ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$

The detailed version, **Part 5**

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In memory of Professor Alexander Buchstab

Content

§0. Foreword

I continue the presentation of the details of the proof of the Theorem announced in [35] and proved in [31], [32]. In Section 1, I prove a Theorem (see Lemma 5.5), which gives sufficient conditions for the coefficient $a_0(\nu)$ in the linear difference equation

$$\sum_{k=0}^{n} a_k(\nu)y(\nu+k)=0$$

to be equal to 0. I have decided to select that Theorem in a separate part because of its general character. In Section 2, I apply the general Theorem proved in Section 1 to the situation investigated in this series.

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§1. Ballad.

Lemma 5.1. *Let* $N\in\mathbb{N}-1, m\in\mathbb{N}$. *Let*

$$R(z)=F(z)+\sum_{i=1}^{\infty}\sum_{k=0}^{N}\beta_{i,k}(z+k)^{-i}, \tag{1}$$

where $F(z) \in \mathbb{C}[z], \beta_{i,k} \in \mathbb{C}$ *for* $i \in \mathbb{N}$ *and* $k = 0, ..., N,$

$$\beta_{i,k} = 0$$

for $i \in \mathbb{N} + m$ *and* $k = 0, ..., N$. *Let*

$$\beta_i^{\vee}(z) = \sum_{k=0}^{N} \beta_{i,k} z^k,$$

where $i \in \mathbb{N}$. *Let further* $s \in \mathbb{N} - 1$ *and*

$$z^s R(z) = F_s(z) + \sum_{i=1}^{\infty} \sum_{k=0}^{N} \beta^*_{s,i,k} (z+k)^{-i}, \tag{2}$$

where $F_s(z) \in \mathbb{C}[z], \beta_{s,i,k} \in \mathbb{C}$ *for* $i \in \mathbb{N}$ *and* $k = 0, ..., N,$

$$\beta^*_{s,i,k} = 0$$

for $i \in \mathbb{N} + m$ *and* $k = 0, ..., N$. *Let*

$$\beta^*_{s,i}(z) = \sum_{k=0}^{N} \beta^*_{s,i,k} z^k.$$

Then

$$\beta^*_{s,i}(z) = \sum_{l=0}^{s} \binom{s}{l} ((-\delta)^l \beta^{\vee}_{s+i-l})(z), \tag{3}$$

where δ *denotes the operator* $z \frac{\partial}{\partial z}$.

Proof. Clearly, the equality (3) is true for $s = 0$. In view of (2),

$$z^{s+1} R(z) = zF_s(z) + \sum_{i=1}^{\infty} \sum_{k=0}^{N} \beta^*_{s,i,k} ((z+k)^{-i+1} - k(z+k)^{-i}).$$

Therefore

$$F_{s+1}(z) = zF_s(z) + \beta^*_{s,1}(1),$$

$$\beta^*_{s+1,i}(z) = -\delta \beta^*_{s,i}(z) + \beta^*_{s,i+1}(z). \tag{4}$$

Clearly, $\beta^*_{0,i}(z) = \beta^\vee_i(z)$, where $i \in \mathbb{N}$; therefore, taking in (4) $s = 0$ we obtain (3) for $s = 1$. Let the equality (3) is true for some $s \in \mathbb{N} - 1$. In view of (4),

$$\beta^*_{s+1,i}(z) = -\delta\beta^*_{s,i}(z) + \beta^*_{s,i+1}(z) =$$

$$\left(\sum_{l=0}^{s} \binom{s}{l} (-\delta)^{l+1} \beta^\vee_{s+i-l}(z)\right) + \sum_{l=0}^{s} \binom{s}{l} (-\delta)^l \beta^\vee_{s+i+1-l}(z) =$$

$$\left(\sum_{l=1}^{s+1} \binom{s}{l-1} ((-\delta)^l \beta^\vee_{s+i+1-l}(z)\right) + \sum_{l=0}^{s} \binom{s}{l} (-\delta)^l \beta^\vee_{s+i+1-l}(z) =$$

$$\sum_{l=0}^{s+1} \binom{s+1}{l} (-\delta)^l \beta^\vee_{s+1+i-l}(z). \blacksquare$$

Lemma 5.2. *Let* $s \in \mathbb{N} - 1$

$$(x)_s = \prod_{i=0}^{s-1} (x+i),$$

where x *is a independent variable. Then the following equality holds in* $\mathbb{C}[x, y]$:

$$(x+y)_s = \sum_{l=0}^{s} \binom{s}{l} (x)_l (y)_{s-l}. \tag{6}$$

Proof. For $s = 0$ and for $s = 1$ the equality (6) is true. Let (6) is true for some $s \in \mathbb{N} - 1$. Then

$$(x+y)_{s+1} = (x+y+s)(x+y)_s = (x+y+s) \sum_{l=0}^{s} \binom{s}{l} (x)_l (y)_{s-l} =$$

$$\sum_{l=0}^{s} \binom{s}{l} (x)_l (y)_{s-l} ((x+l) + (y+s-l)) =$$

$$\left(\sum_{l=0}^{s} \binom{s}{l} (x)_{l+1} (y)_{s-l}\right) + \sum_{l=0}^{s} \binom{s}{l} (x)_l (y)_{s+1-l} =$$

$$\left(\sum_{l=1}^{s+1} \binom{s}{l-1} l (x)_l (y)_{s+1-l}\right) + \sum_{l=0}^{s} \binom{s}{l} (x)_l (y)_{s+1-l} =$$

$$\sum_{l=0}^{s+1}\binom{s}{l}(x)_l(y)_{s+1-l}. \blacksquare$$

Let $\mathfrak{E} = (\mathbb{C}[z])^{\mathbb{N}}$ be a space of all the sequences of elements contained in $\mathbb{C}[z]$. On this space act two commuting operators: the operator $\delta = z\frac{\partial}{\partial z}$ of the coordinate-wise differentiation, transforming each sequence

$$(p_1(z), p_2(z), ...) \tag{7}$$

into the sequence

$$(\delta p_1(z), \delta p_2(z), ...) \tag{8}$$

and the shift operator ∇^* transforming each sequence (7) into the sequence

$$(p_2(z), p_3(z), ...). \tag{9}$$

On the space $\mathfrak{E}$ acts also the operator $z^l(\frac{\partial}{\partial z})^l$ transforming each sequense (7) into the sequences

$$\left(z^l\left(\left(\frac{\partial}{\partial z}\right)^l p_1\right)(z), z^l\left(\left(\frac{\partial}{\partial z}\right)^l p_2\right)(z), ...\right) \tag{10}$$

I we take in (6) $x = -\delta$, $y = \nabla^*$, then we obtain the equality

$$(\nabla^* - \delta)_s = \sum_{l=0}^{s}\binom{s}{l}(-\delta)_l(\nabla^*)_{s-l}. \tag{11}$$

Lemma 5.3. Let $l \in \mathbb{N} - 1$. Then

$$(-\delta)_l = (-z)^l\left(\frac{\partial}{\partial z}\right)^l. \tag{12}$$

Proof. For $l = 0$ and $l = 1$ assertion of the Lemma is evident. Let $l \in \mathbb{N} + 1$ and assrtion of the Lemma is true for $l - 1$. Then

$$(-\delta)_{l+1} = (-\delta + l)(-z)^l\left(\frac{\partial}{\partial z}\right)^l =$$

$$-(-1)^l l(z)^l\left(\frac{\partial}{\partial z}\right)^l + l((-z)^l\left(\frac{\partial}{\partial z}\right)^l + (-z)(-z)^l\left(\frac{\partial}{\partial z}\right)^{l+1} =$$

$$(-z)^{l+1}\left(\frac{\partial}{\partial z}\right)^{l+1}. \blacksquare$$

In view of (12),

$$((-\delta)_l p)\Big|_{z=1} = \left((-1)^l (z)^l \left(\frac{\partial}{\partial z}\right)^l p\right)\Big|_{z=1},$$

for any $p \in \mathbb{Q}[z]$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k (z-1)^k \in \mathbb{C}[[z]]$$

Let us consider the operator ∂^*, for which

$$(\partial^* f)(z) = \sum_{k=0}^{\infty} a_{k+1}(z-1)^k \in \mathbb{C}[[z]].$$

Clearly,

$$((\partial^*)^n f)\Big|_{z=1} = \left((z)^n \left(\frac{\partial}{\partial z}\right)^n f\right)\Big|_{z=1} \Big/ n! \tag{13}$$

for any $f \in \mathbb{C}[[z]]$ and any $n \in \mathbb{N}-1$. The operator ∂^* acts coordinate-wise on the space $\mathfrak{E}$ of all the sequences (7), transforming each sequence (7) into the sequence

$$(\partial^* p_1(z), \partial^* p_2(z), ...) \tag{14}$$

and commutes with the above operator ∇^*.

Lemma 5.4. *Let the conditions of the Lemma 1 is fulfilled and*

$$R(t) = P(t)/Q(t),$$

where

$$P(t) \in \mathbb{C}(t), Q(t) \in \mathbb{C}(t), \deg(Q) - \deg(P) = r \geq 1$$

and the quotient of the leading coefficients of the polynomials P and Q is equal to the number γ. Then

$$\beta_{r-1,1}(1) = \gamma, \beta_{s,1}(1) = 0$$

for $0 \leq s \leq r-2$.

Proof. The assertion of the Lemma follows from the equality

$$\beta_{s,1}(1) = \operatorname{Res}_{z=\infty}(z^s R),$$

where $s = 0, ..., r-1$.

Lemma 5.5 ([32], Lemma 9). *Let* $\delta_1 > 0$, $\varepsilon_1 > 0, n \in \mathbb{N}$ *and for any* $\nu \in [1/\varepsilon_1] + \mathbb{N}$ *are given*

$$P(z:\nu) \in C[z]\diagdown\{0\},\ Q(z:\nu) \in C[z]\diagdown\{0\},\ R(z;\nu) = P(z;\nu)/Q(z;\nu),$$

$$N(\nu) \in \mathbb{N}-1,\ m(\nu) \in \mathbb{N},\ a_k(z:\nu) \in \mathbb{C}[z],$$

where $k = 0, \ldots n$. *Let further* γ_ν *is equal to the result of division of the leading coefficient of the polynomial* $P(z;\nu)$ *by the leading coefficient of the polynomial* $Q(z;\nu)$,

$$r(\nu) = \deg(Q(z;\nu)) - deg(P(z;\nu)) > 1,\ r(\nu) < r(\nu+1)$$

where $\nu \in [1/\varepsilon_1] + \mathbb{N}$,

$$z^s R(z;\nu) = F_s(z;\nu) + \sum_{i=1}^{\infty}\sum_{k=0}^{N(\nu)} \beta^*_{s,i,k}(\nu)(z+k)^{-i},$$

where $s \in \mathbb{N}-1$ *and the following conditions are fulfilled:*

$$F_s(z;\nu) \in \mathbb{C}[z],\ \beta^*_{s,i,k}(\nu) = 0,$$

where $s \in \mathbb{N}-1$, $i \in m(\nu)+\mathbb{N}$, $k = 0, \ldots, N(\nu)$, *Let*

$$\beta^*_{s,i}(z;\nu) = \sum_{k=0}^{N(\nu)} \beta^*_{s,i,k}(\nu) z^k$$

where $s \in \mathbb{N}-1$, $i \in \mathbb{N}-1$, $\nu \in [1/\varepsilon_1] + \mathbb{N}$, *and let the following equality holds:*

$$\sum_{k=0}^{n} a_k(z:\nu)\beta^*_{0,i}(z;\nu+k) = 0, \tag{15}$$

where $i \in \mathbb{N}-1$ *and* $\nu \in [1/\varepsilon_1]+\mathbb{N}$, *i.e. for each, for each* $i \in \mathbb{N}-1$ *te polynomials* $\beta^*_{0,i}(z:\nu)$ *satisfy to the equality*

$$\sum_{k=0}^{n} a_k(z:\nu)y(\nu+k) = 0, \tag{16}$$

where $\nu \in [1/\varepsilon_1] + \mathbb{N}$. *Then*

$$a_0(1;\nu) = 0,$$

for $\nu \in [1/\varepsilon_1] + \mathbb{N}$.

Proof. Let us consider the $\mathbb{N}$-dimensional vector

$$\mathfrak{b}(z;\nu) = (\beta^*_{0,1}(z;\nu),\, \beta^*_{0,2}(z;\nu),\, \beta^*_{0,3}(z;\nu),\, \dots), \tag{17}$$

$i - th$ coordinate of which is equal to $\beta^*_{0,i}(z;\nu)$. In view of (15),

$$\sum_{k=0}^{n} a_k(z:\nu)\mathfrak{b}(z;\nu+k) = 0, \tag{18}$$

Let

$$\mathfrak{y}_s(z;\nu) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k ((\partial^*)^k (\nabla^*)_{s-k}\mathfrak{b})(z;\nu)k!, \tag{19}$$

where $s \in \mathbb{N} - 1$, $\nu \in [1/\varepsilon_1] + \mathbb{N}$, and let $y^*_s(z;\nu)$ be the first coordoinate of the vector $\mathfrak{y}^*_s(z;\nu)$. Let $m \in [1/\varepsilon_1] + \mathbb{N}$. Let us to prove the following

Preparatory Assertion. *The equality*

$$\sum_{k=0}^{n} a_k(z:\nu) y^*_s(\nu+k) = 0 \tag{20}$$

holds for any $s \in [0, r(m)-1] \cap \mathbb{Z}$; *moreover*

$$y^*_{r(m)-1}(1;m) \neq 0,$$

$$y^*_{r(m)-1}(1;\nu) = 0$$

for $\nu \in \mathbb{N} + m$, *and*

$$y^*_s(1;\nu) = 0$$

for $s \in [0,\, r(m)-2] \cap \mathbb{Z}$, $\nu \in \mathbb{N} + m - 1$.

To prove the Preparatory Assertion we use induction on s. Since $y^*_0(1,\nu) = \beta^*_{0,1}(1,\nu) = 0$ for $\nu \in \mathbb{N} - 1$ and $0 < r(m) - 1$, it follows that for $s = 0$ the Preparatory Assertion is true. Let $s \in [1,\, r(m)-1] \cap \mathbb{Z}$ and the Preparatory Assertion is true for $s-1$. Since $s-1 < r(m)-1$, it follows, according the induction assumption, that $y^*_{s-1}(1,\nu) = 0$ for $\nu \in \mathbb{N} + m - 1$; therefore

$$y^*_{s-1}(z,\nu)/(z-1) = (z-1)^{-1}(y^*_{s-1}(z,\nu) - y^*_{s-1}(1,\nu)) = \partial^* y^*_{s-1}(z;\nu)$$

satisfies to the equation (16) for $\nu \in \mathbb{N} + m - 1$, and, according to (19) with $s-1$ instead s, coincides with the first coordinate of the vector

$$(z-1)^{-1}(\mathfrak{y}_{s-1}(z,\nu) - \mathfrak{y}_{s-1}(1,\nu)) = \partial^* \mathfrak{y}_{s-1}(z;\nu) =$$

$$\sum_{k=0}^{s-1}\binom{s-1}{k}(-1)^k((\partial^*)^{k+1}(\nabla^*)_{s-1-k}\mathfrak{b})(z;\nu)k! =$$

$$\sum_{k=1}^{s}\binom{s-1}{k-1}(-1)^{k-1}((\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(z;\nu)(k-1)! =$$

$$\sum_{k=1}^{s}\binom{s-1}{k-1}(-1)^{k-1}(k!(\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(z;\nu)/k =$$

$$s^{-1}\sum_{k=1}^{s}\binom{s}{k}(-1)^{k-1}(k!(\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(z;\nu) =$$

$$-s^{-1}\sum_{k=0}^{s}\binom{s}{k}(-1)^{k}(k!(\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(z;\nu)+$$

$$s^{-1}((\nabla^*)_s\mathfrak{b})(z;\nu),$$

where, in view of (19), the first coordinate of the sum

$$\sum_{k=0}^{s}\binom{s}{k}(-1)^{k}(k!(\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(z;\nu)$$

is equal to $y^*_s(z;\nu)$ and, in view of (17), the first coordinate of the vector

$$((\nabla^*)_s\mathfrak{b})(z;\nu)$$

is equal to $\beta^*_{0,s}(z;\nu)$. Therefore

$$\partial^* y^*_{s-1}(z;\nu) = -s^{-1}y^*_s(z;\nu) + s^{-1}\beta^*_{0,s}(z;\nu),$$

$$y^*_s(z;\nu) = -s\partial^* y^*_{s-1}(z;\nu) + s\beta^*_{0,s}(z;\nu). \tag{21}$$

Since both $-s\partial^* y^*_{s-1}(z;\nu)$ and $s\beta^*_{0,s}(z;\nu)$ satisfy to the equation (16) for $\nu \in \mathbb{N}+m-1$, it follows, in view of (20), that $y^*_s(z;\nu)$ also satisfies to the equation (16) for $\nu \in \mathbb{N}+m-1$. The number $y^*_s(1;\nu)$ coincides with the first coordinate of the vector (19) at the point $z=1$, i.e. with the first coordinate of the vector

$$\sum_{k=0}^{s}\binom{s}{k}(-1)^{k}((\partial^*)^{k}(\nabla^*)_{s-k}\mathfrak{b})(1;\nu)k!. \tag{22}$$

But $(\nabla^* - \delta)_s$ is a linear combinaion of the operators $(\nabla^* - \delta)^k$, $k = 0, \dots, s$ with entire positve coefficients and the coefficient equal to 1 at $(\nabla^* - \delta)^s$. According to the Lemma 5.1, the first coordinate of the vector $((\nabla^* - \delta)^k \mathfrak{b})(1, \nu)$ for $k = 0, \dots, s$ is equal to $\beta^*_{k,1}(1; \nu)$. According to the Lemma 5.4, $\beta^*_{k,1}(1; \nu) = 0$, if $k < r(\nu) - 1$, and $\beta^*_{k,1}(1; \nu) = \gamma_\nu$, if $k = r(\nu) - 1$; in particular, if $s < r(m) - 1$, $\nu = m$ or if $s = r(m) - 1$, $\nu > m$ all the $k = 0, \dots, s$ are less than $r(\nu) - 1$. Therefore in these cases $y^*_s(1, \nu) = 0$. Let now $\nu = m$, $s = r(m) - 1$. Then $k < r(\nu) - 1$, if $k < s$ and $k = r(m) - 1 = r(\nu) - 1$, if $\nu = m$, $s = r(m) - 1$, $k = s$. Therefore $y^*_{r(m)-1}(1, m) = \gamma_m \neq 0$. So, the Preparatory Assertion is proved.

Let $m \in [1/\varepsilon_1] + \mathbb{N}$. Then

$$y^*_{r(m)-1}(1, m) = \gamma_m \neq 0,\ y^*_{r(m)-1}(1, \nu) = 0,$$

if $\nu \in \mathbb{N} + m$, and $y^*_{r(m)-1}(1, \nu)$ is a solution of the equation (16) for $\nu \in \mathbb{N} + m - 1$. Therevore

$$a_0(1; m)\gamma(m) = \sum_{k=0}^{n} a_k(1; m) y^*_{r(m)-1}(m + k) = 0. \blacksquare$$

§2. An application of the Lemma 5.5.

Let us consider the difference equation (3.1.106) in [37], i.e. the following difference equation:

$$\sum_{j=0}^{6} q_j(z; \nu^{-1}) x_{\nu+j} = 0, \tag{23}$$

where $q_j(z; \nu^{-1}) \in \mathbb{Q}[z; \nu^{-1}]$ for $j = 0, \dots, 6$ have no common divisors with exeption of different from zero constants. The function $(R_0(t; \nu))^3$, where $R_0(t; \nu)$ is defined in (2.6) of the paper [35], plays the role of the function $R(z; \nu) = P(z; \nu)/Q(z; \nu)$ of the Lemma 5.5 now; so we can take

$$P(z; \nu) = ((\nu\Delta)!)^3 \prod_{\kappa=\nu+1}^{\nu\Delta} (z - \kappa)^3,$$

$$Q(z; \nu) = ((\nu(\Delta - 1))!)^3 \prod_{\kappa=0}^{\nu\Delta} (z - \kappa)^3,$$

in the Lemma 5.5. Then $((\nu\Delta)!)^3/((\nu(\Delta - 1))!)^3$, plays the role of γ_ν in the Lemma 5.5,

$$1 < r(\nu) = \deg(Q(z; \nu)) - deg(P(z; \nu)) = 3\nu + 3 < r(\nu + 1) = 3\nu + 6,$$

where $\nu \in \mathbb{N}-1$. The value $\nu\Delta$ plays the role of $N(\nu)$ in the Lemma 5.5. now, the number 3 plays the role of $m(\nu)$ in the Lemma 5.5. now, the number $\varepsilon_2(F)$ defined in [37] plays the role of ε_1 in the Lemma 5.5. now. The defined in (2.21)-(2.23) of [35] polynomials $\alpha(z;\nu)/(-z)^\nu$, $\beta(z;\nu)/(-z)^\nu$ and $\gamma(z;\nu)/(-z)^\nu$ play the role respectively of $\beta^*_{0,1}(z;\nu)$, $\beta^*_{0,2}(z;\nu)$ and $\beta^*_{0,3}(z;\nu)$, ...) in the Lemma 5.5. It was proved in the end of the paper [37] that

$$x_\nu = \alpha^*_0(z;\nu) = \alpha^*(z;\nu), x_\nu = \alpha^*_1(z;\nu) = \beta^*(z;\nu), x_\nu = \alpha^*_2(z;\nu) = \gamma^*(z;\nu), \tag{24}$$

and defined in (2.24), (2.26) and (2.28) of the paper [35] polynomials

$$x_\nu = \alpha^*_3(z;\nu) = \phi^*(z;\nu),\ x_\nu = \alpha^*_4(z;\nu) = \psi^*(z;\nu),\ x_\nu = \alpha^*_5(z;\nu) = \xi^*(z;\nu) \tag{25}$$

with $\nu \in \mathbb{N}+[1/\varepsilon_2(F)],\ |z|>1$ are solutions of the equation (22); the uniqueness theorem shows that x_ν in (24) and (25) are solutions of the equation (23) for $\nu \in \mathbb{N}+[1/\varepsilon_2(F)],\ z\in\mathbb{C}$. Therefore $y_\nu = \beta^*_{0,i}(z;\nu)$ with $i=1,2,3,\ \nu\in\mathbb{N}+[1/\varepsilon_2(F)],\ z\in\mathbb{C}$ are solutions of the equation

$$\sum_{j=0}^{6}(-z)^{6-i}q_j(z;\nu^{-1})y_{\nu+j}=0,$$

where $q_j(z;\nu^{-1})$ with $j=0,\ \dots\ ,6$ are coefficients of the equation (23). Applying the Lemma 5.5, we see that

$$q_0(1;\nu^{-1})=0 \tag{26}$$

for $\nu\in\mathbb{N}+[1/\varepsilon_2(F)]$. Let

$$q^\vee_j(\nu^{-1}) = q_{j+1}((1,\nu-1)^{-1}),$$

where $j=0,\dots 5$ and $\nu\in\mathbb{N}+[1/\varepsilon_2(F)]+1$. In view of (26), (24) and (25)

$$x_\nu = \alpha^*_k(1;\nu), \tag{27}$$

where $k=0,1,3,4,5$ and $\nu\in\mathbb{N}+[1/\varepsilon_2(F)]+1$, is a solution of the difference equation

$$\sum_{j=0}^{5}q^\vee_j(\nu^{-1})x_{\nu+j}=0. \tag{28}$$

According to the Lemma 4.1.3 in [39], $q_0^{\vee}(\nu^{-1})q_5^{\vee}(\nu^{-1}) \neq 0_{\mathbb{Q}[\nu^{-1}]}$. Therefore there exists $\varepsilon_5 > 0$ such that

$$q_0^{\vee}(\nu^{-1})q_5^{\vee}(\nu^{-1}) \neq 0$$

for $\nu \in [1/\varepsilon_5] + \mathbb{N}$. Let

$$q_j^{\wedge *}(\nu^{-1}) = q_j^{\vee}(\nu^{-1})/q_5^{\vee}(\nu^{-1}),$$

where $j = 0, \ldots, 5$ and $\nu \in [1/\varepsilon_5] + \mathbb{N}$. Then polynomials (27) compose a basis of the space of the solutions of the equation

$$\sum_{j=0}^{4} q_j^{\wedge *}(\nu^{-1})y(\nu + j) + y(\nu + 5) = 0, \tag{29}$$

where $y(\nu) = x_\nu$, $\nu \in [1/\varepsilon_5] + \mathbb{N}$, and $q_0^{\wedge *}(\nu^{-1}) \neq 0$ for $\nu \in [1/\varepsilon_5] + \mathbb{N}$.

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