NILPOTENT ORBITS, PRIMITIVE IDEALS, AND CHARACTERISTIC CLASSES

Walter Borho

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str.26
5300 Bonn 3

Bergische Universität -
GH Wuppertal
Gaußstr. 20
5600 Wuppertal 1

MPI 86-63

NILPOTENT ORBITS, PRIMITIVE IDEALS, AND CHARACTERISTIC CLASSSES

Walter BORHO

Bergische Universität - GH Wuppertal, Gaußstr. 20, 5600 Wuppertal 1, FR Germany, and Max-Planck-Institut für Mathematik, Gottfried-Claren-Str. 26, 5300 Bonn 3, FR Germany.

CONTENTS

1. INTRODUCTION

1.1 Let G be a semisimple complex Lie group, say linear algebraic, and connected. The present report deals with the following three, a priori fairly unrelated subjects: (i) The geometry of unipotent conjugacy classes of G, or equivalently, of nilpotent orbits in the Lie algebra $\underline{g}$ of G; (ii) the classification of primitive ideals in the universal enveloping algebra $U(\underline{g})$, say with trivial central character for simplicity; and (iii) characteristic classes in $H^*(X)$ of certain bundles on the "flag variety" X, that is the "universal" complete homogeneous space for G. My main point will be to report from recent joint work with J.-L. Brylinski and R. MacPherson [BBM1,2,3] how (iii) can be used as a tool to get insight into both (i) and (ii) simultaneously, and to understand their relation.

For some time, especially in the seventies and early eighties, these subjects developed more or less independently, each beeing studied for its own sake, by its own specific methods. Extensive work has been done by many authors, and remarkable theories have been developed on the three subjects, making each of them individually into a highly cultivated area of mathematical research. Since they have been reviewed individually on various former occasions, I feel free here to focus attention on some of the fascinating relations between these subjects. For more back-ground on the individual subjects, the reader may consult for instance the Lecture Notes by Steinberg [St], Slodowy [Sl], or Spaltenstein [Sp] on (i), the books by Dixmier [Di] and Jantzen [Ja] on (ii), and say [Hi], [Fu] in combination with [BBM1,2,3] concerning (iii).

1.2 The first hint, suggesting that there must be some deep relation between (i) and (ii), became apparent from the fundamental work of T.A. Springer [S] in 1976, resp. A. Joseph [J1,2] a few years later, on (i) resp. (ii): Their results came down to closely relating (i) resp. (ii) to the same kind of objects, namely irreducible Weyl group representations. A careful comparison of Springer's and Joseph's correspondences, ultimately extended by D. Barbasch and D. Vogan [BV1,2] to exhaustive explicit case by case calculations, confirmed some superficial parallelities on one hand, but also exhibited some intriguing discrepancies on the other hand, so that the real relation remained a mystery for some years.

This situation was considered unsatisfactory by some people, including myself. Strictly speaking, in my case, this challenge dates back already to my 1976 exposê at the sêminaire Bourbaki [B1], where I first suggested to relate (i) to (ii) via associated varieties, then reported Jantzen's conjectural partial anticipation of Joseph's theory in case $G = SL_n$ (see loc. cit. 2.9. resp. 5.9), and finally learnt from Springer about his new theory; consequently, it was inevitable for me to wonder how these pieces may fit together into a common frame-work. The solution of this puzzle took me not only some time, but also some new advanced methods, as well as two good friends to teach me how to use them. Much of my joint work with Brylinski or MacPherson was stimulated by the challenge of this puzzle, and is finally involved in the solution reported here. We use the intersection homology approach to (i), as developed in joint work with MacPherson [BM1,2], and the $\mathcal{D}$-module approach to (ii), as developed in joint work with Brylinski [BB1,2], and we combine them using equivariant K-theory (on T*X) as a unifying concept, in order to obtain a common frame-work for the simultaneous study of (i) and (ii) in terms of (iii), as elaborated in joint work with both [BBM1,2,3]. Let me also refer at this point to related work of V. Ginsburg [Gi].

1.3 The purpose of this report is two-fold: First, to popularize the "puzzle" mentioned above (section 2), and second to state our solution (section 3 and 4). In section 2, I tried to illustrate the problem in an intelligible way for the non-expert, using $G = SL_n$ as a standard example. Note that this is not only a courtesy to the reader not familiar with semisimple Lie group theory, but it also avoids here almost totally those "intriguing discrepancies" mentioned above; this will make our problem (to explain the coincidences between Springer's and Joseph's correspondence) particularly clear and persuasive. On the other hand, those "discrepancies" and their explanations for the other simple Lie groups are precisely what fascinates the real gourmet in Lie theory most of all. So in some sense, the experts may consider it a loss of good taste, that I have sacrificed such points radically for the sake of popularity; I hope that they will forgive me.

In sections 3 and 4, I tried to formulate an essential part of our results with as little effort as possible. I spent some care in defining important concepts, but essentially no proofs are included here. However, I added here a final section with comments on the general strategy of proof (section 5), offering at least some flavour of equivariant K-theory and its use in this context.

Remark. This report is essentially identical with my address to the International Congress of Mathematicians at Berkeley, except for the augmentation by a fifth section, concerning methods.

2. REVIEW ON SPRINGER'S AND JOSEPH'S WEYL GROUP REPRESENTATIONS

2.1 Basic notation. We fix a Borel subgroup B in G, and a maximal torus T in B. We denote by $\underline{t} \subset \underline{b} \subset \underline{g}$ the Lie algebras of $T \subset B \subset G$, by $W = N_G(T)/T$ the Weyl group, and by $X = G/B$ the flag variety.

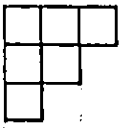
2.2 Standard example. To simplify the present review, I shall sometimes restrict (in the present chapter only) to $G = SL_n$ as a standard example. The reader not familiar with general semisimple Lie group theory may anyway prefer to think in terms of this example through out this paper: Then G is the group $SL(n,\mathbb{C})$ of complex n by n matrices $(n \geq 2)$ with determinant 1, and T resp. B are the subgroups of its diagonal resp. upper triangular matrices. The Lie algebra $\underline{g} = \underline{sl}(n,\mathbb{C})$ consists then of all complex n by n matrices of trace 0, with Lie bracket $[x,y] = xy - yx$ the commutator of matrices, and $\underline{t}$ resp. $\underline{b}$ consists of all traceless diagonal resp. upper triangular matrices. The Weyl group W is the symmetric group S_n in this case, acting on T and $\underline{t}$ by permuting the n eigenvalues of a diagonal matrix. The flag variety $X = G/B$ may in this case be defined alternatively in the original sense: Its points $F \in X$ may be thought of as real "flags" in $\mathbb{C}^n$; that is F is an ascending chain of complex subspaces $F_1 \subset F_2 \subset \ldots \ldots \subset F_n$ of $\mathbb{C}^n$ such that F_i has dimension i.

2.3 Nilpotent orbits. Let N denote the "nilpotent cone" in $\underline{g}$, that is the closed subvariety of all ad-nilpotent elements, which is a cone in $\underline{g}$. (A cone in a vector space is a subset closed under homotheties.) Under the adjoint action of G on $\underline{g}$, N decomposes into a finite number of orbits, called "nilpotent orbits".

In case $G = SL_n$, these orbits are just the conjugacy classes of nilpotent complex n by n matrices. Recall that the set N/G of nilpotent orbits is here in bijection to the set $P(n)$ of partitions of n (theory of Jordan normal form), the "parts" being just the sizes of Jordan blocks. For example, the nilpotent orbit $\mathcal{O}$ in $\underline{sl}_6$ generated by the nilpotent matrix

$$X = \left[\begin{array}{ccc|cc|c} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & 0 & & & \\ \hline & & & 0 & 1 & \\ & & & & 0 & \\ \hline & & & & & 0 \end{array}\right] \qquad \text{(all other entries zero)}$$

corresponds to the partition $\lambda = (3,2,1)$, which is also denoted

$\lambda =$ 

(notation of "Young diagrams").

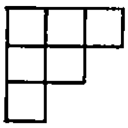
2.4 Primitive ideals. By definition, a primitive ideal J in $U(\underline{g})$ is the kernel of some irreducible representation of $\underline{g}$, or in other words: the annihilator of some simple (left) $U(\underline{g})$-module L, notation $J = \mathrm{Ann}\, L$. The center of $U(\underline{g})$ necessarily acts by a character on L, which we assume here to be "trivial", or equivalently, this means $J \subset \underline{g}\, U(\underline{g})$. Let X_0 denote the set of all such primitive ideals. This set is finite. More precisely, one defines a surjection $W \longrightarrow X_0$ as follows: The module L above can always be chosen in a particularly nice way, as a so-called simple highest weight module (theorem of Duflo), and the highest weights in question here come from a single regular W orbit in $\underline{t}^*$ (Harish-Chandra isomorphism), so the modules in question can be suitably indexed by Weyl group elements $w \in W$, and that surjection $W \longrightarrow X_0$ can then be described $w \longmapsto \mathrm{Ann}\, L_w =: J_w$.

2.5 Combinatorial description. In case $G = SL_n$, the set X_0 of primitive ideals with trivial central character is in bijection to the set $\mathcal{T}(n)$ of all tableaux of size n (theory of Joseph's Goldie rank polynomials, see [Ja]).

Here a "tableau" is the combinatorial object also familiar as a "Young standard tableau" from the representation theory of symmetric groups. An example of a tableau of size $m = 6$ is

$$\tau = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & \multicolumn{1}{|c}{} \\ \cline{1-2} 6 & \multicolumn{1}{|c}{} & \\ \cline{1-1} \end{array} \;;$$

its "shape" is the partition

$\lambda = (3,2,1) =$  ,

and there are exactly 16 tableaux of this same shape. There is a canonical, easily calculable map $T\colon W = S_n \longrightarrow \mathcal{T}(n)$, producing from each permutation w a tableau $T(w)$ (Robinson-Schensted algorithm). For example, the above tableau τ is produced as $T(w)$ from the permutation $w = 216453$, and from 15 other permutations.

Now the bijection between tableaux and primitive ideals, $\mathcal{T}(n) \longrightarrow X_0$,

$\tau \longmapsto J_\tau$, may be described as follows (Joseph): $J_\tau = \mathrm{Ann}\ L_w$ for any permutation w of tableau $\tau = T(w)$.

2.6 Associated varieties. There is also a map $X_0 \longrightarrow N/G$ from primitive ideals to nilpotent orbits. In case $G = SL_n$, in view of the preceding combinatorial descriptions of the two sets $X_0, N/G$, the reader will find it not hard to guess what this map should do in combinatorial terms: It sends the primitive ideal $J = J_\tau$ corresponding to a tableau τ to the nilpotent orbit $\mathcal{O} = \mathcal{O}_\lambda$ corresponding to the shape λ of τ.

But how do we produce directly, and in general, a nilpotent orbit from a primitive ideal J? For this purpose, we identify the associated graded ring of $U(\underline{g})$ with the symmetric algebra $S(\underline{g})$, and interpret it as ring of polynomial functions on $\underline{g}^* = \underline{g}$ (using first the Poincaré-Birkhoff-Witt theorem, and second the Killing form). Then we can define the associated variety of J as the zero set $V(\mathrm{gr}J) \subset \underline{g}$ of the associated graded ideal $\mathrm{gr}\ J \subset S(\underline{g})$. It turns out that this associated variety is irreducible [BB1,2], see also [J3], and contained in N; hence it contains a unique dense orbit $\mathcal{O}$. So the desired map $J \longmapsto \mathcal{O}$ is given by $V(\mathrm{gr}J) = \bar{\mathcal{O}}$. Let me comment here that this relation was suggested [B1] before, but completely proven [BB1] only after Joseph invented his classification theory of primitive ideals (2.9).

2.7 Link to Weyl group representations, illustrated in case $G = SL_n$. Young diagrams, and tableaux, the combinatorial data which we encountered above in the classification of nilpotent orbits resp. primitive ideals, both have a well-known significance in the (Frobenius') theory of representations of the symmetric group: There is a bijection, denoted $\lambda \longmapsto \rho_\lambda$, $P(n) \longrightarrow W^\wedge$, from diagrams λ of size n, to (equivalence classes of) irreducible complex representations of $W = S_n$. Moreover, the tableaux τ of a given shape λ correspond bijectively to a linear basis for the representation ρ_λ. In view of this, the results for $G = SL_n$ reviewed in 2.3, 2.5, may persuade you to make the following guess:

To a nilpotent orbit $\mathcal{O}$, there should correspond an irreducible representation $\rho_{\mathcal{O}}$ of the Weyl group, and to the collection of primitive ideals J with associated variety $\bar{\mathcal{O}}$ should correspond a basis of this same representation. It turns out that this is actually true, not only in case $G = SL_n$, but in general, and the main purpose of my report is to explain how such correspondences can be constructed. I shall next introduce the correspondences of Springer and Joseph, which can be made to do this job, although only with some major effort to verify that the two representations are actually the same. An alternative version, were this difficulty disappears, is then stated in theorems 3.4 and 4.4 below.

2.8 Springer's correspondence. Springer [S] attaches to each nilpotent orbit $\mathcal{O}$ an irreducible representation $\rho_{\mathcal{O}}$ of W as follows: Let $u \in \mathcal{O}$ represent the orbit, and let $X^u \subset X$ be the subvariety of all "flags" respected by u. Then W acts linearly on the homology groups $H_*(X^u)$. (For simplicity, here and below take complex coefficients for (co-)homology groups etc.). For $G = SL_n$, this action is irreducible on the homology group $H_{2d}(X^u)$ of highest degree $(2d = 2\dim X^u)$, and this defines $\rho_{\mathcal{O}}$; this is Springer's explicit, geometrical realization of the bijection $N/G \longrightarrow W\hat{}$ described only combinatorially in 2.7.

For G arbitrary, one has to replace $H_{2d}(X^u)$ by its invariants under the action of the isotropy group $G_u \subset G$ of u to define Springer's representation $\rho_{\mathcal{O}}$, and one obtains only an injection $N/G \longrightarrow W\hat{}$ by mapping $\mathcal{O}$ to $\rho_{\mathcal{O}}$.

2.9 Joseph's Goldie rank polynomials. Joseph [J1] attaches to each primitive ideal $J \in X_0$ a polynomial function p_J on the Cartan subalgebra $\underline{t}$. His construction proceeds in two steps. The first is, to replace J by the whole infinite family $(J_\mu)_{\mu \in Z}$ of "translated" primitive ideals J_μ, with variable central character, depending on a parameter μ which varies in a Zariski dense subset Z of $\underline{t}^*$ ("translation principle" of [BJ]). The second step is to consider the Goldie rank $\text{rk}\ U(\underline{g})/J_\mu$ as a function of μ, and to prove that this extends to a polynomial function on $\underline{t}$; this polynomial, by definition, is p_J. (For the reader not familiar with non-commutative Noetherian ring theory, let me also add the definition of Goldie rank: The quotient ring $U(\underline{g})/J_\mu$ has a complete ring of fractions, which is simple Artinian (Goldie's theorem), hence is a matrix ring over some skew fields (Wedderburn-Artin); then $\text{rk}\ U(\underline{g})/J_\mu$ is the rank of this matrix ring.)

2.10 Joseph's correspondence. Next, Joseph attaches to the primitive ideal $J \in X_0$ the W-submodule generated by p_J in $S(\underline{t}^*)$, and proves that the corresponding W-representation, denoted $\sigma(J)$, is irreducible. This defines Joseph's correspondence $X_0 \longrightarrow W\hat{}$, $J \longmapsto \sigma(J)$. Moreover, if J' ranges over all primitive ideals such that $\sigma(J') = \sigma(J)$, then the corresponding Goldie rank polynomials provide a basis for the representation $\sigma(J)$. The reader will find an excellent exposition of this beautiful theory of Joseph in Jantzen's book [Ja].

2.11 Comparison. The intriguing question, how the correspondences of Springer resp. Joseph relate to each other, on which I commented several times before, can be made precise at this point as follows: Does the correspondence from primitive ideals to nilpotent orbits via associated varieties (2.5) combine with Springer's and Joseph's correspondences to a commutative triangle? Or in other words: Is $\sigma(J)$ equivalent to $\rho_{\mathcal{O}}$, if J corresponds to $\mathcal{O}$? In case $G = SL_n$, the explicit combinatorial descriptions given above prove that this is true. Barbasch and Vogan have verified this as a matter of fact for all cases, by an enormous amount of explicit calculations in [BV1,2]. More conceptual reasons are offered by Hotta and Kashiwara [HK] and below.

3. CHARACTERISTIC CLASS APPROACH TO NILPOTENT ORBITS

In these next two sections, I shall now sketch the simultaneous, uniform approach to both subjects, (i) nilpotent orbits, and (ii) primitive ideals, in terms of (iii) characteristic classes on the flag variety X, as suggested in my recent joint work with Brylinski and MacPherson. For the elaboration of more details, I refer to our original papers [BBM1-3].

3.1 Characteristic classes of cone bundles. The concept of a cone bundle K on X generalizes that of a vector bundle: The bundle map $K \longrightarrow X$ is assumed to be a locally trivial fibration of K by cones (in vector spaces). To extend the theory of Chern classes of vector-bundles, Fulton and MacPherson [Fu] introduced the notion of Segre class $s(K)$ of a cone bundle. It may be characterized by two axioms (1) for a vector-bundle K, $s(K) = c(K)^{-1}$ is the inverse of the total Chern class $c(K)$, and (2) $s(K)$ is functorial under proper push-forwards.

Now to define our characteristic class $Q(K)$ in $H^*(X)$, for any cone subbundle K of codimension d in the cotangent bundle T^*X, we multiply its Segre class by the total Chern class of T^*X, and take the lowest (degree 2d) homogeneous term of the product, notationally:

$$Q(K) := [c(T^*X)s(K)]^{2d}$$

3.2 Springer's resolution of the nilpotent cone. Another key ingredient for our construction is the famous Springer map $\pi: T^*X \longrightarrow N$, which will allow us to pass from nilpotent orbits to the geometry of the flag variety. So let me recall here that this remarkable map has a very easy, elegant definition (as a Kostant-Souriau momentum map [BB1]): The natural action of $\underline{g}$ by vector-fields on X defines a morphism $\underline{g} \times X \longrightarrow TX$ into the tangent bundle, and the map π of the cotangent bundle T^*X into $\underline{g}^* = \underline{g}$ (Killing form) is then obtained by transposition and projection. The remarkable point about π is then that its image in $\underline{g}$ is the nilpotent cone N, and that it resolves the singularities of N.

3.3 Construction of characteristic classes from a nilpotent orbit. Starting from a nilpotent orbit $\mathcal{O} \subset N$, we first produce a collection of cone bundles on X by taking the preimage $\mathcal{O}$ under Springer's map π and then decomposing the closure $\overline{\pi^{-1}\mathcal{O}}$ into irreducible components $K_1,\ldots,K_r$. These are in fact cone bundles

$K_i \longrightarrow X$, called orbital for σ. We know that their codimension d in T^*X depends only on σ, more precisely $2d = \mathrm{codim}_N \sigma$ (work of Spaltenstein and Steinberg).

Next, we take for these "orbital" cone bundles $K_1, \ldots, K_r$ the characteristic classes $Q(K_1), \ldots, Q(K_r)$ as defined in 3.1.

3.4 Theorem.

a) The classes $Q(K_1), \ldots, Q(K_r)$ are linearly independent.

b) They form a W-submodule in $H^{2d}(X)$.

c) This W-representation is equivalent to Springer's ρ_σ.

d) It transforms the basis $Q(K_1), \ldots, Q(K_r)$ according to the formulae below.

The following formulae were first obtained by Hotta in a slightly different context [H1], [H2], see also [J4].

3.5 Hotta's transformation formulae. For each simple reflection s of W, and for all $i = 1, \ldots, r$, we have either

$$sQ(K_i) = -Q(K_i),$$

or else

$$sQ(K_i) = \sum_{j=1}^{r} n_{ij}(s)\, Q(K_j),$$

for some matrix of non-negative integers $n_{ii}(s)$ with diagonal entries $n_{ij}(s) = 1$. Moreover, there is a geometrical interpretation for these integers, for which I refer to our resp. Hotta's original papers. For example, this says that $n_{ij}(s) = 0$ unless K_j intersects K_i in codimension ≤ 1.

3.6 Algebraic construction of our characteristic classes. For the reader with less inclination for geometrical elegance, but with a preference for abstract algebra, let me also offer here an alternative, more algebraic definition of the classes $Q(K)$ defined geometrically in 3.1. This definition refers to the following ingredients:

- Borel's description of $H^*(X)$ in terms of polynomials on the Cartan subalgebra $\underline{t}$, by a W-equivariant isomorphism $\beta: H^*(X) \xrightarrow{\sim} S(\underline{t}^*)^{\mathrm{H}}$ onto the space of all W-harmonic polynomials on $\underline{t}$.
- The Chern character $ch: K(X) \longrightarrow H^*(X)$, attaching to an (algebraic) vector-bundle on X its total Chern class (an isomorphism here).
- The Grothendieck group $K(X)$ of classes of (algebraic) vector-bundles on X, or equivalently the Grothendieck group of the category of all coherent sheaves of $\mathcal{O}_X$-modules.
- The analogous Grothendieck-group $K(T^*X)$, and its isomorphism σ^* with $K(X)$ induced by the zero-section $\sigma: X \hookrightarrow T^*X$ [Fu].

Given these ingredients, we may attach to any closed subvariety K of codimension d in T^*X a cohomology class $Q(K)$ in $H^{2d}(X)$, which we identify with a degree d homogeneous polynomial on $\underline{t}$ via $\beta(H^{2d}(X) = S^d(\underline{t}^*)^{\mathfrak{h}})$. The construction proceeds as follows: Take the class $[\mathcal{O}_K]$ determined by the structure sheaf of K in $K(T^*X)$, and apply the composition of the maps

$$K(T^*X) \xrightarrow[\sim]{\sigma^*} K(X) \xrightarrow[\sim]{ch} H^*(X) \xrightarrow[\sim]{\beta} S(\underline{t}^*)^{\mathfrak{h}};$$

finally take the lowest degree term. So formally, this alternative, algebraic definition of $Q(K)$ reads:

$$Q(K) := [\beta\ ch\ \sigma^*[\mathcal{O}_K]]^d.$$

For a <u>cone bundle</u> K it can be shown that this definition coincides with that in terms of Segre class as stated in 3.1.

4. CHARACTERISTIC CLASS OF A PRIMITIVE IDEAL

<u>4.1</u> <u>Characteristic variety of a primitive ideal</u>. Starting from a primitive ideal $J \in X_0$, we first construct a cone-subbundle in T^*X as follows. Later, we shall define the characteristic class of J as the characteristic class of this cone bundle (4.3).

Consider the quotient $U(\underline{g})/J$ as a left $\underline{g}$-module M, and take the corresponding sheaf of modules

$$\mathcal{M} := \mathcal{D}_X \otimes_{U(\underline{g})} M$$

over the sheaf $\mathcal{D}_X$ of rings of differential operators on X (Beilinson-Bernstein localization of M on X); then the <u>characteristic variety</u> resp. <u>cycle</u> of $\mathcal{M}$ are well-defined notions from the general theory of $\mathcal{D}$-modules, denoted $Ch(\mathcal{M})$ resp. $\mathcal{Ch}(\mathcal{M})$ here. By definition, $Ch(\mathcal{M})$ is a closed subvariety in T^*X, and $\mathcal{Ch}(\mathcal{M})$ is a formal integer linear combination

$$\mathcal{Ch}(\mathcal{M}) = \sum_i m_i\ [V_i],$$

in which each irreducible component V_i of the characteristic variety occurs with some well-defined positive multiplicity m_i. Now characteristic varieties are (by construction) fibred by cones, and in the present case, this fibration is locally trivial on X (as a consequence of the stability of J under the adjoint G-action, and the G-homogeneity of X). So $Ch(\mathcal{M})$, and hence its components V_i, are actually <u>cone bundles</u> over X.

<u>4.2</u> <u>Relation to nilpotent orbits</u>. By my work with Brylinski [BB2], these cone bundles are actually <u>orbital</u> for some nilpotent orbit $\mathcal{O}$. In more detail, by loc.cit. Springer's map π maps $Ch(\mathcal{M})$ onto the associated variety $V(grJ)$, which is the

closure of a nilpotent orbit, as reported already in 2.6. In the sequel $\mathcal{O}$ denotes this nilpotent orbit determined by J. It follows that $Ch(M)$ is contained in $\pi^{-1}\overline{\mathcal{O}}$, and now some tricky dimension arguments show that each component V_i is even contained in $\pi^{-1}\mathcal{O}$ as one of the irreducible components, hence is one of the orbital cone bundles $K_1, \ldots, K_r$ (notation 3.3). - Hence we may write our characteristic cycle as well as

$$Ch(M) = \sum_{i=1}^{r} z_i \, [K_i]$$

where we admit some of the "multiplicities" z_i to be zero.

4.3 Definition. Now we define the characteristic class of J in $H^*(X)$ as the characteristic class of its characteristic variety by:

$$P(J) := Q(Ch(M)) := \sum_{i=1}^{r} z_i \; Q(K_i).$$

Here $Q(K_i)$ is the characteristic class of a cone bundle as defined in 3.1 (or 3.6).

4.4 Theorem: Let $J_1, \ldots J_r \in X_0$ be the collection of primitive ideals corresponding to a nilpotent orbit $\mathcal{O}$ (that is with associated variety $\overline{\mathcal{O}}$). Then:

a) The characteristic classes $P(J_1), \ldots, P(J_r)$ are linearly independent.
b) They span a W-submodule in $H^{2d}(X)$. (Note $2d = \operatorname{codim}_N \mathcal{O}$.)
c) This W-representation is equivalent to Springer's $\rho_{\mathcal{O}}$.
d) The class $P(J_i)$ is proportional to Joseph's Goldie rank polynomial of J_i, that is $\beta P(J_i) = \gamma p_{J_i}$ for some scalar $0 \neq \gamma \in \mathbb{Q}$.

5. THE EQUIVARIANT K-THEORY SET UP FOR PROOFS

Let me conclude this report with the following comments concerning the general strategy of our uniform proofs of theorems 3.4 and 4.4. The purpose of these comments is to indicate the rôle played by the formalism of equivariant K-theory, and to sketch a few ideas which are crucial in our approach. I refer to my original papers with Brylinski and MacPherson for detailed expositions of proofs, and also for more elaborated statements of these and further results.

5.1 Refinement to the G-equivariant level. Our definition of characteristic classes $Q(K)$ as given in section 3 refers only to the geometrical structure of the cone bundle K. This is fine from the point of view of elegance of results, since it allows purely geometrical interpretations. However, from the point of view of proof of some of our results, it is more convenient to take also account of the additional structure on an orbital cone bundle K provided by the group action. The advantage of our algebraic construction of the characteristic class $Q(K)$ given in 3.6 is that it applies almost word by word to the definition of a more refined notion of "equivariant characteristic class", denoted $Q_G(K)$, as well. To be a little more precise, this is defined for any G-stable closed subvariety K in T^*X as a homogeneous polynomial on $\underline{t}$ as follows: Take the class $[\mathcal{O}_K]$ determined by the structure sheaf on K in the Grothendieck group $K_G(T^*X)$ of the category of G-equivariant coherent sheaves on T^*X, and then apply to it the following chain of homomorphisms

$$K_G(T^*X) \xrightarrow{\sigma_G^*} K_G(X) \xrightarrow{ch_G} H_G^{even}(X) \xrightarrow{\beta_G} \hat{S}(\underline{t}^*),$$

which refines the completely analogous one in subsection 3.6 to the G-equivariant level. Here the "equivariant Chern character" ch_G maps the equivariant K-group of the flag variety X into the even degree part of the equivariant cohomology group of X, which in turn by the "equivariant Borel picture" is identified with the ring $\hat{S}(\underline{t}^*)$ of formal power series on $\underline{t}$. Finally, define $Q_G(K)$ as the lowest degree term of the power series manufactured from K by this procedure.

5.2 Relating the equivariant to the geometric level. Now that we haven taken account of the additional G-structure on K we may look at the process of "forgetting" it again; this provides a canonical "forgetful" homomorphism $K_G(X) \longrightarrow K(X)$, which corresponds to projecting a power series F on $\underline{t}$ to its W-harmonic part $F^\natural$, as is easy to see. It turns out that the lowest degree term $Q_G(K)$ does not project to 0, or has non-zero harmonic part. This fact reflects an equality of the codimension of support of a coherent sheaf on X and the degree of the corresponding element in $K_G(X)$ with respect to Grothendieck's γ-filtration, which is true in this case, though a little delicate to prove. We conclude then that our characteristic class $Q(K)$ can be reobtained from the G-equivariant version $Q_G(K)$ just by taking the W-harmonic part:

$$Q(K) = Q_G(K)^\natural.$$

5.3 Reduction to the T-equivariant level. A key technique of our approach consists in switching from G-equivariant K-theory on the flag manifold to T-equivariant K-theory on a vector-space E as follows. Starting from a G-equivariant coherent sheaf on the cotangent bundle T^*X, we first restrict the group action from G to T, and then restrict the sheaf to the single fibre E of T^*X at the base point, which is fixed by T, to obtain a T-equivariant sheaf on E; on the other hand, we restrict the sheaf on T^*X to the zero section X, as we did already in 5.1. These restriction processes give isomorphisms

$$K_G(X) \cong K_G(T^*X) \cong K_T(E). \qquad (*)$$

The point of these manipulations is now that equivariant K-theory of a linear torus action can be carried out very conveniently in terms of calculations with formal characters, as I shall explain a little more precisely below, and that we can reduce our problems from the context of G- equivariant K-theory on X into this more convenient setting. Furthermore, as explained in subsection 5.2, the link to our previous geometrical considerations is made by means of the homomorphisms

$$K_G(X) \longrightarrow K(X) \xrightarrow{ch} H^*(X).$$

In conclusion, this exhibits our technique of translating statements from a context most convenient for computational manipulations (formal characters) into a context most convenient for geometrical interpretation (cohomology of the flag variety), and vice versa. This is one of the crucial ideas underlying the strategy of proofs in [BBM3].

5.4 Formal characters. Since T acts linearly on E, the zero point is T-stable. So the inclusion $\iota: \{0\} \longrightarrow E$ induces a functorial ring homomorphism ι^* of $K_T(E)$ into $K_T(0)$. But $K_T(0)$ is nothing else but $R(T)$, the representation ring of T, which may also be considered as the group algebra of the character group $X(T)$ of T. The map $\iota^*: K_T(E) \longrightarrow R(T)$ turns out to be an isomorphism, which may be described explicitely as follows.

Let F be a T-equivariant coherent sheaf on E. Then $M = \Gamma(E,F)$ is a finitely generated $S(E^*)$-module equipped with an equivariant T-action, and so it decomposes into a direct sum of weight spaces M_x, where $x \in X(T)$. Now one can define the formal character of M as usual as a formal sum

$$ch(M) = \sum_{x\in X(T)} (\dim M_x)\,[x].$$

(Here one has to note that the weight multiplicities $\dim M_x$ are all finite because of the positivity of the weights of E.) We may multiply such expressions in an obvious way by elements of $R(T)$, for instance by

$$\Delta := \prod_x (1-x),$$

where the product is extended over the positive roots (the weights of T in E). Then the desired formula for ι^* reads as follows:

Proposition: $\iota^*[F] = \Delta\, ch(M)$.

In particular, $ch(M)$ may be considered as an element in the fraction field of $R(T)$. It is easy to deduce from this formula the

Corollary: ι^* is an isomorphism of $K_T(E)$ onto $R(T)$.

5.5 A formula for characteristic classes in terms of formal characters. We consider $R(T)$ as a subring of the ring $\hat{S}(\underline{t}^*)$ of formal power series on $\underline{t}$. This is done by making a character x of T correspond to the exponential of its differential:

$$e^{dx} = \sum_{n\geq 0} \frac{1}{n}(dx)^n = x.$$

If $P \in \hat{S}(\underline{t}^*)$ is any power series, then we denote $[P]^d$ its degree d homogeneous term, which is of course a homogeneous polynomial.

Theorem: Let K be an orbital cone bundle. Let $M = \Gamma(E, \mathcal{O}_K)$, the ring of the regular functions on its fibre over the base point, considered as a T- equivariant $S(E^*)$-module.

Then

a) As a formal power series on $\underline{t}$, $\Delta\, ch(M)$ has its lowest nonzero homogeneous term in degree

$d := \operatorname{codim}_{T^*X} K$.

b) The equivariant characteristic class of K (as a polynomial on $\underline{t}$) is given by the formula

$Q_G(K) = [\Delta\, ch(M)]^d$.

c) The characteristic class of K is given by the formula

$Q(K) = ([\Delta\, ch(M)]^d)^\natural$

as the harmonic part of the lowest degree term of $\Delta\, ch(M)$.

5.6 Conclusive remarks on the proof of theorems 3.4 and 4.4. This is the desired explicit expression for our characteristic classes in terms of the formal characters. Since this expression relates our characteristic classes to the "character polynomials" as studied in the previous literature by Joseph, Jantzen, Vogan, and others, it enables us to prove parts d) and c) of theorems 3.4 and 4.4. In case of theorem 4.4, one has to use the work on characteristic varieties of primitive ideals in [BB2] and some $\mathcal{D}$-module theory as an additional ingredient.

To make the identification with Springer's representations, that is to prove part c) of the theorems 3.4 and 4.4, we do not need the equivariant level but we work directly on the geometrical level, using as additional main ingredient the work on intersection homology of closures of nilpotent orbits in [BM1,2].

References:

[BV1] Barbasch, D. - Vogan, D.: Primitive ideals and orbital integrals in complex classical groups; Math. Ann. 259 (1982), 153-199.

[BV2] Barbasch, D. - Vogan, D.: Primitive ideals and orbital integrals in complex exceptional groups; J. Algebra 80 (1983), 350-382.

[B1] Borho, W.: Recent advances in enveloping algebras of semisimple Lie algebras; Séminaire Bourbaki 1976, Springer LNM 677 (1978), exposé 489.

[BJ] Borho, W. - Jantzen, J.C.: Über primitive Ideale in der Einhüllenden einer halbeinfachen Lie-Algebra; Invent. Math. 39 (1977), 1-53.

[BB1] Borho, W. - Brylinski, J.L.: Differential operators on homogeneous spaces I; Invent. Math. 69 (1982), 437-476.

[BB2] Borho, W. - Brylinski, J.L.: Differential operators on homogeneous spaces III; Invent. Math. 80 (1985), 1-68.

[BBM1] Borho, W. - Brylinski, J.L. - MacPherson, R.: A note on primitive ideals and characteristic classes; in: Geometry Today, Birkhäuser: Progress in Math. 60 (1985), 11-20.

[BBM2] Borho, W. - Brylinski, J.L. - MacPherson, R.: Springer's Weyl group representations through characteristic classes of cone bundles; IHES preprint M/85/70, Dec. 1985.

[BBM3] Borho, W. - Brylinski, J.L. - MacPherson, R.: Equivariant K-theory approach to nilpotent orbits; IHES preprint M/86/13, March 1986.

[BM1] Borho, W. - MacPherson, R.: Représentations des groupes de Weyl et homologie d'intersection pour les variétés nilpotentes; C.R. Acad. Sci. Paris (A) 292 (1981), 707-710.

[BM2] Borho, W. - MacPherson, R.: Partial resolutions of nilpotent varieties; in: Analyse et Topologie sur les Espaces Singuliers, Soc. Math. de France, Astérisque 101 (1983), 23-74.

[Di] Dixmier, J.: Algèbres enveloppantes; Paris: Gauthier Villars 1974.

[Fu] Fulton, W.: Intersection theory, Springer: Berlin-Heidelberg- New York - Tokio 1984.

[Gi] Ginsburg, V.: $\underline{g}$-modules, Springer's representations and bivariant Chern classes; Advances Math. 59 (1986).

[H] Hiller, H.: Geometry of Coxeter groups; Res. Notes in Math. 54, Pitman: Boston-London-Melbourne 1982.

[H1] Hotta, R.: On Joseph's construction of Weyl group representations; Tohoku Math. J. 36 (1984), 49-74.

[H2] Hotta, R.: On Springer's representations, J. Fac. Sci., Univ. of Tokyo, IA 28 (1982), 836-876.

[HK] Hotta, R. - Kashiwara, M.: The invariant holonomic system on a semisimple Lie algebra; Invent. Math. 75 (1984), 327-358.

[Ja] Jantzen, J.C.: Einhüllende Algebren halbeinfacher Lie-Algebren; Springer: Berlin-Heidelberg-New York-Tokio 1983.

[J1] Joseph, A.: Goldie rank in the enveloping algebra of a semisimple Lie algebra I,II; J. of Algebra 65 (1980), 269-306

[J2] Joseph, A.: Kostant's problem, Goldie rank, and the Gelfand-Kirillov conjecture; Invent. Math. 56 (1980), 191-213.

[J3] Joseph, A.: On the associated variety of a primitive ideal; J. of Algebra 93 (1985), 509-523.

[J4] Joseph, A.: On the variety of a highest weight module; J. of Algebra 88 (1984), 238-278.

[Sl] Slodowy, P.: Simple singularities and simple algebraic groups; Springer LNM 815 (1980).

[Sp] Spaltenstein, N.: Classes unipotentes et sous-groupes de Borel; Springer LNM 946 (1982).

[St] Steinberg, R.: Conjugacy classes in algebraic groups; Springer LNM 366 (1974).

[S] Springer, T.A.: Trigonometric sums, Green functions of finite groups, and representations of Weyl groups; Invent. Math. 36 (1976), 173-207.