DEFORMED VIRASORO and $\mathcal{W}$ ALGEBRAS from ELLIPTIC ALGEBRAS

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In memoriam M. SAVELIEV

Abstract

Deformed W_n-algebras in the classical (i.e. Poisson structures) and in the quantum cases (exchanges algebras) are constructed from the elliptic algebras $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$.

When I am thinking about Misha, and this has been happening rather frequently these recent months, it is his enthusiasm, his communicative energy which come first in my mind. Something was very important to Misha: that was friendship. We, scientists, have the priviledge to possess friends, real friends among our colleagues, and Misha was a friend. Something else was fundamental for Misha: his family,i.e. his parents, his wife, his daughter. As Professor Manin said this morning, summarising Misha's behaviour very simply and elegantly, Misha was "a good son, a good husband, a good father". Misha's parents died about three years ago, and I remember how he was sad, when meeting him in Georgia in september 96, to inform me about his father and mother death a few months before. Since then, he was concentrating all his forces to protect his wife and his daughter, the two Svetlanas, as he used to say! He was so proud of his gifted daughter: a good musician, a good painter, a brilliant student. I wish to present to them my most respectful thoughts.

Misha was a great expert in Integrable Systems. He was deeply interested in developing models, what he did so nicely in the framework of Toda theories. I learnt a lot from him on Toda theories and also on continuous Lie algebras, subjects on which we collaborated in the 90's. Misha was highly convinced by the importance of symmetries and specially interested in the use of infinite symmetries, among them by the $\mathcal{W}$ algebras. The landscape of continuous symmetries and particularly of infinite dimensional symmetries is becoming richer and richer these years, and I dare to think that Misha would have been interested in the modest seminar I will present now.

The talk which follows is the result of a collaboration started about two years ago, and still in progress, with Jean Avan from Paris, Luc Frappat from Annecy and Marco Rossi from Pisa and to-day in Durham. Our results [1, 2, 3] concern the construction, from the quantum elliptic algebras $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ of deformed Virasoro and more generally deformed $\mathcal{W}$ algebras. To some extent, our approach can be seen as a kind of Sugawara construction in the deformed case. One has to remark that our method differs from others [4, 5, 6] in two essential ways: first, it uses the algebras $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$; secondly, computation are directly performed from the abstract generators of the elliptic algebra, without help of a special realisation, which is the case in the above quoted papers based in particular on the q-deformed bosonisation of the quantum group $\mathcal{U}_q(\widehat{sl}(N)_c)$.

After introducing the algebra $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ [7, 8], we first prove the existence of an extended center in $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ at $c = -N$ and compute the Poisson structures on this center. These structures are identified as q-deformed $Vir_q(sl(N))$ algebras. We then show the existence of closed (quadratic) exchange algebras whenever $(-p^{\frac{1}{2}})^{NM} = q^{-c-N}$ for any integer $M \in \mathbb{Z}$. These algebras admit a classical limit (commuting algebras) at $p = q^{Nh}$ with $h \in \mathbb{Z}\backslash\{0\}$. The related Poisson structures include for h even the Poisson structures in [4]. The exchange algebras therefore realize new quantizations of these Poisson structures. When h is odd, by contrast, this classical limit takes a form different from the initial $Vir_q(sl(N))$ structures. This emphasizes the key role of the initial 3-parameter structure $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ in allowing for an intermediate quantum q-deformed Virasoro algebra. When computing the mode expansion of these structures (both classical and quantum), a "sector-type" labeling is needed due to the singularity behaviour of the structure function viewed as an analytic continuation and therefore exhibiting different formal series expansions corresponding to different convergent series expansions in distinct domains of the complex plane. Finally, we briefly describe the essential features of the extension of our computation to higher spin fields. We conclude by mentioning developments and projects, and pointing out the important role played by Drinfeld twists in providing a synthetic view of deformed algebras.

1 The elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$

We start by defining the R-matrix of the $\mathbb{Z}_N$-vertex model ($\mathbb{Z}_N \equiv \mathbb{Z}/N\mathbb{Z}$) [7, 8]:

$$\widetilde{R}(z,q,p) = \frac{z^{2/N-2}}{\kappa(z^2)} \frac{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau)}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\xi+\zeta,\tau)} \sum_{(\alpha_1,\alpha_2)} W_{(\alpha_1,\alpha_2)}(\xi,\zeta,\tau)\ I_{(\alpha_1,\alpha_2)} \otimes I^{-1}_{(\alpha_1,\alpha_2)},$$

where $z = e^{i\pi\xi}$, $q = e^{i\pi\zeta}$, $p = e^{2i\pi\tau}$ and $(\alpha_1,\alpha_2) \in \mathbb{Z}_N \times \mathbb{Z}_N$.
ϑ are the Jacobi theta functions with rational characteristics $(\gamma_1,\gamma_2) \in \mathbb{Z}/N \times \mathbb{Z}/N$:

$$\vartheta\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}(\xi,\tau) = \sum_{m\in\mathbb{Z}} \exp\Big(i\pi(m+\gamma_1)^2\tau + 2i\pi(m+\gamma_1)(\xi+\gamma_2)\Big).$$

The normalization factor is given by:

$$\frac{1}{\kappa(z^2)} = \frac{(q^{2N}z^{-2};p,q^{2N})_\infty\,(q^2z^2;p,q^{2N})_\infty\,(pz^{-2};p,q^{2N})_\infty\,(pq^{2N-2}z^2;p,q^{2N})_\infty}{(q^{2N}z^2;p,q^{2N})_\infty\,(q^2z^{-2};p,q^{2N})_\infty\,(pz^2;p,q^{2N})_\infty\,(pq^{2N-2}z^{-2};p,q^{2N})_\infty}.$$

The functions $W_{(\alpha_1,\alpha_2)}$ are defined as follows:

$$W_{(\alpha_1,\alpha_2)}(\xi,\zeta,\tau) = \frac{1}{N}\,\frac{\vartheta\left[\begin{array}{c}\frac{1}{2}+\alpha_1/N\\ \frac{1}{2}+\alpha_2/N\end{array}\right](\xi+\zeta/N,\tau)}{\vartheta\left[\begin{array}{c}\frac{1}{2}+\alpha_1/N\\ \frac{1}{2}+\alpha_2/N\end{array}\right](\zeta/N,\tau)}\,,$$

and the matrices $I_{(\alpha_1,\alpha_2)}$ by $I_{(\alpha_1,\alpha_2)} = g^{\alpha_2}\,h^{\alpha_1}$ where $g_{ij} = \omega^i\delta_{ij}$, $h_{ij} = \delta_{i+1,j}$ are $N\times N$ matrices (the addition of indices being understood modulo N) and $\omega = e^{2i\pi/N}$.
The R-matrix $\widetilde{R}$ is $\mathbb{Z}_N$-symmetric:

$$\widetilde{R}^{a+s\,,\,b+s}_{c+s\,,\,d+s} = \widetilde{R}^{a\,,\,b}_{c\,,\,d} \qquad a,b,c,d,s\in\mathbb{Z}_N\,.$$

We introduce the "gauge-transformed" matrix:

$$R(z,q,p) = (g^{\frac{1}{2}}\otimes g^{\frac{1}{2}})\widetilde{R}(z,q,p)(g^{-\frac{1}{2}}\otimes g^{-\frac{1}{2}})\,.$$

It satisfies the following properties:
– Yang–Baxter equation: $R_{12}(z)\,R_{13}(w)\,R_{23}(w/z) = R_{23}(w/z)\,R_{13}(w)\,R_{12}(z)$,
– unitarity: $R_{12}(z)R_{21}(z^{-1}) = 1$,
– crossing symmetry: $R_{12}(z)^{t_2}R_{21}(q^{-N}z^{-1})^{t_2} = 1$,
– antisymmetry: $R_{12}(-z) = \omega\,(g^{-1}\otimes\mathbb{I})\,R_{12}(z)\,(g\otimes\mathbb{I})$,
– quasi-periodicity: $\widehat{R}_{12}(-p^{\frac{1}{2}}z) = (g^{\frac{1}{2}}hg^{\frac{1}{2}}\otimes\mathbb{I})^{-1}\left(\widehat{R}_{21}(z^{-1})\right)^{-1}(g^{\frac{1}{2}}hg^{\frac{1}{2}}\otimes\mathbb{I})$,
where $\widehat{R}_{12}(x) = \tau_N(q^{1/2}x^{-1})\,R_{12}(x)$ and $\tau_N(z) = z^{\frac{2}{N}-2}\,\frac{\Theta_{q^{2N}}(qz^2)}{\Theta_{q^{2N}}(qz^{-2})}$. The function $\tau_N(z)$ is periodic with period q^N: $\tau_N(q^Nz) = \tau_N(z)$ and satisfies $\tau_N(z^{-1}) = \tau_N(z)^{-1}$.

We now define the elliptic quantum algebra $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ [7, 8]. It is an algebra of operators $L_{ij}(z) \equiv \sum_{n\in\mathbb{Z}} L_{ij}(n)\,z^n$ where $i,j\in\mathbb{Z}_N$:

$$L(z) = \left(\begin{array}{ccc} L_{11}(z) & \cdots & L_{1N}(z)\\ \vdots & & \vdots\\ L_{N1}(z) & \cdots & L_{NN}(z)\end{array}\right).$$

The q-determinant is given by ($\varepsilon(\sigma)$ being the signature of the permutation σ):

$$q\text{-}\det L(z) \equiv \sum_{\sigma\in\mathfrak{S}_N}\varepsilon(\sigma)\prod_{i=1}^{N}L_{i,\sigma(i)}(zq^{i-N-1})\,.$$

$\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$ is defined by imposing the following constraints on $L(z)$:

$$\widehat{R}_{12}(z/w)\,L_1(z)\,L_2(w) = L_2(w)\,L_1(z)\,\widehat{R}^*_{12}(z/w)\,,$$

where $L_1(z)\equiv L(z)\otimes\mathbb{I}$, $L_2(z)\equiv\mathbb{I}\otimes L(z)$ and $\widehat{R}^*_{12}(z,q,p)\equiv\widehat{R}_{12}(z,q,p^*=pq^{-2c})$.
The q-determinant is in the center of $\mathcal{A}_{q,p}(\widehat{gl}(N)_c)$ and one sets

$$\mathcal{A}_{q,p}(\widehat{sl}(N)_c) = \mathcal{A}_{q,p}(\widehat{gl}(N)_c)/\langle q\text{-}\det L - q^{\frac{c}{2}}\rangle\,.$$

2 The center of $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$

We set $L^+(z) \equiv L(q^{c/2}z)$ and $L^-(z) \equiv g^{\frac{1}{2}}hg^{\frac{1}{2}}\,L(-p^{\frac{1}{2}}z)\,(g^{\frac{1}{2}}hg^{\frac{1}{2}})^{-1}$. Let us define the following operator in $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$:

$$t(z) \equiv \mathrm{Tr}(\mathcal{L}(z)) \equiv \mathrm{Tr}\left(L^+(q^{c/2}z)\,L^-(z)^{-1}\right).$$

From the defining relations of the elliptic algebra and the properties of the matrix $\widehat{R}$ (especially the quasi-periodicity property), one gets the following theorems:

Theorem 1: At the critical level $c=-N$, the operators generated by $t(z)$ lie in the center of the algebra $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$:

$$\big[t(z), L^+(w)\big] = \big[t(z), L^-(w)\big] = 0\,.$$

Theorem 2: At the critical level $c=-N$, the operators generated by $t(z)$ close an Abelian algebra:

$$[t(z), t(w)] = 0\,.$$

This allows us to define a Poisson structure on the center of $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ at $c=-N$. When $c \neq -N$, one has:

$$\begin{aligned} t(z)t(w) &= \mathcal{Y}(w/z)^{i_1i_2}_{j_1j_2}\;\mathcal{L}(w)^{j_2}_{i_2}\;\mathcal{L}(z)^{j_1}_{i_1} \\ &= t(w)t(z) + (c+N)\left(\frac{d\mathcal{Y}}{dc}\left(\frac{w}{z}\right)\right)^{i_1i_2}_{j_1j_2}\;\mathcal{L}(w)^{j_2}_{i_2}\;\mathcal{L}(z)^{j_1}_{i_1}\Big|_{cr} + \cdots \end{aligned}$$

The Poisson structure is then defined by:

$$\{t(z), t(w)\} = \left(\frac{d\mathcal{Y}}{dc}\left(\frac{w}{z}\right)\right)^{i_1i_2}_{j_1j_2}\;\mathcal{L}(w)^{j_2}_{i_2}\,\mathcal{L}(z)^{j_1}_{i_1}\Big|_{cr}\,.$$

Decomposing $\mathcal{Y}(x)$ as $\mathcal{Y}(x) = T(x)\,\mathcal{M}(x)$ where

$$T(x) = \frac{\tau_N(q^{\frac{1}{2}}x^{-1})\tau_N(q^{\frac{1}{2}-c}x)}{\tau_N(q^{\frac{1}{2}}x)\tau_N(q^{\frac{1}{2}-c}x^{-1})}$$

and

$$\mathcal{M}(x) = \left(\left(R_{21}(x)\,R_{21}(q^{c+N}x)^{-1}\,R_{12}(x^{-1})^{-1}\right)^{t_2}\,R_{12}(q^cx^{-1})^{t_2}\right)^{t_2},$$

one gets

$$\frac{d\mathcal{Y}}{dc}(x)\Big|_{cr} = \frac{dT}{dc}(x)\Big|_{cr}\,\mathbb{I}_2\otimes\mathbb{I}_2 + \frac{d\mathcal{M}}{dc}(x)\Big|_{cr},$$

since $T(x)_{cr} = 1$ and $\mathcal{M}(x)_{cr} = \mathbb{I}_2\otimes\mathbb{I}_2$ at the critical level $c=-N$.

The properties of the matrix R_{12} imply the vanishing of the derivative of $\mathcal{M}$ with respect to c at the critical level and it follows that:

$$\{t(z), t(w)\} = \frac{dT}{dc}\left(\frac{w}{z}\right)\Big|_{cr}\,t(z)\,t(w) \equiv f(w/z)\,t(z)\,t(w)$$

where

$$
\begin{aligned}
f(x) &= -2\ln q \left[\sum_{l\geq 0}\left(\frac{2x^2q^{2Nl}}{1-x^2q^{2Nl}}-\frac{x^2q^{2Nl+2}}{1-x^2q^{2Nl+2}}-\frac{x^2q^{2Nl-2}}{1-x^2q^{2Nl-2}}\right)\right.\\
&\qquad \left.-\frac{x^2}{1-x^2}+\tfrac{1}{2}\,\frac{x^2q^2}{1-x^2q^2}+\tfrac{1}{2}\,\frac{x^2q^{-2}}{1-x^2q^{-2}}-(x\leftrightarrow x^{-1})\right].
\end{aligned}
$$

The Poisson structure on the center of $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ at $c=-N$ *is not p depending.*

We now define the Poisson structure for modes of the generating function $f(x)$. The modes of $t(z)$ are defined in the sense of formal series expansions: $t_m = \displaystyle\oint_C \frac{dz}{2i\pi z}\, z^{-m}\, t(z)$, where C is a contour encircling the origin.
However, the function $f(x)$ has poles for $x=\pm q^{\pm Nl}$ and $x=\pm q^{\pm Nl\pm 1}$ with $l\geq 0$, therefore

$$
\oint_{C_1}\frac{dz}{2\pi i z}\oint_{C_2}\frac{dw}{2\pi i w} \neq \oint_{C_2}\frac{dz}{2\pi i z}\oint_{C_1}\frac{dw}{2\pi i w}\,.
$$

The relative position of the contours C_1 and C_2 must be specified in order to have an unambiguous result for the Poisson bracket. Moreover, the antisymmetry of the Poisson bracket is only guaranteed at the mode level by an explicit symmetrization with respect to the position of the contours C_1 and C_2. The mode Poisson bracket is thus defined as:

$$
\begin{aligned}
\{t_n,t_m\} &= \tfrac{1}{2}\left(\oint_{C_1}\frac{dz}{2\pi i z}\oint_{C_2}\frac{dw}{2\pi i w}+\oint_{C_2}\frac{dz}{2\pi i z}\oint_{C_1}\frac{dw}{2\pi i w}\right)\\
&\qquad\qquad z^{-n}w^{-m}\, f(w/z)\, t(z)\, t(w)
\end{aligned}
$$

where C_1 and C_2 are circles of radii R_1 and R_2 and one chooses $R_1>R_2$.

The Poisson bracket depends on the relative positions of the contours C_1 and C_2, circles of radii R_1 and R_2. Thus we introduce the notion of sectors: the sector (k) is defined by $R_1/R_2 \in \left]|q|^{-P(k)}, |q|^{-P(k+1)}\right[$ where $\{P(k)\}$ is the ordered set of powers of q^{-1} where the poles of f are located: $\{P(k)\}=\{0,1;N-1,N,N+1;\dots;Nl-1,Nl,Nl+1;\dots\}$.

In the case of $sl(N)$, for $R_1/R_2\in]1,|q|^{-1}[$ (first sector), one gets the following formula:

$$
\{t_n,t_m\}_{k=0} = -2(q-q^{-1})\ln q\sum_{r\in\mathbb{Z}}\frac{[(N-1)r]_q[r]_q}{[Nr]_q}\, t_{n-2r}\, t_{m+2r}
$$

where $[r]_q$ are q-numbers: $[r]_q\equiv \dfrac{q^r-q^{-r}}{q-q^{-1}}$.

In the case of $sl(2)$, for $R_1/R_2 \in]|q|^{\pm k},|q|^{\pm(k+1)}[$, it is possible to get a nice compact formula for any sector:

$$
\begin{aligned}
\{t_n,t_m\}_k &= (-1)^{k+1}2\ln q\oint_{C_1}\frac{dz}{2\pi i z}\oint_{C_2}\frac{dw}{2\pi i w}\\
&\qquad \sum_{s\in\mathbb{Z}}\frac{q^{(2k+1)s}-q^{-(2k+1)s}}{q^s+q^{-s}}\,(\frac{z}{w})^{2s}\, z^{-n}w^{-m}\, t(z)\, t(w)\,.
\end{aligned}
$$

It follows that one obtains in this way a family of Poisson structures indexed by $k\in\mathbb{Z}$, the sector $k=0$ corresponding to the Poisson bracket in [4].

To realize deformed $\mathcal{W}_N$ Poisson structures, we need to introduce generating functions for the higher spin objects. Having at our disposal only one commuting generating function $t(z)$, we are led to define shifted products, although with the same generator. We define therefore the higher spin objects for $1 \le i \le N-1$ as:

$$s_i(z) = \prod_{u=-(i-1)/2}^{(i-1)/2} t(q^u z)$$

The Poisson algebra reads as:

$$\{s_i(z), s_j(w)\} = \sum_{u=-(i-1)/2}^{(i-1)/2} \sum_{v=-(j-1)/2}^{(j-1)/2} f\Big(q^{v-u}\frac{w}{z}\Big)\, s_i(z)\, s_j(w)\,.$$

As before, one can introduce the notion of sectors for the Poisson brackets of the mode expansions, now labelled by $k(i,j)$ [3].
For the sector $k=0$, one gets:

$$\begin{aligned}\{s_i(n), s_j(m)\} &= -\sum_{r\in\mathbb{Z}} \frac{[(N-\max(i,j))r]_q[\min(i,j)r]_q}{[Nr]_q} \\ &\quad\times 2(q-q^{-1})\ln q\, s_i(n-2r)s_j(m+2r)\,. \qquad (*)\end{aligned}$$

Any Poisson structure in a given sector $k(i,j)$ can be obtained from (*) by adding to the r-dependent structure coefficient contributions from the relevant singularities of the structure function, i.e. power series expansions of terms $\delta(\pm q^{\pm P_{ij}(s)} w/z)\, s_i(z)\ s_j(w)$ for $1 \le s \le k(i,j)$.

3 $\mathcal{W}_{q,p}(sl(N))$ exchange algebras

3.1 q-Virasoro exchange algebras

While for $c=-N$, $t(z)$ and $L(w)$ commute, for $(-p^{\frac{1}{2}})^{NM} = q^{-c-N}$ ($M \in \mathbb{Z}$), $t(z)$ and $L(w)$ realize an *exchange algebra*:

$$t(z)\, L(w) = F(M, w/z)\, L(w)\, t(z)$$

where

$$F(M,x) = \begin{cases} q^{2M(N-1)} \displaystyle\prod_{k=0}^{NM-1} \frac{\Theta_{q^{2N}}(x^{-2}p^{-k})\,\Theta_{q^{2N}}(x^2p^k)}{\Theta_{q^{2N}}(x^{-2}q^2p^{-k})\,\Theta_{q^{2N}}(x^2q^2p^k)} & \text{for } M>0\,, \\ q^{-2|M|(N-1)} \displaystyle\prod_{k=1}^{N|M|} \frac{\Theta_{q^{2N}}(x^{-2}q^2p^{k})\,\Theta_{q^{2N}}(x^2q^2p^{-k})}{\Theta_{q^{2N}}(x^{-2}p^{k})\,\Theta_{q^{2N}}(x^2p^{-k})} & \text{for } M<0\,. \end{cases}$$

It follows that one gets the following theorem:
Theorem 3: On the 2d-surface $\Sigma_{N,M} = \{(p,q)\,|\,(-p^{\frac{1}{2}})^{NM} = q^{-c-N}\}$ of the 3d-space of parameters q,p,c, the operators generated by $t(z)$ close a quadratic algebra:

$$t(z)\, t(w) = \mathcal{Y}_{N,p,q,M}\Big(\frac{w}{z}\Big)\, t(w)\, t(z)$$

where

$$\mathcal{Y}_{N,p,q,M}(x) = \begin{cases} \displaystyle\prod_{k=1}^{NM} \frac{\Theta^2_{q^{2N}}(x^2p^{-k})\,\Theta_{q^{2N}}(x^2q^2p^k)\,\Theta_{q^{2N}}(x^2q^{-2}p^k)}{\Theta^2_{q^{2N}}(x^2p^k)\,\Theta_{q^{2N}}(x^2q^2p^{-k})\,\Theta_{q^{2N}}(x^2q^{-2}p^{-k})} & \text{for } M>0\,, \\ \displaystyle\prod_{k=1}^{N|M|-1} \frac{\Theta^2_{q^{2N}}(x^2p^{-k})\,\Theta_{q^{2N}}(x^2q^2p^k)\,\Theta_{q^{2N}}(x^2q^{-2}p^k)}{\Theta^2_{q^{2N}}(x^2p^k)\,\Theta_{q^{2N}}(x^2q^2p^{-k})\,\Theta_{q^{2N}}(x^2q^{-2}p^{-k})} & \text{for } M<0\,. \end{cases}$$

For $N = 2$, one has

$$\begin{aligned} &\mathcal{Y}_{2,p,q,M}(xq^2) = \mathcal{Y}_{2,p,q,M}(x)\,, \\ &\mathcal{Y}_{2,p,q,M}(x)\,\mathcal{Y}_{2,p,q,M}(xq) = 1\,. \end{aligned}$$

One recovers the same periodicity properties satisfied by structure function $\varphi(x)$ of [5] and [6] in the case of $sl(2)$ (when $M = 1$, one makes the change of variable $q^2 \to p$, $p \to q$, $x^2 \to x$, then $\mathcal{Y}_{2,p,q,1}(x) \to \varphi^2(x)$ of [6]).
For $N > 2$, one has

$$\begin{aligned} &\mathcal{Y}_{N,p,q,M}(xq^N) = \mathcal{Y}_{N,p,q,M}(x)\,, \\ &\mathcal{Y}_{N,p,q,M}(x)\,\mathcal{Y}_{N,p,q,M}(xq)\,\ldots\,\mathcal{Y}_{N,p,q,M}(xq^{N-1}) = 1\,. \end{aligned}$$

3.2 $\mathcal{W}_{q,p}(sl(N))$ exchange algebras

The generalization to higher spin generators in the quantum case follows that of the classical case, although with a required notion of ordering between individual $t(z)$-generators:

$$s_i(z) = \prod^{\curvearrowleft}_{\frac{i-1}{2} \geq u \geq -\frac{i-1}{2}} t(q^u z)\,.$$

The exchange algebra takes now the form:

$$s_i(z)s_j(w) = \prod_{u=-\frac{i-1}{2}}^{\frac{i-1}{2}} \prod_{v=-\frac{j-1}{2}}^{\frac{j-1}{2}} \mathcal{Y}_{N,p,q,M}\left(q^{v-u}\frac{w}{z}\right)\, s_j(w)\, s_i(z)\,.$$

Furthermore one can verify that $s_N(z)$ commutes with all other generators; hence we are justified in restricting i to $1, \ldots, N-1$.

3.3 Classical limit

Theorem 4: On the surface $\Sigma_{N,M}$, when $p = q^{Nh}$ with $h \in \mathbb{Z}\backslash\{0\}$:

$$\mathcal{Y}_{N,p,q,M} = 1\,.$$

Hence $t(z)$ realizes an Abelian subalgebra in $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ (though it is not in the center of $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ in general).

The result of the previous theorem allows us to define Poisson structures on the corresponding Abelian algebras generated by $t(z)$. They are obtained as limits of the exchange algebra of

Theorem 3 when $p = q^{Nh}$ with $h \in \mathbb{Z}\backslash\{0\}$. Setting $q^{Nh} = p^{1-\beta}$, the h-labeled Poisson structure is defined by:

$$\{t(z), t(w)\}^{(h)} \equiv \lim_{\beta\to 0} \frac{1}{\beta} \left(t(z)t(w) - t(w)t(z)\right).$$

For h even, $\{t(z), t(w)\}^{(h)}$ coincides with the Poisson structure of the critical case $c = -N$, while for h odd, $\{t(z), t(w)\}^{(h)}$ gives new Poisson structures (for $sl(N)$ when $N > 2$). Explicitly, one has the following expression:

$$\{t(z), t(w)\}^{(h)} = f_h(w/z)\, t(z)\, t(w)$$

with

$$\begin{aligned}
f_h(x) &= \mathcal{N}_{odd} \Bigg[\sum_{\ell\geq 0} E(\tfrac{NM}{2})(E(\tfrac{NM}{2})+1) \left(\frac{2x^2q^{2N\ell}}{1-x^2q^{2N\ell}} - \frac{x^2q^{2N\ell+2}}{1-x^2q^{2N\ell+2}} \right.\\
&\quad \left. - \frac{x^2q^{2N\ell-2}}{1-x^2q^{2N\ell-2}} \right) + E(\tfrac{NM+1}{2})^2 \left(\frac{2x^2q^{2N\ell+N}}{1-x^2q^{2N\ell+N}} - \frac{x^2q^{2N\ell+N+2}}{1-x^2q^{2N\ell+N+2}} \right.\\
&\quad \left. - \frac{x^2q^{2N\ell+N-2}}{1-x^2q^{2N\ell+N-2}} \right) - \tfrac{1}{2}E(\tfrac{NM}{2})(E(\tfrac{NM}{2})+1) \left(\frac{2x^2}{1-x^2} - \frac{x^2q^2}{1-x^2q^2} \right.\\
&\quad \left. - \frac{x^2q^{-2}}{1-x^2q^{-2}} \right) - (x \leftrightarrow x^{-1}) \Bigg] \qquad \text{for } h \text{ odd}\\
&= \mathcal{N}_{even} \Bigg[\sum_{\ell\geq 0} \left(\frac{2x^2q^{2N\ell}}{1-x^2q^{2N\ell}} - \frac{x^2q^{2N\ell+2}}{1-x^2q^{2N\ell+2}} - \frac{x^2q^{2N\ell-2}}{1-x^2q^{2N\ell-2}} \right)\\
&\quad -\tfrac{1}{2} \left(\frac{2x^2}{1-x^2} - \frac{x^2q^2}{1-x^2q^2} - \frac{x^2q^{-2}}{1-x^2q^{-2}} \right) - (x \leftrightarrow x^{-1}) \Bigg] \qquad \text{for } h \text{ even}
\end{aligned}$$

where $\mathcal{N}_{odd} = 2Nh\ln q$, $\mathcal{N}_{even} = N^2M(NM+1)h\ln q$ and the notation $E(n)$ means the integer part of the number n.

The mode Poisson brackets in the sector $k = 0$ is given by:

$$\begin{aligned}
\{t_n, t_m\}^{(h)} &= -2(q-q^{-1})\ln q \sum_{r\in\mathbb{Z}} \left(-E(\tfrac{NM+1}{2})^2 \frac{[r]_q^2}{[Nr]_q} \right.\\
&\quad \left. + E(\tfrac{NM}{2})(E(\tfrac{NM}{2})+1) \frac{[(N-1)r]_q[r]_q}{[Nr]_q} \right) t_{n-2r}\, t_{m+2r}\\
&\qquad \text{for } h \text{ odd}\\
&= -2(q-q^{-1})\ln q \sum_{r\in\mathbb{Z}} \frac{[(N-1)r]_q[r]_q}{[Nr]_q}\, t_{n-2r}\, t_{m+2r}\\
&\qquad \text{for } h \text{ even}
\end{aligned}$$

Since the initial non-abelian structure for $t(z)$ is closed, the exchange algebras

$$t(z)\, t(w) = \mathcal{Y}_{N,p,q,M}(w/z)\, t(w)\, t(z)$$

are *inequivalent quantizations* with respect to p of the Poisson bracket which defines $Vir_q(sl(N))$: $t(z)$ generates a $Vir_{q,p}(sl(N))$ algebra.

The generalization for the higher spin generators $s_i(z)$ presents no conceptual difficulty. The following results are obtained:
– for h even, $\{s_i(z), s_j(w)\}^{(h)}$ coincides with the Poisson structure of the critical case $c = -N$. The Poisson bracket is again identical to the "core" contribution in [4] (i.e. without lower-spin extensions).
– for h odd, $\{s_i(z), s_j(w)\}^{(h)}$ gives rise to two new types of quadratic Poisson structures (that is q-deformed classical $\mathcal{W}_N$-algebras) whether $i + j \leq N$ or $i + j > N$.

4 Conclusion

Here above is summarized the construction, from elliptic algebras, of the deformed Virasoro and $\mathcal{W}$ algebras in the classical case as well as in the quantum case. We can mention that a study for consistent central extensions, providing in particular general results for the simplest cases of $Vir_q(sl(2))$ has very recently been achieved in ref. [11]. Also, a direct determination of abstract generators for q-deformed $\mathcal{W}_N$ algebras has been carried out. Such an universal construction of $\mathcal{W}_{q,p}$ algebras is achieved in [12].

An important application of the above developed constructions is the determination of integrable models submitted to such symmetries. It is known that the trigonometric Calogero–Moser model is related to the Virasoro algebra and the Ruijsenaars–Schneider model to the q-deformed Virasoro algebra considered in ref. [5]. One expects the existence of generalized Ruijsenaars–Schneider models the symmetries of which would be given by the just described q-Virasoro exchange algebras. For such a program, the bosonization of these algebras appears as an essential tool, which is still missing.

Let us conclude by remarking that the Yang-Baxter equation admits two classes of elliptic solutions, denoted vertex-type and face-type solutions, and to which respectively correspond the elliptic algebras $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ and $\mathcal{B}_{q,\lambda}(\mathcal{G})$, $\mathcal{G}$ being any simple Lie algebra. Each of these algebras can be obtained from the quantum affine algebra $\mathcal{U}_q(\mathcal{G}_c)$ by performing a Drinfeld twist [8]. It might also be interesting to note that the second kind of well known algebra deformation, the Yangian $Y(\mathcal{G})$,can be incorporated in this scheme, via its double, in the following way. The scaling limit q=1 of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl}(N)_c)$ with $p = q^2\eta, \eta$ fixed, leads to the deformed centrally extended double Yangian structure $\mathcal{D}Y_r(\widehat{sl}(N))_c$,in which a treatment analogous to the one presented in the previous sections and leading to exchange subalgebras can be performed (see [13] and ref.therein). Taking the η inf. limit then provides the centrally extended double Yangian $\mathcal{D}Y(\widehat{sl}(N))_c$. We have summarised such connections between these algebras in the following tableau. We hope it will give a more synthetic view in the landscape of deformed algebras.

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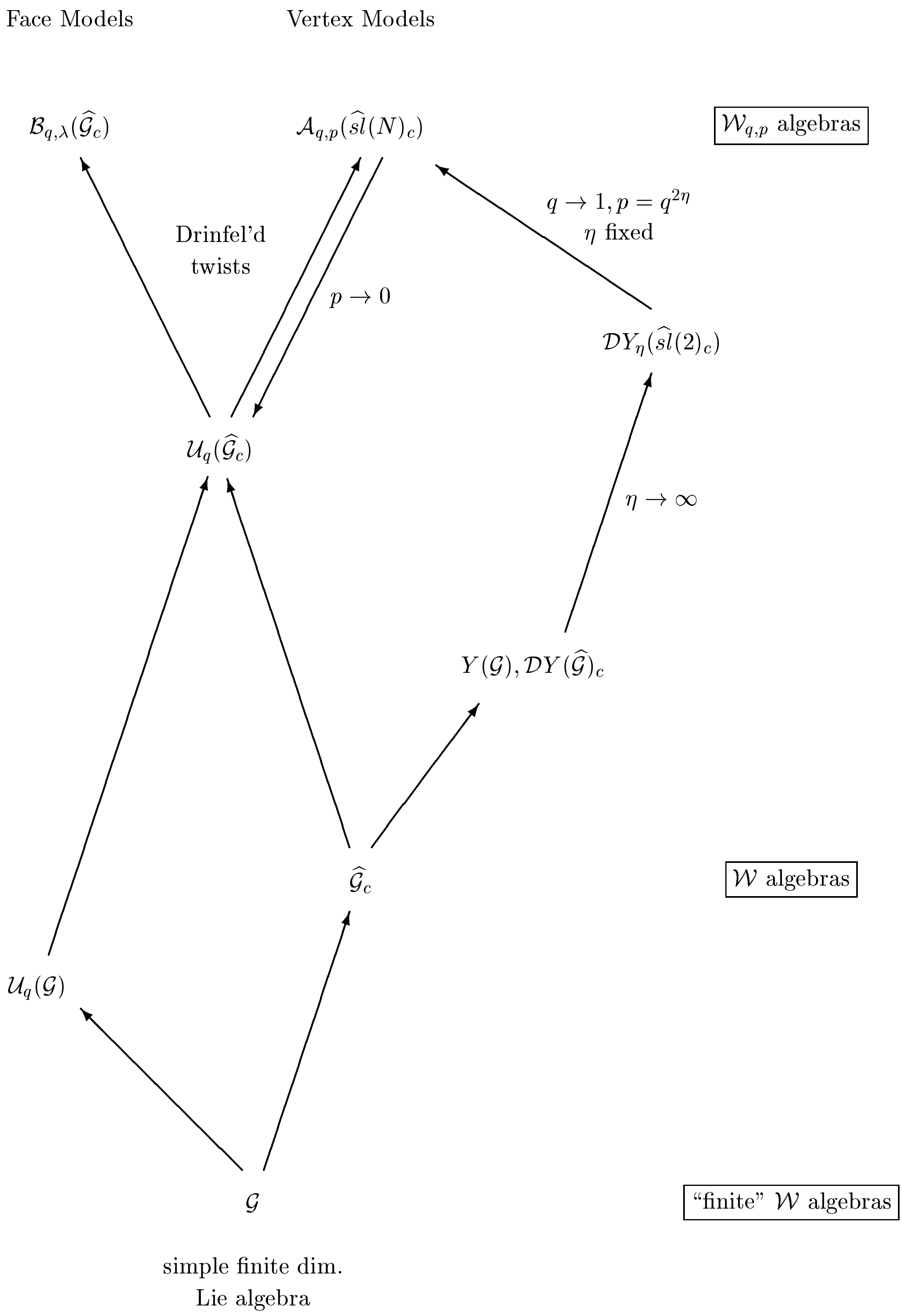

A schematic view of deformed algebras in the set of current continuous symmetries

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