Singular hermitian metrics on vector bundles

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Dedicated to the memory of Michael Schneider

Abstract

We introduce a notion of singular hermitian metrics (s.h.m.) for holomorphic vector bundles and define positivity in view of L^2-estimates. Associated with a suitably positive s.h.m. there is a (coherent) sheaf 0-th kernel of a certain d''-complex. We prove a vanishing theorem for the cohomology of this sheaf. All this generalizes to the case of higher rank known results of Nadel for the case of line bundles. We introduce a new semi-positivity notion, t-nefness, for vector bundles, establish some of its basic properties and prove that on curves it coincides with ordinary nefness. We particularize the results on s.h.m. to the case of vector bundles of the form $E = F \otimes L$, where F is a t-nef vector bundle and L is a positive (in the sense of currents) line bundle. As applications we generalize to the higher rank case 1) Kawamata-Viehweg Vanishing Theorem, 2) the effective results concerning the global generation of jets for the adjoint to powers of ample line bundles, and 3) Matsusaka Big Theorem made effective.

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0 Introduction

In this study I introduce a notion of singular hermitian metrics (*s.h.m.*) on holomorphic vector bundles over complex manifolds. The original motivation was to explore the possibility of employing, in the setting of vector bundles, the new transcendental techniques developed by Demailly and Siu in order to study global generation problems for (adjoint) line bundles. The notes [10] are an excellent introduction to these techniques and to the results in the literature. One can consult the lucid notes [13] for an algebraic counterpart to these techniques.

Let me discuss the case of line bundles. Let X be a non-singular projective manifold of dimension n, L and E be an ample and a nef line bundle on X, respectively, a be a non-negative integer and m be a positive one.
Under which conditions on a and m will the line bundle

$$\mathfrak{P} := K_X^{\otimes a} \otimes L^{\otimes m} \otimes E$$

be generated by its global sections (free)?
More generally, we can ask for conditions on a and m under which the simultaneous generation of the higher jets of $\mathfrak{P}$ at a prescribed number of points on X is ensured.

It is clear that $m \gg 0$ answers the question. However, how big m should be could depend, *a priori*, on X. For example, Matsusaka Big Theorem asserts that $L^{\otimes m}$ is very ample for every $m \geq M := M(n, L^n, K_X \cdot L^{n-1})$. An effective value for M has been recently determined in [25] and [11]; see also [14] for the case of surfaces.

The presence of the canonical line bundle, i.e. $a > 0$, changes dramatically the shape of the lower bound on m. Fujita's Conjecture speculates that $K_X \otimes L^{\otimes m}$ should be free as soon as $m \geq n+1$. This conjecture is true for $n \leq 4$ by the work of Reider, Ein-Lazarsfeld and Kawamata. In the papers [1] and [28] it is proved that $m \geq \frac{1}{2}(n^2+n+2)$ gives freeness.

Effective results depending only on n are proved for $a \geq 1$ by several authors. The seminal paper is [9] where it is proved, by (differential-geometric-)*analytic methods*, that $K_X^{\otimes 2} \otimes L^{\otimes m}$ is very ample for all $m \geq 12n^n$. Then followed the paper [18], where a similar result is proved using *algebraic-geometric methods.* Since then, several papers have appeared on the subject. The reader may consult the following references to compare the various results and techniques: [10] (an account of the analytic approach with a rather complete bibliography), [13] (an account of the algebraic approach and of how many of the analytic instruments may be re-tooled and made algebraic), [1] and [28] (freeness; written in the analytic language, but apt to be completely translated into the algebraic language after observations by Kollár [19], §5; see also [27]), [11], [26] and [24] (very ampleness; analytic), [25] and [11] (an effective version of Matsusaka Big Theorem; analytic).

An extra nef factor E plays a minor role and all of the results quoted above hold in its presence. This simple fact was the starting point of my investigation.

Question 0.0.1 *Can we obtain effective results on a and m for the global generation of the vector bundles $\mathfrak{P}$ by assuming that E is a suitably semi-positive vector bundle of rank r? More generally, can we obtain similar results about the simultaneous generation of the higher jets of $\mathfrak{P}$ at a prescribed number of points on X ?*

I expected that the statements in the aforementioned literature concerning the line bundles $\mathfrak{P}$ with the nef line bundle E should carry over, *unchanged*, to the case in which E is a nef vector bundle.

On a projective manifold a nef line bundle can be endowed with hermitian metrics whose curvature forms can be made to have arbitrarily small negative parts (cf. Definition 3.1.2). In the analytic context this fact can be used to make the presence of a nef line bundle E harmless. The same is true in the algebraic context because of the the numerical properties of nefness.

A natural algebraic approach to the case of higher rank is to consider an analogous question for the tautological line bundle ξ of the projectivized bundle $\pi : \mathbb{P}(E) \to X$. The results I obtain with the algebraic approach are for $\mathfrak{P} \otimes \det E$; compare Remark 5.2.5 with the sample effective global generation result presented below;

see [5]. On the analytic side, the problem is that the nefness of a vector bundle E does not seem to be linked to a curvature condition on E itself.

As far as Question 0.0.1 is concerned, nefness does not seem to give enough room to work analytically with higher rank vector bundles.

Instead I introduce, for every vector bundle E and every positive integer t, the notion of *t-nefness* which is a new semi-positivity concept for vector bundles. In some sense it is in between the algebraic notion of nefness and the differential-geometric notion of t-semipositivity. It is a natural higher rank curvature analogue of the aforementioned characterization of nef line bundles. The property of t-nefness is checked by considering tensors in $T_X \otimes E$ of rank at most t; such tensors have ranks never bigger than $N := \min(\dim X, \operatorname{rank} E)$. Incidentally, $(t+1)$-nefness implies t-nefness for all positive integers t, 1-nefness implies nefness and I do not know whether nefness implies 1-nefness.

Though, as I show in Theorem 3.3.1, on curves 1-nefness is equivalent to nefness, the notion of t-nefness is rather difficult to check in an algebraic context. However, see Example 3.1.4 for a list of nef bundles which I know to be N-nef or from which it is easy to obtain N-nef bundles (e.g. nef bundles on curves, nef line bundles, flat bundles, nef bundles on toric or abelian varieties, the tangent bundles of low-dimensional Kähler manifolds with nef tangent bundles, pull-backs, etc.).

Assuming that E is N-nef, I prove for the vector bundles $\mathfrak{P}$ the same statements as the ones in the literature for the line bundle case; see Theorem 5.2.2. Moreover, if E is 1-nef, then the same results hold replacing E by $E \otimes \det E$. The scheme of the proofs is the same as in the rank one case (see Proposition 2.2.2, §5.2, and of course [10], §5 and §8). However, at each and every step we need higher rank analogues of the analytic package developed for the line bundle case by Demailly and Nadel: regularization, L^2-estimates, coherence of relevant sheaves and vanishing theorems. For the purpose of proving these effective results for the vector bundles $\mathfrak{P}$, one would have to make precise the notion of singular hermitian metrics with positivity and prove their relevant properties in a special case: the one of a hermitian vector bundle twisted by a line bundle endowed with a singular metric. Then one would have to prove the relevant vanishing theorems. All this can be done by building on [7], §5 and §9.

However, I felt that it should be worthwhile to develop a general theory of *singular hermitian metrics* on vector bundles with special regards towards positivity.

Inspired by the case of line bundles, in this paper I develop such a theory and obtain as an application the effective results mentioned above. To get a flavor of the results let me state (5.2.2.1′), which constitutes an answer to Question 0.0.1 (see Remark 5.2.3 for a geometric interpretation of these kind of results):

Effective global generation. *Let E be N-nef. Then $K_X \otimes L^{\otimes m} \otimes E$ is globally generated by its global sections for all $m \geq \frac{1}{2}(n^2+n+2)$. Moreover, if E is 1-nef, then the previous statement is true if we replace E by $E \otimes \det E$.*

The paper is organized as follows.
§**1** fixes the notation. §**2** is devoted to s.h.m. which are defined in §2.1. The case of line bundles is discussed in §2.2. In §2.3 we introduce the sheaf $\mathcal{E}(h)$ which generalizes Nadel multiplier ideal sheaf. In §2.4 we define positivity for s.h.m. and study some of its properties. §**3** revolves about the notion of t-nefness. The definition and the basic properties are to be found in §3.1 and §3.2, respectively. §3.3 is devoted to the proof of Theorem 3.3.1 which ensures that on curves the algebraic-geometric notion of nefness can be characterized by the differential-geometric notion of 1-nefness. §3.4 consists of a footnote to [12], Theorem 1.12: ampleness for a vector bundle E can be characterized by a curvature condition on a system of metrics on *all* symmetric powers S^pE of E, though positivity may occur only for $p \gg 0$. §**4** is devoted to vanishing theorems. The basic one is Theorem 4.1.2, a generalization of Nadel Vanishing Theorem; Proposition 4.1.3 asserts that $\mathcal{E}(h)$ is coherent in the presence of suitable positivity. §4.2 links t-nefness and positivity via vanishing; see Theorem 4.2.3. Theorem 4.2.4 is a generalization of Kawamata-Viehweg Vanishing Theorem. §**5** contains the effective results concerning the vector bundles $\mathfrak{P}$. §5.1 contains, for the reader's convenience, a summary of the results of Anghern-Siu and Siu concerning special s.h.m. on line bundles which, transplanted to N-nef vector bundles, will provide the global generation of jets. We also offer the simple Lemma 5.1.2, which constructs metrics with similar properties starting from free line bundles. §5.2 contains our effective results concerning the vector bundles $\mathfrak{P}$; see Theorem 5.2.2.

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1 Notation and preliminaries

Our basic reference for the language of complex differential geometry is [17]. Sufficient and more self-contained references are [7], §2 and [10], §3.

All manifolds are second countable, connected and complex; the dimension is the complex one. All vector bundles are holomorphic. The term *hermitian metric* always refers to a hermitian metric of class $\mathcal{C}^2$. A *hermitian bundle* (E, h) is the assignment of a vector bundle E together with a hermitian metric h on it.

Duality for vector bundles is denoted by the symbol "$*$" and $\mathrm{End}(E)$ is the vector bundle of endomorphisms of E. We often do not distinguish between vector bundles and associated sheaves of holomorphic sections; at times, we employ simultaneously the additive and multiplicative notation for line bundles.

- $d = d' + d''$ denotes the natural decomposition of the exterior derivative d into its $(1,0)$ and $(0,1)$ parts; d'' denotes also the usual operator associated with a vector bundle E.

If (E, h) is a rank r hermitian vector bundle on a manifold X of dimension n, then we denote by:

- $D_h(E)$ the associated *hermitian connection* which is also called the *Chern connection*;
- $\Theta_h(E) = iD_h^2(E)$ the associated *curvature tensor*;

in particular, if L is a line bundle with a metric h, represented locally on some open set U by $e^{-2\varphi}$, then we have $\Theta_h(L)_{|U} = 2id'd''\varphi$;

- $\tilde{\Theta}_h(E)$ the associated hermitian form on $T_X \otimes E$.

If θ is a hermitian form on a complex vector space V, we denote $\theta(v,v)$ by $\theta(v)$; if in addition, θ is positive definite, then we denote $\theta(v)$ by $|v|^2_\theta$.

- $\mathrm{Herm}_h(V)$ is the set of endomorphisms α of a hermitian vector space (V,h) such that $h(\alpha(v),w) = h(v,\alpha(w))$, $\forall v,\, w \in V$.

Given Θ, a real $(1,1)$-form with values in $\mathrm{Herm}_h(E)$, we denote the associated hermitian form on $T_X \otimes E$ by Θ_h, or by Θ, if no confusion is likely to arise. The hermitian form $\tilde{\Theta}_h(E)$ will be denoted from now on by $\Theta_h(E)$. If ω is a real $(1,1)$-form, e.g. the one associated with a hermitian metric on X, then $\omega \otimes \mathrm{Id}_E$ has values in $\mathrm{Herm}_h(E)$ and we denote the associated hermitian form by $\omega \otimes \mathrm{Id}_{E_h}$ so that $\omega \otimes \mathrm{Id}_{E_h}(t\otimes e) = \omega(t,it)|e|^2_h$, $\forall x \in X$, $\forall t \in T_{X,x}$ and $\forall e \in E_x$.

The rank of a tensor. Let V and W be complex vector spaces of finite dimensions r and s, respectively, $v = \{v_i\}_{i=1}^r$ and $w = \{w_\alpha\}_{\alpha=1}^s$ be bases for V and W, respectively; tensor products are taken over $\mathbb{C}$.
Every tensor $\tau \in V\otimes W$ defines two linear maps $\alpha_\tau : W^* \to V$ and $\beta_\tau : V^* \to W$; moreover, we can write $\tau = \sum_{i\alpha} \tau_{i\alpha} v_i \otimes w_\alpha$ and associate with τ the $r\times s$ matrix $||\tau_{i\alpha}||$. The integer $\rho(\tau) := \mathrm{rank}(\alpha_\tau) = \mathrm{rank}(\beta_\tau) = \mathrm{rank}||\tau_{i\alpha}||$ is called the *rank* of the tensor τ.
Tensors of rank zero or one are called *decomposable*; they have the form $\tau = v\otimes w$, for some $v \in V$ and $w\in W$. For any non-zero tensor $\tau \in V\otimes W$ we have that $1 \leq \rho(\tau) \leq \min(r,s)$. In particular, if either $r=1$, $s=1$, or both, then every tensor $\tau \in V\otimes W$ is decomposable.

Inequalities associated with the rank. Given two hermitian forms θ_1 and θ_2 on $V\otimes W$, we can compare them on tensors of various rank. Let t be any positive integer. We write $\theta_1 \geq_t \theta_2$ if the hermitian form $\theta_1 - \theta_2$ is semi-positive definite on all tensors in $V\otimes W$ of rank $\rho \leq t$. If $\theta \geq_t 0$, then $\theta \geq_{t'} 0$ for every $t' \leq t$. If $\theta_1 \geq_{\min(r,s)} \theta_2$, then $\theta_1 \geq_t \theta_2$ for every t. The symbol $>_t$ can be defined analogously and it enjoys similar properties.

These considerations and this language are easily transferred to vector bundles.

2 Singular hermitian metrics on vector bundles

In this section we define singular hermitian metrics on vector bundles, discuss the case of line bundles, introduce the sheaf $\mathcal{E}(h)$ and define positivity.

2.1 The definition of singular hermitian metrics

Let X be a manifold of dimension n, E be a rank r vector bundle over X and $\bar{E}$ the conjugate of E. Let h be a section of the smooth vector bundle $E^*\otimes \bar{E}^*$ with measurable coefficients, such that h is an almost everywhere (a.e.) positive definite hermitian form on E; we call such an h a *measurable metric* on E. A measurable metric h on E induces naturally measurable metrics on E^*, on any tensor representation of E, e.g. $T^\alpha E$, $S^\beta E$, $\wedge^\gamma E$ etc., on any quotient bundle of E, etc.

In practice these metrics h occur as *degenerate metrics* of some sorts, e.g. h is a hermitian metric outside a proper analytic subset Σ of X, so that the curvature tensor is well-defined outside Σ.

We are interested in those h for which the curvature tensor has a global meaning. We propose the following simple-minded definition.

Definition 2.1.1 (s.h.m.) *Let X, E and h be as above and $\Sigma \subseteq X$ be a closed set of measure zero. Assume that there exists a sequence of hermitian metrics h_s such that:*

$$\lim_{s\to\infty} h_s = h \qquad \textit{in the } \mathcal{C}^2-\textit{topology on } X\setminus\Sigma.$$

We call the collection of data (X, E, Σ, h, h_s) a singular hermitian metric (s.h.m.) on E. We call $\Theta_h(E_{|X\setminus\Sigma})$ the curvature tensor of (X, E, Σ, h, h_s) and we denote it by $\Theta_h(E)$. $\Theta_h(E)$ has continuous coefficients and values in $\mathrm{Herm}_h(E)$ away from Σ; we denote the a.e.-defined associated hermitian form on $T_X \otimes E$ by the same symbol $\Theta_h(E)$.

If no confusion is likely to arise, we indicate a s.h.m by (E, h) or simply by h.

The guiding principle which subtends this definition can be formulated as follows.

Assume that we would like to prove a property P for h which is true for all metrics h' of class $\mathcal{C}^2$ in the presence of a certain curvature condition C on h'; if h has the required property C and we can find hermitian metrics h_s which regularize h "maintaining" C, then P holds for all h_s and we can try to prove, using limiting arguments, that P holds for h.

This principle has been successfully exploited in [7], §5; see §2.2 for a brief discussion. We will take this principle as the definition of positivity; see Definition 2.4.1 and Proposition 4.1.1, where P is the solution to the d''-problem with L^2-estimates and C is "positivity."

Because of the convergence in the $\mathcal{C}^2$-topology, the notion of s.h.m. is well behaved under the operations of taking quotients, dualizing, forming direct sums, taking tensor products, forming tensor representations, etc.

2.2 Discussion of the line bundle case: curvature current, positivity, Nadel Ideal, Nadel Vanishing Theorem, and the production of sections

We now remark that the singular metrics on line bundles to be found in the literature are s.h.m. We also discuss some of the relevant features of these metrics in the presence of positivity. Basic references for what follows are [10], §5, [7], §9 and §5. A technical remark: for the mere purpose of being consistent with Definition 2.1.1, in what follows we assume that plurisubharmonic (psh) functions are $\mathcal{C}^2$ outside a closed set of measure zero. In all the applications one uses *algebraic singular metrics* as in [11], so that this condition is automatically satisfied. However, all the theory described below and its applications work without this restriction; see also [8], §3.

Note that in what follows we can replace the hermitian line bundle (L, h_0) by a hermitian vector bundle (E, h_0) by operating minor changes.

A *singular metric* on a line bundle L over a manifold X is, by definition, a metric of the form $h = h_0 e^{-2\varphi}$, where h_0 is a hermitian metric on L and φ is a locally integrable function on X. We shall always assume that X is Kähler and that φ is *almost psh*, i.e. it can be written, locally on X, as the sum $\varphi = \alpha + \psi$, where α is a local function of class $\mathcal{C}^2$ and ψ is a local psh function. By taking $d'd''$ in the sense of distributions, we can define the associated *curvature* $(1,1)$-*current*:

$$T := \Theta_{h_0}(L) + 2id'd''\varphi_{ac} + 2id'd''\varphi_{sing},$$

where $2id'd''\varphi_{\mathrm{ac}}$ and $2id'd''\varphi_{sing}$ are the absolutely continuous and singular part of $2id'd''\varphi$, respectively; $2id'd''\varphi_{\mathrm{ac}}$ has locally integrable coefficients and $2id'd''\varphi_{\mathrm{sing}}$ is supported on some closed set Σ of measure zero. A regularizing-approximating result of Demailly's exhibits these singular metrics on line bundles as s.h.m. by constructing the necessary regularizing hermitian metrics $\{h_s\}_{s=1}^{\infty}$. We have $\Theta_h(L) = \Theta_{h_0}(L) + id'd''\varphi_{ac}$. Similar considerations hold for metrics dual to metrics as above.

Example 2.2.1 (Cf. [10], Example 3.11 and [11], page 246) Let $D = \sum m_i D_i$ be a divisor with coefficients $m_i \in \mathbb{Z}$. The associated line bundle carries a singular metric with curvature current $T = 2\pi \sum m_i [D_i]$ where the $[D_i]$ are the currents of integration over the subvarieties D_i.

These currents are positive if and only if all $m_i \geq 0$. More generally, given a finite number of non-trivial holomorphic sections of a multiple of a line bundle L, we can construct a s.h.m. on L. This metric will be singular only at the common zeroes of the sections in question.

The *Nadel ideal* $\mathcal{I}(h)$ (see §2.3) *is coherent.* This is an essential feature in view of the use of this ideal in conjunction with Riemann-Roch Theorem.

Let ω be a Kähler metric on a weakly pseudoconvex manifold X. Assume that $\Theta_h(L) \geq \epsilon\omega$ as a $(1,1)$-current, for some positive and continuous function ϵ on X. Then we have *Nadel Vanishing Theorem*: $H^q(X, K_X \otimes L \otimes \mathcal{I}(h)) = 0, \quad \forall q > 0$; see [22]. This can be seen as a consequence of the solution to the d''-problem for (L,h) with L^2-estimates; see [10], §5.

As an easy consequence of Nadel Vanishing Theorem we have the following result which lays the basis for the effective results for the global generation of adjoint line bundles etc. See [10], Corollary 5.12.

Proposition 2.2.2 *Let (X,ω) be as above and $\mathcal{L}$ be a line bundle over X equipped with a s.h.m. h such that $\Theta_h(L) \geq \epsilon\omega$ for some continuous and positive function ϵ on X. Assume that p is a positive integer and that $s_1, \ldots, s_p$ are non-negative ones. Let $x_1, \ldots, x_p$ be distinct isolated points of the complex space $V(\mathcal{I}(h))$ such that $\mathcal{I}(h) \subseteq \mathfrak{m}_{x_i}^{s_i+1}$. Then there is a surjective map*

$$H^0(X, K_X + \mathcal{L}) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{i=1}^{p} \mathcal{O}(K_X + \mathcal{L}) \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{s_i+1}.$$

Once the analytic package (definition of s.h.m., regularization-approximation, solution of d'' with L^2-estimates, coherence of Nadel ideal and Nadel Vanishing Theorem) has been developed, in order to solve the global generation problem one needs s.h.m. as in Proposition 2.2.2. This requires hard work and it has been done by Anghern-Siu, Demailly, Siu and Tsuji. The coherence and the vanishing theorem are utilized together with a clever use of Noetherian Induction.

We are about to provide a similar analytic package for the case of vector bundles.

2.3 The subsheaf $\mathcal{E}(h)$ associated with a measurable metric (E,h)

If h is a measurable metric on E and e is a measurable section of E, then the function $|e|_h$ is measurable.

Definition 2.3.1 *Let h be a measurable metric on E.*
Let $\mathcal{I}(h)$ be the analytic sheaf of germs of holomorphic functions on X defined as follows:

$$\mathcal{I}(h)_x := \{f_x \in \mathcal{O}_{X,x} :\ |f_x e_x|_h^2 \text{ is integrable in some neighborhood of } x, \forall\, e_x \in E_x\}.$$

Analogously, we define an analytic sheaf $\mathcal{E}(h)$ by setting:

$$\mathcal{E}(h)_x := \{e_x \in E_x \ :\ |e_x|_h^2 \text{ is integrable in some neighborhood of } x\}.$$

Remark 2.3.2 It is easy to show, using the triangle inequality, that $\mathcal{I}(h) \otimes E \subseteq \mathcal{E}(h)$.

We call $\mathcal{I}(h)$ the *multiplier ideal* of (E,h). Note that if E is a line bundle together with a measurable metric h, then $\mathcal{E}(h) = \mathcal{I}(h) \otimes E$.

There are other subsheaves of E, associated with a measurable metric h.

Given any measurable metric h on a vector bundle E, the tautological line bundle $\xi := \mathcal{O}_{\mathbb{P}(E)}(1)$ inherits a natural measurable metric $\mathfrak{h}$, the quotient metric of the surjection $\pi^*E \to \xi$;

here $\pi : \mathbb{P}(E) \to X$ is the structural morphism of the projectivized bundle and we are using Grothendieck's notation. We thus get two sheaves $\mathcal{I}(\mathfrak{h})$ and $\xi \otimes \mathcal{I}(\mathfrak{h})$. If we apply π_*, then we get two other subsheaves of E.

In summary, associated with (E, h) there are four subsheaves of E:

$$\mathcal{I}(h) \otimes E \subseteq \mathcal{E}(h), \quad \pi_* \mathcal{I}(\mathfrak{h}) \otimes E \quad \text{and} \quad \pi_* \ \xi \otimes \mathcal{I}(\mathfrak{h}).$$

Remark 2.3.3 The inclusion above may be strict. In fact, consider the vector bundle $\Delta \times \mathbb{C}^2$, where (Δ, z) is the unit disk in $\mathbb{C}^1$; define a s.h.m. by setting $h = \operatorname{diag}(e^{-2\log|z|}, e^{-4\log|z|})$. Then one checks that $\mathcal{I}(h) = z^2 \cdot \mathcal{O}_\Delta$ and that $\mathcal{E}(h) = z \cdot \mathcal{O}_\Delta \oplus z^2 \cdot \mathcal{O}_\Delta$. The same example shows that $\mathcal{E}(h)$ is not in general equal to neither $\pi_* \ \xi \otimes \mathcal{I}(\mathfrak{h})$, nor $\pi_* \mathcal{I}(\mathfrak{h}) \otimes E$. In fact, a direct computation shows that: $\mathcal{I}(\mathfrak{h}) = \pi^*(z)$. We have $\mathcal{I}(h) \otimes E \subset \mathcal{E}(h) \subset \pi_* \ \mathcal{I}(\mathfrak{h}) \otimes \xi = \pi_* \ \mathcal{I}(\mathfrak{h}) \otimes E$.

What is, among the four sheaves above, the "right" object to look at? To answer this question we consider:

the complex $(\mathfrak{L}^\bullet, d'')$. Let h be a measurable metric on a vector bundle E and ω be a hermitian metric on X. By following the standard conventions in [30], we obtain a metric with measurable coefficients for the fibers of $T_X^{p,q^*} \otimes E$; we denote this metric again by h. We define a complex $(\mathfrak{L}^\bullet, d'')$ of sheaves on X as follows. This complex is independent of the choice of ω.
Let $\mathfrak{L}^q$ be the sheaf of germs of (n, q)-forms u with values in E and square-integrable coefficients such that $|u|_h^2$ is locally integrable, $d''u$ is defined in the sense of distributions with square-integrable coefficients and $|d''u|_h^2$ is locally integrable.

The kernel of d'' in degree zero is $K_X \otimes \mathcal{E}(h)$ (cf. [17], page 380). A solution to the d''-problem with L^2-estimates for (E, h) would imply the vanishing of the higher cohomology of $K_X \otimes \mathcal{E}(h)$. See Theorem 4.1.2.

If we are aiming at vanishing theorems as in the line bundle case, the sheaf $\mathcal{E}(h)$ seems to be the right object to look at.

2.4 Positivity

As is well-known, the curvature tensor $\Theta_h(L)$ of a hermitian line bundle (L, h) is decomposable and can be identified with a real $(1,1)$-form on X. This latter is a positive $(1,1)$-form if and only if the hermitian form $\Theta_h(L)$ is positive on $T_X \otimes L$.

It is therefore natural to define positivity for singular metrics on line bundles using the notion of *positive currents* according to Lelong; see [20], §2.

In the higher rank case the curvature tensor is not, in general, decomposable. We introduce a notion of positivity which incorporates what is needed to obtain L^2-estimates-type results.

Let ω be a hermitian metric on X, θ be a hermitian form on T_X with continuous coefficients and (X, E, Σ, h, h_s) be a s.h.m.; in particular, the curvature tensor and the curvature form $\Theta_h(E)$ are defined a.e. (i.e. outside of Σ) and have measurable coefficients.

Definition 2.4.1 ($\geq_t^\mu$; compare with [7], §5.) *Let things be as above and t be a positive integer. We write:*

$$\Theta_h(E) \geq_t^\mu \theta \otimes \mathrm{Id}_{E_h}$$

if the following requirements are met.
There exist a sequence of hermitian forms θ_s on $T_X \otimes E$ with continuous coefficients, a sequence of continuous functions λ_s on X and a continuous function λ on X subject to the following requirements:
(2.4.1.1) $\forall x \in X$*:* $|e_x|_{h_s} \leq |e_x|_{h_{s+1}}$, $\forall s \in \mathbb{N}$ *and* $\forall e_x \in E_x$*;*

(2.4.1.2) $\theta_s \geq_t \theta \otimes \mathrm{Id}_{E_{h_s}}$*;*
(2.4.1.3) $\Theta_{h_s}(E) \geq_t \theta_s - \lambda_s \omega \otimes \mathrm{Id}_{E_{h_s}}$*;*
(2.4.1.4) $\theta_s \to \Theta_h(E)$ *a.e. on* X*;*
(2.4.1.5) $\lambda_s \to 0$ *a.e. on* X*;*
(2.4.1.6) $0 \leq \lambda_s \leq \lambda$, $\forall s$.

Conditions (2.4.1.1) and (2.4.1.6) are needed to apply Lebesgue's theorems on monotonic and dominated convergence. In order to obtain L^2-estimates-type results, we also need the remaining four conditions to make precise the sought-for control of the curvature by the regularizing and approximating metrics h_s.

Remark 2.4.2 As an application of the L^2-estimates, we will see that if $\Theta_h(E) \geq^{\mu}_N \theta \otimes \mathrm{Id}_{E_h}$, for some continuous θ, then the sheaf $\mathcal{E}(h)$ is coherent; see Proposition 4.1.3.

Example 2.4.3 If (E, h) is a hermitian bundle with $\Theta_h(E) \geq_t \theta \otimes \mathrm{Id}_{E_h}$, then it is easy to exhibit h as a s.h.m. such that $\Theta_h(E) \geq^{\mu}_t \theta \otimes \mathrm{Id}_{E_h}$; just set $h_s := h$ $\forall s$, etc.

Example 2.4.4 Let (E, h) be a vector bundle together with a *continuous* s.h.m metric. Under certain positivity conditions on the current $id'd''h^*$which is defined on the total space of E^* (see [21], §7.1) we can exhibit h as a s.h.m. with positivity in the sense of Definition 2.4.1. This is achieved in two steps. In the first one h^* is regularized by using Riemannian convolution coupled with the parallel transport associated with an arbitrary hermitian metric on E^* (see [21], Lemme 7.2). In the second one the resulting metrics are modified so that they have the prescribed properties; this technical modification follows ideas in [7], §8. Details will appear elsewhere.

Example 2.4.5 Let $h = h_0 e^{-2\varphi}$ be a singular metric on a line bundle L with $T \geq \theta$ as currents where θ is a continuous and real $(1, 1)$-form. [7], Théorème 9.1, exhibits these data as a s.h.m. h with $\Theta_h(L) \geq^{\mu}_1 \theta \otimes \mathrm{Id}_{L_h}$.
Conversely, if we have a s.h.m. h with $\Theta_h(L) \geq^{\mu}_1 \theta \otimes \mathrm{Id}_{L_h}$, then we have $\Theta_{h_s}(L) \geq \theta_s - \lambda_s \omega$ and $T \geq T_{ac} \geq \theta$.

Remark 2.4.6 The existence of a s.h.m. h on a line bundle L for which $\Theta_h(L) \geq^{\mu}_1 0$ does not imply that L is nef. See [12], Remark 1.6.
What is true is that if L is nef, then L will admit a metric $h = h_0 e^{-2\varphi}$ with h_0 a hermitian metric on L and φ almost psh such that $\Theta_h(L) \geq^{\mu}_1 0$. This can be seen by using [12], Proposition 1.4, [8], Proposition 3.7 and [7], Théorème 9.1.
Similar remarks hold for big line bundles on projective manifolds (cf. [10], Proposition 6.6).

The following lemma is elementary.

Lemma 2.4.7 *Let* (E, Σ_E, h, h_s) *and* (F, Σ_F, g, g_s) *be s.h.m. on two vector bundles* E *and* F *over* X*,* σ_1 *and* σ_2 *be two real* $(1, 1)$*-forms with continuous coefficients such that* $\Theta_h(E) \geq^{\mu}_{t_1} \sigma_1 \otimes \mathrm{Id}_{E_h}$ *and* $\Theta_g(F) \geq^{\mu}_{t_2} \sigma_2 \otimes \mathrm{Id}_{F_g}$.
Then $H := h \otimes g$ *on* $E \otimes F$ *can be seen as a s.h.m. by setting* $H_s := h_s \otimes g_s$ *and*

$$\Theta_H(E \otimes F) \geq^{\mu}_{\min(t_1, t_2)} (\sigma_1 + \sigma_2) \otimes \mathrm{Id}_{(E \otimes F)_H}.$$

Note that if the rank of F is one, then $\min(t_1, t_2) = t_1$.

We now prove that positivity is inherited by quotient metrics.

Lemma 2.4.8 *Let* (X, E, Σ, h, h_s) *be a s.h.m such that* $\Theta_h(E) \geq^{\mu}_t \theta \otimes \mathrm{Id}_{E_h}$*,* $\phi : E \to Q$ *be a surjection of vector bundles with kernel* K*. Then* Q *admits a s.h.m.* $(Q, \Sigma' \subseteq \Sigma, q_s, q)$ *such that* $\Theta_q(Q) \geq^{\mu}_1 \theta \otimes \mathrm{Id}_{Q_q}$.

Proof. Consider the dual exact sequence $0 \to Q^* \to E^* \to K^* \to 0$. Each hermitian metric h_s^* defines by restriction a hermitian metric q_s^* on Q^*; analogously we get $q^* := h^*_{|Q^*}$. Clearly (Q^*, Σ', q, q_s) is a s.h.m for an appropriate $\Sigma' \subseteq \Sigma$. For every s we have that $\Theta_{q_s^*}(Q^*) = \Theta_{h_s^*}(E^*)_{|Q^*} + i\beta_s^* \wedge \beta_s$, where β_s is a $(1,0)$-form with values in $\mathrm{Hom}(Q^*, K^*)$, $\mathcal{C}^1$ coefficients and β^* is its adjoint. Moreover $i\beta_s^* \wedge \beta_s \leq_1 0$; see [7], Lemme 6.6. The statement follows easily by dualizing again, which has the effect of transposing and changing the signs. □

More generally, a s.h.m. on E "with positivity" will induce s.h.m. "with positivity" on $T^\alpha E$, $S^\beta E$ and $\wedge^\gamma E$. We leave the various formulations and elementary proofs to the reader.

3 t-nef vector bundles

3.1 The definition of t-nefness

Let X be a compact manifold of dimension n, ω be a hermitian metric on X, E be a vector bundle of rank r on X, $N := \min(n, r)$ and L be a line bundle on X. Every tensor in $T_X \otimes E$ has rank $\rho \leq N$.

There are notions of semi-positivity associated with every positive integer t. The standard one is the following.

Definition 3.1.1 (t-semi-positive vector bundle) *We say that a vector bundle E is t-semi-positive, if E admits a hermitian metric h such that $\Theta_h(E) \geq_t 0$.*

Note that E is 1-semi-positive if and only if it is Griffiths-semi-positive, and that E is N-semi-positive if and only if it is Nakano-semi-positive. A similar remark holds for strict inequalities.

In algebraic geometry the most natural semi-positivity concept is *nefness*. A differential-geometric characterization of this concept can be given as follows.

Definition 3.1.2 (Nef line bundle and nef vector bundle) *We say that L is nef if for every $\epsilon > 0$ there exists a hermitian metric h_ϵ on L such that $\Theta_{h_\epsilon}(L) \geq -\epsilon\omega$ as $(1,1)$-forms or, equivalently, if $\Theta_{h_\epsilon}(L) \geq_1 -\epsilon\omega \otimes \mathrm{Id}_{L_{h_\epsilon}}$ as hermitian forms on $T_X \otimes L$.*
We say that E is nef if the tautological line bundle $\xi := \mathcal{O}_{\mathbb{P}(E)}(1)$ is nef.

Note that the compactness of Ximplies that the definitions given above are independent of the choice of ω. The same holds true for all the other definitions given below which involve a choice of ω.

If X is projective, then Definition 3.1.2 is equivalent to the usual one: *L is nef if $L \cdot C \geq 0$, for every integral curve C in X*; see [10], Proposition 6.2.

Unfortunately a nef line bundle is not necessarily 1-semi-positive ($\geq_1 0$). See [12], Example 1.7, where an example is given of a nef rank two vector bundle E on an elliptic curve such that the nef tautological line bundle ξ on $\mathbb{P}(E)$ is not $\geq_1 0$. Moreover, E is not $\geq_1 0$ (otherwise ξ would be $\geq_1 0$); this shows that even on curves nefness and Griffiths-semi-positivity do not coincide. Recall Theorem 1.12, [12], which states that nefness of a vector bundle E can be characterized by the presence of a system of hermitian metrics on all bundles $S^\alpha E$ such that they are suitably semi-positive for $\alpha \gg 0$. I do not know if nefness can be characterized in terms of hermitian metrics on the vector bundle itself.

The two facts above and the need to express semi-positivity in terms of curvature have motivated my introducing the notion of t-nefness.

Definition 3.1.3 (t-nef vector bundle) *We say that a vector bundle E is t-nef if for every $\epsilon > 0$ there exists a hermitian metric h_ϵ on E such that $\Theta_{h_\epsilon}(E) \geq_t -\epsilon\omega \otimes \mathrm{Id}_{E_{h_\epsilon}}$.*

Every flat vector bundle is N-nef.
If E is t-semi-positive, then E is t-nef. As pointed out above, the converse is not true in general; see [12], Example 1.7.
If E is t-semi-positive (t-nef, respectively), then E is t'-semi-positive (t'-nef, respectively), for every t' such that $1 \leq t' \leq t$.
By definition, a line bundle is nef if and only if it is 1-nef. A 1-nef vector bundle is nef as we will see in Proposition 3.2.4. The converse is true on curves as we will see in Theorem 3.3.1. We do not know whether the converse is true or false when $\dim X \geq 2$. This problem is the analogue of Griffiths' question: does ampleness imply Griffiths-positivity?
We have checked that if E is nef and is the tangent bundle of a compact complex surface or of a compact Kähler threefold, then E is 1-nef. This is done by using the classification results contained in [12] and Proposition 3.2.4. According to conjectures in [12], the same should be true for compact Kähler manifolds of arbitrary dimension.
(**From nefness to 1-nefness**) On special manifolds, such as toric and abelian varieties, we have that if E is nef, then $E \otimes \det E$ is 1-nef; see [21], §7.2.1. By the following paragraph, if E is a rank r nef vector bundle on such a variety, then $E \otimes (\det\, E)^{\otimes r+2}$ is N-nef.
(**From 1-nefness to N-nefness**) On any compact manifold, if E is 1-nef, then $E \otimes \det E$ is N-nef. See [6].

Example 3.1.4 (**Some N-nef vector bundles**) The results mentioned above and the ones of sections §3.2 and §3.3 give us the following list of examples.

1) A nef vector bundle over a curve is N-nef. A nef line bundle is t-nef for every t.
2) A flat vector bundle is N-nef.
3) If X is a special manifold such as a toric or an abelian variety and E is nef of rank r, then $E \otimes \det E$ is 1-nef and $E \otimes (\det E)^{\otimes r+2}$ is N-nef.
4) If X is a Kähler manifold of dimension $n \leq 3$ with nef tangent bundle T_X, then T_X is 1-nef and $K_X^{\otimes -1} \otimes T_X$ is N-nef.
5) Every Nakano-semipositive vector bundle is N-nef (the converse is not true). If E is a Griffiths-semipositive vector bundle, then $E \otimes \det E$ is N-nef.
6) The extension of two t-nef vector bundles is t-nef. Positive tensor representations of a t-nef vector bundle are t-nef. If E_1 and E_2 are t-nef, then $E_1 \otimes E_2$ is t-nef. If E is t-nef and L is a nef line bundle, then $E \otimes L$ is t-nef.
7) If E is 1-nef, then $E \otimes \det E$ is N-nef.
8) If $f : X \to Y$ is a morphism and E is a t-nef vector bundle on Y, then f^*E is t-nef. If $t = 1$ and, either f is finite and surjective, or X is projective and f has equidimensional fibers, then the converse is true.

At this point the following question is only natural.

Question 3.1.5 *Is every nef vector bundle 1-nef?*

3.2 Basic properties of t-nefness

Let us list and prove some basic properties of t-nef vector bundles. We start with functorial ones.

Proposition 3.2.1 *Let $f : X \to Y$ be a holomorphic map, where X and Y are compact manifolds and E is a vector bundle on Y.*

(1) *If E if t-nef, then f^*E is t-nef.*

(2) *Assume that f is surjective and that the rank of E is one. Then f^*E is* 1*-nef (= nef) if and only if E is* 1*-nef (= nef).*

(3) *Assume that f is finite and surjective, that Y (and thus X) is Kähler and let E be of any rank. Then f^*E is* 1*-nef if and only if E is* 1*-nef.*

Proof. (1). Let ω and ω' be two hermitian metrics on X and Y, respectively. Let A be a positive constant such that $A\omega \geq f^*\omega'$. Fix $\epsilon > 0$ and let $\epsilon' := \frac{\epsilon}{A}$. Let h' be a hermitian metric on E such that $\Theta_{h'}(E) \geq_t -\epsilon'\omega' \otimes \mathrm{Id}_{E_{h'}}$. Endow f^*E with the pull-back metric $h := f^*h'$. The claim follows from the formula $\Theta_h(f^*E) = f^*\Theta_{h'}(E)$.

(2). See [12], Proposition 1.8.ii for the case of equidimensional fibers and [23] for the general statement.

(3) It follows easily from [21], §7.1: assign to E the appropriate trace metrics and regularize. $\square$

Remark 3.2.2 As pointed out in Example 2.4.4, the regularizing metrics in (3) can be chosen to satisfy favorable conditions towards L^2-estimates.

Question 3.2.3 Can we drop the assumption of finiteness from (3). A. J. Sommese has pointed that the answer is positive when X is projective and the fibers are equidimensional: slice X with sufficiently ample general divisors to reduce to the case in which the morphism is finite. Is (3) true if we replace 1-nef by t-nef, with $t > 1$?

Proposition 3.2.4 *Let X, E and r be as above. Then:*

(1) *Let $E \to Q$ be a surjection of vector bundles. If E is* 1*-nef, then Q is* 1*-nef.*

(2) *If E is* 1*-nef, then E is nef.*

(3) *If S^mE is* 1*-nef, then E is nef.*

(4) *Let $0 \to K \to E \to Q \to 0$ be an exact sequence of vector bundles. If K and Q are t-nef, then E is t-nef.*

(5) *Let $E = E_1 \oplus E_2$. The vector bundle E is t-nef if and only if E_1 and E_2 are t-nef.*

(6) *Let F be another vector bundle. Assume that E and F are t-nef and t'-nef respectively; then $E \otimes F$ is $\min(t, t')$-nef.*

(7) *Assume that E is t-nef.*
Then S^mE and $\wedge^l E$ are t-nef for all $m \geq 0$ and for $0 \leq l \leq r$.
Moreover, $\Gamma^a E$ is t-nef, where $\Gamma^a E$ is the irreducible tensor representation of $Gl(E)$ of highest weight $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$, with $a_1 \geq \ldots a_r \geq 0$.

(8) *Let $0 \to E \to E' \to \tau \to 0$ be an exact sequence with E' a vector bundle on X and τ a sheaf, quotient of a* 1*-nef vector bundle E''. If E is* 1*-nef, then so is E'.*

(9) *Let $0 \to K \to E \to Q \to 0$ be an exact sequence of vector bundles. If E and $\det Q^*$ are* 1*-nef, then K is* 1*-nef.*

(10) *Assume that $\det E$ is hermitian flat; the vector bundle E is t-nef if and only if E^* is t-nef.*

(11) *Let E be* 1*-nef and $s \in \Gamma(E^*)$. Then s has no zeroes.*

Proof. Fix, once and for all, ω a hermitian metric on X.

(1). Let $\epsilon > 0$ and h_ϵ be a hermitian metric on E with $\Theta_{h_\epsilon}(E) \geq_1 -\epsilon\omega \otimes \mathrm{Id}_{E_{h_\epsilon}}$. Endow Q with the quotient metric h'_ϵ; Q can be seen as a smooth sub-bundle of E via the $\mathcal{C}^\infty$ orthogonal splitting of $E \to Q$ determined by h_ϵ, so that $h_{\epsilon|Q} = h'_\epsilon$. It is well-known (e.g. [7], Lemme 6.6) that $\Theta_{h'_\epsilon}(Q) \geq_1 \Theta_{h_\epsilon}(E)_{|Q_{h'_\epsilon}}$ and it is clear that $\mathrm{Id}_{E_{h_\epsilon}|Q} = \mathrm{Id}_{Q_{h'_\epsilon}}$. The claim follows.

(2). Let $\pi : \mathbb{P}(E) \to X$ be the canonical projection. By virtue of 3.2.1.1 we have that π^*E is 1-nef; (1) and the canonical surjection $\pi^*E \to \mathcal{O}_{\mathbb{P}(E)}(1)$ imply that this latter line bundle is 1-nef.

(3) π^*S^mE is 1-nef by 3.2.1.1, so that $\mathcal{O}_{\mathbb{P}(E)}(m)$, being a quotient of π^*S^mE, is 1-nef, by (1). It follows that $\mathcal{O}_{\mathbb{P}(E)}(1)$ is 1-nef and thus nef.

(4). Fix a $\mathcal{C}^\infty$ vector bundle isomorphism $\Phi : E \to K \oplus Q$. Let and $\epsilon > 0$. By assumption, there are metrics $h_{K,\epsilon}$ and $h_{Q,\epsilon}$ such that $\Theta_{h_{K,\epsilon}}(K) \geq_t -\frac{\epsilon}{3}\omega \otimes \mathrm{Id}_{K_{h_{K,\epsilon}}}$ and $\Theta_{h_{Q,\epsilon}}(Q) \geq_t -\frac{\epsilon}{3}\omega \otimes \mathrm{Id}_{Q_{h_{Q,\epsilon}}}$. Fix an arbitrary positive real number $\rho > 0$ and consider the automorphism $\phi_\rho : Q \to Q$ defined by multiplication by the factor ρ^{-1}.
Let $\Phi_\rho := (\mathrm{Id}_K \oplus \phi_\rho) \circ \Phi : E \to K \oplus Q$; denote the first component of Φ_ρ by $\Phi_{K,\rho}$ and the second one by $\Phi_{Q,\rho}$
Define a hermitian metric on E by setting $h_{\epsilon,\rho} := \Phi^*_{K,\rho}h_{K,\epsilon} \oplus \Phi^*_{Q,\rho}h_{Q,\epsilon}$. Its associated Chern connection has the form:

$$D_{h_{\epsilon,\rho}} = \begin{pmatrix} D_{h_{K,\epsilon}} & -\beta^*_\rho \\ \beta_\rho & D_{h_{Q,\epsilon}} \end{pmatrix},$$

where $\beta_\rho = \rho\beta_1$ is a $(1,0)$-form with values in $\mathrm{Hom}(K, Q)$. By calculating D^2 we see that

$$\Theta_{h_{\epsilon,\rho}}(E) \geq_t -\frac{2}{3}\epsilon\omega \otimes \mathrm{Id}_{E_{h_{\epsilon,\rho}}} + O(\rho)\omega \otimes \mathrm{Id}_{E_{h_{\epsilon,\rho}}}.$$

The claim follows by recalling that X is compact and by taking ρ sufficiently small.

(5). The "if" part follows from (4). The converse follows by observing that if E has a metric h, then each E_i inherits a metric h_i for which $D_{h_i} = D_{h|E_i}$. The same holds for the curvature tensors.

(6). The proof is immediate once one recalls the formula for the curvature of the tensor product of two hermitian metrics: $\Theta_{h_1 \otimes h_2}(E_1 \otimes E_2) = \Theta_{h_1}(E_1) \otimes \mathrm{Id}_{E_2} + \mathrm{Id}_{E_1} \otimes \Theta_{h_2}(E_2)$.

(7). The tensor powers $T^n(E)$ are t-nef by virtue of (6). $S^n(E)$ and $\wedge^n(E)$ are both direct summands of $T^n(E)$ so that they are t-nef by (5). Recall that Γ^aE is a direct summand of the vector bundle $\otimes_{i=1}^r S^{a_i}(\wedge^iE)$ which is t-nef by what above, (5) and (6).

(8). The "pull-back" construction gives the following commutative diagram of coherent sheaves:

$$\begin{array}{ccccccccc} 0 & \to & E & \to & E' & \to & \tau & \to & 0 \\ & & \uparrow \mathrm{Id}_E & & \uparrow q & & \uparrow & & \\ 0 & \to & E & \to & E''' & \to & E'' & \to & 0, \end{array}$$

where q is surjective. Since E and E'' are locally free and 1-nef, so is E''' by (4). Since q is surjective, it follows that E' is 1-nef by (1).
The proofs of (9) and (10) are the same as in the nef case; the proof of (11) is in fact easier. The reader can consult [12]. □

Remark 3.2.5 It is easy to show, using (6), that if E is t-nef and L is a positive line bundle, then $E \otimes L$ admits a hermitian metric h with curvature $\Theta_h(E \otimes L) >_t 0$. In particular, if E is N-nef, then $E \otimes L$ is Nakano-positive. A similar remark holds for the symbol $\geq^\mu_t$; see Lemma 4.2.1.

Remark 3.2.6 As far as (1) above is concerned, it is not true that if E is t-nef, then Q is t-nef. In fact, consider the canonical surjection $\mathcal{O}^3_{\mathbb{P}^2} \to T_{\mathbb{P}^2}(-1)$: $\mathcal{O}^3_{\mathbb{P}^2}$ is 2-nef, but if $T_{\mathbb{P}^2}(-1)$ were 2-nef, then $T_{\mathbb{P}^2} = T_{\mathbb{P}^2}(-1) \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ would be $>_2 0$, i.e. Nakano-positive and this is a contradiction. This example also shows that 1-nefness is strictly weaker than 2-nefness. We do not know whether (8) is false when we replace 1 by t.

3.3 Nefness and t-nefness on curves

It is an outstanding problem in Hermitian differential geometry to determine whether an ample vector bundle is Griffiths-positive. In [29], Umemura proves that on curves ampleness and Griffiths-positivity coincide. As is was pointed out to me by N.M. Kumar, the part of the argument that needs a result analogue to Proposition 3.2.4.8 is omitted in [29].

We now prove that on curves nefness and 1-nefness coincide: the algebraic notion of nefness can be characterized in differential-geometric terms. Recall that Example 1.7, [12], implies that even over a curve, a nef vector bundle is not necessarily Griffiths-semi-positive.

Theorem 3.3.1 *Let X be a nonsingular projective curve and E be a vector bundle of rank r on X. The following are equivalent.*

(i) E is 1-nef;

(ii) E is nef;

(iii) every quotient bundle of E, and in particular E, has non-negative degree.

Proof. $(i) \Rightarrow (ii)$. This is Proposition 3.2.4.2.

$(ii) \Rightarrow (iii)$. In fact they are equivalent by [4], Proposition 1.2.7.

$(iii) \Rightarrow (i)$. We divide the proof in three cases, according to whether $g = 0$, $g = 1$ or $g \geq 2$. Let d be the degree of E. By assumption $d \geq 0$.

If the genus $g(X) = 0$, then E splits into a direct sum of line bundles and the statement follows easily.

Let $g(X) = 1$. It is enough to consider the case when E is indecomposable. Let us first assume that $d \geq r$. By [2], Lemma 11, E admits a maximal splitting $(L_1, \ldots, L_r)$ with L_i ample line bundles on X. It follows that E could then be constructed inductively from (ample $=$) positive line bundles by means of extensions. A repeated use of Proposition 3.2.4.4 would allow us to conclude. We may thus assume, without loss of generality, that $0 \leq d < r$. If $r = 1$, then E is either ample or hermitian flat; in both cases we are done. We now proceed by induction on the rank of E. Assume that we have proved our contention for every vector bundle of rank strictly less than r. By [2], Lemma 15 and Theorem 5, E sits in the middle of an exact sequence:

$$0 \to A \to E \to B \to 0,$$

where B, being a quotient of E, enjoys property (iii) and A is either a trivial vector bundle (if $d > 0$) or a hermitian flat line bundle (if $d = 0$). In any case A is clearly 1-nef and B is 1-nef by the induction hypothesis. We can apply 3.2.4.4 and conclude that E is 1-nef. This proves the case $g(X) = 1$.

We now assume that $g(X) \geq 2$. The proof will be by induction on r. If $r = 1$, then we are done since $\deg E \geq 0$ implies that either E is ample or it is hermitian flat. Assume that we have proved our assertion for every vector bundle of rank strictly less than r.

There are two cases.

In the first one we suppose that E contains a non-trivial vector sub-bundle K which is 1-nef. Consider the exact sequence of coherent shaves:

$$0 \to K \to E \to Q := E/K \to 0.$$

There are two sub-cases. In the first one we assume that Q is locally free. By assumption every quotient vector bundle of Q, being in turn a quotient bundle of E, has positive degree. The induction hypothesis forces Q to be 1-nef. Proposition 3.2.4.4 allows us to conclude that E is 1-nef as well.
In the second sub-case $Q \simeq F \oplus \tau$, with F locally free and τ has zero-dimensional support; in particular there is a surjection $\mathcal{O}_X^m \to \tau$. If K' is the kernel of the surjection $E \to F$, then we have the exact sequence

$$0 \to K \to K' \to \tau \to 0,$$

so that, by 3.2.4.8, K' is 1-nef and we are reduced to the first sub-case.
In the second case we are allowed to assume that E does not contain properly any non-trivial vector bundle K which is 1-nef.

CLAIM. E is stable.
To prove this we start a new proof by induction. Let K be a vector bundle contained in E, neither trivial nor equal to E. Since (iii) implies that $\deg(E) \geq 0$, to prove that E is stable it is enough to show that $\deg K < 0$. Seeking a contradiction, let us assume that $\deg K \geq 0$. Let s be the rank of K. If $s = 1$, since K is not 1-nef by the working assumption of this second case, we see that $\deg K < 0$ (otherwise K would be either ample or hermitian flat) and we have reached a contradiction if $s = 1$. Assume that, for every non-trivial $K'' \subseteq E$ with rank strictly less than s, $\deg K'' < 0$. Each sub-bundle of K has, by this second inductive hypothesis, negative degree. Since we are assuming that $\deg K \geq 0$, it follows that every quotient of K, including K itself has non-negative degree, so that, by the first induction hypothesis, K is 1-nef and we have reached a contradiction for every s: the degree of K must be negative, E is stable and the claim is proved.
Since $\det E$ has non-negative degree by assumption, $\det E$ is 1-nef. Since E is stable, for any hermitian metric h on $\det E$ of curvature $\Theta_h(\det E)$, a standard calculation (see [29], Lemma 2.3) yields a hermitian metric H on E of curvature $\frac{1}{r}\Theta_h(\det E) \otimes \mathrm{Id}_E$. This proves that E is 1-nef also in the second case. □

3.4 A differential-geometric characterization of ampleness for vector bundles

We now prove a characterization of ampleness by means of curvature properties which is a simple consequence of [12], Theorem 1.12.

Proposition 3.4.1 *Let X be a compact manifold equipped with a hermitian metric ω and E be a vector bundle on X. Then E is ample if and only if there exists a sequence of hermitian metrics h_m on S^mE such that*

(i) *the sequence of metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by the surjective morphisms*

$$\pi^* S^m E \to \mathcal{O}_{\mathbb{P}(E)}(m)$$

converges uniformly to a hermitian metric h of positive curvature on $\mathcal{O}_{\mathbb{P}(E)}(1)$ and

(ii) *there exist $\eta > 0$ and $m_0 \in \mathbb{N}$ such that $\forall m \geq m_0$:*

$$\Theta_{h_m}(S^m(E)) \geq_1 m\eta\omega \otimes \mathrm{Id}_{S^m E_{h_m}}.$$

Remark 3.4.2 If E is ample, then the fact that some metrics h_m with property (ii) exist for all $m \gg 0$ is a well known consequence of [16], theorems F and A. The point made by the statement above is that the metrics h_m are constructed on *all* symmetric powers $S^m(E)$, and that they are all built starting from a suitable metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$; see [8], Theorem 4.1.

Proof. The proof of the implication "$\Leftarrow$" follows easily from (i): $\mathcal{O}_{\mathbb{P}(E)}(1)$ is positive by the existence of h so that E is ample.
For the reverse implication "$\Rightarrow$" we argue as follows. Fix a hermitian metric ω' on $\mathbb{P}(E)$. The ampleness of E implies the ampleness of $\mathcal{O}_{\mathbb{P}(E)}(1)$, which then admits a hermitian metric h of positive curvature; the compactness of X ensures us that there exist $\alpha > 0$ and $A > 0$ such that

$$\Theta_h(\mathcal{O}_{\mathbb{P}(E)}(1)) \geq \alpha\omega' \geq \alpha A \pi^* \omega.$$

Define $\eta := \frac{2}{3}\alpha A$. We are now in the position of using [8], Theorem 4.1 with $v := \frac{3}{2}\eta\omega$ and $\epsilon := \frac{1}{2}\eta$. □

4 Vanishing theorems

In this section we link the positivity of h to the vanishing of the cohomology of $K_X \otimes \mathcal{E}(h)$.

4.1 The basic L^2-estimate and vanishing theorem, and the coherence of $\mathcal{E}(h)$

Following Demailly, [7], §5, we say that a s.h.m. (X, E, Σ, h, h_s) is *t-approximable* if $\Theta_h(E) \geq_t^\mu 0$ (cf. Definition 2.4.1). We denote the space of (p,q)-forms with values in E and coefficients which are locally square-integrable by $L^2_{p,q}(X, E, \mathrm{loc})$. As usual, $n = \dim X$, r is the rank of E and $N := \min(n, r)$.

Proposition 4.1.1 (See [7], Théorème 5.1) *Let (X, ω) be Kähler, where either ω is complete or X is weakly pseudoconvex. Assume that (E, h) is a s.h.m. with the property that $\Theta_h(E) \geq^\mu_{n-q+1} \epsilon\omega \otimes \mathrm{Id}_{E_h}$, where ϵ is a non-negative and continuous function on X and $q > 0$ is a positive integer.*
Let $g \in L^2_{n,q}(X, E, \mathrm{loc})$ be such that

$$d''g = 0, \qquad \int_X |g|^2_h \, dV_\omega < +\infty \qquad \text{and} \qquad \int_X \frac{1}{\epsilon}|g|^2_h \, dV_\omega < +\infty.$$

Then there exists $f \in L^2_{n,q-1}(X, E, \mathrm{loc})$ such that

$$d''f = g \qquad \text{and} \qquad \int_X |f|^2_h dV \leq \frac{1}{q}\int_X \frac{1}{\epsilon}|g|^2_h \, dV_\omega.$$

Sketch of proof. Théorème 5.1 states something slightly different but it is immediate to recover the statement of the proposition. We merely point out, for the reader's convenience, the minor changes to be implemented to obtain the above statement. The notation is from [7].
The assumption $\Theta_h(E) \geq^\mu_{n-q+1} \epsilon\omega \otimes \mathrm{Id}_{E_h}$ has two consequences. The former is that h is $n-q+1$-approximable. The latter is that, by virtue of [7], Lemme 3.2 (3.4):

$$|g|^2_{\Theta_h(E)} \leq \frac{1}{q\epsilon}|g|^2_h \quad a.e.$$

We can apply the aforementioned theorem and conclude. □

The following generalizes Nadel Vanishing Theorem. It is an easy consequence of the proposition above.

Theorem 4.1.2 *Let (X,ω) be Kähler with X weakly pseudoconvex. Assume that (E,h) is a s.h.m. such that $\Theta_h(E) \geq_N^\mu \epsilon\omega \otimes \mathrm{Id}_{E_h}$ for some positive and continuous function ϵ. Then, $H^q(X, K_X \otimes \mathcal{E}(h)) = 0$, $\forall q > 0$.*

Proof. The complex $(\mathfrak{L}^\bullet, d'')$ of §2.3 is exact by Proposition 4.1.1 applied to small balls. This complex is therefore an acyclic resolution of $K_X \otimes \mathcal{E}(h)$ whose cohomology is isomorphic to the cohomology of the complex of global sections of $(\mathfrak{L}^\bullet, d'')$. This latter cohomology is trivial for every positive value of q by Proposition 4.1.1 (modify the metric as in [10], Proposition 5.11). □

We now prove that if h is suitably positive, then $\mathcal{E}(h)$ is coherent. The line bundle case is due to Nadel.

Proposition 4.1.3 *Let X be a complex manifold, (X, E, Σ, h, h_s) be a s.h.m., and θ be a continuous real $(1,1)$-form on X such that $\Theta_h(E) \geq_N^\mu \theta \otimes \mathrm{Id}_{E_h}$. Then $\mathcal{E}(h)$ is coherent.*

Proof. We make the necessary changes from the line bundle case (cf. [10], Proposition 5.7).
Note that the condition $\Theta_h(E) \geq_N^\mu \theta \otimes \mathrm{Id}_{E_h}$ implies that $h \geq h_1$ a.e.
The statement being local, we may assume that X is a ball centered about the origin in $\mathbb{C}^n$ with holomorphic coordinates (z), that E is trivial and that θ has bounded coefficients. Let ω be the $(1,1)$-form associated with the euclidean metric on X. Let $\mathfrak{S}$ be the vector space of holomorphic sections f of E such that $\int_X |f|_h^2\, d\lambda < \infty$, where $d\lambda$ is the Lebesgue measure on $\mathbb{C}^n$. Consider the natural evaluation map $ev : \mathfrak{S} \otimes_{\mathbb{C}} \mathcal{O}_X \to E$. The sheaf $\mathfrak{E} := Im(ev)$ is coherent by Noether Lemma (cf. [15], page 111) and it is contained in $\mathcal{E}(h)$.
We want to prove that $\mathcal{E}(h)_x = \mathfrak{E}_x$ for all $x \in X$. In view of Nakayama's Lemma, [3], Corollary 2.7, it is enough to show that $\mathfrak{E}_x + \mathfrak{m}_x^\gamma \cdot \mathcal{E}(h)_x = \mathcal{E}(h)_x$ for some $\gamma \geq 1$.

STEP I. Assume that we could prove that:

(•) $\qquad \mathfrak{E}_x + \mathcal{E}(h)_x \cap \mathfrak{m}_x^l \cdot E_x = \mathcal{E}(h)_x$ for every positive integer l.

By the Artin-Rees Lemma, [3], Corollary 10.10, there would be a positive integer $k = k(x)$ such that

$$\mathcal{E}(h)_x = \mathfrak{E}_x + \mathcal{E}(h)_x \cap \mathfrak{m}_x^l \cdot E_x \subseteq \mathfrak{E}_x + \mathfrak{m}_x^{l-k} \cdot \mathcal{E}(h)_x \subseteq \mathfrak{E}_x + \mathfrak{m}_x \cdot \mathcal{E}(h)_x \subseteq \mathcal{E}(h)_x$$

for all $l \geq k$. All symbols "$\subseteq$" could be replaced by equalities and we could conclude that $\mathfrak{E}_x = \mathcal{E}(h)_x$ by Nakayama's Lemma as above.

STEP II. We now prove (•).
Let f be a germ in $\mathcal{E}(h)_x$ and σ be a smooth cut-off function such that is identically 1 around x and that has compact support small enough so that σf is smooth on X.
For every positive integer l define a strictly psh function $\varphi_l := (n+l)\ln|z-x| + C|z|^2$ where C is a positive constant chosen so that $2id'd''(C|z|^2) + \theta \geq \epsilon\omega$, for some positive constant ϵ.
Define a metric on E by setting $H_l := he^{-2\varphi_l}$. Since both $\ln|z-x|$ and $|z|^2$ are psh, we can apply the results of [7], §9 to φ and deduce, with the aid of Lemma 2.4.7, that H_l is a s.h.m. on E with $\Theta_{H_l}(E) = \Theta_h(E) + 2id'd''\varphi_l \otimes \mathrm{Id}_E$ and such that $\Theta_{H_l}(E) \geq_N^\mu \epsilon\omega \otimes \mathrm{Id}_{E_{H_l}}$.
Consider the smooth $(0,1)$- form $g := d''(\sigma f)$ which has compact support and is identically zero around x. The function $|z-x|^{-2n-2l}$ is continuous outside x. It follows that:

$$\int_X |g|_{H_l}^2\, d\lambda = \int_X |g|_h^2\, |z-x|^{-2n-2l}\, e^{-2C|z|^2}\, d\lambda < \infty.$$

We solve, for every index l, the equation $d''u = g$ with L^2-estimates relative to H_l using Proposition 4.1.1. We obtain a set of solutions u_l such that

$$\int_X |u_l|_{H_l}^2\, d\lambda = \int_X |u_l|_h^2\, |z-x|^{-2n-2l}\, e^{-2C|z|^2}\, d\lambda < \infty.$$

Since the factor $e^{-2C|z|^2}$ does not affect integrability we get that

$$\int_X |u_l|_h^2 \, |z-x|^{-2n-2l} \, d\lambda < \infty.$$

Since $d''(\sigma f - u_l) = 0$ and $h \leq H_l$, we see that $\sigma f - u_l =: F_l \in \mathfrak{E}$ (cf. [17] page 380). The germ $u_{l,x} = f - F_{l,x}$ is holomorphic. Since $h \geq h_1$ and h_1 is continuous, there is a positive constant B such that:

$$\int_X B \, |u_l|^2 \, |z-x|^{-2n-2l} \, d\lambda \leq \int_X |u_l|_h^2 \, |z-x|^{-2n-2l} \, d\lambda < \infty.$$

Let $u_l^{\{j\}}$ be the j-th coordinate function of u_l, $j = 1, \ldots, r$. By a use of Parseval's formula (cf. [10], 5.6.b) we see that $u_l^{\{j\}} \in \mathfrak{m}_x^l$ for every index j. It follows that $(\bullet)$ holds and we are done. □

4.2 t-nefness and vanishing

We now show how to use Theorem 4.1.2 to infer the vanishing of cohomology in the case of a N-nef vector bundle twisted by a line bundle which can be endowed with a positive s.h.m.

The following is an elementary consequence of Lemma 2.4.7:

Lemma 4.2.1 *Let E be a t-nef vector bundle on a compact manifold X, ω be a hermitian metric on X, θ be a real $(1,1)$-form with continuous coefficients and (F, g, g_s) be a vector bundle endowed with a s.h.m. such that $\Theta_g(F) \geq_t^\mu \theta \otimes \mathrm{Id}_{F_g}$.*
Then for every constant $\eta > 0$ there is a s.h.m. H_η on $E \otimes F$ for which:

$$\Theta_{H_\eta}(E \otimes F) \geq_t^\mu (\theta - \eta\omega) \otimes \mathrm{Id}_{(E\otimes F)_{H_\eta}}.$$

Moreover, if F is a line bundle and $\Theta_g(F) \geq \theta$ as $(1,1)$-forms, then the same conclusion holds.

Lemma 4.2.2 *Let (F, h_F) be a hermitian vector bundle on a manifold X and (L, h_L) be a line bundle on X endowed with a singular metric h_L as in* §2.2. *Consider the vector bundle $E := F \otimes L$ endowed with the measurable metric $h := h_F \otimes h_L$.*
Then $\mathcal{E}(h) = \mathcal{I}(h_L) \otimes E$ and $\mathcal{E}(h)$ is coherent.

Proof. The statement $\mathcal{E}(h) = \mathcal{I}(h_L) \otimes E$ is local on X so that we may assume that X is a ball in $\mathbb{C}^n$, that F and L are trivial, that $h_L = e^{-2\varphi}$ with φ almost psh and that h_F has bounded coefficients.
Let us first prove that $\mathcal{I}(h_L)_x \otimes E_x \subseteq \mathcal{E}(h)_x$. Let $e_x \in E_x$ and $f_x \in \mathcal{I}(h_L)_x$. Since h_F is continuous, we have that $|f_x e_x|_h^2 = |f_x|^2 |e_x|_{h_F}^2 e^{-2\varphi}$ is locally integrable.
Let us prove the reverse inclusion $\mathcal{E}(h)_x \subseteq \mathcal{I}(h_L)_x \otimes E_x$ for every x in X. There exists a constant $\tilde\epsilon > 0$ such that $h_F \geq \tilde\epsilon \Delta$, where Δ is the standard euclidean metric on the fibers of E. Fix $x \in X$. Assume that $\mathcal{E}(h)_x \ni e_x =< f_1, \ldots, f_r >$. Then $|e_x|_h^2 = |e_x|_{h_F}^2 e^{-2\phi} \geq \tilde\epsilon \sum |f_i|^2 e^{-2\varphi}$. As the left hand side of the inequality is integrable around x, so is each summand on the right. This proves the reverse inclusion. To conclude recall that $\mathcal{I}(h_L)$ is coherent (or apply Proposition 4.1.3). □

The following result is the key to the proofs of the effective statements to be found in §5. See Ex. 3.1.4 for examples of N-nef vector bundles.

Theorem 4.2.3 *Let (X, ω) be as in* Theorem 4.1.2, *and (F, h_F), (L, h_L) and (E, h) be as in* Lemma 4.2.2.

If $\Theta_h(E) \geq^{\mu}_N \epsilon\omega \otimes \mathrm{Id}_E$ *for some positive and continuous function* ϵ*, then*

$$H^q(X, K_X \otimes F \otimes L \otimes \mathcal{I}(h_L)) = H^q(X, K_X \otimes \mathcal{E}(h)) = 0, \qquad \forall\, q > 0.$$

Moreover, if X *is compact,* F *is* N*-nef and* (L,h) *is such that* $\Theta_h(L) \geq \epsilon\omega$*, for some positive constant* ϵ*, then the same conclusion holds.*

Proof. By Lemma 4.2.2, we have that $\mathcal{E}(h) = \mathcal{I}(h_L) \otimes E$. We conclude in view of Theorem 4.1.2. The case of X compact is a special case after Lemma 4.2.1. □

The following is not needed in the sequel. We include it since it is a generalization of Kawamata-Vieheweg Vanishing Theorem (K-V) and it can be proved along the lines of [10], 6.12 by using Theorem 4.2.3 instead of Nadel Vanishing Theorem. The "1-nef" case follows easily from K-V Theorem and the Leray spectral sequence by looking at the projectivization of E. The statement in the "N-nef" case seems new for $0 < q < \operatorname{rank} E$ and the vanishing in the complementary range follows from K-V and Le-Potier spectral sequence. See [10] for the particular language employed in the statement below.

Theorem 4.2.4 *Let* (X, E, F) *be the datum of:* X *a projective manifold,* E *a* N*-nef vector bundle on* X*,* F *a line bundle on* X *such that some positive multiple* mF *can be written as* $mF = L + D$*, where* L *is a nef line bundle and* D *is an effective divisor. Then*

$$H^q(X, K_X \otimes E \otimes F \otimes \mathcal{I}(\frac{1}{m}D)) = 0 \qquad \text{for } q > \dim X - \nu(L),$$

where $\nu(L)$ *is the numerical dimension of* L *and* $\mathcal{I}(\frac{1}{m}D)$ *is the multiplier ideal of the singular local weights associated with the* $m-$*roots of the absolute values of local equations for* D*.*
As a special case, we have that if F *is a nef line bundle, then*

$$H^q(X, K_X \otimes E \otimes F) = 0 \qquad \text{for } q > \dim X - \nu(F);$$

in particular, if F *is nef and big, then*

$$H^q(X, K_X \otimes E \otimes F) = 0 \qquad \text{for } q > 0.$$

5 Effective results

5.1 Special s.h.m. on line bundles after Anghern-Siu, Demailly, Siu and Tsuji

The following proposition is at the heart of the effective base-point-freeness, point-separation and jet-separation results in [1], [28], [26], [24] and [11]; it provides us with the necessary s.h.m. which we transplant to the vector bundle case and use in connection with Theorem 4.2.3.

First we need to fix some notation.

Let F be a rank r vector bundle on a complex manifold X and p be any positive integer. We say that *the global sections of* F *generate simultaneous jets of order* $s_1, \ldots, s_p \in \mathbb{N}$ *at arbitrary* p *distinct points of* X if the natural maps

$$H^0(X, F) \to \bigoplus_{\imath=1}^{p} \mathcal{O}(F)_{x_i} \otimes \mathcal{O}_X/\mathfrak{m}_{x_i}^{s_i+1}$$

are surjective for every choice of p distinct points $x_1, \ldots, x_p$ in X.

We say that *the global sections of F separate arbitrary p distinct points of X* if the above holds with all $s_i = 0$.

Assume that X is compact. Let $V := H^0(X,F)$ and $h^0 := h^0(X,F) := \dim_{\mathbb{C}} H^0(X,F)$. Consider $G := G(r,h^0)$ the Grassmannian of r-dimensional quotients of V, $\mathfrak{Q}$ the universal quotient bundle of G and $\mathfrak{q}$ the determinant of $\mathfrak{Q}$.
As soon as F is generated by its global sections (which corresponds to the above conditions being met for $p=1$ and $s_1=0$), we get a morphism $f : X \to G$ assigning to each $x \in X$ the quotient $F_x \otimes k(x)$ and such that $F \simeq f^*\mathfrak{Q}$. The Plücker embedding defined by $\mathfrak{q}$ gives a closed embedding into the appropriate projective space $\iota : G \to \mathbb{P}$. We obtain a closed embedding $\hat{f} := \iota \circ f : X \to \mathbb{P}$. It is clear that:
- V separates arbitrary 2 points of X iff f is bijective birational onto its image;
- If V separates arbitrary pairs of points of X and generates jets of order 1 at an arbitrary point of X, then f is a closed embedding.

Given n, p and $\{s_1,\dots,s_p\}$ as above let us define the following integers:

$$m_1(n,p) := \frac{1}{2}(n^2 + 2pn - n + 2),$$

$$m_2(n,p;s_1,\dots,s_p) = 2n\sum_{i=1}^{p} B(3n+2s_i-3,n) + 2pn + 1,$$

where $B(a,b)$ denotes the usual binomial coefficient,

$$m_3(n,p;s_1,\dots,s_p) = (pn + \sum_{i=1}^{p} s_i)\, m_1(n,1)$$

and

$$m_4(n) = (n+1)\, m_1(n,1).$$

Proposition 5.1.1 *Let X be a projective manifold of dimension n and L be an ample line bundle on X. Fix a Kähler form ω on X.*

(5.1.1.1) (Cf. [1] and [28].) *Let p be a positive integer. Assume that $m \geq m_1(n,p)$.*
Then for any set of p distinct points $\{x_1,\dots x_p\}$ of X, there exists a nonempty subset $J_0 \subset \{1,\dots,p\}$ with the following property:
there exist $\epsilon > 0$, a s.h.m. h for mL with $\Theta_h(mL) \geq_1^{\mu} \epsilon\omega \otimes \mathrm{Id}_{L_h}$ and with the property that the multiplier ideal $\mathcal{I}(h)$ of h is such that the closed subscheme given by $\mathcal{I}(h)$ has the points x_i as isolated points $\forall i \in J_0$ and contains all the points $\{x_i\}$.

(5.1.1.2) (Cf. [24]; see also [11].) *Fix a positive integer p and a sequence of non-negative integers $\{s_1,\dots,s_p\}$. Assume that $m \geq m_2(n,p;s_1,\dots,s_p)$.*
Then for any set of p distinct points $\{x_1,\dots x_p\}$ of X there exist $\epsilon > 0$, a s.h.m. h for $K_X + mL$ with $\Theta_h(K_X+mL) \geq_1^{\mu} \epsilon\omega \otimes \mathrm{Id}_{L_h}$ and with the property that the multiplier ideal $\mathcal{I}(h)$ satisfies $\mathcal{I}(h)_{x_i} \subseteq \mathfrak{m}_{x_i}^{s_i+1}$, for every $1 \leq i \leq p$, and is such that the closed subscheme given by $\mathcal{I}(h)$ has all the points x_i as isolated points.

The easy lemma that follows is probably well-known and makes precise a well-understood principle: *it is easy to go from global generation to the generation of higher jets.* Though the presence of the nef line bundle M is redundant in the statement, we use it because of the application of this lemma to the case of higher rank.

Lemma 5.1.2 (From freeness to the generation of jets) *Let X, n, p and $\{s_1, \ldots, s_p\}$ be as above, F, A and M be line bundles on X such that F is ample and generated by its global sections, A is ample and M is nef. Then the global sections of $K_X + (pn + \sum_{i=1}^p s_i)F + A + M$ generate simultaneous jets of order $s_1, \ldots, s_p$ at arbitrary distinct points $x_1, \ldots, x_p$ of X.*
Moreover, $K_X + (n+1)F + A + M$ is very ample.

Proof. Fix ω a hermitian metric on X and g a hermitian metric on A with positive curvature $\Theta_h(A) \geq \frac{3}{2}\epsilon\omega$ for some $\epsilon > 0$.
Since F is ample and the linear system $|F|$ is free of base-points, for every index i there are n sections $\{\sigma_{ij}\}_{j=1}^n$ of F such that their common zero locus is zero-dimensional at x_i.
Define a s.h.m. h on $(pn + \sum_i s_i)F$ by first defining metrics h_i on $(n+s_i)F$:

$$h_i^{-1} := \left[\sum_{j=1}^{n} |\sigma_{ij}^2|\right]^{n+s_i}$$

and then by multiplying them together

$$h := \prod_{i=1}^{p} h_i.$$

Since M is nef, one can choose a hermitian metric l on it such that $\Theta_l(M) \geq -\frac{1}{2}\epsilon\omega$.
Define a metric H on $(pn + \sum_{i=1}^p s_i)F + A + M$ by setting

$$H := h \otimes g \otimes l.$$

We have that $\Theta_{h_i}((n+s_i)F) \geq 0$, $\forall i$ so that $\Theta_h((np + \sum s_i)F) \geq 0$. It follows that $\Theta_H \geq \epsilon\omega$.
Since g and l are continuous, $\mathcal{I}(H) = \mathcal{I}(h)$.
By virtue of [10], Lemma 5.6.b, we have that $\mathcal{I}(H)_{x_i} = \mathcal{I}(h)_{x_i} \subseteq \mathfrak{m}_{x_i}^{s_i+1}$, $\forall i$ and that the scheme associated with $\mathcal{I}(H)$ is zero-dimensional at all the points x_i. We conclude by Proposition 2.2.2.
The second part of the statement is [1], Lemma 11.1 (the proof of which contains minor inaccuracies but it is correct). □

5.2 Effective results on vector bundles

We now see how to "transplant" the metrics of Proposition 5.1.1 to vector bundles and how to use the results of §4 to prove effective results for the vector bundles of the form $\mathfrak{P}$ as in the Introduction.

Let us remark that the lower bounds on m given in the various statements of the theorem that follows are only indicative. Any improvement of these bounds in the line bundle case that can be obtained using strictly positive singular metrics would give an analogous improvement in the vector bundle case; see [24], Proposition 5.1 for example.

Let n, p, $\{s_1, \ldots, s_p\}$ and the various m_i be as in section §5.1. Assume that E is a rank r vector bundle on X and let $N := \min\{n, r\}$.

Remark 5.2.1 See Ex. 3.1.4 for examples of N-nef vector bundles.

Theorem 5.2.2 *Let X be a projective manifold of dimension n, E be N-nef, A and L be ample line bundles on X.*

(5.2.2.1) *If $m \geq m_1(n,p)$, then the global sections of $K_X \otimes E \otimes (mL)$ separate arbitrary p distinct points of X;*

(5.2.2.1′) *if* $m \geq \frac{1}{2}(n^2+n+2)$*, then* $K_X \otimes E \otimes (mL)$ *is generated by its global sections;*

(5.2.2.2) *if* $m \geq m_2(n,p;s_1,\dots,s_p)$*, then the global sections of* $2K_X \otimes E \otimes (mL)$ *generate simultaneous jets of order* $s_1,\dots,s_p \in \mathbb{N}$ *at arbitrary* p *distinct points of* X*;*

(5.2.2.2′) *if* $m \geq m_2(n,1;1)$*, then the global sections of* $2K_X \otimes E \otimes (mL)$ *separate arbitrary pair of points of* X *and generate jets of order* 1 *at an arbitrary point of* X*.*

(5.2.2.3) *if* $m \geq m_3(n,p;s_1,\dots,s_p)$*, then the global sections of* $(pn+\sum s_i+1)K_X \otimes E \otimes (mL) \otimes A$ *generate simultaneous jets of order* $s_1,\dots,s_p \in \mathbb{N}$ *at arbitrary* p *distinct points of* X*;*

(5.2.2.4) *if* $m \geq m_4(n)$*, then the global sections of* $(n+2)K_X \otimes E \otimes (mL) \otimes A$ *separate arbitrary pair of points of* X *and generate jets of order* 1 *at an arbitrary point of* X*;*

(5.2.2.5) *the global sections of* $E \otimes (mL)$ *separate arbitrary pair of points of* X *and generate jets of order* 1 *at an arbitrary point of* X *as soon as*

$$m \geq C_n (L^n)^{3^{(n-2)}} (n+2+\frac{L^{n-1}\cdot K_X}{L^n})^{3^{(n-2)}(\frac{n}{2}+\frac{3}{4})+\frac{1}{4}},$$

where $C_n = (2n)^{\frac{3^{(n-1)}-1}{2}} (n^3-n^2-n-1)^{3^{(n-2)}(\frac{n}{2}+\frac{3}{4})+\frac{1}{4}}$*.*

Remark 5.2.3 Let us give a geometric interpretation to, say, (5.2.2.2′). We employ the notation of §5.1. Let (X,E,L,m) be as in (5.2.2.2′). Let $E' := E \otimes (K_X + mL)$ and $L' = (r+1)(2K_X + mL) + \det E$; note that $h^0 := h^0(X,E') = \chi(X,E')$ and that L' is very ample. Then there is a closed embedding

$$\phi := f \times g : X \longrightarrow G(r,h^0) \times \mathbb{P}^n$$

such that $E \simeq f^*(\mathfrak{Q} \otimes \mathfrak{q}) \otimes g^*\mathcal{O}_{\mathbb{P}^n}(-1)$, $\deg \hat{f}(X) = (\det E')^n$ and g is finite surjective with $\deg g = L'^n$.
Let $\{X_i, E_i, L_i\}_{i\in I}$ be a set of triplets as above. If we can bound from above h^0_i, $\deg \hat{f}_i(X_i)$ and $L_i'^n$, then we can find embeddings $\phi_i : X_i \to G \times \mathbb{P}^n$ with $G = G(r, \max_I(h^0_i))$ such that the relevant invariants are bounded from above. This applies, for example, to the set of flat vector bundles of fixed rank over a (family of) projective manifold(s), to the set of all nef vector bundles of fixed rank over curves of fixed genus, to the set of projective surfaces with nef tangent bundles, etc. By virtue of Remark 5.2.5, a similar remark holds, more generally, for nef vector bundles.

Proof of Theorem 5.2.2. Note that (5.2.2.1′) and (5.2.2.2′) are special cases of (5.2.2.1) and (5.2.2.2), respectively. We shall prove (5.2.2.1) and (5.2.2.2) in detail to illustrate the method. The remaining three assertions are left to the reader and can be proved using the same method with the aid of Lemma 5.1.2 for the second and third to last, and with the guideline of [11], 4.7 for the last one.

Proof of (5.2.2.1). We follow closely [1]. The proof is by induction on p. Let $p=1$. Let $x \in X$ be arbitrary. By (5.1.1.1) we have a strictly positive s.h.m. h on mL such that x is an isolated point of the scheme associated with $\mathcal{I}(h)$. By virtue of Theorem 4.2.3, $H^1(X, K_X \otimes E \otimes (mL) \otimes \mathcal{I}(h)) = 0$ and the following surjections imply the case $p=1$:

$$H^0(X, K_X \otimes E \otimes (mL)) \twoheadrightarrow H^0(X, K_X \otimes E \otimes (mL) \otimes \mathcal{O}_X/\mathcal{I}(h))$$
$$\twoheadrightarrow H^0(X, K_X \otimes E \otimes (mL) \otimes \mathcal{O}_X/\mathfrak{m}_x).$$

Let us assume that (5.2.2.1) is true for all integers $\rho \leq p-1$ and prove the case $\rho = p$. Let h be as in (5.1.1.1) and $\mathcal{I}(h)$ be its multiplier ideal. By virtue of Theorem 4.2.3, we have that $H^1(K_X \otimes E \otimes (mL) \otimes \mathcal{I}(h)) = 0$. Let $\mathcal{J}$ be the ideal sheaf on X which agrees with $\mathcal{I}(h)$ on $X \setminus J_0$

and which agrees with $\mathcal{O}_X$ on J_0. Relabel the points so that $J_0 = \{1, \ldots, l\}$. By tensoring the exact sequence

$$0 \to \mathcal{I}(h) \to \mathcal{J} \to \mathcal{J}/\mathcal{I}(h) \to 0$$

with $K_X \otimes E \otimes mL$ we get the surjection:

$$H^0(X, K_X \otimes E \otimes (mL) \otimes \mathcal{J}) \longrightarrow\!\!\!\!\!\rightarrow \bigoplus_{i=1}^{l} \mathcal{O}(K_X \otimes E \otimes (mL))_{x_i} \otimes \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}$$

which implies that we can choose sections $a_{1,j} \in H^0(X, K_X \otimes E \otimes (mL))$ vanishing at $x_2, \ldots, x_p$, but generating the stalk $(K_X \otimes E \otimes (mL))_{x_1}$. We now apply the induction hypothesis to the set of $p-1$ points $\{x_2, \ldots, x_p\}$. By repeating the above procedure, and keeping in mind that at each stage we may have to relabel the points, we obtain sections $\{a_{i,j_i}\} \in H^0(X, K_X \otimes E \otimes (mL))$, $\forall\, 1 \leq i \leq p$ vanishing at $\{x_{i+1}, \ldots, x_p\}$ but generating the stalk $(K_X \otimes E \otimes (mL))_{x_i}$. Given any point x_i, with $1 \leq i \leq r$, and any vector $w \in (K_X \otimes E \otimes (mL))_{x_i} \otimes \mathcal{O}_{X,x_i}/m_{x_i}$ it is now easy to find a linear combination of the sections a_{i,j_i} which is w at x_i and zero at all the other $p-1$ points. This proves (5.2.2.1).

Proof of (5.2.2.2). We fix the integers p, $s_1, \ldots, s_p$ and p arbitrary distinct points on X. We take a singular metric h on $K_X + mL$ with $m \geq m_1$ for which the associated multiplier ideal $\mathcal{I}(h)$ has the properties ensured by (5.1.1.2). Theorem 4.2.3 gives us the vanishing of $H^1(K_X \otimes K_X \otimes E \otimes (mL) \otimes \mathcal{I}(h))$ which, in turn, gives the wanted surjection in view of the obvious surjections

$$\mathcal{O}_{X,x_i}/\mathcal{I}(h)_{x_i} \to \mathcal{O}_{X,x_i}/\mathfrak{m}_{x_i}^{s_i+1}, \quad \forall\, 1 \leq i \leq p.$$

□

Remark 5.2.4 Both statements in Proposition 5.1.1 have counterparts entailing not powers mL of an ample line bundle L, but directly an ample line bundle $\mathfrak{L}$ which has "intersection theory" large enough. See [1], Theorem 0.3, [11] Theorem 2.4.b and [24].
As a consequence one has statements similar to the ones of Theorem 5.2.2 with mL substituted by an ample line bundle $\mathfrak{L}$ with intersection theory large enough; we omit the details.

Remark 5.2.5 Let X, n, E, L be as in this section except that E is only assumed to be nef. Using algebraic techniques we can see that $K_X \otimes E \otimes \det E \otimes L^m$ is globally generated for $m \geq \frac{1}{2}(n^2 + n + 2)$. Statements involving higher jets can be proved as well. Details will appear in [5].

Question 5.2.6 Let X, E and L be as above. Is the vector bundle $K_X \otimes E \otimes L^{\otimes m}$ generated by global sections for every $m \geq \frac{1}{2}(n^2 + n + 2)$?

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