{4,-1}= \tag{85}$$

$$\begin{pmatrix}-6200 & 4496 & 2868 & 0\\ 2384 & -1732 & -1104 & 0\\ -724 & 528 & 336 & 0\\ 512 & -380 & -240 & 0\end{pmatrix},$$

it follows that

$$V_2^{\sim *}(1)T_{8,-1}V_2^{\sim *}(1)=0E_8. \tag{86}$$

Since

$$A_1T_{4,-1}A_2T_{4,-1}=-B_1T_{4,-1}C_2T_{4,-1}= \tag{87}$$

$$\begin{pmatrix}32112 & 102084 & 87216 & -29736\\ -21158 & -67256 & -57444 & 19624\\ 13000 & 41320 & 35280 & -12080\\ -7242 & -23016 & -19644 & 6744\end{pmatrix},$$

$$A_1T_{4,-1}B_2T_{4,-1}=-B_1T_{4,-1}D_2T_{4,-1}= \tag{88}$$

$$\begin{pmatrix}-73344 & -26364 & 0 & 0\\ 48346 & 17376 & 0 & 0\\ -29720 & -10680 & 0 & 0\\ 16566 & 5952 & 0 & 0\end{pmatrix},$$

$$C_1T_{4,-1}A_2T_{4,-1}=-D_1T_{4,-1}C_2T_{4,-1}= \tag{89}$$

$$\begin{pmatrix}3488 & 11084 & 9456 & -3256\\ -1342 & -4264 & -3636 & 1256\\ 408 & 1296 & 1104 & -384\\ -290 & -920 & -780 & 280\end{pmatrix},$$

$$C_1T_{4,-1}B_2T_{4,-1}=-D_1T_{4,-1}D_2T_{4,-1}= \tag{90}$$

$$\begin{pmatrix}-7984 & -2868 & 0 & 0\\ 3074 & 1104 & 0 & 0\\ -936 & 336 & 0 & 0\\ 670 & -240 & 0 & 0\end{pmatrix},$$

it follows that

$$V_2^{\sim *}(1)T_{8,-1}V_2^{\sim *}(1)=0E_8. \tag{91}$$

Since

$$A_1T_{4,-1}A_3T_{4,-1}=-B_1T_{4,-1}C_3T_{4,-1}= \tag{92}$$

$$\begin{pmatrix} 8788 & 35152 & 52728 & 35152 \\ -5792 & -23168 & -34752 & -23168 \\ 3560 & 14240 & 21360 & 144240 \\ -1984 & -7936 & -11904 & -7936 \end{pmatrix},$$

(93) $$A_1T_{4,-1}B_3T_{4,-1} = -B_1T_{4,-1}D_3T_{4,-1} =$$

$$\begin{pmatrix} 8788 & 0 & 0 & 0 \\ -5972 & 0 & 0 & 0 \\ 3560 & 0 & 0 & 0 \\ -1984 & 0 & 0 & 0 \end{pmatrix},$$

(94) $$C_1T_{4,-1}A_3T_{4,-1} = -D_1T_{4,-1}C_3T_{4,-1} =$$

$$\begin{pmatrix} 956 & 3824 & 5736 & 3824 \\ -368 & -1472 & -2208 & -1472 \\ 112 & 448 & 672 & 448 \\ -80 & -320 & -480 & -320 \end{pmatrix},$$

(95) $$C_1T_{4,-1}B_3T_{4,-1} = -D_1T_{4,-1}D_3T_{4,-1} =$$

$$\begin{pmatrix} 956 & 0 & 0 & 0 \\ -368 & 0 & 0 & 0 \\ 112 & 0 & 0 & 0 \\ -80 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

(96) $$V_2^{\sim *}(1)T_{8,-1}V_2^{\sim *}(3) = 0E_8.$$

Since

(97) $$A_2T_{4,-1}A_0T_{4,-1} = -B_2T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} 20745 & 37692 & -20917 & -55568 \\ -13164 & -23913 & 13288 & 35273 \\ 7731 & 14040 & -7815 & -20724 \\ -4080 & -7407 & 4132 & 10943 \end{pmatrix},$$

(98) $$A_2T_{4,-1}B_0T_{4,-1} = -B_2T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} 14663 & 38492 & -3755 & -9880 \\ -9308 & -24431 & 2384 & 6271 \\ 5469 & 14352 & -1401 & -3684 \\ -2888 & -7577 & 740 & 1945 \end{pmatrix},$$

(99) $$C_2T_{4,-1}A_0T_{4,-1} = -D_2T_{4,-1}C_0T_{4,-1} =$$

$$\begin{pmatrix} 1845 & 3348 & -1873 & -4952 \\ -660 & -1197 & 672 & 1773 \\ 159 & 288 & -163 & -428 \\ 24 & 45 & -20 & -61 \end{pmatrix},$$

(100) $$C_2T_{4,-1}B_0T_{4,-1} = -D_2T_{4,-1}D_0T_{4,-1} =$$

$$\begin{pmatrix} 1307 & 3428 & -335 & -880 \\ -468 & -1227 & 120 & 315 \\ 113 & 296 & -29 & -76 \\ 16 & 43 & -5 & -11 \end{pmatrix},$$

it follows that

(101) $$V_2^{\sim*}(2)T_{8,-1}V_2^{\sim*}(0) = 0E_8.$$

Since

(102) $$A_2T_{4,-1}A_1T_{4,-1} = -B_2T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} 45288 & 109622 & 15248 & -129468 \\ -28743 & -69572 & -9671 & 82176 \\ 16884 & 40866 & 5676 & -48276 \\ -8913 & -21572 & -2993 & 25488 \end{pmatrix},$$

(103) $$A_2T_{4,-1}B_1T_{4,-1} = -B_2T_{4,-1}D_1T_{4,-1} =$$

$$\begin{pmatrix} -62992 & 47030 & 29640 & 0 \\ 39979 & -29852 & -18813 & 0 \\ -23484 & -17538 & 11052 & 0 \\ 12397 & -9260 & -5835 & 0 \end{pmatrix},$$

(104) $$C_2T_{4,-1}A_1T_{4,-1} = -D_2T_{4,-1}C_1T_{4,-1} =$$

$$\begin{pmatrix} 4032 & 9858 & 1352 & -11532 \\ -1143 & -3492 & -483 & 4128 \\ 348 & 842 & 116 & -996 \\ 51 & 124 & 19 & -144 \end{pmatrix},$$

(105) $$C_2T_{4,-1}B_1T_{4,-1} = -D_2T_{4,-1}D_1T_{4,-1} =$$

$$\begin{pmatrix} -5608 & 4190 & 2640 & 0 \\ 2007 & 1500 & -945 & 0 \\ -484 & 362 & 228 & 0 \\ -71 & 52 & 33 & 0 \end{pmatrix},$$

it follows that

(106) $$V_2^{\sim*}(2)T_{8,-1}V_2^{\sim*}(1) = 0E_8.$$

Since

$$A_2T_{4,-1}A_2T_{4,-1} = -B_2T_{4,-1}C_2T_{4,-1} = \tag{107}$$

$$\begin{pmatrix} 35765 & 113420 & 96030 & -34780 \\ -22700 & -71987 & -60948 & 22078 \\ 13335 & 42288 & 35802 & -12972 \\ -7040 & -22325 & -18900 & 6850 \end{pmatrix},$$

$$A_2T_{4,-1}B_2T_{4,-1} = -B_2T_{4,-1}D_2T_{4,-1} = \tag{108}$$

$$\begin{pmatrix} -82795 & -29640 & 0 & 0 \\ 52552 & 18813 & 0 & 0 \\ -30873 & -11052 & 0 & 0 \\ 16300 & 5835 & 0 & 0 \end{pmatrix},$$

$$C_2T_{4,-1}A_2T_{4,-1} = -D_2T_{4,-1}C_2T_{4,-1} = \tag{109}$$

$$\begin{pmatrix} 3185 & 10100 & 8550 & -3100 \\ -1140 & -3615 & -3060 & 1110 \\ 275 & 872 & 738 & -268 \\ 40 & 127 & 108 & -38 \end{pmatrix},$$

$$C_2T_{4,-1}B_2T_{4,-1} = -D_2T_{4,-1}D_2T_{4,-1} = \tag{110}$$

$$\begin{pmatrix} -7375 & -2640 & 0 & 0 \\ 2640 & 945 & 0 & 0 \\ -637 & -228 & 0 & 0 \\ -92 & -33 & -240 & 0 \end{pmatrix},$$

it follows that

$$V_2^{\sim *}(2)T_{8,-1}V_2^{\sim *}(2) = 0E_8. \tag{111}$$

Since

$$A_2T_{4,-1}A_3T_{4,-1} = -B_2T_{4,-1}C_3T_{4,-1} = \tag{112}$$

$$\begin{pmatrix} 9880 & 39520 & 59280 & 39520 \\ -6271 & -25084 & -37626 & -25084 \\ 3684 & 14736 & 22104 & 14736 \\ -1945 & -7780 & -11670 & -7780 \end{pmatrix},$$

$$A_2T_{4,-1}B_3T_{4,-1} = -B_2T_{4,-1}D_3T_{4,-1} = \tag{113}$$

$$\begin{pmatrix} 9880 & 0 & 0 & 0 \\ -6271 & 0 & 0 & 0 \\ 3684 & 0 & 0 & 0 \\ -1945 & 5835 & 0 & 0 \end{pmatrix},$$

(114) $$C_2T_{4,-1}A_3T_{4,-1} = -D_2T_{4,-1}C_3T_{4,-1} =$$
$$\begin{pmatrix} 880 & 3520 & 5280 & 3520 \\ -315 & -1260 & -1890 & -1260 \\ 76 & 304 & 456 & 304 \\ 11 & 44 & 66 & 44 \end{pmatrix},$$

(115) $$C_2T_{4,-1}B_3T_{4,-1} = -D_2T_{4,-1}D_3T_{4,-1} =$$
$$\begin{pmatrix} 880 & 0 & 0 & 0 \\ -315 & 0 & 0 & 0 \\ 76 & 0 & 0 & 0 \\ 11 & 0 & -240 & 0 \end{pmatrix},$$

it follows that

(116) $$V_2^{\sim *}(2)T_{8,-1}V_2^{\sim *}(3) = 0E_8.$$

Since

(117) $$A_3T_{4,-1}A_0T_{4,-1} = -B_3T_{4,-1}C_0T_{4,-1} =$$
$$\begin{pmatrix} -2856 & -5355 & 2380 & 7259 \\ 1824 & 3420 & -1520 & -4636 \\ -1080 & -2025 & 900 & 2745 \\ 576 & 1080 & -480 & -1464 \end{pmatrix},$$

(118) $$A_3T_{4,-1}B_0T_{4,-1} = -B_3T_{4,-1}D_0T_{4,-1} =$$
$$\begin{pmatrix} -1904 & -5117 & 476 & 1309 \\ 1216 & 3268 & -304 & -836 \\ -720 & -1935 & 180 & 495 \\ 384 & 1032 & -96 & -264 \end{pmatrix},$$

(119) $$C_3T_{4,-1}A_0T_{4,-1} = -D_3T_{4,-1}C_0T_{4,-1} =$$
$$\begin{pmatrix} -264 & -495 & 220 & 671 \\ 96 & 180 & -80 & -244 \\ -24 & -45 & 20 & 61 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(120) $$C_3T_{4,-1}B_0T_{4,-1} = -D_3T_{4,-1}D_0T_{4,-1} =$$
$$\begin{pmatrix} -176 & -473 & 44 & 121 \\ 644 & 172 & -16 & -44 \\ -16 & -43 & 4 & 11 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

$$V_3^{\sim*}(3)T_{8,-1}V_2^{\sim*}(0) = 0E_8. \tag{121}$$

Since

$$A_3T_{4,-1}A_1T_{4,-1} = -B_3T_{4,-1}C_1T_{4,-1} = \tag{122}$$

$$\begin{pmatrix} -6069 & -14756 & -2261 & 17136 \\ 3876 & 9424 & 1444 & -10944 \\ -2295 & -5580 & -855 & 6480 \\ 1224 & 2976 & 456 & -3456 \end{pmatrix},$$

$$A_3T_{4,-1}B_1T_{4,-1} = -B_3T_{4,-1}D_1T_{4,-1} = \tag{123}$$

$$\begin{pmatrix} 8449 & -6188 & -3927 & 0 \\ -5396 & 3952 & 2508 & 0 \\ 3195 & -2340 & -1485 & 0 \\ -1704 & 1248 & 792 & 0 \end{pmatrix},$$

$$C_3T_{4,-1}A_1T_{4,-1} - D_3T_{4,-1}C_1T_{4,-1} = \tag{124}$$

$$\begin{pmatrix} -561 & -1364 & -209 & 1584 \\ 204 & 496 & 76 & -576 \\ -51 & -124 & -19 & 144 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3T_{4,-1}B_1T_{4,-1} = -D_3T_{4,-1}D_1T_{4,-1} = \tag{125}$$

$$\begin{pmatrix} 781 & -572 & -363 & 0 \\ -284 & -208 & 132 & 0 \\ 71 & -52 & -33 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

$$V_3^{\sim*}(3)T_{8,-1}V_2^{\sim*}(1) = 0E_8. \tag{126}$$

Since

$$A_3T_{4,-1}A_2T_{4,-1} = -B_3T_{4,-1}C_2T_{4,-1} = \tag{127}$$

$$\begin{pmatrix} -4760 & -155113 & -12852 & 4522 \\ 3040 & 9652 & 8208 & -2888 \\ -1800 & -5715 & -4860 & 1710 \\ 960 & 3048 & 2592 & -912 \end{pmatrix},$$

(128)
$$A_3T_{4,-1}B_2T_{4,-1} = -B_3T_{4,-1}D_2T_{4,-1} =$$
$$\begin{pmatrix} 10948 & 3927 & 0 & 0 \\ -6992 & -2508 & 0 & 0 \\ 4140 & 1485 & 0 & 0 \\ -2208 & -792 & 0 & 0 \end{pmatrix},$$

(129)
$$C_3T_{4,-1}A_2T_{4,-1} - D_3T_{4,-1}C_2T_{4,-1} =$$
$$\begin{pmatrix} -440 & -1397 & -1188 & 418 \\ 160 & 508 & 432 & -152 \\ -40 & -127 & -108 & 38 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

(130)
$$C_3T_{4,-1}B_2T_{4,-1} = -D_3T_{4,-1}D_2T_{4,-1} =$$
$$\begin{pmatrix} 10121 & 363 & 0 & 0 \\ -368 & -132 & 0 & 0 \\ 92 & 33 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

it follows that

(131)
$$V_3^{\sim *}(3)T_{8,-1}V_2^{\sim *}(2) = 0E_8.$$

Since

(132)
$$A_3T_{4,-1}A_3T_{4,-1} = -B_3T_{4,-1}C_3T_{4,-1} =$$
$$\begin{pmatrix} -1309 & -5236 & -7854 & -5236 \\ 836 & 3344 & 5016 & 3344 \\ -495 & -1980 & -2970 & -1980 \\ 264 & 1056 & 1584 & 1056 \end{pmatrix},$$

(133)
$$A_3T_{4,-1}B_3T_{4,-1} = -B_3T_{4,-1}D_3T_{4,-1} =$$
$$\begin{pmatrix} -1309 & 0 & 0 & 0 \\ 836 & 0 & 0 & 0 \\ -495 & 0 & 0 & 0 \\ 264 & 0 & 0 & 0 \end{pmatrix},$$

(134)
$$C_3T_{4,-1}A_3T_{4,-1} - D_3T_{4,-1}C_3T_{4,-1} =$$
$$\begin{pmatrix} -121 & -484 & -726 & -484 \\ 44 & 176 & 264 & 176 \\ -11 & -44 & -66 & -44 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_3T_{4,-1}B_3T_{4,-1} = -D_3T_{4,-1}D_3T_{4,-1} = \begin{pmatrix} -121 & 0 & 0 & 0 \\ 44 & 0 & 0 & 0 \\ -11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{135}$$

it follows that

$$V_3^{\sim *}(3)T_{8,-1}V_3^{\sim *}(3) = 0E_8. \tag{136}$$

So, we see that

$$V_i^{\sim *}(3)T_{8,-1}V_k^{\sim *}(3) = 0E_8 \tag{137}$$

for $\{i, k\} \subset \{0, 1, 2, 3, 4\}$. Therefore (17) holds for $l = 2$. So the equalities (16) – (18) hold for l=2, and we had made full test of the equality $A_2(z,\nu)A_2(z,-\nu) = -\nu^{14}E_4$. The equality (18) substantinates the test, which was made for the eaquality $A_2(z,\nu)A_2(z,-\nu) = -\nu^{14}E_4$ in the Part 3 and in the section 4.3.

Arithmetical applications will be given in the next parts.

§3.2. Corrections to the previous parts of this paper.

The equalities (5), (6) in Part 3 should be replaced by the equalities (5), (6) in this part respectively. On the page 4, line 3 after equality (15) should be

$$H_l^*(k) = R_l^*(k) - V_l^*(k),$$

instead of

$$H_l^*(k) = R_l^\vee(k) - V_l^*(k),$$

In Part 3 on page 14 the equality

$$V_2^*(9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 608 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -832 & 608 & 0 & 0 & 0 & 0 & 0 & 0 \\ 320 & -992 & -160 & 0 & 0 & 0 & 0 & 0 \\ -32 & 384 & -96 & -160 & 0 & 0 & 0 & 0 \\ 0 & -32 & 96 & 128 & 32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 \end{pmatrix}$$

must stand, instead of

$$V_2^*(9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 00 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 608 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -832 & 608 & 0 & 0 & 0 & 0 & 0 & 0 \\ 320 & -992 & 160 & 0 & 0 & 0 & 0 & 0 \\ -32 & 384 & -96 & -160 & 0 & 0 & 0 & 0 \\ 0 & -32 & 96 & 128 & 32 & 0 & 0 & 0 \\ 0 & 0 & 0 & 32 & 64 & 32 & 0 & 0 \end{pmatrix};$$

the equality

$$V_2^*(10) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -360 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 408 & -360 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & 400 & -8 & 0 & 0 & 00 & 0 & 0 \\ 8 & -104 & 8 & -8 & 0 & 0 & 0 & \\ 0 & 0 & -48 & -96 & -56 & 0 & 0 & 0 \end{pmatrix},$$

must stand, instead of

$$V_2^*(10) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -360 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 408 & -360 & 0 & 0 & 0 & 0 & 0 & 0 \\ -120 & 400 & -8 & 0 & 0 & 00 & 0 & 0 \\ 8 & -104 & 8 & -80 & 0 & 0 & 0 & \\ 0 & 0 & -48 & -96 & -56 & 0 & 0 & 0 \end{pmatrix},$$

the equality

$$V_2^*(11) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 192 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -160 & 192 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & -96 & 64 & 0 & 0 & 0 & 0 & 0 \\ 0 & 32 & 64 & 64 & 0 & 0 & 0 & 0 \end{pmatrix},$$

must stand, instead of

$$V_2^*(11) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 192 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -160 & 192 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & -9664 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 32 & 64 & 64 & 0 & 0 & 0 & 0 \end{pmatrix};$$

the equality

$$V_2^*(12) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -88 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & -88 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -16 & -56 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

must stand, instead of

$$V_2^*(12) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 40 & -88 & 0 & 0 & 0 & 0 & 0 & 0 \\ -8 & -16 & -56 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In Part 3 the relation

$$V_l^*(3+2l+\kappa) \in \mathfrak{R}_{4+2l}(N^\wedge)^\kappa \cap (N^\wedge)^\kappa \mathfrak{R}_{4+2l}$$

for $l = 0, 1, 2, \kappa = 0, .., , 3+2l$. must stand, instead of relations (58) – (59), and the equality

$$V_0^*(3) = \begin{pmatrix} 16 & 0 & 0 & 0 \\ -8 & 16 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

must stand, instead of the equality

$$V_0^*(3) = \begin{pmatrix} 16 & 0 & 0 & 0 \\ -8 & 16 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{pmatrix}.$$

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