QUANTIZATION OF SYMPLECTIC DYNAMICAL r-MATRICES AND THE QUANTUM COMPOSITION FORMULA

ANTON ALEKSEEV AND DAMIEN CALAQUE

Abstract. In this paper we quantize symplectic dynamical r-matrices over a possibly nonabelian base. The proof is based on the fact that the existence of a star-product with a nice property (called strong invariance) is sufficient for the existence of a quantization. We also classify such quantizations and prove a quantum analogue of the classical composition formula for coboundary dynamical r-matrices.

Introduction

Let $\mathfrak{h}\subset\mathfrak{g}$ be an inclusion of Lie algebras and $H\subset G$ the corresponding inclusion of Lie groups. Let $U\subset\mathfrak{h}^*$ be an invariant open subset and let $Z\in(\wedge^3\mathfrak{g})^{\mathfrak{g}}$. A *(coboundary) dynamical r-matrix* is a $\mathfrak{h}$-equivariant map $r:U\to\wedge^2\mathfrak{g}$ satisfying the *(modified) classical dynamical Yang-Baxter equation*

$$\frac{1}{2}[r(\lambda),r(\lambda)]-\sum_i h_i\wedge\frac{\partial r}{\partial\lambda^i}(\lambda)=Z\quad(\lambda\in U)\,,$$

where $(h_i)_i$ and $(\lambda^i)_i$ are dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively. Then (following [14, 6])

$$\pi_r:=\pi_{lin}+\sum_i\frac{\partial}{\partial\lambda^i}\wedge\overrightarrow{h_i}+\overrightarrow{r(\lambda)}\,,\tag{0.1}$$

together with Z, defines a H-invariant ($\mathfrak{g}$-)quasi-Poisson structure on $M=U\times G$. Here π_{lin} is the linear Poisson structure on $U\subset\mathfrak{h}^*$.
Such a dynamical r-matrix is called *symplectic* if the $\mathfrak{g}$-quasi-Poisson manifold (M,π_r,Z) is symplectic, i.e. if $\pi_r^\#:T^*M\to TM$ is invertible and $Z=0$.

By a *dynamical twist quantization* of $r(\lambda)$ we mean a $\mathfrak{h}$-equivariant map $J=1\otimes1+O(\hbar):U\to\otimes^2U\mathfrak{g}[[\hbar]]$ satisfying the *semi-classical limit condition*

$$J(\lambda)-J^{2,1}(\lambda)=\hbar r(\lambda)+O(\hbar^2)\quad(\lambda\in U)$$

and the *modified dynamical twist equation*

$$J^{12,3}(\lambda)*_{PBW}J^{1,2}(\lambda+\hbar h^{(3)})=\Phi^{-1}J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda)\quad(\lambda\in U)\,.$$

where $\Phi\in(\otimes^3U\mathfrak{g})^{\mathfrak{g}}[[\hbar]]$ is an associator quantizing Z, of which we know the existence from [5, Proposition 3.10]. Recall that $*_{PBW}$ is the Poincaré-Birkhoff-Witt star-product given on polynomial functions as the pull-back of the usual product in $U(\mathfrak{h}_\hbar)$[1] by the symmetrization map $\mathrm{sym}:S(\mathfrak{h})[[\hbar]]\to U(\mathfrak{h}_\hbar)$. We also made use of the following notations:

$$J^{12,3}(\lambda):=(\Delta\otimes\mathrm{id})(J(\lambda))\quad\text{and}$$

$$J^{1,2}(\lambda+\hbar h^{(3)}):=\sum_{k\geq0}\frac{\hbar^k}{k!}\sum_{i_1\cdots i_k}\frac{\partial^kJ}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}(\lambda)\otimes h_{i_1}\cdots h_{i_k}\,.$$

In this paper we prove the following generalization of [13, Theorem 5.3] to the case of a nonabelian base:

[1] $\mathfrak{h}_\hbar=\mathfrak{h}[[\hbar]]$ with bracket $[,]_\hbar:=\hbar[,]_{\mathfrak{h}}$.

Theorem 0.1. *Any symplectic dynamical r-matrix admits a dynamical twist quantization (with $\Phi = 1$).*

Furthermore, two dynamical twist quantizations $J_1, J_2 : U \to \otimes^2 U\mathfrak{g}[[\hbar]]$ of r are said *gauge equivalent* if there exists a $\mathfrak{h}$-equivariant map $T = 1 + O(\hbar) : U \to U\mathfrak{g}[[\hbar]]$ such that

$$T^{12}(\lambda) *_{PBW} J_1(\lambda) = J_2(\lambda) *_{PBW} T^1(\lambda + \hbar h^{(2)}) *_{PBW} T^2(\lambda)\,.$$

In this context we also prove the following generalization of [13, Section 6] to the case of a nonabelian base, asserting that dynamical twist quantizations are classified by the *second dynamical r-matrix cohomology* (see Definition 3.2):

Theorem 0.2. *Let r be a symplectic dynamical r-matrix. Then the space of gauge equivalence classes of dynamical twist quantizations of r (with $\Phi = 1$) is an affine space modeled on $H^2_r(U, \mathfrak{g})[[\hbar]]$.*

A class of examples of symplectic dynamical r-matrices is given by *nondegenerate reductive splittings*. A reductive splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (with $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$) is called nondegenerate if there exists $\lambda \in \mathfrak{h}^*$ for which $\omega(\lambda) \in \wedge^2\mathfrak{m}^*$ defined by $\omega(\lambda)(x, y) =< \lambda, [x, y]_{|\mathfrak{h}} >$ is nondegenerate. Then $r^{\mathfrak{m}}_{\mathfrak{h}}(\lambda) := -\omega(\lambda)^{-1} \in \wedge^2\mathfrak{m} \subset \wedge^2\mathfrak{g}$ defines a symplectic dynamical r-matrix on the invariant open subset $\{\lambda | \mathrm{det}\omega(\lambda) \neq 0\} \subset \mathfrak{h}^*$ (see [11, Proposition 1] or [14, Theorem 2.3]; see also [7, Proposition 1.1] for a more algebraic proof). Moreover, one can use it to "compose" dynamical r-matrices (see [11, Proposition 1] and [7, Proposition 0.1]):

Proposition 0.3 (The composition formula)**.** *Assume that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ is a nondegenerate reductive splitting and let $r^{\mathfrak{m}}_{\mathfrak{t}} : \mathfrak{t}^* \supset V \to \wedge^2\mathfrak{h}$ be the corresponding symplectic dynamical r-matrix. If $\rho : \mathfrak{h}^* \supset U \to \wedge^2\mathfrak{g}$ is a dynamical r-matrix with $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$, then*

$$\theta_\rho := r^{\mathfrak{m}}_{\mathfrak{t}} + \rho_{|\mathfrak{t}^*} : \mathfrak{t}^* \supset U \cap V \longrightarrow \wedge^2\mathfrak{g}$$

is a dynamical r-matrix with the same Z.

We prove that one can "quantize" the map $\rho \mapsto \theta_\rho$:

Theorem 0.4. *With the hypothesis of Proposition 0.3, there exists a map*

$$\Theta : \{\text{Dynamical twist quantizations of } \rho\} \longrightarrow \{\text{Dynamical twist quantizations of } \theta_\rho\}$$

which keeps the associator Φ fixed.

The paper is organized as follows.
In Section 1 we first recall basic facts about quasi-Poisson manifolds, compatible quantizations and related results. We then give a sufficient condition for the existence of dynamical twist quantizations.
In Section 2 we give a very short proof of Theorem 0.1, using a result stating the existence of a quantum momentum map (see also [10, 12]) which is based on Fedosov's well-known globalization procedure [8, 9]. We start the section with a summary of basic ingredients of Fedosov's construction.
In Section 3 we prove Theorem 0.2 using a variant of the well-established classification of star-products on a symplectic manifold by formal series with coefficients in the second De Rham cohomology group of the manifold.
In Section 4 we prove a quantum analogue of the composition formula for classical dynamical r-matrices. We start with a new proof of the classical composition formula (Proposition 0.3) using (quasi-)Poisson reduction. We then derive its quantum counterpart (Theorem 0.4) using quantum reduction.

Notations. We denote by $\mathcal{O}_{\mathfrak{h}^*} = S(\mathfrak{h})$ polynomial functions on $\mathfrak{h}^*$, $\mathcal{O}_U = C^\infty(U) \supset \mathcal{O}_{\mathfrak{h}^*}$ smooth functions on $U \subset \mathfrak{h}^*$, and $\mathcal{O}_G$ the ring of smooth functions on the (eventually formal) Lie group G. For any element $x \in \mathfrak{g}$ we denote by $\overrightarrow{x}$ (resp. $\overleftarrow{x}$) the corresponding left (resp.

right) invariant vector field on G.

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1. A sufficient condition for the existence of a dynamical twist quantization

Let $r : \mathfrak{h}^* \supset U \to \wedge^2\mathfrak{g}$ be a coboundary dynamical r-matrix. Denote by π_r the bivector field on $M = U \times G$ given by (0.1).

1.1. Quasi-Poisson manifolds and their quantizations.

Recall from [1, 2] that a *($\mathfrak{g}$-) quasi-Poisson manifold* is a manifold X together with a $\mathfrak{g}$-action $\rho : \mathfrak{g} \to \mathfrak{X}(X)$, an invariant bivector field $\pi \in \Gamma(X, \wedge^2 TX)^{\mathfrak{g}}$ and an element $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ such that

$$[\pi, \pi] = \rho(Z)\,. \tag{1.1}$$

Let $\{f, g\} :=< \pi, df \wedge dg >$ $(f, g \in \mathcal{O}_X)$ be the corresponding *quasi-Poisson bracket*. Then equation (1.1) is equivalent to

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} =< \rho(Z), df \wedge dg \wedge dh > \quad (f, g, h \in \mathcal{O}_X)\,.$$

One can define the *quasi-Poisson cochain complex* of (X, π, Z) as follows: k-cochains are $C^k_\pi(X) = \Gamma(X, \wedge^k TX)^{\mathfrak{g}}$ and the differential is $d_\pi = [\pi, -]$. The fact that $d_\pi \circ d_\pi = 0$ follows from an easy calculation:

$$d_\pi \circ d_\pi(x) = [\pi, [\pi, x]] = \frac{1}{2}[[\pi, \pi], x] = \frac{1}{2}[\rho(Z), x] = 0 \quad (x \in C^k_\pi(X))\,.$$

Let us now fix an associator $\Phi \in (\otimes^3 U\mathfrak{g})^{\mathfrak{g}}[[\hbar]]$ quantizing Z (we know it exists from [5, Proposition 3.10]). Following [6, Definition 4.4], by a *quantization* of a given quasi-Poisson manifold (X, π, Z) we mean a series $* \in \mathrm{Bidiff}(X)^{\mathfrak{g}}[[\hbar]]$ of invariant bidifferential operators such that

- $f * g = fg + O(\hbar)$ for any $f, g \in \mathcal{O}_X$,
- $f * g - g * f = \hbar\{f, g\} + O(\hbar^2)$ for any $f, g \in \mathcal{O}_X$, and
- if we write $m_*(f \otimes g) := f * g$ for $f, g \in \mathcal{O}_X$, and $\widetilde{\Phi} := S^{\otimes 3}(\Phi^{-1})$, then[2]

$$m_* \circ (m_* \otimes \mathrm{id}) = m_* \circ (\mathrm{id} \otimes m_*) \circ \rho^{\otimes 3}(\widetilde{\Phi})\,. \tag{1.2}$$

Here, S denotes the antipode of $U\mathfrak{g}$.

One has a natural notion of *gauge transformation* for quantizations. It is given by an element $Q = \mathrm{id} + O(\hbar) \in \mathrm{Diff}(X)^{\mathfrak{g}}[[\hbar]]$ that act on $*$'s in the usual way:

$$f *^{(Q)} g := Q^{-1}(Q(f) * Q(g)) \quad (f, g \in \mathcal{O}_X)\,.$$

More precisely, if $(\Phi, *)$ is a quantization of (Z, π) then $(\Phi, *^{(Q)})$ is also. In this case we say that $*$ and $*^{(Q)}$ are *gauge equivalent*.

[2] We thank Pavel Etingof for pointing to us that one has to use $\widetilde{\Phi}$ instead of Φ in this definition.

1.2. **Classical and quantum momentum maps.** Let (X, π, Z) be $\mathfrak{g}$-quasi-Poisson manifold and let $\mathcal{G}$ be a Lie algebra with Lie group $\mathbf{G}$. A *momentum map* is a smooth $\mathfrak{g}$-invariant map $\mu : M \to \mathcal{G}^*$ such that $\mu_*\pi = \pi_{lin}$ and for which the corresponding infinitesimal action $\mathcal{G} \to \mathfrak{X}(X); x \mapsto \{\mu^*x, -\}$ integrates to a right action of $\mathbf{G}$.

Let us describe the reduction procedure with respect to a given momentum map μ. First of all $\mathbf{G}$ acts on $\mu^{-1}(0)$ and hence one can define the reduced space $X_{red} := \mu^{-1}(0)/\mathbf{G}$. Let us assume that it is smooth (this is the case when 0 is a regular value and $\mathbf{G}$ acts freely); its function algebra is $\mathcal{O}_{X_{red}} = \mathcal{O}_X^{\mathcal{G}}/(\mathcal{O}_X^{\mathcal{G}} \cap \mathcal{I}_0)$, where $\mathcal{I}_0$ is the ideal generated by $\mathrm{im}(\mu^*)$. Since μ is $\mathfrak{g}$-invariant then $\mathfrak{g}$ acts on $\mu^{-1}(0)$. Moreover the $\mathfrak{g}$-action and the $\mathcal{G}$-action commute (because π and μ are $\mathfrak{g}$-invariant), consequently $\mathfrak{g}$ also acts on X_{red}. Now observe that $\mathcal{O}_X^{\mathcal{G}} = \{f \in \mathcal{O}_X | \{f, \mathcal{I}_0\} \subset \mathcal{I}_0\}$, therefore the quasi-Poisson bracket $\{,\}$ naturaly induces a quasi-Poisson bracket (with the same Z) on $\mathcal{O}_{X_{red}}$. In other words, X_{red} inherits a structure of a quasi-Poisson manifold from the one of X.

Now assume that we are given a quantization $(*, \Phi)$ of the quasi-Poisson manifold (X, π, Z). By a *quantum momentum map* quantizing μ we mean a map of algebras

$$\mathbf{M} = \mu^* + O(\hbar) : (\mathcal{O}_{\mathcal{G}^*}[[\hbar]], *_{PBW}) \longrightarrow (\mathcal{O}_X[[\hbar]], *)$$

taking its values in $\mathfrak{g}$-invariant functions, and such that for any $f \in \mathcal{O}_X$ and any $x \in \mathcal{G}$ one has $[\mathbf{M}(x), f]_* = \hbar\{\mu^*x, f\}$.

Remark 1.1. One only needs to know $\mathbf{M}$ on linear functions $x \in \mathcal{G}$.

Let us describe the quantum reduction with respect to a given quantum momentum map $\mathbf{M}$. First denote by $\mathcal{I}$ the right ideal generated by $\mathrm{im}(\mathbf{M})$ in $(\mathcal{O}_X[[\hbar]], *)$ and observe that its normalizer is $\mathcal{O}_X^{\mathfrak{h}}[[\hbar]]$. Therefore $\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \cap \mathcal{I}$ is a two-sided ideal in $\mathcal{O}_X^{\mathfrak{h}}[[\hbar]]$, and we can define the reduced algebra $\mathcal{A} := \mathcal{O}_X^{\mathfrak{h}}[[\hbar]]/\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \cap \mathcal{I}$. Since $\mathcal{I} \cong \mathcal{I}_0[[\hbar]]$ then $\mathcal{A} \cong \mathcal{O}_{X_{red}}[[\hbar]]$ (as $\mathbb{R}[[\hbar]]$-modules). It is easy to see that the induced product on $\mathcal{O}_{X_{red}}[[\hbar]]$, together with Φ, gives a quantization of the quasi-Poisson structure on X_{red}.

A quantization $*$ of a quasi-Poisson manifold (M, π, Z) with a momentum map $\mu : M \to \mathcal{G}^*$ for which μ^* itself defines a quantum momentum map (it will be $\mathbf{M} = U(\mu^*) \circ \mathrm{sym}$) is called *strongly ($\mathcal{G}$-)invariant.*

1.3. **Compatible quantizations.** Let us first observe that π_r defines a quasi-Poisson structure on $M = U \times G$. Here the action is $\mathfrak{g} \ni x \mapsto \overleftarrow{x} \in \mathfrak{X}(M)$ (it generates left translations). Remark that since $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$ then $\overleftarrow{Z} = \overrightarrow{Z}$. Moreover, the natural map $M \to \mathfrak{h}^*, (\lambda, g) \mapsto \lambda$ is a momentum map and the corresponding right H-action is given by $(\lambda, g) \cdot h := (\mathrm{Ad}_h^*\lambda, gh)$. Conversely,

Proposition 1.2 ([14], Proposition 2.1). *A function $\rho \in \mathcal{O}(M, \wedge^2\mathfrak{g})$ is a coboundary classical dynamical r-matrix if and only if*

$$\pi = \pi_{lin} + \sum \frac{\partial}{\partial\lambda^i} \wedge \overrightarrow{h_i} + \overrightarrow{\rho(\lambda)}$$

defines a $\mathfrak{g}$-quasi-Poisson structure on $U \times G$.

Proof. The proof given in [14] is for the case when $Z = 0$, but it admits a straightforward generalization. □

Following Ping Xu ([14]), by a *compatible quantization* of π_r we mean a quantization $*'$ which is such that for any $u, v \in \mathcal{O}_{\mathfrak{h}^*}$ and any $f \in \mathcal{O}_G$, $u *' v = u *_{PBW} v$, $f *' u = fu$ and

$$u *' f = \sum_{k \geq 0} \frac{\hbar^k}{k!} \sum_{i_1, \ldots, i_k} \frac{\partial^k u}{\partial\lambda^{i_1} \cdots \partial\lambda^{i_k}} \overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} \cdot f \,. \tag{1.3}$$

Proposition 1.3. *There is a bijective correspondance between compatible quantizations of π_r and dynamical twist quantizations of r.*

Proof. Let $*'$ be a compatible quantization of π_r. Since $*'$ is G-invariant then for all $f,g\in\mathcal{O}_G$ one has

$$(f*'g)(\lambda)=\overrightarrow{J(\lambda)}(f,g)\quad(\lambda\in\mathfrak{h}^*)$$

with $J:U\to\otimes^2U\mathfrak{g}[[\hbar]]$. Moreover

Lemma 1.4. *$*'$ is strongly $\mathfrak{h}$-invariant*[3].

Proof of the lemma. Let $f=gu$ ($g\in\mathcal{O}_G$ and $u\in\mathcal{O}_{\mathfrak{h}^*}$) on $U\times G$. Then for any $h\in\mathfrak{h}$ one has

$$\begin{aligned}
h*'f-f*'h &= h*'(gu)-(gu)*'h=h*'(g*'u)-(g*'u)*'h\\
&= (h*'g)*'u-g*'(u*'h)\qquad\qquad(\Phi\text{ acts trivialy})\\
&= (g*'h+\hbar(\overrightarrow{h}\cdot g))*'u-g*'(u*'h)\\
&= g*'([h,u]_{*'})+\hbar(\chi_h\cdot g)*'u=g([h,u]_{*_{PBW}})+\hbar(\chi_h\cdot g)u\\
&= \hbar(g(\chi_h\cdot u)+(\chi_h\cdot g)u)=\hbar(\chi_h\cdot f)
\end{aligned}$$

Hence for any $f\in\mathcal{O}_M$, $h*f-f*h=\hbar(\chi_h\cdot f)$. □

Therefore using [14, Proposition 3.2] one obtains that J is H-equivariant. The following lemma ends the first part of the proof:

Lemma 1.5. *J satisfies the dynamical twist equation.*

Proof of the lemma. Let us define $\mathbf{L}:\mathfrak{g}\ni x\mapsto\overrightarrow{x}$ and $\mathbf{R}:\mathfrak{g}\ni x\mapsto\overleftarrow{x}$, and denote by $m^{(n)}:\mathcal{O}_M^{\otimes n}\to\mathcal{O}_M\,;\,f_1\otimes\cdots\otimes f_n\mapsto f_1\cdots f_n$ the standard n-fold product of functions. A computation in [14] emphases the fact that for all $f,g,h\in\mathcal{O}_G$, one has[4]

$$m_{*'}\circ(m_{*'}\otimes\mathrm{id})(f\otimes g\otimes h)=\overrightarrow{J^{12,3}(\lambda)*_{PBW}J^{1,2}(\lambda+\hbar h^{(3)})}(f\otimes g\otimes h)$$

and

$$m_{*'}\circ(\mathrm{id}\otimes m_{*'})(f\otimes g\otimes h)=\overrightarrow{J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda)}(f\otimes g\otimes h)\,.$$

Therefore,

$$\begin{aligned}
m_{*'}\circ(\mathrm{id}\otimes m_{*'})\circ\mathbf{R}^{\otimes3}(\widetilde{\Phi})(f\otimes g\otimes h) &= m^{(3)}\big(\mathbf{L}^{\otimes3}(J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda))\mathbf{R}^{\otimes3}(\widetilde{\Phi})(f\otimes g\otimes h)\big)\\
&= m^{(3)}\big(\mathbf{R}^{\otimes3}(\widetilde{\Phi})\mathbf{L}^{\otimes3}(J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda))(f\otimes g\otimes h)\big)\\
&= \overleftarrow{S^{\otimes3}(\Phi^{-1})}\big(\mathbf{L}^{\otimes3}(J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda))(f\otimes g\otimes h)\big)\\
&= \overrightarrow{\Phi^{-1}}\big(\mathbf{L}^{\otimes3}(J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda))(f\otimes g\otimes h)\big)\\
&= \overrightarrow{\Phi^{-1}J^{1,23}(\lambda)*_{PBW}J^{2,3}(\lambda)}(f\otimes g\otimes h)\,,
\end{aligned}$$

where the equality before the last one follows from the invariance of Φ. This ends the proof of the lemma. □

[3]In particular $*'$ is H-invariant. It was not noticed in [14], where the definition of compatible star-products includes this H-equivariance property. The lemma claims that it comes for free (like in the classical situation).

[4]The reader must pay attention to the following important remark: for any $P\in\otimes^nU\mathfrak{g}$ we denote by $\overrightarrow{P}$ (resp. $\overleftarrow{P}$) the corresponding left (resp. right) invariant multidifferential operator, while $\mathbf{L}^{\otimes n}(P)$ (resp. $\mathbf{R}^{\otimes n}(P)$) is an element in $\otimes^n\mathrm{Diff}(G)^{G_{left}}$ (resp. $\otimes^n\mathrm{Diff}(G)^{G_{right}}$). Namely, $\overrightarrow{P}=m^{(n)}\circ\big(\mathbf{L}^{\otimes n}(P)\big)$.

Conversely, let $J=\sum_\alpha f_\alpha A_\alpha\otimes B_\alpha$ be a dynamical twist quantization of r ($f_\alpha\in\mathcal{O}_U[[\hbar]]$ and $A_\alpha,B_\alpha\in U\mathfrak{g}$). Following [14] we define a G-invariant product $*'$ on $\mathcal{O}_M[[\hbar]]$ by

$$g_1*'g_2:=\sum_{k\geq 0,\alpha}\frac{\hbar^k}{k!}\sum_{i_1,\dots,i_k}f_\alpha *_{PBW}(\overrightarrow{A_\alpha}\cdot\frac{\partial^k g_1}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}})*_{PBW}(\overrightarrow{B_\alpha}\overrightarrow{h_{i_1}}\dots\overrightarrow{h_{i_k}}\cdot g_2)\,.$$

One can check by direct computations that $\mathfrak{h}$-equivariance of J implies strong $\mathfrak{h}$-invariance of $*'$, and that the dynamical twist equation implies equation (1.2). □

Remark 1.6. Since $\mathcal{O}_{\mathfrak{h}^*}=S(\mathfrak{h})$ is generated as a vector space by h^n, $h\in\mathfrak{h}$ and $n\in\mathbb{N}$, then one can rewrite condition (1.3) as

$$h^n*'f=\sum_{k=0}^n\hbar^kC_n^k(\overrightarrow{h}^k\cdot f)h^{n-k}\,.$$

We saw in Lemma 1.4 that a compatible quantization always satisfies the strongly $\mathfrak{h}$-invariance condition. In what follows we show that this condition is actually sufficient for the existence of a compatible quantization.

1.4. A sufficient condition for the existence of a compatible quantization.

Proposition 1.7. *Assume that we are given a strongly $\mathfrak{h}$-invariant quantization $*$ of π_r on M. Then there exists a gauge equivalent compatible quantization $*'$ of π_r. Therefore there exists a dynamical twist quantization J of r.*

Proof. First observe that $h*h'-h'*h=\hbar[h,h']_{\mathfrak{h}}=[h,h']_{\mathfrak{h}_\hbar}$. Therefore we have an algebra morphism

$$a:U(\mathfrak{h}_\hbar)\longrightarrow(\mathcal{O}_M,*)\,.$$

Then define the algebra morphism $Q:\mathcal{O}_{\mathfrak{h}^*\times G}=S(\mathfrak{h})\otimes\mathcal{O}_G\longrightarrow\mathcal{O}_M$ as follows:

$$Q(fu)=f*a(\mathrm{sym}(u))\quad(u\in S(\mathfrak{h}),\,f\in\mathcal{O}_G)\,,$$

where sym : $S(\mathfrak{h})[[\hbar]]\longrightarrow U(\mathfrak{h}_\hbar)$ is the isomorphism sending h^n to h^n for any $h\in\mathfrak{h}$. Thus $Q(h^n\otimes f)=f*\underbrace{h*\cdots*h}_{n\ times}$, and since $*$ can be expressed as a series $m_0+O(\hbar)$ of bidifferential operators on M then Q can be expressed as a series $\mathrm{id}+O(\hbar)$ of differential operators on M. Moreover it is obviously $\mathfrak{g}$-invariant (since $*$ is), consequently we have a new quantization $*'$ of π_r, gauge equivalent to $*$, defined as follows: for any $f,g\in\mathcal{O}_M$,

$$f*'g=Q^{-1}(Q(f)*Q(g))\,.$$

Let us now check that $*'$ satisfies all Xu's properties for compatible quantizations.

- for any $u,v\in S(\mathfrak{h})$,

$$\begin{aligned}u*'v&=Q^{-1}\big(a(\mathrm{sym}(u))*a(\mathrm{sym}(v))\big)\\&=Q^{-1}\big(a(\mathrm{sym}(u)\mathrm{sym}(v))\big)\\&=Q^{-1}\big(a(\mathrm{sym}(u*_{PBW}v))\big)=u*_{PBW}v\end{aligned}$$

- let $u\in S(\mathfrak{h})$ and $f\in\mathcal{O}_G$, then $f*'u=Q^{-1}\big(f*a(\mathrm{sym}(u))\big)=fu$. Let us now compute $u*'f$; we can assume that $u=h^n$, $h\in\mathfrak{h}$, and then

$$\begin{aligned}u*'f&=Q^{-1}\big(a(\mathrm{sym}(u))*f\big)=Q^{-1}\big(\underbrace{h*\cdots*h}_{n\ times}*f\big)\\&=Q^{-1}\big(\sum_{k=0}^nC_n^k\hbar^k(\overrightarrow{h}^k\cdot f)*\underbrace{h*\cdots*h}_{n-k\ times}\big)\\&=\sum_{k=0}^n\hbar^kC_n^k(\overrightarrow{h}^k\cdot f)h^{n-k}\end{aligned}$$

- Since $*$ is a H-invariant star-product, then Q is a H-invariant gauge equivalence. Therefore $*'$ is also H-invariant.

The proposition is proved. □

Remark 1.8. The gauge transformation Q constructed above obviously satisfies $Q(h) = h$ for any $h \in \mathfrak{h}$.

2. Quantization of symplectic dynamical r-matrices

In this section we prove Theorem 0.1. We start by recalling Fedosov's construction of star-products on a symplectic manifold (for more details we refer to [8, 9]).

2.1. **Fedosov's star-products.** Let (M,ω) a symplectic manifold and denote by $\pi = \omega^{-1}$ the corresponding Poisson bivector. Then its tangent bundle TM inherits a Poisson structure $\tilde{\pi}$ expressed locally as

$$\tilde{\pi} = \pi^{ij}(x)\frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial y^j}\,,$$

where y^i's are coordinates in the fibers. This Poisson structure is regular and constant on the symplectic leaves which are the fibers T_xM of the bundle. Therefore it is quantized by the series of fiberwize bidifferential operators $\exp(\hbar\tilde{\pi})$. It defines an associative product $\circ$ on sections of $W = \hat{S}(T^*M)[[\hbar]]$ that naturally extends to $\Omega^*(M,W)$. The center of $(\Omega^*(M,W),\circ)$ consists of forms that are constant in the fibers, i.e. lying in $\Omega^*(M)[[\hbar]]$.

By assigning the degree $2k+l$ to sections of $\hbar^k S^m(T^*M)$ there is a natural decreasing filtration

$$W = W_0 \supset W_1 \supset \cdots \supset W_i \supset W_{i+1} \supset \cdots \supset \mathcal{O}_M\,.$$

Now fix (once and for all) a torsion free connection ∇ on M with Christoffel's symbols Γ_{ij}^k. One can assume without loss of generality that it is symplectic (see [9, Section 2.5]). Then consider

$$\partial : \Omega^*(M,W) \to \Omega^{*+1}(M,W)$$

its induced covariant derivative. In Darboux local coordinates we have

$$\partial = d + \frac{1}{\hbar}[\Gamma, -]_\circ\,,$$

where $\Gamma = -\frac{1}{2}\Gamma_{ijk}y^iy^jdx^k$ is a local 1-form with values in W ($\Gamma_{ijk} = \omega_{il}\Gamma_{jk}^l$). One has

$$\partial^2 = -\frac{1}{\hbar}[R, -]_\circ$$

where $R = \frac{1}{4}R_{ijkl}y^iy^jdx^k \wedge dx^l$, and $R_{ijkl} = \omega_{im}R_{jkl}^m$ being the curvature tensor of the symplectic connection ∇.

Let us consider more general derivations of $(\Omega^*(M,W),\circ)$ of the form

$$D = \partial - \delta + \frac{1}{\hbar}[r, -]_\circ$$

where $r \in \Omega^1(M,W)$ and $\delta = \frac{1}{\hbar}[\omega_{ij}y^idx^j, -]_\circ$. A simple calculation yields that

$$D^2 = -\frac{1}{\hbar}[\Omega, -]_\circ$$

where $\Omega = \omega + R + \delta r - \partial r - \frac{1}{\hbar}r^2 \in \Omega^2(M,W)$ is called the *Weyl curvature* of D. In particular D is flat (i.e. $D^2 = 0$) if and only if $\Omega \in \Omega^2(M)[[\hbar]]$ (i.e. is a central 2-form), and in this case Bianchi identity for ∇ implies that $d\Omega = D\Omega = 0$.

In computing D^2 one sees that $\delta : \Omega^*(M,W_k) \to \Omega^{*+1}(M,W_{k-1})$ has square zero and that the torsion freeness of ∇ implies $\delta\partial + \partial\delta = 0$. Then we define a homotopy operator $\kappa : \Omega^*(M,W_k) \to \Omega^{*-1}(M,W_{k+1})$ on monomials $a \in \Omega^p(M,S^q(T^*M))$: if $p+q \neq 0$ then

$$\kappa(a) = \frac{1}{p+q}y^i\partial_{dx^i}a$$

and else $\kappa(a)=0$. One easily check that $\kappa^2=0$ and $\delta\kappa+\kappa\delta=\mathrm{id}-\sigma$ where

$$\sigma : W \to C^\infty(M)[[\hbar]], a \mapsto a_{|dx^i=y^i=0}$$

is the projection onto 0-forms constant in the fibers.

Theorem 2.1 (Fedosov)**.** *For any closed 2-form $\Omega=\omega+O(\hbar)\in Z^2(M)[[\hbar]]$ there exists a unique $r\in\Omega^1(M,W_3)$ such that $\kappa(r)=0$ and*

$$D=\partial-\delta+\frac{1}{\hbar}[r,-]_\circ$$

has Weyl curvature Ω and is therefore flat.

Proof. First observe that $\Omega=\omega+R+\delta r-\partial r-\frac{1}{\hbar}r^2$ with $\kappa(r)=0$ if and only if

$$r=\kappa(\Omega-\omega-R+\partial r+\frac{1}{\hbar}r^2)\,. \tag{2.1}$$

Since ∂ preserves the filtration and κ raises its degree by 1 then $\kappa(\Omega-\omega-R)\in\Omega^1(M,W_3)$ and the sequence $(r_n)_{n\geq 3}$ defined by the iteration formula

$$r_{n+1}=r_0+\kappa(\partial r_n+\frac{1}{\hbar}r_n^2) \tag{2.2}$$

with $r_3=\kappa(\Omega-\omega-R)$ converges to a unique element $r\in\Omega^1(M,W_3)$ which is a solution of equation (2.1). We have proved the existence.

Conversely, for any solution $r\in\Omega^1(M,W_3)$ of (2.1) define $r_k=r \bmod W_{k+1}$. Then $r_3=\kappa(\Omega-\omega-R)$ and the sequence $(r_n)_{n\geq 3}$ satisfies (2.2). Unicity is proved. ☐

Such a flat derivation D is called a *Fedosov connection (of ∇-type)*. The previous theorem claims that they are in bijection with series Ω of closed two forms starting with ω.

Theorem 2.2 (Fedosov)**.** *If D is a Fedosov connection then for any $f_0\in C^\infty(M)[[\hbar]]$ there exists a unique D-closed section $f\in\Gamma(M,W)$ such that $\sigma(f)=f_0$. Hence σ establishes an isomorphism between $Z^0_D(W)$ and $C^\infty(M)[[\hbar]]$.*

Proof. Let $f_0\in C^\infty(M)[[\hbar]]$. One has $D(f)=0$ with $\sigma(f)=f_0$ if and only if

$$f=f_0+\kappa(\partial f+\frac{1}{\hbar}[r,f]_\circ)\,. \tag{2.3}$$

Like in the proof of Theorem 2.1 we can solve (uniquely) this equation with the help of an iteration formula: $f_{n+1}=f_0+\kappa(\partial f_n+\frac{1}{\hbar}[r,f_n]_\circ)$. ☐

Then $f*g=\sigma(\sigma^{-1}(f)\circ\sigma^{-1}(g))$ defines a star-product on $C^\infty(M)[[\hbar]]$ that quantizes (M,ω). A star-product constructed this way is called a *Fedosov star-product (of ∇-type)* and is uniquely determined, once ∇ is fixed, by its *characteristic 2-form*

$$\omega_\hbar:=\frac{1}{\hbar}(\Omega-\omega)\in Z^2(M)[[\hbar]]\,.$$

Moreover one can easily prove the following

Lemma 2.3 ([4])**.** *Let $\omega_\hbar^{(i)}=\sum_{k>0}\hbar^{k-1}\omega_k^{(i)}\in Z^2(M)[[\hbar]]$ $(i=1,2)$ and denote by $*_i$ the Fedosov star-product with characteristic two-form $\omega_\hbar^{(i)}$. If $\omega_\hbar^{(1)}=\omega_\hbar^{(2)}+O(\hbar^k)$ then*

$$*^{(1)}-*^{(2)}=\hbar^{k+1}\pi^{\#}(\omega_k^{(1)}-\omega_k^{(2)})+O(\hbar^{k+2})\,.$$

2.2. **Fedosov's construction in the presence of symmetries.** Let (M,ω) a symplectic manifold. Let us prove two results on the compatibility of Fedosov's construction with group actions and hamiltonians vector fields.

Proposition 2.4 (Fedosov)**.** *Assume that a group G acts on (M,ω) by symplectomorphisms and is equipped with a G-invariant torsion free connection. Then for any $\omega_\hbar \in Z^2(M)^G[[\hbar]]$ the corresponding Fedosov star-product is G-invariant.*

Proof. First observe that starting from a G-invariant torsion free connection ∇ one can assume without loss of generality that it is symplectic (see the proof of Proposition 5.2.2 in [9], where all expressions become obviously G-invariant).

Then, being a symplectomorphism of (M,ω), any elements $g \in G$ acts via its differential dg on $(TM,\tilde{\pi})$ as a Poisson automorphism linear in the fibers. Then its dual map $g^* : T^*M \to T^*M$ defined by $< g^*\xi, x >=< \xi, dg(x) >$ extends to W as an automorphism.

Finally, we need to prove that g^* preserves the Fedosov connection with Weyl curvature $\Omega = \omega + \omega_\hbar$. On one hand the automorphism g^* commutes with ∂ (since ∇ is assumed to be G-invariant) and so $g^*R = R$. On the other hand g^* also commutes with δ and κ, thus if r is a solution of equation (2.1) with $\kappa(r) = 0$ then so is g^*r. By uniqueness $g^*r = r$. We are done. □

Proposition 2.5 (Fedosov)**.** *Let $H \in \mathcal{O}_M$ such that $\chi = \{H, \cdot\}$ preserves a torsion free connection on M. Then for any $\omega_\hbar \in Z^2(M)[[\hbar]]$ such that $\iota_\chi\omega_\hbar = 0$ the corresponding Fedosov star-product $*$ satisfies $H * f - f * H = \hbar(\chi \cdot f)$ for any $f \in \mathcal{O}_M$.*

Proof. First observes that $L_\chi\omega_\hbar = (d\iota_\chi + \iota_\chi d)\omega = 0$. Therefore, the infinitesimal version of the previous proof ensure us that the Fedosov connection D with Weyl curvature $\Omega = \omega + \omega_\hbar$ is L_X-equivariant. Hence in local Darboux coordinates it writes $D = d + \frac{1}{\hbar}[\gamma, -]_\circ$ with $L_\chi\gamma = 0$, and $\Omega = -d\gamma - \frac{1}{\hbar}\gamma^2$. Let us compute

$$\begin{aligned} D(H - \iota_\chi\gamma) &= dH - d\iota_\chi\gamma - \frac{1}{\hbar}[\gamma, \iota_\chi\gamma]_\circ = \iota_\chi\Omega + \iota_\chi d\gamma + \frac{1}{\hbar}[\iota_\chi\gamma, \gamma]_\circ \\ &= \iota_\chi(\Omega + d\gamma + \frac{1}{\hbar}\gamma^2) = 0 \end{aligned}$$

Since $\sigma(H - \iota_\chi\gamma) = \sigma(H) = H$, it means that $\sigma^{-1}(H) = H - \iota_\chi\gamma$ in local Darboux coordinates. Consequently, for any $f \in \mathcal{O}_M[[\hbar]]$

$$\begin{aligned} H * f - f * H &= \sigma\big([H - \iota_\chi\gamma, \sigma^{-1}(f)]_\circ\big) = \sigma\big(-[\iota_\chi\gamma, \sigma^{-1}(f)]_\circ\big) \\ &= \sigma\big(-\iota_\chi\hbar(D-d)\sigma^{-1}(f)\big) = \hbar\sigma\big(\iota_\chi d\sigma^{-1}(f)\big) \\ &= \hbar\sigma\big(L_\chi\sigma^{-1}(f) - d\iota_\chi\sigma^{-1}(f)\big) = \hbar L_\chi(f) = \hbar(\chi \cdot f) \end{aligned}$$

The proposition is proved. □

2.3. **Proof of Theorem 0.1.** Let $r : U \to \wedge^2\mathfrak{g}$ a symplectic dynamical r-matrix. A basis $\mathcal{B}$ of vector fields on $M = U \times G$ is given by $\mathcal{B} = (\dots, \partial_{\lambda^i}, \dots, \dots, \overrightarrow{e_i}, \dots)$ where $(\lambda^i)_i$ is a base of $\mathfrak{h}^*$ and $(e_i)_i$ is a base of $\mathfrak{g}$. Then one defines a torsion free connection ∇ on M as

$$\nabla_b X = \frac{1}{2}[b, X] \qquad \big(b \in \mathcal{B},\ X \in \mathfrak{X}(M)\big)\,.$$

Remark that $[\chi_h, b] \in \mathrm{span}_{\mathbb{R}}\mathcal{B}$ for any $b \in \mathcal{B}$. Therefore it follows immediatly from the Jacobi identity that ∇ is $\mathfrak{h}$-invariant: for all $X, Y \in \mathfrak{X}(M)$ and $h \in \mathfrak{h}$,

$$[\chi_h, \nabla_X Y] = \nabla_{[\chi_h, X]}Y + \nabla_X[\chi_h, Y]\,.$$

Thus from Proposition 2.5 the Fedosov star-product $*$ with the trivial characteristic 2-form is strongly $\mathfrak{h}$-invariant. Moreover ∇ is obviously G-invariant, hence Proposition 2.4 implies that $*$ is also G-invariant.

Finally, we apply Proposition 1.7 to construct a compatible quantization of π_r. We are done. □

3. Classification

Let $r : \mathfrak{h}^* \supset U \to \wedge^2\mathfrak{g}$ a dynamical r-matrix. Denote by π_r the corresponding H-invariant $\mathfrak{g}$-quasi-Poisson structure (0.1) on $M = U \times G$ (together with $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$).

3.1. Strongly invariant equivalences and obstructions.

By a *strongly invariant equivalence* between two strongly $\mathfrak{h}$-invariant quantizations of π_r we mean a H-invariant equivalence Q (nameky, $Q = \mathrm{id} + O(\hbar) \in \mathrm{Diff(M)}^{G\times H}[[\hbar]]$) satisfying $Q(h) = h$ for any $h \in \mathfrak{h} \subset \mathcal{O}_M$. We will now developp an analogue of the usual obstruction theory in this context.

Let us denote by b the Hochschild coboundary operator for cochains on the (commutative) algebra $\mathcal{O}_M$. We start with the following result which is a variant of a standard one.

Proposition 3.1. *Suppose that $*_1$ and $*_2$ are two strongly invariant quantizations of π_r:*

$$f *_i g = \sum_{k\geq 0} \hbar^k C_k^i(f,g) \qquad (i = 1,2)\,.$$

Assume that $*_1$ and $*_2$ coincide up to order n, i.e. $C_k^1 = C_k^2$ if $k \leq n$. Then

(1) *there exists $B \in \Gamma(M, \wedge^2 TM)^{G\times H}$ and $E \in \mathrm{Diff}(M)^{G\times H}$ such that $B(h,-) = 0$ and $E(h) = 0$ if $h \in \mathfrak{h} \subset \mathcal{O}_{\mathfrak{h}^*}$, $[\pi_r, B] = 0$, and satisfying*

$$(C_{n+1}^1 - C_{n+1}^2)(f,g) = B(f,g) + (bE)(f,g) \qquad (f, g \in \mathcal{O}_M)\,;$$

(2) *there exists $P \in \mathrm{Diff}(M)^{G\times H}$ such that $C_1 = \pi_r + bP$ and $P(h) = 0$ for $h \in \mathfrak{h}$;*
(3) *if $B = [\pi_r, X]$, $X \in \mathfrak{X}(M)^{G\times H}$ such that $X(h) = 0$, then the strongly invariant equivalence $Q = 1 + \hbar^n X + \hbar^{n+1}(E + [X,P])$ transforms $*_1$ into another strongly invariant star-product which coincides with $*_2$ up to order $n+1$.*

Proof. We use a similar argument as in [13, 3, 4].

(1) It is well-known that $b(C_{n+1}^1 - C_{n+1}^2) = 0$. Hence we may write

$$C_{n+1}^1 - C_{n+1}^2 = B + b(E_0)$$

where $B \in \Gamma(M, \wedge^2 TM)^{G\times H}$ is the skew-symmetric part of $C_{n+1}^1 - C_{n+1}^2$ and $E_0 \in \mathrm{Diff}(M)$. Moreover, one knows (see e.g. [4]) that $[\pi_r, B] = 0$, and it follows directly from the strong $\mathfrak{h}$-invariance property for $*_1$ and $*_2$ that $B(h,-) = 0$ if $h \in \mathfrak{h}$.

Since $U \times G$ admits a $G\times H$-invariant connection and $b(E_0)$ is obviously $G\times H$-invariant, then according to [3, Proposition 2.1] we can assume that E_0 is $G\times H$-invariant. In particular E_0 is G-invariant and hence $E_0(f)$, $f \in \mathcal{O}_U$, is a function on U only. Thus we can define a H-invariant vector field $\vec{v}$ on U as follows: $< dh, \vec{v} >= E_0(h)$ for any $h \in \mathfrak{h} \subset \mathcal{O}_U$. Now $E := E_0 - \vec{v}$ satisfies all the required properties and $b(E) = b(E_0) - b(\vec{v}) = b(E_0)$.

(2) It is standard that $C_1 = \pi_r + b(P_0)$. By repeating a similar argument as in (1) we can prove that P_0 can be chosen so that $P_0 = P \in \mathrm{Diff}(M)^{G\times H}$ and satisfies $P(h) = 0$ for any $h \in \mathfrak{h}$.

The third statement (3) follows from an easy (and standard) calculation. □

This proposition means that obstructions to strongly invariant equivalences are in the second cohomology group of the subcomplex $C^\infty(U, \wedge^*\mathfrak{g})^{\mathfrak{h}}$ in the H-invariant quasi-Poisson cochain complex of (M, π_r, Z). On such cochains the (quasi-)Poisson coboundary operator $[\pi_r, -]$ reduces to $[r, -]$.

Definition 3.2. The cohomology $H_r^*(U, \mathfrak{g})$ of this cochain complex is called the *dynamical r-matrix cohomology* associated to $r : U \to \wedge^2\mathfrak{g}$.

3.2. **Classification of strongly invariant star-products.** Now assume that the quasi-Poisson manifold (M,π_r) is actually symplectic and denote by ω_r the symplectic form; it is $G\times H$-invariant and satisfies $\iota_{\chi_h}\omega=0$ for any $h\in\mathfrak{h}\subset\mathcal{O}_{\mathfrak{h}^*}$. The $G\times H$-invariant isomorphism

$$\pi_r^\#:T^*M\tilde{\longrightarrow}TM$$

extends to a $G\times H$-invariant isomorphism of cochain complexes

$$(\Omega^*(M),d)\tilde{\longrightarrow}(\Gamma(M,\wedge^*TM),[\pi_r,-])$$

which restricts to an isomorphism

$$(\Omega^*_{\mathfrak{h}}(M)^G,d)\tilde{\longrightarrow}(C^\infty(U,\wedge^*\mathfrak{g})^{\mathfrak{h}},[r,-])\,,$$

where $\Omega^*_{\mathfrak{h}}(M):=\{\alpha\in\Omega^*(M)^H|\iota_{\chi_h}\alpha=0\,,\forall h\in\mathfrak{h}\}$.

Let us fix once and for all a symplectic $G\times H$-invariant connection ∇ on M (we know it exists) and remember from the previous section that for any $\omega_\hbar\in\hbar\Omega^2(M)^{G\times H}[[\hbar]]$ such that $d\omega_\hbar=0$ there exists a (unique) $G\times H$-invariant Fedosov star-product $*$ with characteristic 2-form $\omega_\hbar$. Moreover, if $\omega_\hbar\in\Omega^2_{\mathfrak{h}}(M)[[\hbar]]$ then Proposition 2.5 implies that $*$ is strongly $\mathfrak{h}$-invariant. Therefore we can associate a strongly invariant quantization of π_r (which is actually a Fedosov star-product) to any closed two form $\omega_\hbar\in\Omega^2_{\mathfrak{h}}(M)^G[[\hbar]]$.

In the rest of the section, all Fedosov star-products are assumed to be of ∇-type and G-invariant (since they quantize the $\mathfrak{g}$-quasi-Poisson structure π_r).

Theorem 3.3. *Two strongly invariant Fedosov star-products are equivalent by a strongly invariant equivalence if and only if their characteristic 2-forms lies in the same cohomology class in* $H^{G,2}_{\mathfrak{h}}(M)[[\hbar]]$.

Proof. Let $*_0$ and $*_1$ two strongly invariant Fedosov star-products with respective characteristic 2-form $\omega_\hbar^{(0)}$ and $\omega_\hbar^{(1)}$.

First assume that $\omega_\hbar^{(0)}=\omega_\hbar^{(1)}+d\alpha$ for some $\alpha=\sum_k\hbar^k\alpha^{(k)}\in\Omega^1_{\mathfrak{h}}(M)^G[[\hbar]]$, and define $\omega_\hbar(t)=\omega_\hbar^{(0)}+td\alpha$. Let $D_t=\partial-\delta+\frac{1}{\hbar}[r(t),-]$ be the Fedosov differential with Weyl curvature $\Omega(t)=\omega+\hbar\omega_\hbar(t)$. Let $H(t)\in\Gamma(M,W)$ be the solution of the equation

$$D_tH(t)=-\alpha+\dot r(t)$$

with $\sigma(H(t))=0$. Then $H(t)$ is $G\times H$-invariant since D_t, α and $r(t)$ are. According to [9, Theorem 5.5.3] the solution of the Heisenberg equation

$$\frac{dF(t)}{dt}+[H(t),F(t)]_\circ=0\,,\quad F(0)=f$$

establishes an isomorphism $Z^0_{D_0}(W)\to Z^0_{D_1}(W)$, $f\mapsto F(1)$ and then the corresponding series of differential operators $Q:(\mathcal{O}_M[[\hbar]],*_0)\to(\mathcal{O}_M[[\hbar]],*_1)$ is obviously $G\times H$-invariant. Remember from the proof of Proposition 2.5 that in local Darboux coordinates the Fedosov differential writes $D_t=d+\frac{1}{\hbar}[\gamma(t),-]_\circ$ and $\sigma_t^{-1}(h)=h-\iota_{\chi_h}\gamma(t)$ if $h\in\mathfrak{h}$. Now remark that $\dot\gamma(t)=\dot r(t)$ and that $\iota_{\chi_h}r(t)$ is independant of t. Hence $\sigma_t^{-1}(h)$ does not depend on t and thus $Q(h)=h$.

Conversely, assume that $*_0$ and $*_1$ are related by a strongly invariant equivalence with $[\omega_\hbar^{(0)}]\neq[\omega_\hbar^{(1)}]$ in $H^{G,2}_{\mathfrak{h}}(M)[[\hbar]]$. Write $\omega_\hbar^{(i)}=\sum_{k>0}\hbar^{k-1}\omega_k^{(i)}$ $(i=0,1)$ and denote by l the lowest integer for which $[\omega_l^{(0)}]\neq[\omega_l^{(1)}]$ in $H^{G,2}_{\mathfrak{h}}(M)$. Let us then define

$$\omega_\hbar^{(2)}=\sum_{0<k<l}\hbar^{k-1}\omega_k^{(0)}+\sum_{k\geq l}\hbar^{k-1}\omega_k^{(1)}$$

and $*_2$ the strongly invariant Fedosov star-product with characteristic 2-form $\omega_\hbar^{(2)}$. Since $[\omega_\hbar^{(2)}]=[\omega_\hbar^{(1)}]\in H^{G,2}_{\mathfrak{h}}(M)[[\hbar]]$ then $*_2$ is equivalent to $*_1$, and hence to $*_0$, by a strongly invariant equivalence. Then we deduce from Lemma 2.3 that $C_{l+1}^2 - C_{l+1}^0 = \pi_r^\#(\omega_l^{(1)} - \omega_l^{(0)})$, where C_k^i's $(i = 0, 2)$ are the cochains defining $*_i$. Thus it follows from Proposition 3.1 that $\omega_l^{(1)} - \omega_l^{(0)}$ is exact and we obtain a contradiction. $\square$

Theorem 3.4. *Any strongly invariant quantization of π_r is equivalent to a strongly invariant Fedosov star-product by a strongly invariant equivalence. Therefore the set of strongly invariant star-products quantizing π_r up to strongly invariant equivalences is an affine space modeled on $H_{\mathfrak{h}}^{G,2}(M)[[\hbar]] = H_r^2(U, \mathfrak{g})$.*

Proof. We follow [4, Proposition 4.1].

Let $*$ be an arbitrary strongly invariant quantization of π_r. Denote by $*_0$ the strongly invariant Fedosov star-product with the trivial characteristic 2-form, which coincides with $*$ up to order 0. Moreover, the skew-symmetric part of the first order term in $* - *_0$ vanishes, hence it follows from Proposition 3.1 that there exists a strongly invariant equivalence $Q^{(0)} = 1 + \hbar Q_0$ that transforms $*$ into a new strongly invariant quantization $*^{(0)}$ which coincides with $*_0$ up to order 1. Now the skew-symmetric part of the second order term in $*^{(0)} - *_0$ yields a closed form $\omega_1 \in \Omega_{\mathfrak{h}}^2(M)^G$.

Denote by $*_1$ the strongly invariant Fedosov star-product with characteristic 2-form ω_1. Lemma 2.3 tells us that the skew-symmetric part of the second order term in $*^{(0)} - *_1$ vanishes, hence it follows from Proposition 3.1 that there exists a strongly invariant equivalence $Q^{(1)} = 1 + \hbar^2 Q_1$ that transforms $*^{(0)}$ into a new strongly invariant quantization $*^{(1)}$ which coincides with $*_1$ up to order 2.

Repeating this procedure we get a sequence of strongly invariant equivalences $Q^{(k)} = 1 + \hbar^{k+1} Q_k$ $(k \geq 0)$ and a sequence of closed forms ω_k $(k > 0)$ such that the strongly invariant quantization $*^{(k)}$ obtained from $*$ by applying successively $Q^{(0)}, \ldots, Q^{(k)}$ coincides up to order $k+1$ with the strongly invariant Fedosov star-product $*_k$ with characteristic 2-form $\omega_1 + \cdots + \hbar^{k-1}\omega_k$.

Finally, the strongly invariant equivalence $Q := \cdots Q^{(2)} Q^{(1)} Q^{(0)}$ transform $*$ into the strongly invariant Fedosov star-product with characteristic 2-form $\omega_\hbar = \sum_{k>0} \hbar^{k-1}\omega_k$. $\square$

3.3. **Classification of dynamical twist quantizations.** Let T be a gauge equivalence of dynamical twist quantizations J_1 and J_2 of r. One can view T as an element in $\mathrm{Diff}(U \times G)^{G \times H}[[\hbar]]$ such that $T(u) = u$ for any $u \in \mathcal{O}_{\mathfrak{h}^*}$. Moreover if we denote by $*_i'$ the compatible quantization of π_r corresponding to J_i $(i = 1, 2)$ then it follows from an easy calculation that

$$T(f *_1' g) = T(f) *_2' T(g) .$$

Conversely, any $G \times H$-invariant gauge equivalence T from $*_1'$ to $*_2'$ which is such that $T(u) = u$ for any $u \in \mathcal{O}_{\mathfrak{h}^*}$, that we will call from now a *compatible equivalence*, obviously gives rise to a gauge equivalence of the dynamical twist quantizations J_1 and J_2.

Therefore, the set of dynamical twist quantization of r up to gauge equivalences is in bijection with the set of compatible quantizations of π_r up to compatible equivalences.

Remember from Proposition 1.7 and Remark 1.8 that any strongly invariant quantization is equivalent to a compatible one by a strongly invariant equivalence. Moreover the PBW star-product has the following nice property: for any $h \in \mathfrak{h}$, $h^{*_{PBW} n} = h^n$. Hence any strongly invariant equivalence between two compatible quantizations is actually a compatible equivalence. Consequently:

Proposition 3.5. *There is a bijection*

$$\frac{\{\textit{strongly invariant quantizations of } \pi_r\}}{\textit{strongly invariant equivalences}} \longleftrightarrow \frac{\{\textit{compatible quantizations of } \pi_r\}}{\textit{compatible equivalences}}$$

End of the proof of Theorem 0.2. Assume that the dynamical r-matrix is symplectic. Then Theorem 0.2 follows from Proposition 3.5 and Theorem 3.4. □

4. The quantum composition formula

In this section we assume that $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{m}$ is a nondegenerate reductive splitting and we denote by $r_{\mathfrak{t}}^{\mathfrak{m}} : \mathfrak{t}^* \supset V \to \wedge^2\mathfrak{h}$ the corresponding symplectic dynamical r-matrix. Let $p : \mathfrak{h} \to \mathfrak{t}$ be the $\mathfrak{t}$-invariant projection along $\mathfrak{m}$. For any function f on $\mathfrak{h}^*$ with values in a $\mathfrak{h}$-module L we write $f_{|\mathfrak{t}^*}$ for the function $f \circ p^*$ on $\mathfrak{t}^*$ with values in L viewed as a $\mathfrak{t}$-module; in particular if f is $\mathfrak{h}$-invariant then $f_{|\mathfrak{t}^*}$ is $\mathfrak{t}$-invariant.

4.1. **The classical composition formula (proof of Proposition 0.3).** Let $\rho : U \to \wedge^2\mathfrak{g}$ be a dynamical r-matrix with $Z \in (\wedge^3\mathfrak{g})^{\mathfrak{g}}$. Then $\pi := \pi_{r_{\mathfrak{t}}^{\mathfrak{m}}} + \pi_\rho$ defines a $\mathfrak{g}$-quasi-Poisson structure (with the same Z) on the manifold $X = V \times H \times U \times G$ which is

- H-invariant with respect to left multiplication on H,
- H-invariant with respect to the right action on $U \times G$,
- T-invariant with respect to the right action on $V \times H$.

The right diagonal H-action, given by $(\tau, x, \lambda, y) \cdot q = (\tau, q^{-1}x, \mathrm{Ad}_q^*\lambda, yq)$, actually comes from a momentum map:

$$\begin{array}{rcl} \mu : X & \longrightarrow & \mathfrak{h}^* \\ (\tau, x, \lambda, y) & \longmapsto & \lambda - \mathrm{Ad}_{x^{-1}}^*(p^*\tau) \,. \end{array}$$

Consequently we can apply the reduction with respect to μ. The right H-invariant smooth map

$$\begin{array}{rcl} \psi : X = V \times H \times U \times G & \longrightarrow & M := U \cap V \times G \\ (\tau, x, \lambda, y) & \longmapsto & (\tau, yx) \end{array}$$

restricts to a diffeomorphism $\mu_2^{-1}(0)/H \to M$ with inverse given by

$$(\tau, y) \longmapsto \overline{(\tau, 1, p^*\tau, y)} \quad (\tau \in U \cap V \,,\, y \in G)\,.$$

Remark 4.1. From an algebraic viewpoint, we have an injective map of commutative algebras $\psi^* : \mathcal{O}_M \to \mathcal{O}_X$ with values in $\mathcal{O}_X^{\mathfrak{h}} = \mathcal{O}_{X/H}$ and such that, composed with the projection $\mathcal{O}_X^{\mathfrak{h}} \to \mathcal{O}_X^{\mathfrak{h}}/(\mathcal{O}_X^{\mathfrak{h}} \cap < \mathrm{im}(\mu^*) >) = \mathcal{O}_{\mu^{-1}(0)/H}$, it becomes an isomorphism.

Since ψ is obviously left G-invariant then it remains to show that the induced $\mathfrak{g}$-quasi-Poisson structure on M is $\pi_{\rho_{|\mathfrak{t}^*} + r_{\mathfrak{t}}^{\mathfrak{m}}}$. Let $t, t' \in \mathfrak{t} \subset \mathcal{O}_{\mathfrak{t}^*}$ and $f, g \in \mathcal{O}_G$. First of all we have

$$\{\psi^*t, \psi^*t'\}_X = \{t, t'\}_X = [t, t'] = \psi^*[t, t']\,,$$

hence $\{t, t'\}_M = [t, t']$. Then

$$\{\psi^*t, \psi^*f\}_X = \{t, f(yx)\}_X = \overrightarrow{t}^H \cdot (f(yx)) = (\overrightarrow{t} \cdot f)(yx) = \psi^*(\overrightarrow{t} \cdot f)\,.$$

The third equality follows from the left H-invariance of $\overrightarrow{t}^H$. Thus $\{t, f\}_M = \overrightarrow{t} \cdot f$. Finally

$$\begin{array}{rcl} \{\psi^*f, \psi^*g\}_X(\tau, x, \lambda, y) & = & \overrightarrow{r_{\mathfrak{t}}^{\mathfrak{m}}(\tau)}^H \cdot (f(yx), g(yx)) + \overrightarrow{\rho(\lambda)}^G \cdot (f(yx), g(yx)) \\ & = & \big(\overrightarrow{r_{\mathfrak{t}}^{\mathfrak{m}}(\tau)} \cdot (f, g)\big)(yx) + \big(\overrightarrow{\rho(\mathrm{Ad}_x^*\lambda)} \cdot (f, g)\big)(yx)\,. \end{array}$$

Therefore, when restricting to $\mu_2^{-1}(0)$ one obtains

$$\begin{array}{rcl} \{\psi^*f, \psi^*g\}_X(\tau, x, \mathrm{Ad}_{x^{-1}}^*(p^*\tau), y) & = & \big(\overrightarrow{r_{\mathfrak{t}}^{\mathfrak{m}}(\tau)} \cdot (f, g)\big)(yx) + \big(\overrightarrow{\rho(p^*\tau)} \cdot (f, g)\big)(yx) \\ & = & \psi^*\big(\overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}} + \rho_{|\mathfrak{t}^*})} \cdot (f, g)\big)\,. \end{array}$$

Therefore $\{f, g\}_M = \overrightarrow{(r_{\mathfrak{t}}^{\mathfrak{m}} + \rho_{|\mathfrak{t}^*})} \cdot (f, g)$. This ends the proof of Proposition 0.3. □

4.2. **Quantization of the momentum map μ.** Let us first consider $(V \times H, \pi_{r_{\mathfrak{t}}^{\mathfrak{m}}})$. There is a momentum map

$$\begin{aligned} \nu : V \times H &\longrightarrow \mathfrak{h}^* \\ (\tau, x) &\longmapsto -\mathrm{Ad}_{x^{-1}}(p^*\tau) \end{aligned}$$

with corresponding right H-action on $V \times H$ given by $(\tau, x) \cdot q = (\tau, q^{-1}x)$.

Like in subsection 2.3 one has a T-invariant and H-invariant torsion free connexion on $V \times H$, therefore from Proposition 2.5 the corresponding Fedosov star-product $*$ is both strongly $\mathfrak{h}$-invariant and strongly $\mathfrak{t}$-invariant[5].

Then Proposition 1.7 tells us that there exists a strongly $\mathfrak{t}$-invariant (and H-invariant) equivalence Q such that $*' := *^{(Q)}$ is a compatible quantization of $\pi_{r_{\mathfrak{t}}^{\mathfrak{m}}}$. Consequently we can define the following algebra morphism:

$$\mathbf{N} := Q^{-1} \circ U(\nu^*) \circ \mathrm{sym} : (\mathcal{O}_{\mathfrak{h}^*}[[\hbar]], *_{PBW}) \to (\mathcal{O}_{V\times H}[[\hbar]], *') \,.$$

It is obviously a quantization of the Poisson map ν and, moreover, for any $h \in \mathfrak{h}$ and any $f \in \mathcal{O}_{V\times H}$ one has

$$[\mathbf{N}(h), f]_{*'} = Q^{-1}\big([\nu^*h, Q(f)]_*\big) = Q^{-1}(\hbar\{\nu^*h, Q(f)\}) = \hbar\{\nu^*h, f\} \,.$$

In other words, $\mathbf{N}$ is a quantum momentum map quantizing ν.

Let us now assume that we know a dynamical twist quantization $J(\lambda) : U \to \otimes^2 U\mathfrak{g}[[\hbar]]$ of $\rho(\lambda)$ (with some associator Φ) and denote by $*_J$ the corresponding compatible quantization of π_ρ on $U \times G$. Together with $*'$ it induces a quantization $*'_J$ of $\pi_{r_{\mathfrak{t}}^{\mathfrak{m}}} + \pi_\rho$ on X (with the same Φ).

Remark 4.2. Actually $*'_J$ is the compatible quantization corresponding to the dynamical twist quantization $\mathbf{J}(\tau,\lambda) := J_{\mathfrak{t}}^{\mathfrak{m}}(\tau)J(\lambda) : (\mathfrak{t}\oplus\mathfrak{h})^* \supset V \times U \longrightarrow \otimes^2 U(\mathfrak{h}\oplus\mathfrak{g})[[\hbar]]$ of the dynamical r-matrix $\mathbf{r}(\tau,\lambda) := r_{\mathfrak{t}}^{\mathfrak{m}}(\tau) + \rho(\lambda) : V \times U \longrightarrow \wedge^2(\mathfrak{h}\oplus\mathfrak{g})$. Here $J_{\mathfrak{t}}^{\mathfrak{m}}$ is the dynamical twist quantizing $r_{\mathfrak{t}}^{\mathfrak{m}}$.

For any $f \in \mathcal{O}_{\mathfrak{h}^*}$ we define $\mathbf{M}(f) := (Q \otimes \mathrm{inc}) \circ \Delta(f) \in (\mathcal{O}_{V\times H} \otimes \mathcal{O}_{U\times G})[[\hbar]] = \mathcal{O}_X[[\hbar]]$. Here $\mathrm{inc} : \mathcal{O}_{\mathfrak{h}^*} \hookrightarrow \mathcal{O}_{U\times G}$ is the natural inclusion and $\Delta : \mathcal{O}_{\mathfrak{h}^*} \to \mathcal{O}_{\mathfrak{h}^*} \otimes \mathcal{O}_{\mathfrak{h}^*} = \mathcal{O}_{\mathfrak{h}^*\times\mathfrak{h}^*}$ is defined by $\Delta(f)(\lambda_1, \lambda_2) = f(\lambda_1 + \lambda_2)$.

Proposition 4.3. *The algebra morphism*

$$\mathbf{M} : (\mathcal{O}_{\mathfrak{h}}[[\hbar]], *_{PBW}) \longrightarrow (\mathcal{O}_X[[\hbar]], *'_J)$$

is a quantum momentum map quantizing μ.

4.3. **Quantization of the composition formula (proof of Theorem 0.4).** Let us assume that J is a dynamical twist quantization of ρ and keep the notations of the previous subsection.

Denote by $\mathcal{I}$ the right ideal generated by $\mathrm{im}(\mathbf{M})$ in $(\mathcal{O}_X[[\hbar]], *'_J)$ and consider the reduced algebra $\mathcal{A} := \mathcal{O}_X^{\mathfrak{h}}[[\hbar]]/\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \cap \mathcal{I}$. Let $\Psi = \psi^* + O(\hbar)$ be the composition of $\psi^* : \mathcal{O}_M[[\hbar]] \to \mathcal{O}_X^{\mathfrak{h}}[[\hbar]]$ with the projection $\mathcal{O}_X^{\mathfrak{h}}[[\hbar]] \to \mathcal{A} \cong \mathcal{O}_{\mu^{-1}(0)/H}[[\hbar]]$. It is obviously bijective and G-invariant (since ψ^* is), therefore it defines a quantization $\tilde{*}$ of the quasi-Poisson structure $\pi_{r_{\mathfrak{t}}^{\mathfrak{m}} + \rho_{|\mathfrak{t}^*}}$. We end the proof of Theorem 0.4 using the following proposition:

Proposition 4.4. *$\tilde{*}$ is a compatibe quantization.*

Proof. First of all for any $u, v \in \mathcal{O}_{\mathfrak{t}^*}$ one has

$$(\psi^*u) *'_J (\psi^*v) = u *'_J v = u *_{PBW} v = \psi^*(u *_{PBW} v) \,.$$

Consequently $u \tilde{*} v = \Psi^{-1}(\Psi(u) \cdot_{\mathcal{A}} \Psi(v)) = u *_{PBW} v$.

[5] Remind that we also have a momentum map $V \times H \to \mathfrak{t}^*; (\tau, x) \mapsto \tau$ with corresponding right T-action given by $(\tau, x) \cdot b = (\mathrm{Ad}_b^*(\tau), xb)$.

Then let $u \in \mathcal{O}_{\mathfrak{t}^*}$ and $f \in \mathcal{O}_G$. On one hand

$$(\psi^* f) *'_J \psi^* u = (f(yx)) *'_J u = f(yx)u = \psi^*(fu)$$

and thus $f \tilde{*} u = fu$. On the other hand for $u = t^n$ $(t \in \mathfrak{t})$ one has

$$\psi^*(t^n) *'_J (\psi^* f) = (t^n) *'_J (f(yx)) = \sum_{k=0}^{n} \hbar^k C_n^k \big((\overrightarrow{t}^H)^k \cdot (f(yx)) \big) t^{n-k}$$

$$= \sum_{k=0}^{n} \hbar^k C_n^k (\overrightarrow{t}^k \cdot f)(yx) t^{n-k} = \psi^* \Big(\sum_{k=0}^{n} \hbar^k C_n^k (\overrightarrow{t}^k \cdot f) t^{n-k} .$$

Therefore $t^n \tilde{*} f = \sum_{k=0}^{n} \hbar^k C_n^k (\overrightarrow{t}^k \cdot f) t^{n-k}$. The proposition is proved. □

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