CODING PARKING FUNCTIONS BY PAIRS OF PERMUTATIONS

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Abstract. We introduce a new class of admissible pairs of triangular sequences and prove an bijection between the set of admissible pairs of triangular sequences of length n and the set of parking functions of length n.

1. Introduction

A sequence $p = (p_0, \dots, p_{n-1})$ is called a *parking function* if it is majorated by a permutation, that is, there exists a permutation (one-to-one mapping) σ of the set $\{0, 1, \dots, n-1\}$ such that $p_0 \le \sigma(0), \dots, p_{n-1} \le \sigma(n-1)$. The sequence $p = (p_0, \dots, p_{n-1})$ is a parking function if and only if for every $s = 0, \dots, n-1$ it contains at least $s+1$ terms p_i satisfying the inequality $p_i \le s$. The set of parking functions with n terms will be denoted PF_n.

Parking functions are a popular subject in combinatorics. Taking their name from a problem of car parking along a one-way street (see [1]), they attracted attention after the following theorem had been proved:

Theorem 1 (Kreweras, [3], 1977)**.** *For every k, $0 \le k \le n(n-1)/2$, there exists a one-to-one correspondence κ_n between the set PF_n and the set of T_n trees with $n+1$ numbered vertices such that if $p_0 + \dots + p_n = u$ then the tree $D = \kappa_n(p)$ has exactly $n(n-1)/2 - u$ inversions. In particular, the total number of parking functions is equal to the total number of trees, that is, $(n+1)^{n-1}$.*

A *tree* here means a connected graph without cycles, whose vertices are numbered $0, 1, \dots, n$. We say that a pair of vertices (i, j) forms an inversion if $i < j$ but the path joining the vertex i with the vertex 0 passes through j. The paper [3] contains an explicit construction of the correspondence involved. (Note that Kreweras's *suites majeures* differ formally from parking functions we have defined: $(q_0, \dots, q_{n-1})$ is a suite majeure if $(n - q_0, \dots, n - q_{n-1})$ is a parking function.)

The permutation group Σ_n acts in a natural way on the set PF_n. Consider a vector space of dimension $(n+1)^{n-1}$ with the basis e_p whose elements are numbered by the parking functions $p \in \mathrm{PF}_n$. This space carries a natural linear representation of Σ_n; denote this representation $\mathcal{P}_n$. Define a *weight* $w(p)$ of the parking function p as $w(p) = n(n-1)/2 - (p_0 + \dots + p_{n-1})$. The permutation group action preserves the weight, and therefore $\mathcal{P}_n$ becomes a graded representation.

Another instance of parking functions (and the main inspiration of this paper) is the following theorem conjectured first in [1] and proved later in a series of works by the same author, see [2] and references therein. Consider a natural action of the permutation group Σ_n on the direct product $V_n = (\mathbb{C}^2)^n$. Let $\mathbb{C}[V_n]$ be the ring of polynomials on V_n, and $J_n \subset \mathbb{C}[V_n]$ be the ideal generated by Σ_n-invariant polynomials of positive degree. The factor $R_n = \mathbb{C}[V_n]/J_n$ is called a module of diagonal harmonics. It is a doubly-graded module: if one denotes arguments of the polynomial $f \in \mathbb{C}[V_n]$ as $x_1, y_1, \dots, x_n, y_n$ (x_i, y_i being coordinates in the i-th copy of $\mathbb{C}^2$) then the gradings are the total degree of f with respect to all x_i and its total degree with respect to all y_i. Either grading makes R_n a graded representation of Σ_n.

Theorem 2 (Haiman, [2], 2000)**.** *R_n is isomorphic, as a graded representation of Σ_n, to the representation $\mathcal{P}_n$ tensored by the sign representation ϵ_n.*

In particular, the dimension of the homogeneous component of R_n of the grading k is equal to the number of parking functions p with $p_0 + \dots + p_{n-1} = n(n-1)/2 - k$, or to the number of trees with $n+1$ numbered vertex having exactly k inversions.

Note that in fact the representation R_n is bi-graded, but Theorem 2 ignores the second grading. There are explicit formulas for dimensions of the bihomogeneous components of R_n (see [2]) but they have nothing to do with trees and parking functions. Nevertheless, the theorem suggests that the representation $\mathcal{P}_n$ also can be made doubly graded — that is, the sets of parking functions and trees should carry the second grading, yet unknown, different from the weight defined above.

The aim of our project was to find an elementary approach to the above bi-grading. Unfortunatelly we failed to define the second grading. This paper is a description of steps made in this direction, some of them being rigorously proved statements, and some, numerical observations. We start at sections 2 and 3 with a combinatorial construction encoding parking functions by pairs of permutations satisfying some admissibility condition. The last section contains some data shading light on the relation of this construction to Theorem 2 above.

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2. Main definitions

Throughout this paper we use the following conventions:

1. All the numbers mentioned are integers, unless otherwise stated.
2. All sequences are indexed starting from zero; giving numbers to elements of a finite set, we also start from zero.

2.1. **Permutations and triangular sequences.** We call a sequence $a = (a_0, \dots, a_{n-1})$ *triangular* if an inequality $0 \le a_i \le i$ is satisfied for all $i = 0, \dots, n-1$. Denote A_n the set of all triangular sequences of length n. Apparently, the cardinality of A_n is $n!$; this allows to put it into a one-to-one correspondence with the set Σ_n of all permutations of the set $\{0, \dots, n-1\}$.

There are several explicit constructions for this correspondence. We will usually use the correspondence $\alpha_n : A_n \to \Sigma_n$ defined inductively as follows. If $n = 1$, there is only one triangular sequence, only one permutation, and only one correspondence α_1 between them. Now let α_{n-1} be defined and let $\sigma' = (\sigma'(0), \dots, \sigma'(n-2)) = \alpha_{n-1}(a_0, \dots, a_{n-2})$. Take the number $n-1$ and insert it into the line $(\sigma'(0), \dots, \sigma'(n-2))$ between $\sigma'(n-2-a_n)$ and $\sigma'(n-1-a_n)$; the resulting sequence will represent the permutation $\sigma = \alpha_n(a_0, \dots, a_{n-1})$. Formally,

$$\sigma(i) = \begin{cases} \sigma'(i), & \text{if } i \le n-2-a_n, \\ n-1, & \text{if } i = n-1-a_n, \\ \sigma'(i-1), & \text{if } i \ge n-a_n. \end{cases}$$

It is easy to see that α_n is indeed a one-to-one correspondence. The inverse mapping can be described by the following rule: if $a = \alpha_n^{-1}(\sigma)$ then a_i equals the number of $j > i$ such that $\sigma(j) < \sigma(i)$. A pair (i, j) such that $j > i$ but $\sigma(j) < \sigma(i)$ is called an *inversion* of the permutation σ; the total number of inversions of the permutation $\alpha_n(a)$ is thus equal to $a_0 + \dots + a_{n-1}$.

2.2. **Triangular sequences and parking functions.** Consider a pair of triangular sequences $k = (k_0, \dots, k_{n-1}), l = (l_0, \dots, l_{n-1}) \in A_n$ such that $l_s \le k_s$ for all $s = 0, \dots, n-1$. Define a sequence $\beta_n(k, l) = p = (p_0, \dots, p_{n-1})$ by induction as follows. Let β_{n-1} be defined, and $(p'_0, \dots, p'_{n-1}) = \beta_{n-1}(k', l')$ where $k' = (k_0, \dots, k_{n-2}), l' = (l_0, \dots, l_{n-2}) \in A_{n-2}$. Now take the number k_{n-1} and insert it into the line $(p'_0, \dots, p'_{n-1})$ between positions number $k_{n-1} - l_{n-1} - 1$ and $k_{n-1} - l_{n-1}$; the resulting sequence will be $\beta_n(k, l)$. Formally, if $p = \beta_n(k, l)$ then

$$p_i = \begin{cases} p'_i, & \text{if } i \le k_{n-1} - l_{n-1} - 1, \\ k_{n-1}, & \text{if } i = k_{n-1} - l_{n-1}, \\ p'_{i-1}, & \text{if } i \ge k_{n-1} - l_{n-1} + 1. \end{cases}$$

It is easy to see that $\beta_n(k, l)$ is a parking function for all k, l.

An equivalent description of this algorithm is as follows: let us have, first, n empty positions numbered from 0 to $n-1$, left to right. Take k_{n-1} and place it to the position number $k_{n-1} - l_{n-1}$. Then re-number empty positions using numbers from 0 to $n-2$, skipping the occupied position. Then take k_{n-2} and place it to the empty position whose (new) number is $k_{n-2} - l_{n-2}$. Again, re-number the empty positions using numbers from 0 to $n-3$, etc.

3. Admissible pairs of triangular sequences

Let, again, $k = (k_0, \dots, k_{n-1})$ and $l = (l_0, \dots, l_{n-1})$ be triangular sequences. We say that a pair of integers (i, j), $0 \le i < j \le n-1$ forms an *irregular position* for a pair $k, l \in A_n$ if $l_i > l_j$ and $k_j \le i$. The pair $k, l \in A_n$ is called *admissible* if $l_s \le k_s$ for all $s = 0, \dots, n-1$, and no irregular positions exist. Denote $\mathrm{Adm}_n \subset A_n \times A_n$ the set of all admissible pairs.

The next statement is the main proved result of the paper.

Theorem 3. *The mapping β_n provides a one-to-one correspondence between the sets Adm_n and PF_n.*

Corollary. *There are $(n+1)^{n-1}$ admissible pairs of triangular sequences.*

To prove Theorem 3 we need two lemmas. Let $p \in \mathrm{PF}_n$ be a parking function, and r be a number such that p_r is the maximal term of the sequence p; if there are several such terms, take the smallest r possible. Let (k, l) be an admissible pair such that $p = \beta_n(k, l)$, and let s be a number such that $k_s = p_r$ is sent to position r by the mapping β_n (in other words, $s = \sigma_r$ where $\sigma = \alpha_n(k - l)$, cf. Section 2).

Consider the set U of all i, $p_r \le i \le n-1$, such that for all j, $p_r \le j \le i$, the inequality $l_j \le (n-1-r) + (p_r - i)$ takes place.

Lemma 1. $s = \max U$.

Proof. By the choice of s, we have $k_j \le k_s = p_r$ for every $j > s$. Now if $l_j < l_s$ then $k_j \le k_s \le s$, and (s, j) is an irregular position. For an admissible pair (k, l) no such positions exist, and so $l_j \ge l_s$ for all $j > s$. This inequality means that the mapping β_n sends every term k_j, $j > s$, to a position left of (less than) r, and therefore $r = (n-1-s)+(k_s-l_s) = (n-1-s)+(p_r-l_s)$. So, $l_s = (n-1-r)+(p_r-s)$.

Now let j be such that $p_r \le j \le s-1$. Again, we have $l_j \le l_s$, because (k, l) is an admissible pair. Therefore, $l_j \le (n-1-r)+(p_r-s)$ which means that $s \in U$.

Suppose there exists $t \in U$ such that $t > s$. Then $p_r \le s \le t-1$, and there holds the inequality $l_s = (n-1-r)+p_r-s \le (n-1-r)+p_r-t$ — a contradiction. □

Now delete the r-th element of the parking function p obtaining a sequence p'. Also, delete the s-th elements from sequences k and l resulting in k' and l', respectively.

Lemma 2. 1. *The sequence p' is a parking function: $p' \in \mathrm{PF}_{n-1}$.*
2. *The sequences k' and l' are triangular and form an admissible pair: $(k', l') \in \mathrm{Adm}_{n-1}$.*
3. $\beta_n(k', l') = p'$.

Proof. 1. Let σ be a permutation of the set $\{0, \dots, n-1\}$ majorating the sequence p. Since p_r is the maximal term of p, then without loss of generality $\sigma(r) = n-1$. Deleting the r-th term from σ one obtains a permutation σ' of the set $\{0, \dots, n-2\}$ majorating p'.

2. If $j < s$ then $k'_j = k_j \le j$. As we noticed in the proof of Lemma 1, $k_j \le k_s$ for all $j \ge s$, and therefore $k'_j = k_{j-1} \le s \le j$ for such j, too. Thus, the sequence k' is triangular. The inequalities $l'_i \le k'_i$ imply that the sequence l' is also triangular. Every irregular position (i, j) for (k', l') would be irregular for (k, l), too, and thus $(k', l') \in \mathrm{Adm}_{n-1}$.

3. Evident. □

Proof of Theorem 3. 1. Existence — prove that for every parking function $p \in \mathrm{PF}_n$ there exists an admissible pair $(k, l) \in \mathrm{Adm}_n$ such that $p = \beta_n(k, l)$. Use the induction by n, the base $n = 1$ being evident. To make the induction step, define the number r and the parking function $p' \in \mathrm{PF}_{n-1}$ as described in the beginning of this section. By induction hypothesis, there exists a pair $(k', l') \in \mathrm{Adm}_{n-1}$ such that $p' = \beta_{n-1}(k', l')$. Let U be the set of all $i, 0 \le i \le n-1$, such that for all $j, p_r \le j \le i-1$, the inequality $l'_j \le n-1-r+p_r-i$ takes place. Define the number s as the maximal element of U, assuming $s = 0$ if $U = \varnothing$. Now insert the terms $k_s = p_r$ and $l_s = (n-1-r)+p_r-s$ into k' and l' getting k and l, respectively. We are to prove now that (k, l) is an admissible pair of triangular sequences satisfying $\beta_n(k, l) = p$.

By the choice of s, the inequality $k_s = p_r \le s$ holds, which means that the sequence k is triangular. Prove that $l_s \le k_s$ (triangularity of l would follow). This inequality is equivalent to $n-1-r \le s$, so if $n-1-r \le p_r$ then it follows from the previous one.

Suppose $n-1-r > p_r$. Then for every $j, p_r \le j \le n-1-r$, one has $l'_j \le k'_j < p_r + 1 = (n-1-r)+p_r-(n-2-r)$. This means that the number $n-1-r$ belongs to the set U, and therefore, again, $n-1-r \le s$. So, $l_s \le k_s$ and l is triangular.

Prove now that $\beta_n(k,l) = p$. Show first that $l_j \geq l_s$ for all $j > s$. Suppose that $l_j < l_s$ for some $j > s$. By the induction hypothesis, (k',l') is a admissible pair, and therefore $(s, j-1)$ is not an irregular position for it. The inequality $k'_{j-1} = k_j \geq s+1 > p_r$ is impossible, so, $l_{s+1} = l'_s \leq l'_{j-1} = l_j$, and therefore $l_{s+1} < l_s = (n-1-r) + p_r - s$, or $l'_s \leq (n-1-r) + p_r - (s+1)$. By the choice of s, the number $s+1$ is not an element of the set U. This means that there exists t, $p_r \leq t < s$, such that $l_t > l_{s+1}$. The inequality $k'_s \geq t+1 > p_r$ is impossible, so in this case $(t-1, s)$ is an irregular position for (k',l') — a contradiction.

So, $l_j \geq l_s$ for all $j > s$. Since $k_s = p_r$ is the maximal term of the sequence p, we have also $k_j \leq k_s$ all $j > s$. This implies that the mapping β_n sends all the k_j with $j > s$ to positions left of r, and therefore $k_s = p_r$ is sent to position r. It follows now from the induction hypothesis ($p' = \beta_{n-1}(k',l')$) that $p = \beta_n(k,l)$.

We proved already that the sequences k and l are triangular, and $l_i \leq k_i$ for all $i = 0, \dots, n-1$. Prove that there are no irregular positions for k, l and therefore $(k,l) \in \mathrm{Adm}_n$. Let (u,v) be such a position; consider several cases:

Case 1. $u < v < s$. Then (u,v) is irregular for (k',l'), too — a contradiction. The same argument applies to cases $u < s < v$ (the position $(u, v-1)$) and $s < u < v$ (the position $(u-1, v-1)$).

Case 2. $u = s < v$. This is impossible because, as we proved earlier, $l_v \leq l_s$.

Case 3. $u < v = s$. This means that $p_r \leq u$ and thus $l'_u = l_u > l_s = (n-1-r) + p_r - s$, which is impossible by the definition of the set U.

2. Uniqueness. Again, use induction by n, the base $n = 1$ being evident. Let $\beta_n(k^{(1)}, l^{(1)}) = \beta_n(k^{(2)}, l^{(2)}) = p \in \mathrm{PF}_n$ where $(k^{(1)}, l^{(1)})$ and $(k^{(2)}, l^{(2)})$ are admissible pairs. Choose the number r as above, and let s_1, s_2 be numbers such that the mapping β_n applied to pairs $(k^{(1)}, l^{(1)})$ and $(k^{(2)}, l^{(2)})$ sends $k^{(1)}_{s_1}$ and $k^{(2)}_{s_2}$, respectively, to position r. Let $\tilde{k}^{(1)}$ and $\tilde{l}^{(1)}$ be sequences obtained by deletion of the s_1-th term from $k^{(1)}$ and $l^{(1)}$, and similarly $\tilde{k}^{(2)}$ and $\tilde{l}^{(2)}$. By Lemma 2, $\beta_{n-1}(k^{(1)}, l^{(1)}) = \beta_{n-1}(k^{(2)}, l^{(2)})$, and by the induction hypothesis, $\tilde{k}^{(1)} = \tilde{k}^{(2)}$, $\tilde{l}^{(1)} = \tilde{l}^{(2)}$.

The pair $(k^{(1)}, l^{(1)})$ is admissible, and $k^{(1)}_i \leq k^{(1)}_{s_1}$ for all i. It implies that $l^{(1)}_j \geq l^{(1)}_{s_1}$ for all $j > s_1$ and $l^{(1)}_j \leq l^{(1)}_{s_1}$ for all j, $p_r \leq j \leq s_1 - 1$. The same is true for $l^{(2)}$. As we know, the sequences $l^{(1)}$ and $l^{(2)}$ become the same after deletion of the s_1-th and the s_2-th terms, respectively. Hence, if $l^{(1)}_{s_1} > l^{(2)}_{s_2}$ then $s_1 > s_2$, and vice versa.

On the other hand, the mapping β_n sends all the $k^{(1)}_j$ with $j > s_1$ to positions left of r, and therefore $r = (n-1-s_1) + (p_r - l^{(1)}_{s_1})$. A similar equation is true for $l^{(2)}$, hence, $l^{(1)}_{s_1} + s_1 = l^{(2)}_{s_2} + s_2$. So, $s_1 = s_2$ and $l^{(1)}_{s_1} = l^{(2)}_{s_2}$ — uniqueness is proved. □

4. Admissible pairs and diagonal harmonics

Here we present some relation between the construction of Theorem 3 and the module R_n of diagonal harmonics described in Section 1. Let $H_{u,v}$ be the bihomogeneous component of the module R_n of bi-degree (u,v); denote $h_{u,v}$ its dimension.

Define now the four sets $Y_0, Y_1, Y_2, Y_3 \subset A_n$ of triangular sequences as follows.

0. The set Y_0 consists of only one sequence, namely $(0, 0, \dots, 0)$.
1. The set Y_1 consists of sequences $(l_0, \dots, l_{n-1})$ such that $l_0 = \dots = l_{n-2} = 0$ and $l_{n-1} \geq 1$.

2. The set Y_2 consists of sequences $(l_0, \dots, l_{n-1})$ such that $l_0 = \dots = l_{n-3} = 0$ and $l_{n-1} \geq l_{n-2} \geq 1$.
3. Y_3 is a union of two sets, Y_3' and Y_3''. The set Y_3' consists of sequences $(l_0, \dots, l_{n-1})$ such that $l_0 = \dots = l_{n-3} = 0$ and $1 \leq l_{n-2} > l_{n-1}$. The set Y_3'' consists of sequences $(l_0, \dots, l_{n-1})$ such that $l_0 = \dots = l_{n-4} = 0$ and $1 \leq l_{n-3} \leq l_{n-2} \leq l_{n-1} \leq n-2$ (note an additional inequality at the end; triangularity requires only $l_{n-1} \leq n-1$).

Numerical computations made for all $n \leq 7$ (using tables of $h_{u,v}$ taken from [1]) give the following observation:

For all u and $v = 0, 1, 2, 3$ and all $n \leq 7$ the dimension $h_{u,v}$ is equal to the number of admissible pairs (k, l) such that $k_0 + \dots + k_{n-1} = n(n-1)/2 - u$ and $l \in Y_v$.

Thus it can be conjectured that there exists a splitting of the set A_n into a disjoint union: $A_n = \bigsqcup_{v=0}^{n(n-1)/2} Y_v$ such that the statement above holds for all u, v (and all n). The authors, though, know neither a construction of Y_v nor a proof of the conjecture above for small v.

Besides the numerical observations for $n \leq 7$ there are some more facts supporting the conjecture.

1. Let $S \subset \mathbb{C}[x_1, \dots, x_n]$ be the ideal generated by symmetrical polynomials of positive degree. Apparently, $\mathbb{C}[x_1, \dots, x_n]/S$ is a graded module isomorphic to $\bigoplus_u H_{u,0}$. As it is well known, the dimension $h_{u,0}$ of its component of gradung u is equal to the number of permutations having exactly u inversions. On the other hand, for $v = 0$ we have $l = (0, 0, \dots, 0)$, and the admissibility condition does not impose any limitations on k. The results of Section 2 now imply that the observation above is true for $v = 0$ and all n.

2. Apparently, $h_{u,v} = h_{v,u}$, and therefore the dimension $h_{0,v}$ is equal to the number of permutations with v inversions. The conjecture above implies that $h_{0,v}$ equals to the number of admissible pairs (k, l) such that $k_0 + \dots + k_{n-1} = n(n-1)/2$ and $l \in Y_v$. The first equation holds only if $k_i = i$ for all $i = 0, \dots, n-1$. For such k and the pair (k, l) is admissible for every $l \in A_n$. This implies that the number of elements in Y_v should be equal to the number of permutations with v inversions. It is easy to check that this is true for $v = 0, 1, 2, 3$ (and all n).

3. It can be easily checked that the answer for $h_{u,v}$ given by the conjecture satisfies the condition $h_{u,v} = h_{v,u}$ for all n and all $u, v \leq 3$ (that is, in all cases when the conjectural values of both $h_{u,v}$ and $h_{v,u}$ are known).

4. From Theorem 2 we know that $\sum_u h_{u,v}$ equals to the number T_v of trees with v inversions. The conjecture implies then that the total number of admissible pairs (k, l) with $l \in Y_v$ should be equal to T_v, too. A direct computation (see [3] for a formula for T_v) shows that this is true for $v = 0, 1, 2, 3$ (and all n).

The splitting $A_n = \bigsqcup_{v=0}^{n(n-1)/2} Y_v$, if known, would provide the second grading on the sets of admissible pairs. The one-to-one correspondences β_n and κ_n mentioned in Section 1 would allow then to define the grading on the set of parking functions and on the set of trees. (Recall that the first grading for an admissible pair (k, l) equals $n(n-1)/2 - (k_0 + \dots + k_{n-1})$, for a parking function $p = \beta_n(k, l)$ it is $w(p) = n(n-1)/2 - (p_0 + \dots + p_{n-1}) = n(n-1)/2 - (k_0 + \dots + k_{n-1})$, and for a tree $D = \kappa_n(p)$ it is equal, by Theorem 1, to the number of inversions). Thus, by now we are able to describe conjecturally the sets of parking functions and trees of

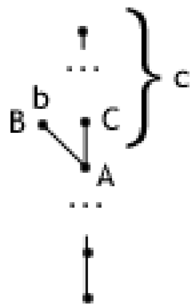

FIGURE 1. Such trees probably have grading 1: $c \geq b$

grading $0, 1, 2$ and 3 using explicit constructions of β_n (see Section 2) and κ_n (see [3]). To exemplify what happens we give here the answers for gradings 0 and 1:

0. Parking functions of grading zero are triangular sequences: $p_s \leq s$ for all $s = 0, 1, \ldots, n-1$. Trees of grading zero are "linear" trees with one branch only.
1. A parking function of grading 1 has exactly one term p_s such that $p_s > s$. After deletion of this term the remaining sequence is triangular (i.e. a parking function of grading 0). A tree of grading 1 has exactly one "branching point" — a vertex A having two children, B and C (all the other non-terminal vertices have one child only). The vertex B is terminal and carries the number b which is less or equal to the length of path joining A with the other terminal vertex (see Figure 1).

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