# On Orbit Configuration Spaces and the Rational Cohomology of $F(\mathbb{R}\mathrm{P}^n, k)$

Miguel A. Xicoténcatl

Abstract. The purpose of this article is to study the orbit configuration space $F_{\mathbb{Z}_2}(S^n, k)$ of $k$-tuples of non-antipodal, distinct points in $S^n$. The integral cohomology of $F_{\mathbb{Z}_2}(S^n, k)$ is computed. In the rational case, the algebra of invariants under the natural action of $(\mathbb{Z}_2)^k$, is isomorphic to the cohomology of $F(\mathbb{R}\mathrm{P}^n, k)$.

## 1. Introduction

Let $M$ be a connected manifold and denote by $F(M, k)$ the ordered configuration space of $k$ distinct points in $M$, as defined in [**5**]. Configuration spaces have proved to be very important in algebraic topology and for this reason they have received a lot attention since their introduction by Fadell and Neuwirth. The cohomology of ordered configuration spaces is very well understood for $M = \mathbb{R}^n, S^n$ and $N \times \mathbb{R}$, see [**3**] and [**4**]. Also, a model for the rational homotopy type of $F(M, k)$ was given recently in [**6**] and [**8**] when $M$ is a smooth complex projective variety. However, not very much is known about the cohomology of $F(M, k)$ for more general manifolds.

Analogues to configuration spaces for the case of manifolds with group actions have been also consider in [**9**]. Namely, if $G$ acts freely on $M$, define the *orbit configuration space* by

$$F_G(M, k) = \{(m_1, \ldots, m_k) \in M^k \mid Gm_i \cap Gm_j = \varnothing \ \text{ if } \ i \neq j\}.$$

where $Gm$ denotes the orbit of $m$ in $M$.

Let $\mathbb{Z}_2$ act on $S^n$ via the antipodal map. In this article we study the orbit configuration space $F_{\mathbb{Z}_2}(S^n, k)$. In particular, we give a complete description of its cohomology algebra with coefficients in $\mathbb{Z}$ and $\mathbb{Q}$, see Theorem 5.2. Similar results in this direction have been obtained independently by E. M. Feichtner. The space $F_{\mathbb{Z}_2}(S^n, k)$ admits also a natural action $(\mathbb{Z}_2)^k$-action and

1991 *Mathematics Subject Classification.* 55R80, 55R05, 20J06.

the quotient is homeomorphic to $F(\mathbb{R}\mathrm{P}^n, k)$. In the rational case, it is also shown that

$$H^*\left(F(\mathbb{R}\mathrm{P}^n, k); \mathbb{Q}\right) \cong H^*\left(F_{\mathbb{Z}_2}(S^n, k); \mathbb{Q}\right)^{(\mathbb{Z}_2)^k}$$

The article is organized as follows. In section 2 we recall some of the basic properties of orbit configuration spaces. Then we specialize to the case of $M = S^n$ and $G = \mathbb{Z}_2$ and proceed to compute the cohomology of $F_{\mathbb{Z}_2}(S^n, k+1)$. This is done in several steps.

The fibre of the projection onto the first coordinate, $\pi\colon F_{\mathbb{Z}_2}(S^n, k+1) \longrightarrow S^n$ is homeomorphic to the $\mathbb{Z}_2$-orbit configuration space $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$, where the $\mathbb{Z}_2$ action on $\mathbb{R}^n - \{0\}$ is given through the involution $\tau(x) = -x/|x|^2$. In section 3 we calculate the cohomology algebra of $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. The relations obtained here are analogues of the 3-term relation $A_{s,u}A_{s,t} - A_{t,u}A_{s,t} + A_{t,u}A_{s,u} = 0$ in the cohomology of $F(\mathbb{R}^n, k)$ as described by F. Cohen [**2**], [**3**]. In addition to these, we obtain an extra relation.

Theorem 1.1. *The integral cohomology of $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is generated, as a graded algebra, by the set $(n-1)$-dimensional classes: $A_{i,0}$, $i = 1, \ldots, k$ and $A_{i,j}$, $\bar{A}_{i,j}$ for $1 \le j < i \le k$, subject to the following relations:*

(a) *For $i = 1, \ldots, k$ and $1 \le t \le s$,*

$$A_{i,0}^2 = A_{s,t}^2 = \bar{A}_{s,t}^2 = 0.$$

(b) *For $1 \le t \le s$,*

$$\begin{aligned}
A_{s,0}A_{s,t} &= A_{t,0}(A_{s,t} - A_{s,0}) \\
A_{s,0}\bar{A}_{s,t} &= (-1)^n A_{t,0}(\bar{A}_{s,t} - A_{s,0}) \\
A_{s,t}\bar{A}_{s,t} &= (-1)^n A_{t,0}(\bar{A}_{s,t} - A_{s,t})
\end{aligned}$$

(c) *For $1 \le u < t < s \le k$,*

$$\begin{aligned}
A_{s,u}A_{s,t} &= A_{t,u}(A_{s,t} - A_{s,u}) \\
A_{s,u}\bar{A}_{s,t} &= (-1)^n (A_{u,0} + A_{t,0} - A_{t,u})(\bar{A}_{s,t} - A_{s,u}) \\
A_{s,t}\bar{A}_{s,u} &= (-1)^n \bar{A}_{t,u}(\bar{A}_{s,u} - A_{s,t}) \\
\bar{A}_{s,u}\bar{A}_{s,t} &= A_{u,0}(\bar{A}_{s,t} - \bar{A}_{s,u}) + (-1)^n (A_{t,0} - A_{t,u})(\bar{A}_{s,t} - \bar{A}_{s,u})
\end{aligned}$$

In sections 4 and 5 we work out all the differentials in the Serre spectral sequence that converges to $H^*F_{\mathbb{Z}_2}(S^n, k+1)$. Finally, in section 6 the $(\mathbb{Z}_2)^k$-action is analyzed for some values of $k$. As a corollary, we get the rational cohomology of the corresponding $F(\mathbb{R}\mathrm{P}^n, k)$.

## 2. Orbit configuration spaces

Let $M$ be a manifold with a free action of a group $G$ and let $F_G(M, k)$ be the corresponding orbit configuration spaces as define in section 1. The main relation to ordinary configuration spaces is given by the following result which was proven in [**9**].

PROPOSITION 2.1. *Let $G$ act on $M$ such that the canonical projection $M \to (M/G)$ is a principal $G$-bundle. Then $G^k$ acts on $F_G(M, k)$ and $F_G(M, k)/G^k \approx F(M/G, k)$.*

In particular, there is a principal $G^k$-bundle: $G^k \longrightarrow F_G(M, k) \longrightarrow F(M/G, k)$ which can be used for homological calculations, as shown here and in section 6. More precisely, notice that the $G^k$-action on $F_G(M, k)$ induces an action on cohomology, and we have

THEOREM 2.2. *Let $G$ be a finite group acting freely on a manifold $M$ and let $\mathbb{F}$ be a field of characteristic $0$ or prime to $|G|$. Then there is an isomorphism of algebras:*

$$H^* \left(F(M/G, k)\,;\, \mathbb{F}\right) \cong H^* \left(F_G(M, k)\,;\, \mathbb{F}\right)^{G^k}$$

*where $(-)^{G^k}$ denotes the module of invariants.*

PROOF. Since $G^k$ acts properly on $F_G(M, k)$, we can apply the spectral sequence for a covering (as given in [**7**] pg. 342 )

$$E_2^{p,q} = H^p \left(G^k \,;\, H^q F_G(M, k)\right)$$

which converges to $H^* F(M/G, k)$. The hypothesis on the characteristic of $\mathbb{F}$, implies that multiplication by $|G|^k$ is an isomorphism in $\mathbb{F}$. Hence the cohomology of $G^k$ (with coefficients in any $\mathbb{F}[G^k]$-module), is concentrated in degree 0 and is isomorphic to submodule of invariants. The Theorem follows. □

A useful result in the study of the homology of orbit configuration spaces is the following analogue of the Fadell and Neuwith fibrations for ordinary configuration spaces. For any natural number $\ell$, let $Q_\ell^G \subset M$ be th union of $\ell$ distinct orbits. The next Lemma was proven in [**9**].

LEMMA 2.3. *Let $M$ be a manifold with a properly discontinuous action of a group $G$ and, such that the orbit space $M/G$ is again a manifold. Then for $\ell \leq k$, the projection $p\colon F_G(M, k) \longrightarrow F_G(M, \ell)$ onto the first $\ell$ coordinates is a locally trivial bundle, with fibre $F_G(M - Q_\ell^G, k - \ell)$.*

For the rest of the paper, we specialize to the case $M = S^n$ with the antipodal action of the group $\mathbb{Z}_2$. In this case, $F_{\mathbb{Z}_2}(S^n, k)$ is the subspace of the $k$-fold product $S^n \times \ldots \times S^n$ which consists of all $k$-tuples $(x_1, \ldots, x_k)$ such that $x_i \neq \pm x_j$.

Identify $S^n$ with $\mathbb{R}^n \cup \{\infty\}$, the one point compactification of $\mathbb{R}^n$, by using the stereographic projection and choose $\infty$ to be the base point. Let $\pi : F_{\mathbb{Z}_2}(S^n, k+1) \to S^n$ be the projection onto the first coordinate. By the Lemma above, $\pi$ is a fibration with fibre $\mathbb{F}_{\mathbb{Z}_2}(S^n - \{\pm e_{n+1}\}, k)$. A further analysis of the stereographic projection shows that the restriction of the $\mathbb{Z}_2$-action to $(S^n - \{\pm e_{n+1}\}) \approx \mathbb{R}^n - \{0\}$ is given by the involution:

$$\tau(x) = -\frac{x}{|x|^2}\ .$$

Therefore we get a fibration

$$F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \longrightarrow F_{\mathbb{Z}_2}(S^n, k+1) \xrightarrow{\pi} S^n \tag{2.1}$$

whose Serre spectral sequence will be studied in sections 4 and 5. For this purpose, we use the rest of this and next section to calculate the cohomology of the fibre of (2.1).

For $k = 1$ we have $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 1) \simeq S^{n-1}$. Assume inductively that $H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1)$ is concentrated in dimensions multiples of $(n-1)$. A second application of Lemma 2.3 yields a fibration

$$\begin{tikzcd}
F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \arrow[d, "\pi_1"'] & (\mathbb{R}^n - \{0\} - Q^{\mathbb{Z}_2}_{k-1}) \arrow[l] \\
F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k-1) &
\end{tikzcd}$$

whose fibre has the homotopy type of a wedge $(\bigvee_{2k-1} S^{n-1})$. From the induction hypothesis, it follows that all differentials in the Serre spectral sequence are trivial, and the cohomology of the total space splits as the tensor product of the cohomology of the base and the cohomology of the fibre. Proceeding inductively we get:

$$H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \cong \bigotimes_{i=1}^{k} H^*(\bigvee_{2i-1} S^{n-1}).$$

Explicit generators in dimension $(n-1)$ are constructed next. For $i = 1, \ldots, k$ and $1 \leq t < s \leq k$, consider the maps

$$p_i,\ p_{s,t},\ \bar{p}_{s,t} : F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \longrightarrow S^{n-1}$$

given by:

$$
\begin{aligned}
p_i(x) &= \frac{x_i}{|x_i|} \\
p_{s,t}(x) &= \frac{x_s - x_t}{|x_s - x_t|} \\
\bar{p}_{s,t}(x) &= \frac{x_s - \tau x_t}{|x_s - \tau x_t|}
\end{aligned}
$$

and use them to define cohomology classes:

$$
\begin{aligned}
A_{i,0} &= p_i^*(\iota_{n-1}) \\
A_{s,t} &= p_{s,t}^*(\iota_{n-1}) \\
\bar{A}_{s,t} &= \bar{p}_{s,t}^*(\iota_{n-1})
\end{aligned}
$$

where $\iota_{n-1}$ is a fixed generator of $H^{n-1}(S^{n-1})$. Then one verifies that all these classes are linearly independent. One possibility is to construct maps $S^{n-1} \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ representing their duals in homology. Thus the classes defined above form a basis for $H^{n-1}F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ and the next theorem follows.

THEOREM 2.4. *The cohomology of $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$ is given additively by:*

$$
H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \cong M_1 \otimes M_2 \otimes \cdots \otimes M_k
$$

*where $M_i$ is the free module generated by the following set of $(n-1)$-dimensional classes $\{A_{i,0}\} \cup \{A_{i,j}, \bar{A}_{i,j}\}_{1 \le j < i}$.*

## 3. The algebra structure in $H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$

In order to prove the relations in Theorem 1.1, we have to compute the products among the generators of each factor $M_i$ in the tensor product above. Since all the $A_{i,j}$'s are spherical classes, we already know that $A_{i,j}^2 = 0$. The calculation can be reduced to the case $k = 3$ as follows. Remember that there is an isomorphism of modules

$$
F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \cong \langle A_{1,0} \rangle \otimes \langle A_{2,0}, A_{2,1}, \bar{A}_{2,1} \rangle \otimes \langle A_{3,0}, A_{3,1}, \bar{A}_{3,1}, A_{3,2}, \bar{A}_{3,2} \rangle
$$

and make the following change of notation. For $|j| < i \le k$ let

$$
A_{i,j} = \begin{cases} A_{i,j} & \text{if } j \ge 0 \\ \bar{A}_{i,-j} & \text{if } j < 0. \end{cases}
$$

Notice that for $1 \le u < t < s \le k$, the projection

$$
\pi_{s,t,u} \colon F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3)
$$

given by $\pi_{s,t,u}(x_1,\ldots,x_k) = (x_u, x_t, x_s)$ is a fibration and admits a section. Therefore $H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3)$ injects into $H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$. Furthermore, the induced map satisfies

$$(\pi_{s,t,u})^* \colon H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3) \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$$

$$\begin{aligned} A_{1,0} &\longmapsto A_{u,0} \\ A_{2,0} &\longmapsto A_{t,0} \\ A_{3,0} &\longmapsto A_{s,0} \\ A_{2,\pm 1} &\longmapsto A_{t,\pm u} \\ A_{3,\pm 1} &\longmapsto A_{s,\pm u} \\ A_{3,\pm 2} &\longmapsto A_{s,\pm t} \end{aligned}$$

Thus, the problem has been reduced to calculate the products among the generators of $H^*F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3)$. And this is approached as follows. Each of the desired products is given by a linear combination

$$A \cdot B = \sum C^{i,j}_{s,t} \;\; A_{i,j} \otimes A_{s,t}$$

where the product $A_{i,j} \otimes A_{s,t}$ runs over the natural basis for $H^{2(n-1)}$ and coefficients $C^{s,t}_{i,j}$ are to be determined. We construct maps

$$F^{i,j}_{s,t} \colon S^{n-1} \times S^{n-1} \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 3)$$

such that the induced morphisms in degree $2n-2$ are precisely the duals of the classes $A_{i,j} \otimes A_{s,t}$. Under this circumstances,

$$C^{i,j}_{s,t} = (F^{i,j}_{s,t})^*(A \cdot B) = (F^{i,j}_{s,t})^*(A) \cdot (F^{i,j}_{s,t})^*(B)$$

and then explicit calculations can be carried out in order to obtain the coefficients. The required maps are constructed as follows. Let $\mathbf{e}$ denote the first canonical unit vector in $\mathbb{R}^n$ and define

$$F^{i,j}_{s,t}(z, w) = (y_1,\, y_2,\, y_3)$$

where $y_1$, $y_2$ and $y_3$ are given by

$$y_1 = \begin{cases} \frac{z}{2} & \text{if } i = 1 \\ \mathbf{e} & \text{otherwise} \end{cases}$$

and for $\ell = 2, 3$

$$y_\ell = \begin{cases} y_j + \frac{z}{2} & \text{if } \ell = i, \quad j > 0 \\ \tau(y_{|j|}) + \frac{z}{2} & \text{if } \ell = i, \quad j < 0 \\ \frac{z}{2} & \text{if } \ell = i, \quad j = 0 \\ y_t + \frac{w}{4} & \text{if } \ell = s, \quad t > 0 \\ \tau(y_{|t|}) + \frac{w}{4} & \text{if } \ell = s, \quad t < 0 \\ \frac{w}{4} & \text{if } \ell = s, \quad t = 0 \\ \ell \cdot \mathbf{e} & \text{if } \ell \neq i, s\ . \end{cases}$$

For example, $F^{1,0}_{2,0} = (\frac{z}{2}, \frac{w}{4}, 3\mathbf{e})$, $\quad F^{2,1}_{3,-1} = (\mathbf{e}, \mathbf{e} + \frac{z}{2}, -\mathbf{e} + \frac{w}{4})$, etc. Then one must study the effect of all these maps on degree $n-1$. This analysis provides all the coefficients for the $k = 3$. The general result follows from naturality.

## 4. The computation of the differentials

In this section we determine the non trivial differentials in the Serre spectral sequence for the fibration (2.1), whose $E_2$-term is given by:

$$E_2^{p,q} = H^p(S^n) \otimes H^q F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \tag{4.1}$$

Since the cohomology of the fibre is generated multiplicatively by the classes in degree $n-1$, it is enough to determine

$$d_n^{0,n-1} \colon E_n^{0,n-1} \longrightarrow E_n^{n,0} = H^n(S^n).$$

Let's denote by $x$ the $n$-dimensional class in $H^*(S^n)$ which corresponds to the natural orientation of $S^n$. Then we have:

Proposition 4.1. *For $0 \le j < i \le k$, the $n$-th differential in the spectral sequence (4.1) is given by:*

$$d_n(A_{i,0}) = d_n(A_{i,j}) = d_n(\bar{A}_{i,j}) = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ 2x & \textit{if } n \textit{ is even}\ . \end{cases}$$

Proof. For $n$ odd, the fibration (2.1) admits a section. Therefore the differential in question vanishes, and the spectral sequence collapses.

For $n$ even, we first reduce the calculation to the cases $k = 1$ and 2. Let $\pi_i : F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \to \mathbb{R}^n - \{0\}$ be the projection onto the $i$-th coordinate. The induced map in cohomology satisfies ${\pi_i}^*(A_{1,0}) = A_{i,0}$. Notice that $\pi_i$ fits into a map of fibrations

$$\begin{array}{ccc}
F_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},k) & \xrightarrow{\pi_i} & \mathbb{R}^n-\{0\} \\
\downarrow & & \downarrow \\
F_{\mathbb{Z}_2}(S^n,k+1) & \xrightarrow{1\times\pi_i} & F_{\mathbb{Z}_2}(S^n,2) \\
\downarrow & & \downarrow \\
S^n & \xrightarrow{id} & S^n
\end{array}$$

Now, a comparison of the corresponding spectral sequences gives $d_n(A_{i,0}) = d_n\ {\pi_i}^*(A_{1,0}) = d_n(A_{1,0})$. Similarly for $i > j$, consider the projections $\pi_{i,j} : F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k) \to F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2)$ with $\pi_{i,j}(x_1, \dots, x_k) = (x_j, x_i)$. These maps satisfy ${\pi_{i,j}}^*(A_{2,1}) = A_{i,j}$ and ${\pi_{i,j}}^*(\bar{A}_{2,1}) = \bar{A}_{i,j}$. Again by naturality we have:

$$d_n(A_{i,j}) = d_n(A_{2,1})$$
$$d_n(\bar{A}_{i,j}) = d_n(\bar{A}_{2,1})$$

Secondly, we show that $d_n(A_{2,1}) = d_n(\bar{A}_{2,1})$. For $x \in \mathbb{R}^n$, define its "conjugate" by $\bar{x} = (x_1, \dots, x_{n-1}, -x_n)$. Let $R : S^n \to S^n$ be the rotation given by the following element of $SO(n+1)$:

$$[R] = \begin{pmatrix} I_{n-1} & 0 \\ 0 & -I_2 \end{pmatrix}$$

and let $a : S^n \to S^n$ be the antipodal map. Consider the map $(a, 1, a) : F_{\mathbb{Z}_2}(S^n, 3) \longrightarrow F_{\mathbb{Z}_2}(S^n, 3)$ and the homotopic map $R^3 \circ (a, 1, a)$. The later induces a map of fibrations:

(4.2)
$$\begin{array}{ccc}
F_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2) & \xrightarrow{\varphi} & F_{\mathbb{Z}_2}(\mathbb{R}^n-\{0\},2) \\
\downarrow & & \downarrow \\
F_{\mathbb{Z}_2}(S^n,3) & \xrightarrow{R^3\circ(a,1,a)} & F_{\mathbb{Z}_2}(S^n,3) \\
\downarrow & & \downarrow \\
S^n & \xrightarrow{R\circ a} & S^n
\end{array}$$

In both columns of (4.2), the fibre over $\infty = e_{n+1} \in S^n = \mathbb{R}^n \cup \{\infty\}$ is the space $F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2)$. The induced map on the fibres is given by

$$\varphi(z, w) = (\frac{\bar{z}}{|z|^2}, -\bar{w})$$

and it has the following effect in cohomology:

$$\begin{aligned}
\varphi^*(A_{1,0}) &= -A_{1,0}\\
\varphi^*(A_{2,0}) &= (-1)^{n+1}A_{2,0}\\
\varphi^*(A_{2,1}) &= (-1)^{n+1}\bar{A}_{2,1}\\
\varphi^*(\bar{A}_{2,1}) &= (-1)^{n+1}A_{2,1}
\end{aligned}$$

These results are obtained by a direct analysis of the Serre spectral sequence for the fibration

$$\mathbb{R}^n - \{0, \pm e_1\} \longrightarrow F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) \longrightarrow \mathbb{R}^n - \{0\}$$

Notice that the map between the bases in (4.2) induces multiplication by $(-1)^{n+1}$ in $H^n(S^n)$. Thus we obtain the following relation in the spectral sequence for the columns of (4.2)

$$\begin{aligned}
d_n(\bar{A}_{2,1}) &= (-1)^{n+1} d_n\ \varphi^*(A_{2,1})\\
&= (-1)^{n+1}(R \circ a)^*\ d_n(A_{2,1})\\
&= d_n(A_{2,1})
\end{aligned}$$

Next, we show that for $n$ even, $d_n(A_{1,0}) = 2x$. Let $V_2(\mathbb{R}^{n+1})$ denote the Stiefel manifold of 2-orthonormal frames in $\mathbb{R}^{n+1}$, and recall from [**9**] that the inclusion of $V_2(\mathbb{R}^{n+1})$ in $F_{\mathbb{Z}_2}(S^n, 2)$ induces an equivalence of bundles between the unit tangent bundle of $S^n$ and fibration (2.1) for $k = 1$. The only differential in the spectral sequence for the first is given by multiplication with the Euler class of $S^n$ which, in this case, is just multiplication by 2.

Finally, we show that for $n$ even, $d_n(A_{2,1}) = d_n(\bar{A}_{2,1}) = 2x$. Consider the inclusion of $F_{\mathbb{Z}_2}(S^n, 3)$ into the ordinary configuration space $F(S^n, 3)$. This map induces a map of fibrations

(4.3)
$$\begin{tikzcd}
F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) \arrow[r, "j"] \arrow[d] & F(\mathbb{R}^n, 2) \arrow[d] \\
F_{\mathbb{Z}_2}(S^n, 3) \arrow[r, "i"] \arrow[d] & F(S^n, 3) \arrow[d] \\
S^n \arrow[r, "id"] & S^n
\end{tikzcd}$$

which gives the usual inclusion $j$ in the fibres. Recall that $F(\mathbb{R}^n, 2) \simeq S^{n-1}$ and denote the generator in dimension $(n-1)$ by $y$. It is also well know [**4**] that in the spectral sequence for the second column $d_n(y) = 2x$. The result will follow from naturality if we show that $j^*(y) = A_{2,1}$. To see this, look at the induced map of fibrations

$$\begin{array}{ccc}
\mathbb{R}^n - \{0, \pm e_1\} & \xrightarrow{j_2} & \mathbb{R}^n - \{e_1\} \\
\downarrow & & \downarrow \\
F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, 2) & \xrightarrow{j} & F(\mathbb{R}^n, 2) \\
\downarrow & & \downarrow \\
\mathbb{R}^n - \{0\} & \xrightarrow{j_1} & \mathbb{R}^n
\end{array}$$

The inclusion of the fibres satisfies ${j_2}^*(y) = A_{2,1}$ and the claim follows.

□

## 5. The integral cohomology of $F_{\mathbb{Z}_2}(S^n, k+1)$

Through all this section, $R$ will denote a principal ideal domain. The most important examples will be $R = \mathbb{Z}$ and $\mathbb{Q}$. We have already proved that for $n$ even:

$$d_n(A_{i,0}) = d_n(A_{s,t}) = d_n(\bar{A}_{s,t}) = 2x$$

for $1 \leq i \leq k$ and $1 \leq t < s \leq k$. We show here that this information suffices to determine all the differentials in the spectral sequence. For $j \geq 0$ let

$$C_j = H^{j(n-1)}\left(F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k); R\right)$$

Then $C_*$ is a graded algebra and it is multiplicatively generated by $C_1$. Recall that a basis for $C_1$, as a free $R$-module, is given by:

$$\mathcal{B} = (A_{1,0}, A_{2,0}, A_{2,1}, \bar{A}_{2,1}, \ldots, A_{k,0}, A_{k,1}, \ldots, \bar{A}_{k,1}, \ldots).$$

At this point it will be convenient to make a change of basis. Set:

$$B_{i,0} = A_{i,0} - A_{1,0} \qquad \text{for} \quad 2 \leq i \leq k$$

$$B_{s,t} = A_{s,t} - A_{1,0} \qquad \text{for} \quad 1 \leq t < s \leq k$$
$$\bar{B}_{s,t} = \bar{A}_{s,t} - A_{1,0}$$

and define $\mathbb{A}$ and $\mathbb{K}$ to be the free $R$-modules generated by $A_{1,0}$ and by the set $\{B_{i,0}, B_{s,t}, \bar{B}_{s,t} \mid 2 \leq i \leq k, \ 1 \leq t < s \leq k\}$, respectively. Then we have $C_1 = \mathbb{A} \oplus \mathbb{K}$ and since $A_{1,0}^2 = 0$, it can be shown by induction that $C_j = \mathbb{A} \cdot \mathbb{K}^{j-1} \oplus \mathbb{K}^j$ for $j \geq 1$. Here $\mathbb{K}^j$ is the $R$-module spanned by the products of length $j$ of elements in $\mathbb{K}$ and $\mathbb{K}^0 = R$. Also, notice that $\mathbb{K}$ is the kernel of the homomorphism:

$$d_n : E_n^{0,n-1} \longrightarrow E_n^{n,0}$$

Moreover, we have:

LEMMA 5.1. *For $j \geq 1$, let $d^j$ denote the differential $d_n : E_n^{0,j(n-1)} \to E_n^{n,(j-1)(n-1)}$. Then*

(a) *$ker(d^j) = \mathbb{K}^j$ for $j \geq 1$.*

(b) *$im(d^j) = 2x \cdot \mathbb{K}^{j-1}$ for $j \geq 1$ and in particular,*

(c) *$im(d^{j+1}) = (2x) \cdot ker(d^j)$.*

PROOF. Recall that $E_n^{0,j(n-1)} = C_j = \mathbb{A} \cdot \mathbb{K}^{j-1} \oplus \mathbb{K}^j$. Since $\mathbb{K} = \ker(d^1)$, it is possible to show by induction that $d_n(\mathbb{K}^i) = 0 \quad \forall i \geq 0$. In the other hand, let $A_{1,0} \cdot k$ be a typical element of $\mathbb{A} \cdot \mathbb{K}^{j-1}$. Then $d_n(A_{1,0} \cdot k) = d_n(A_{1,0}) \cdot k = 2x \cdot k$. This last expression vanishes only when $k$ does and the Lemma follows. □

Therefore:

$$E_{n+1}^{0,j(n-1)} = \mathbb{K}^j \qquad \text{for} \quad j \geq 0,$$

$$E_{n+1}^{n,j(n-1)} = xA_{1,0} \cdot \mathbb{K}^{j-1} \oplus (x \cdot \mathbb{K}^j)_2 \qquad \text{for} \quad j \geq 0$$

where $(-)_2$ : *Free $R$-$\mathcal{M}od \longrightarrow R$-$\mathcal{M}od$* is the "mod 2 reduction" functor, which takes $\bigoplus_i R$ to $\bigoplus_i (R/2R)$. Here $\mathbb{K}^{-1}$ is understood to be 0 and thus we have:

THEOREM 5.2. *The cohomology of $F_{\mathbb{Z}_2}(S^n, k+1)$ is given by*

(a) *For $R = \mathbb{Z}$ or $\mathbb{Q}$ and $n$ odd, there is an isomorphism of algebras:*

$$H^* F_{\mathbb{Z}_2}(S^n, k+1) \cong H^*(S^n) \otimes H^* F_{\mathbb{Z}_2}(\mathbb{R}^n - \{0\}, k)$$

(b) *For $R = \mathbb{Z}$ and $n$ even:*

$$H^q\left(F_{\mathbb{Z}_2}(S^n, k+1); \mathbb{Z}\right) = \begin{cases} \mathbb{K}^j & \text{if } q = j(n-1),\ j \geq 0 \\ xA_{1,0} \cdot \mathbb{K}^{j-2} \oplus (x \cdot \mathbb{K}^{j-1})_2 & \text{if } q = j(n-1)+1,\ j \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(c) *For $R = \mathbb{Q}$ and $n$ even:*

$$H^q\left(F_{\mathbb{Z}_2}(S^n, k+1); \mathbb{Q}\right) = \begin{cases} \mathbb{K}^j & \text{if } q = j(n-1),\ j \geq 0 \\ xA_{1,0} \cdot \mathbb{K}^{j-2} & \text{if } q = j(n-1)+1,\ j \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

REMARK 5.3. In the case when $R = \mathbb{Q}$, a simple calculation shows that

$$\sum_{j=0}^{k} \dim_{\mathbb{Q}}(\mathbb{K}^j)\, t^{j(n-1)} = (1+3t^{n-1})(1+5t^{n-1})\dots[1+(2k-1)t^{n-1}]$$

and thus, for $n$ even the Poincaré polynomial of $F_{\mathbb{Z}_2}(S^n, k+1)$ is given by

$$\prod_{\ell=2}^{k}\left[1+(2\ell-1)t^{n-1}\right] + t^{2n-1}\prod_{\ell=2}^{k}\left[1+(2\ell-1)t^{n-1}\right]$$
$$= (1+t^{2n-1})\prod_{\ell=2}^{k}\left[1+(2\ell-1)t^{n-1}\right]$$

## 6. The $(\mathbb{Z}_2)^k$-action on $F_{\mathbb{Z}_2}(S^n, k)$

Recall that $(\mathbb{Z}_2)^k = \langle \epsilon_1, \dots, \epsilon_k \rangle$ acts coordinate-wise on $F_{\mathbb{Z}_2}(S^n, k)$, namely for every $x = (x_1, \dots, x_k) \in F_{\mathbb{Z}_2}(S^n, k)$, $\epsilon_i(x) = (x_1, \dots, -x_i, \dots, x_k)$. Therefore, the maps $\epsilon_i$ induce a natural $(\mathbb{Z}_2)^k$-action in cohomology, and in the case of rational coefficients it was proved in section 2 that

$$H^*\left(F(\mathbb{R}\mathrm{P}^n, k); \mathbb{Q}\right) \cong H^*\left(F_{\mathbb{Z}_2}(S^n, k); \mathbb{Q}\right)^{(\mathbb{Z}_2)^k}$$

In this section we study the action for some examples. Notice that for $i = 2, \dots, k$ the following diagram commutes:

$$\begin{tikzcd} F_{\mathbb{Z}_2}(S^n, k) \arrow[r, "\epsilon_i"] \arrow[d, "\pi_1"'] & F_{\mathbb{Z}_2}(S^n, k) \arrow[d, "\pi_1"] \\ S^n \arrow[r, "id"] & S^n \end{tikzcd}$$

and then, $\epsilon_2, \dots, \epsilon_k$ induce maps of fibrations. Their action on $H^*F_{\mathbb{Z}_2}(S^n, k)$ is computed by analyzing the actions induced on the cohomology of the base and of the fibre.

The map $\epsilon_1$ also induces a map of fibrations, being $a\colon S^n \to S^n$ the induced map on the bases. Since $a$ is not a pointed map, $\epsilon_1$ does not preserve fibres. Therefore, we need a separate argument for the case $i = 1$. Let $R \in SO(n+1)$ be a rotation in $\mathbb{R}^{n+1}$ that interchanges the points $0, \infty \in S^n$. For instance take $R$ to be as in section 4. The restriction of $R$ to $S^n$ is $\mathbb{Z}_2$-equivariant and moreover, it $\mathbb{Z}_2$-equivariantly isotopic to the identity map $1_{S^n}$. Thus it induces a map:

$$R^k\colon F_{\mathbb{Z}_2}(S^n, k) \longrightarrow F_{\mathbb{Z}_2}(S^n, k)$$

which is homotopic to the identity and such that the following diagram commutes:

$$\begin{tikzcd}
F_{\mathbb{Z}_2}(S^n, k) \arrow[r, "R^k \circ \epsilon_1"] \arrow[d, "\pi_1"'] & F_{\mathbb{Z}_2}(S^n, k) \arrow[d, "\pi_1"] \\
S^n \arrow[r, "R \circ a"] & S^n
\end{tikzcd}$$

where $a\colon S^n \to S^n$ is the antipodal map. Thus $R^k \circ \epsilon_1$ is a map of fibrations, which is homotopic to $\epsilon_1$ and hence it has the same effect in cohomology. Moreover, notice that $R^k \circ \epsilon_1$ preserves fibres. Therefore we can express the action of $\epsilon_1$ on the spectral sequence using the expression for $E_2^{*,*}$ in terms of $H^*(S^n)$ and the cohomology of the fibre over $\infty$.

For $k = 2$, the action of $(\mathbb{Z}_2)^2$ is summarized in Table 1.

TABLE 1

| | $\epsilon_1$ | $\epsilon_2$ |
|---|---|---|
| $x$ | $(-1)^{n+1}x$ | $x$ |
| $A_{1,0}$ | $-A_{1,0}$ | $(-1)^n A_{1,0}$ |

For $n$ odd, $H^* F_{\mathbb{Z}_2}(S^n, 2) = \Lambda(x) \otimes \Lambda(A_{1,0})$ and the invariants appear only on degrees 0 and $n$. Thus:

$$H^*\left(F(\mathbb{R}\mathrm{P}^n, 2); \mathbb{Q}\right) \cong \Lambda(x).$$

For $n$ even and $k = 2$, the module $\mathbb{K}$ defined in last section, vanishes. Thus $H^* F_{\mathbb{Z}_2}(S^n, 2)$ is exterior on the class $x \cdot A_{1,0}$, which Since this class which is invariant under the $(\mathbb{Z}_2)^2$-action. Therefore

$$H^*\left(F(\mathbb{R}\mathrm{P}^n, 2); \mathbb{Q}\right) \cong \Lambda(y_{2n-1}).$$

Remark 6.1. It is interesting to notice that for $n$ even, this method takes care of the problem of the local coefficient system in the Fadell-Neuwirth fibration

$$(\mathbb{R}\mathrm{P}^n - Q_1) \longrightarrow F(\mathbb{R}\mathrm{P}^n, 2) \longrightarrow \mathbb{R}\mathrm{P}^n$$

If the action of $\pi_1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_2$ on the fibre was trivial, we would have

$$\begin{aligned} E_2^{*,*} &\cong H^*(\mathbb{R}\mathrm{P}^n) \otimes H^*(\mathbb{R}\mathrm{P}^{n-1}) \\ &\cong \Lambda(x_{n-1}) \end{aligned}$$

Actually, it can be shown that the $\mathbb{Z}_2$-action on $H^*(\mathbb{R}\mathrm{P}^{n-1})$ is trivial in degree 0 and multiplication by $-1$ on degree $(n-1)$. Therefore, the $E_2$ term is given by the cohomology of $\mathbb{R}\mathrm{P}^n$ with local coefficients

$$E_2^{p,q} = H^p\left(\mathbb{R}\mathrm{P}^n; \underline{H^q(\mathbb{R}\mathrm{P}^{n-1})}\right) \tag{6.1}$$

which can be computed as follows. Let $\mathbb{Q}[\tau]$ denote the group ring of $\mathbb{Z}_2 = \langle\tau\rangle$, with the convention that $\tau^2 = 1$. Recall that the standard complex for $S^n$ with two cells in every dimension is given as

$$\mathbb{Q}[\tau] \xleftarrow{1-\tau} \mathbb{Q}[\tau] \xleftarrow{1+\tau} \mathbb{Q}[\tau] \xleftarrow{1-\tau} \cdots \xleftarrow{1+\tau} \mathbb{Q}[\tau] \longleftarrow 0 \longleftarrow \cdots$$

Applying the functor $Hom_{\mathbb{Q}[\tau]}(-,\mathbb{Q})$, where $\mathbb{Q}$ has the trivial action, yields the complex

$$\mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{\times 0} \cdots \xrightarrow{\times 2} \mathbb{Q} \longrightarrow 0 \longrightarrow \cdots$$

and we obtain

$$E_2^{p,0} \cong \begin{cases} \mathbb{Q} & p = 0 \\ 0 & \text{otherwise}\ . \end{cases}$$

In the other hand, applying $Hom_{\mathbb{Q}[\tau]}(-,\mathbb{Q}(-1))$ we get

$$\mathbb{Q} \xrightarrow{\times 2} \mathbb{Q} \xrightarrow{\times 0} \mathbb{Q} \xrightarrow{\times 2} \cdots \xrightarrow{\times 0} \mathbb{Q} \to 0 \to \cdots$$

and then

$$E_2^{p,n-1} \cong \begin{cases} \mathbb{Q} & p = n \\ 0 & \text{otherwise}\ . \end{cases}$$

Therefore, the spectral sequence (6.1) collapses at $E_2$ and provide us with the same answer for $H^*F(\mathbb{R}\mathrm{P}^n, 2)$.

TABLE 2

| | $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ |
|---|---|---|---|
| $x$ | $(-1)^{n+1}x$ | $x$ | $x$ |
| $A_{1,0}$ | $-A_{1,0}$ | $(-1)^n A_{1,0}$ | $A_{1,0}$ |
| $A_{2,0}$ | $-(A_{2,0}+A_{2,1}+\bar{A}_{2,1})$ | $A_{2,0}$ | $(-1)^n(A_{2,0}+A_{2,1}+\bar{A}_{2,1})$ |
| $A_{2,1}$ | $\bar{A}_{2,1}$ | $\bar{A}_{2,1}$ | $(-1)^{n-1}\bar{A}_{2,1}$ |
| $\bar{A}_{2,1}$ | $A_{2,1}$ | $A_{2,1}$ | $(-1)^{n-1}A_{2,1}$ |

TABLE 3

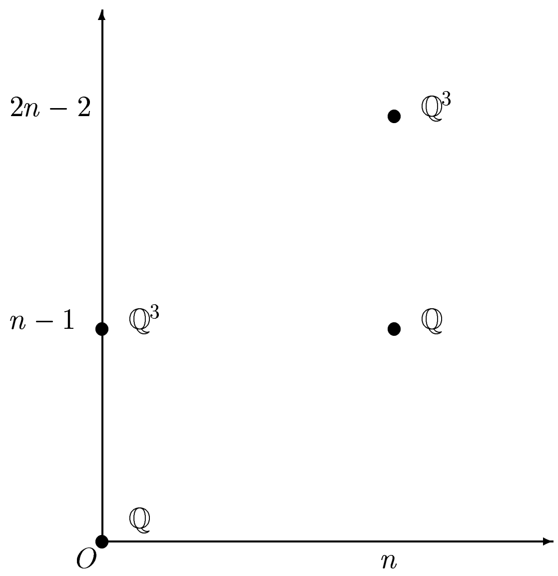
| *dim* | *Basis* | *Invariants* |
|---|---|---|
| 0 | 1 | 1 |
| $n-1$ | $A_{1,0},\quad A_{2,0},\quad A_{2,1},\quad \bar{A}_{2,1}$ | $A_{2,1}+\bar{A}_{1,0}$ |
| $n$ | $x$ | $x$ |
| $2n-2$ | $A_{1,0}A_{2,0},\quad A_{1,0}A_{2,1},\quad A_{1,0}\bar{A}_{2,1}$ | 0 |
| $2n-1$ | $xA_{1,0},\quad xA_{2,0},\quad xA_{2,1},\quad x\bar{A}_{2,1}$ | $x(A_{2,1}+\bar{A}_{2,1})$ |
| $3n-2$ | $xA_{1,0}A_{2,0},\quad xA_{1,0}A_{2,1},\quad xA_{1,0}\bar{A}_{2,1}$ | 0 |

For $k=3$, the $(\mathbb{Z}_2)^3$-action on the $E_2$ term is given in table 2.

For $n$ odd, $H^*F_{\mathbb{Z}_2}(S^n,3)\cong\langle x\rangle\otimes\langle A_{1,0}\rangle\otimes\langle A_{2,0},A_{2,1},\bar{A}_{2,1}\rangle$ and a basis for the invariants is given in table 3. Thus we have

$$H^*F(\mathbb{R}\mathrm{P}^n,3)\cong\Lambda(x)\otimes\Lambda(A_{2,1}+\bar{A}_{2,1}).$$

For $n$ even, the $(\mathbb{Z}_2)^3$-action on $E_2$ induces a well defined action on $E_{n+1}$, which has the following picture:




Let $B_{2,0}$, $B_{2,1}$ and $\bar{B}_{2,1}$ stand for the classes of $(A_{2,0} - A_{1,0})$, $(A_{2,1} - A_{1,0})$ and $(\bar{A}_{2,1} - A_{1,0})$ respectively. Then we have

TABLE 4

| | *Basis* | *Invariants* |
|---|---|---|
| $E_{n+1}^{0,0}$ | $1$ | $1$ |
| $E_{n+1}^{0,n-1}$ | $B_{1,0}, \quad B_{2,1}, \quad \bar{B}_{2,1}$ | $0$ |
| $E_{n+1}^{n,n-1}$ | $[xA_{1,0}]$ | $[xA_{1,0}]$ |
| $E_{n+1}^{n,2n-2}$ | $[xA_{1,0}A_{2,0}], \quad [xA_{1,0}A_{2,1}], \quad [xA_{1,0}\bar{A}_{2,1}]$ | $0$ |

Thus for $n$ even, $H^*F(\mathbb{R}\mathrm{P}^n, 3)$ is an exterior algebra on a class of dimension $(2n-1)$. The results obtained in this section are summarized table 5. More details and computations for the cases $k > 3$ will appear elsewhere.

TABLE 5

| $X$ | $H^*(X;\mathbb{Q})$ | *Poincaré polynomial* |
|---|---|---|
| $\mathbb{R}\mathrm{P}^{2k}$ | $\mathbb{Q}$ | $1$ |
| $F(\mathbb{R}\mathrm{P}^{2k},2)$ | $\Lambda(y_{4k-1})$ | $1+t^{4k-1}$ |
| $F(\mathbb{R}\mathrm{P}^{2k},3)$ | $\Lambda(y_{4k-1})$ | $1+t^{4k-1}$ |
| $\mathbb{R}\mathrm{P}^{2k+1}$ | $\Lambda(x_{2k+1})$ | $1+t^{2k+1}$ |
| $F(\mathbb{R}\mathrm{P}^{2k+1},2)$ | $\Lambda(x_{2k+1})$ | $1+t^{2k+1}$ |
| $F(\mathbb{R}\mathrm{P}^{2k+1},3)$ | $\Lambda(x_{2k+1})\otimes\Lambda(y_{2k})$ | $(1+t^{2k+1})(1+t^{2k})$ |

**Acknowledgments**

During the preparation of this work, the author was a visitor at the Max-Planck-Institut für Mathematik. He would like to thank the institution for its hospitality, support and excellent working environment.

**References**

[1] C.F. Bödigheimer, F.R. Cohen and L. Taylor *On the homology of configuration spaces*, Topology 28 (1989), 111-123.

[2] F.R. Cohen, *On configuration spaces, their homology, and Lie Algebras*, Journal of Pure and Applied Algebra 100 (1995), 19-42.

[3] F.R. Cohen, *The homology of $\mathcal{C}_{n+1}$-spaces, $n \geq 0$*, Lecture Notes in Math., vol. 533, Springer-Verlag (1976), 207–351.

[4] F.R. Cohen and L. Taylor, *Computations of Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces*, Lecture Notes in Math., vol. 657, Springer-Verlag, (1978) 106–143.

[5] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand. 10 (1962), 111-118.

[6] I. Kriz, *On the rational homotopy type of configuration spaces*, Ann. of Math. (2) 139 (1994), no. 2, 227–237.

[7] S. Mac Lane, *Homology*, Grundlehren der Mathematischen Wissenchaften, Springer-Verlag (1963).

[8] B. Totaro, *Configuration spaces of algebraic varieties*, Topology 35 (1996), 1057–1067.

[9] M.A. Xicoténcatl, *Orbit configuration spaces, infinitesismal braid relations in homology and equivariant loop spaces*, Ph.D. Thesis, The University of Rochester (1997).

Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn, Germany.

Depto. de Matemáticas, Cinvestav del IPN, Apdo. Postal 14-740, México, D.F. C.P. 07300, Mexico

*E-mail address*: xico@mpim-bonn.mpg.de, xico@math.cinvestav.mx