# Cohomology of Local Systems and Siegel Modular Forms for Genus 2

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This is a report on joint work with *Carel Faber*.

Let $\rho : \mathrm{GL}(2,\mathbb{C}) \to V$ be a finite-dimensional representation. A (genus 2) *Siegel modular form of weight* $\rho$ is a holomorphic function $f : \mathbb{H}_2 \to V$ satisfying

$$f((A\tau + B)(C\tau + D)^{-1}) = \rho(C\tau + D)f(\tau)$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z})$ and all $\tau \in \mathbb{H}_2$, the Siegel upper half space of degree 2. If $\rho$ is irreducible, say $\rho = \mathrm{Sym}^j \otimes \det^k$ of the standard representation then we say that $f$ has weight $(j,k)$. We let $M_{j,k}$ be the vector space of modular forms of weight $(j,k)$ and $S_{j,k}$ be the subspace of cusp forms. Since the representations of $\mathrm{GL}(2,\mathbb{C})$ form a ring we can form the ring

$$R := \oplus_{j,k} M_{j,k}$$

of (vector valued) Siegel modular forms. It contains the subring $R^0 := \oplus_k M_{0,k}$ of classical Siegel modular forms. Igusa determined generators for $R^0$, but for the larger ring $R$ very little is known. Satoh and Ibukiyama determined generators for the $R^0$-module $\oplus_k M_{2,k}$, but that is about it. But at least there is a formula for the dimension of $M_{j,k}$, due to Tsushima.

In the genus 1 case one can get information on modular forms using the moduli $A_1$ (= $j$-line) of elliptic curves. Let $\pi : X_1 \to A_1$ be the universal elliptic curve. On $A_1$ we have a local system $\mathbb{W} := R^1\pi_*\mathbb{Q}$ of rank 2. We put $\mathbb{W}^k := \mathrm{Sym}^k\mathbb{W}$. Then the Eichler-Shimura isomorphism tells us that

$$H^1_c(A_1, \mathbb{W}^k \otimes \mathbb{C}) \cong \mathbb{C} \oplus S_{k+2} \oplus \overline{S_{k+2}}.$$

Since $A_1$ is defined over $\mathbb{Z}$ we can get information on this cohomology by calculating the traces of Frobenius on the $\ell$- adic variant of the local system $\mathbb{W}^k$ over a finite field $\mathbb{F}_q$. Since knowledge of the cohomology $H^1_c(A_1, \mathbb{W}^k)$ for all $k$ is equivalent to knowing the cohomology of $M_{1,n}$, the moduli of $n$-pointed elliptic curves, for all $n$ (with trivial coefficients), we also could count $\#M_{1,n}(\mathbb{F}_q)$ and thus get information about cusp forms. For example, one finds the formula

$$\#\overline{M_{1,11}}(\mathbb{F}_p) = p^{11} + 2037p^{10} + 213677p^9 + \ldots + 2037p + 1 - \tau(p),$$

where $\tau(p)$ is the $p$-th Fouriercoefficient of the normalized cusp form $\Delta \in S_{12}$ of weight 12 and $\overline{M_{1,11}}$ denotes the Deligne-Mumford compactification.

We define the motivic Euler characteristic $e_c(X)$ of a variety $X$ as

$$e_c(X) := \sum (-1)^i [H_c^i(X)],$$

where the class $[H_c^i(X)]$ is taken in $K_0$ of mixed Hodge structures or of effective Chow motives. The formula above then becomes

$$e_c(\overline{M_{1,11}}) = L^{11} + 2037L^{10} + \ldots + 2037L + 1 - S[12],$$

where we now write $L$ for the Tate motive $h^2(\mathbb{P}^1)$ and $S[k]$ for the motive of cusp forms of weight $k$ on $\mathrm{SL}(2,\mathbb{Z})$. This motive (shown to exist by Scholl) has rank $2\dim S_k$ and weight $k-1$.

We consider the map $H_c^*(A, \mathbb{W}^k) \to H^*(A_1, \mathbb{W}^k)$. The image is denoted by $H_!^*(A_1, \mathbb{W}^k)$ and is called interior cohomology. It satisfies Poincaré duality.

For $k$ even, $k \geq 2$ we now can write $e_c(A_1, \mathbb{W}^k) = -S[k+2] - 1$ and $e_!(A_1, \mathbb{W}^k) = -S[k+2]$.

We now go to genus 2. Let $A_2$ be the moduli stack of principally polarized abelian surfaces. It contains the moduli $M_2$ of genus 2 curves as an open part. Moreover, $A_2$ comes with an universal abelian variety $\pi : X_2 \to A_2$ and with a standard local system $\mathbb{V} := R^1\pi_*\mathbb{Q}$ of rank 4. A pair of integers $(l,m)$ with $l \geq m \geq 0$ determines an irreducible representation of $\mathrm{Sp}(4,\mathbb{Q})$ and in a similar way a local system $\mathbb{V}_{l,m}$ over $A_2$. For example, $\mathbb{V} = \mathbb{V}_{1,0}$ and $\mathbb{V}_{1,1}$ is the primitive part of $\wedge^2\mathbb{V}$.

We are interested in the motivic Euler characteristic

$$e_c(A_2, \mathbb{V}_{l,m}) = \sum_i (-1)^i [H_c^i(A_2, \mathbb{V}_{l,m})].$$

Because of the hyperelliptic involution this vanishes if $l+m$ is odd.

What we did was this. We compiled for every finite field $\mathbb{F}_q$ of cardinality $q \leq 37$ ($q \neq 27$) a list of Weil polynomials (characteristic polynomial of Frobenius on $H^1_{et}(X, \mathbb{Q}_\ell)$) occuring for a curve $X$ of genus 2 defined over $\mathbb{F}_q$ and we determined their frequencies as the curve $X$ ran through $M_2(\mathbb{F}_q)$. Moreover, we complemented this by also treating the curves consisting of two elliptic curves. Since $A_2$ is a stack we need to take into account the order of the automorphism group of the objects that we count.

Independently, we also counted $\#M_{2,n}(\mathbb{F}_p)$ for all primes $p \leq 233$ and all $n$. All this required quite some computer time, to put it mildly.

Since we are really interested in the genus 2 forms we must first identify the parts that come from genus 1. Therefore, we define the *Eisenstein cohomology* by

$$e_{\mathrm{Eis}}(A_2, \mathbb{V}_{l,m}) := e_c(A_2, \mathbb{V}_{l,m}) - e_!(A_2, \mathbb{V}_{l,m}).$$

For genus 1 we have $e_{\mathrm{Eis}}(A_1, \mathbb{W}^k) = -1$ for $k \geq 2$ and even. Eisenstein cohomology for $g \geq 2$ was studied by Harder, Schwermer and Pink. On the basis of our calculations we could guess a formula for the Eisenstein cohomology and we now think we can prove the following.

**Theorem 1** *For $l > m > 0$ the Euler characteristic $e_{\mathrm{Eis}}(A_2, \mathbb{V}_{l,m})$ of the Eisenstein cohomology is*

$$-S[l+3] - s_{l+m+4}L^{m+1} + S[m+2] + s_{l-m+2} \cdot 1 + \begin{cases} 1 & l \quad even \\ 0 & else \end{cases},$$

*with $s_n = \dim S_n$.*

Faltings proved that $H^3(A_2, \mathbb{V}_{l,m})$ and $H^3_c(A_2, \mathbb{V}_{l,m})$ possess a Hodge filtration of weight $\geq l+m+3$ and $\leq l+m+3$; $H^3_!(A_2, \mathbb{V}_{l,m})$ thus gets a Hodge filtration

$$(0) \subset F^{l+m+3} \subset F^{l+2} \subset F^{m+1} \subset F^0 = H^3_!(A_2, \mathbb{V}_{l,m}).$$

Moreover, if the weight $(l, m)$ is regular (i.e. $l > m > 0$) then $H^i_!(A_2, \mathbb{V}_{l,m}) = (0)$ for $i \neq 3$.

The Hodge step $F^{l+m+3}$ has an interpretation in terms of Siegel modular forms

$$F^{l+m+3} \cong S_{l-m,m+3}.$$

But there is still one more contribution from $g = 1$ that we must pin down. It stems from the endoscopic lift. Endoscopic lifting for the symplectic group has been studied by various people, like Kudla, Rallis, Weissauer. On the basis of our calculations we can make a guess for this contribution. The experts on endoscopic lifing should be able to prove this.

**Conjecture 1** *For $(l, m)$ regular the endoscopic contribution is given by*

$$e_{\mathrm{endo}}(A_2, \mathbb{V}_{l,m}) = -s_{l+m+4}S[l-m+2]L^{m+1}.$$

The remainder now comes from genus 2 Siegel modular forms in a "true sense":

$$S[l-m, m+3] := H^3_!(A_2, \mathbb{V}_{l,m}) - H^3_{\mathrm{endo}}(A_2, \mathbb{V}_{l,m})$$

should be a motive of rank $4 \dim S_{l-m,m+3}$ and with a Hodge filtration with steps $l+m+3, l+2, m+1, 0$. In relevant cases we can check this conjecture.

Our results enable us, assuming the conjecture on the endoscopic contribution, to calculate the traces $\lambda(p)$ of the Hecke operator $T(p)$ on $S_{j,k}$ for all $p \leq 37$.

Let us give an example. The first case which is interesting is $S_{6,8}$; here the dimension is 1. At our request Ibukiyama constructed a non-zero form in $S_{6,8}$ as a theta series in 16 variables. But for $g = 2$ it is very difficult to calculate Hecke eigenvalues from Fourier coefficients.

We can calculate the characteristic polynomials at the primes $p = 2$ and $p = 7$. These are

$$X^4 - 204800X^2 + 274877906944$$

and

$$\begin{aligned} X^4 &+ 107822000\, X^3 + 5939548912704750\, X^2 \\ &+ 122905167667730302454600\, X \\ &+ 1299348114471230201171721456 98449. \end{aligned}$$

Finally, I want to illustrate this by giving some Hecke eigenvalues on the 1-dimensional spaces $S_{8,8}$ and $S_{12,6}$.

| $p$ | $\lambda(p)$ on $S_{8,8}$ | $\lambda(p)$ on $S_{12,6}$ |
|---:|---:|---:|
| 2 | $2^6 \cdot 3 \cdot 7$ | $-2^4 \cdot 3 \cdot 5$ |
| 3 | $-2^3 \cdot 3^2 \cdot 89$ | $2^3 \cdot 3^5 \cdot 5 \cdot 7$ |
| 5 | $-2^2 \cdot 3 \cdot 5^2 \cdot 13^2 \cdot 607$ | $2^2 \cdot 3 \cdot 5^2 \cdot 7 \cdot 79 \cdot 89$ |
| 7 | $2^4 \cdot 7 \cdot 109 \cdot 36973$ | $-2^4 \cdot 5^2 \cdot 7 \cdot 119633$ |
| 11 | $2^3 \cdot 3 \cdot 4759 \cdot 114089$ | $2^3 \cdot 3 \cdot 23 \cdot 2267 \cdot 2861$ |
| 13 | $-2^2 \cdot 13 \cdot 17 \cdot 109 \cdot 3404113$ | $2^2 \cdot 5 \cdot 7 \cdot 13 \cdot 50083049$ |
| 17 | $2^2 \cdot 3^2 \cdot 17 \cdot 41 \cdot 1307 \cdot 168331$ | $-2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 47 \cdot 14320807$ |
| 19 | $-2^3 \cdot 5 \cdot 74707 \cdot 9443867$ | $-2^3 \cdot 5 \cdot 7^3 \cdot 19 \cdot 2377 \cdot 35603$ |

## References


[1] Carel Faber, Gerard van der Geer: Sur la cohomologie des Systèmes Locaux sur les Espaces des Modules des Courbes de Genre 2 et des Surfaces Abéliennes. `math.AG/0305094`.