ON TYCHONOFF GROUPS

by

R.I. GRIGORCHUK

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
53225 Bonn
Germany

Dept. of Higher Mathematics
Moscow Institute of Railway
Transportation Engineers
nL Obrastcova 15
Moscow 101475
Russia

MPI 95-12

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R.I. GRIGORCHUK[1]

1. Introduction.

The class AG of amenable groups can be characterized by the property of a group to have a fixed point for any action by affine transformations on convex compact subset of locally convex topological vector space.

Now let us suppose that instead of compact set we have a nonzero cone. What kind of fixed-point theorems may hold in this situation? There is a number of conditions when a selftransformation of a cone has a nonzero fixed point. We will consider the situation when a group acts by affine transformations on convex cone with compact base. The groups for which any such action has an invariant ray are called Tychonoff and were defined (in case of Lie groups) by H. Furstenberg [3]. We investigate the Tychonoff property for abstract groups (with discrete topology) and show that this property is closely related to the property of having a small space of harmonic functions. Another interesting property is established in

Theorem 4.1. *Any infinite finitely generated Tychonoff group is indicable (i.e. can be mapped onto infinite cyclic group).*

This theorem is the first step in an attempt to describe all finitely generated Tychonoff groups.

In the end of the paper we consider bounded actions of groups of subexponential growth on convex cones with compact base and prove a fixed-point theorem for such actions.

2.The definition and some properties of Tychonoff groups.

Let us recall some notions. A selfmap $A : E \to E$ of a topological vector space E is called affine on a convex subset $V \subset E$ if for any $x, y \in V$ and $p, q \geq 0$, $p + q = 1$

$$A(px + qy) = pAx + qAy,$$

A set $K \subset E$ is called a cone if

1. $K + K \subset K$

[1]The results presented here were obtained under the financial support of the Russian Fund for Fundamental Research Grants 93-01-00239, 94-01-00820 and of the International Science Foundation, Grant MVI000.

2. $\lambda K \subset K$ for any number $\lambda \geq 0$
3. $K \cap (-K) = \{0\}$.
The ray in a cone K is any halfline:

$$L_x = \{\lambda_x : \lambda \geq 0\},$$

where $x \in K, \quad x \neq 0$.

A cone K has a compact base if there is a continuous linear functional Φ on E such that $\Phi(x) > 0, \quad$ if $\quad x \in K$, $x \neq 0$ and such that the set

$$B = \{x \in K : \Phi(x) = 1\}$$

is compact. Any such set B is called the base of the cone K.

Definition 2.1. A group G is called Tychonoff if for any action of G by continuous affine transformations on convex cone K with compact base in locally convex topological vector space there is a G-invariant ray.

Let TG be the class of Tychonoff groups. We will see later that $TG \subset AG$, where AG is the class of amenable groups.

We agree that everywhere in this paper K denotes a cone with compact base B determined by a functional Φ.

Examples.

2.2. *Any finite group is Tychonoff.*

If a finite group G acts on a cone K then for any $x \in K$, $x \neq 0$ the nonzero point

$$\xi = \frac{1}{|G|} \sum_{g \in G} gx$$

is G-invariant and so the ray L_ξ is G-invariant as well.

2.3. *Infinite cyclic group* $\mathbf{Z}$ *is Tychonoff.*

If $A : K \longrightarrow K$ is the affine transformation determined by the generator element of a cyclic group, then the transformation

$$\widetilde{A} : B \longrightarrow B$$
$$\widetilde{A}(x) = \frac{A(x)}{\Phi(A(x))}$$

is continuous and by Tychonoff theorem has fixed point $\xi \in B$. The ray L_ξ is $\mathbf{Z}$ - invariant.

Later we will see that nilpotent and in particular commutative groups are Tychonoff. The following example shows that a virtually commutative group need not belong to the class TG.

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2.4. *Infinite dihedral group is not Tychonoff.*

This group is given by one of the following presentations (by means of generators and relations):

$$
\begin{aligned}
G &= < a, b | a^2 = b^2 = 1 > = \\
&= < a, c | a^2 = (ac)^2 = 1 > \qquad (1)
\end{aligned}
$$

where $b = ac$. It is easy to see that G is isomorphic to the group generated by matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

which acts by linear transformations on 2-dimensional vector space. The first quarter

$$K = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$$

is G - invariant and has compact base but there are no invariant rays for such action.

Proposition 2.5. *Factor group of Tychonoff group is Tychonoff.*

This is obvious.

Proposition 2.6. *Let G be directed (by inclusion) union of Tychonoff groups H_i, $i \in I$ (that is $G = \cup_i H_i$ and for any H_i and H_j there is $H_k \supset (H_i \cup H_j)$). Then $G \in TG$.*

$\triangle$ Let G act by affine transformation on a cone K with compact base B and let B_i be the compact nonempty set of traces of G_i - invariant rays on the base B:

$$B_i = \{x \in B : gx = \lambda_g x, \quad g \in G_i, \quad \lambda_g > 0\}.$$

The system $\{B_i\}_{i \in I}$ satisfies the finite intersection property and so the intersection $B_\infty = \cap_{i \in I} B_i$ is nonempty. Any $x \in B_\infty$ determines G - invariant ray L_x. $\triangle$

The next statement was remarked in [3].

Proposition 2.7. *The strict inclusion $TG \subset AG$ holds.*

$\triangle$ Let us prove that TG is a subset of AG. Let $l_\infty(G)$ be the space of bounded functions on G with uniform norm, $l^*_\infty(G)$ be the space of continuous functionals equipped by the weak-* topology and let $B \subset l^*_\infty(G)$ be the set of means on G that is the set of linear positive functionals $m \in l^*_\infty(G)$ such that $m(\mathbf{1}_G) = 1$, where $\mathbf{1}_G$ is constant on G function with value 1.

Now let K be the cone generated by B:

$$K = \{0\} \cup \{x \in l^*_\infty(G) : \lambda x \in B \quad \text{for some} \quad \lambda > 0\}.$$

Then B is the base of cone K, determined by the functional Φ: $\Phi(m) = m(\mathbf{1}_G)$, $m \in l^*_\infty(G)$.

By the Alaoglu theorem this base is compact in the weak - * topology. The group G acts on $l_\infty(G)$ by left shifts: $(L_g f)(x) = f(g^{-1}x)$ and this action in canonical way induces the action on the dual space: $(gm)(f) = m(L_g f), \quad m \in l^*_\infty(G)$.

The cone K is G - invariant, so there is G - invariant ray $L_x, \quad x \in K, \quad x \neq 0$. But the base B is G - invariant as well. Thus $m = B \cap L_x$ is invariant point for action of G and m is left invariant mean on G. So $G \in AG$. The inclusion $TG \subset AG$ is strict, because infinite dihedral group is amenable but not Tychonoff. $\triangle$

An extension of a Tychonoff group by another Tychonoff group need not be a Tychonoff group, as shows the example of infinite dihedral group.

A subgroup of a Tychonoff group also need not be a Tychonoff. The corresponding example will be constructed later.

Now we are going to consider some types of extension preserving the Tychonoff property.

Proposition 2.8. *Let $G = M \times N$, where $M, N \in TG$. Then $G \in TG$.*

$\triangle$ Let G act on K with base B and let $B_0 \subset B$ be nonempty subset determined by the traces of M - invariant rays on B. Let $x_0 \in B_0$ and $mx_0 = \lambda(m)x_0$, $m \in M$, where $\lambda : M \to \mathbb{R}_+$ is some homomorphism. We define K_0 as a (nonzero convex closed) subcone of K consisting of vectors x with the property $mx = \lambda(m)x$. The cone K_0 is N - invariant. In fact, if $x \in K_0$ then

$$mnx = nmx = \lambda(m)nx.$$

Since $N \in TG$ there is N - invariant ray L_ξ which is G - invariant as well. $\triangle$

Corollary 2.9. *Any commutative group is Tychonoff.*

Let us agree that in this paper the term "a character" of a group G will mean any homomorphism $G \to \mathbb{R}_+$ where $\mathbb{R}_+$ is the multiplicative group of positive numbers.

Proposition 2.9. *Let $G = \mathbb{Z} \ltimes_A \mathbb{Z}^d$ be a semidirect product of infinite cyclic group $\mathbb{Z}$ and free abelian group of rank $d \geq 2$, where a generator of $\mathbb{Z}$ acts on $\mathbb{Z}^d$ as the automorphism determined by a matrix $A \in GL_n(\mathbb{Z})$ with the following condition: A has no eigenvalues on the unit circle except probably 1. Then $G \in TG$.*

$\triangle$ Let G act on K with compact base B determined by a functional Φ. Because $\mathbb{Z}^d \in TG$ there is a vector $\xi \in B$ such that for some character $\varphi : \mathbb{Z}^d \to \mathbb{R}_+$ and any $g \in \mathbb{Z}^d$ the equality $g\xi = \varphi(g)\xi$ holds.

Let

$$K_\varphi = \{x \in K : gx = \varphi(g)x, \quad g \in \mathbb{Z}^d\}.$$

Then K_φ is convex subcone of K determined by some nonempty compact base $B_\varphi \subset B$.

If a is a generator of infinite cyclic group $\mathbb{Z}$ then $aK_\varphi = K_{\varphi^a}$, where the action of a on φ is determined by the relation $\quad \varphi^a(g) = \varphi(g^{-1}ag)$.

For any $b \in \mathbb{Z}^d$ and $x \in K_\varphi$

$$bgx = gg^{-1}bgx = \varphi(g^{-1}bg)gx$$

By the Alaoglu theorem this base is compact in the weak - * topology. The group G acts on $l_\infty(G)$ by left shifts: $(L_g f)(x) = f(g^{-1}x)$ and this action in canonical way induces the action on the dual space: $(gm)(f) = m(L_g f), \quad m \in l^*_\infty(G)$.

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Δ Let G act on K with compact base B determined by a functional Φ. Because $\mathbb{Z}^d \in TG$ there is a vector $\xi \in B$ such that for some character $\varphi : \mathbb{Z}^d \to \mathbb{R}_+$ and any $g \in \mathbb{Z}^d$ the equality $g\xi = \varphi(g)\xi$ holds.

Let

$$K_\varphi = \{x \in K : gx = \varphi(g)x, \quad g \in \mathbb{Z}^d\}.$$

Then K_φ is convex subcone of K determined by some nonempty compact base $B_\varphi \subset B$.

If a is a generator of infinite cyclic group $\mathbb{Z}$ then $aK_\varphi = K_{\varphi^a}$, where the action of a on φ is determined by the relation $\varphi^a(g) = \varphi(g^{-1}ag)$.

For any $b \in \mathbb{Z}^d$ and $x \in K_\varphi$

$$bgx = gg^{-1}bgx = \varphi(g^{-1}bg)gx$$

and so we have uniform on $n \in \mathbf{Z}$ upper bound:

$$\varphi(a^{-n}ba^n) = \frac{\Phi(ba^n x)}{\Phi(a^n x)} \le \sup_{y\in B} \frac{\Phi(by)}{\Phi(y)} < \infty.$$

We see that for any $b \in \mathbf{Z}^d$ the sequence $\varphi^{a^n}(b)$ when n ranges over $\mathbf{Z}$ is bounded. Let us show that φ is invariant under A.

Let $a_1, \cdots, a_d$ be a basis of the group $\mathbf{Z}^d$. Any character on $\mathbf{Z}^d$ is determined by the vector $\overline{\chi} = (\chi_1, \cdots, \chi_d)$ of positive numbers: if $g = a_1^{m_1} \dots a_d^{m_d}$ then

$$\chi(g) = \chi_1^{m_1} \cdots \chi_d^{m_d}.$$

Let us consider the additive character $\mu = \log \chi$:

$$\mu(g) = m_1 \log \chi_1 + \cdots + m_d \log \chi_d.$$

The action of an automorphism a on $\mathbf{Z}^d$ corresponds to the mapping $\overline{m} \longrightarrow \overline{m}A$ of integer vectors $\overline{m} = (m_1, \cdots, m_d)$ which determines elements of the group $\mathbf{Z}^d$. Thus

$$\mu^{a^n}(g) = m_1^{(n)} \log \chi_1 + \cdots + m_d^{(n)} \log \chi_d,$$

where

$$(m_1^{(n)}, \cdots, m_d^{(n)}) = (m_1, \cdots, m_d)A^n$$

and

$$\mu^{a^n}(g) = < \overline{\log \chi}, \overline{m}A^n > = < \overline{\log \chi}(A')^n, \overline{m} >$$

where $<,>$ is scalar product and A' is the matrix transpose to A.

Lemma 2.10. *Let A be a linear transformation of $\mathbb{R}^d$ which have no eigenvalues on unit circle except probably 1. If for some x the set of vectors $\{A^n x\}_{n=-\infty}^{+\infty}$ is bounded then $Ax = x$.*

The proof is identical to the proof of lemma 4.1 from [3] and is omitted.

If the sequence of vectors $\{\overline{\log \varphi}(A')^n\}_{n=-\infty}^{+\infty}$ (φ is the character defined above) is bounded then due to the lemma 2.10 and condition of the proposition 2.9 the vector $\overline{\log \varphi}$ is invariant with respect to A'. Thus φ is a - invariant and so the cone K_φ is a - invariant. An arbitrary a - invariant ray in K_φ will be G - invariant as well. $\triangle$

The following three statements are similar to those given above.

Proposition 2.11. *Let $G = N \ltimes H$ be a semidirect product, where N and H are Tychonoff and let N act trivially on the set of characters of a group H. Then $G \in TG$.*

$\triangle$ The cone

$$K_\varphi = \{x \in k : gx = \varphi(g)x, \quad g \in H\},$$

where φ is a character for which there is a vector $\xi \in K$, $\xi \neq 0$, with $g\xi = \varphi(g)\xi$ for any $g \in H$ is N - invariant and so any N - invariant ray in K_φ will be G - invariant. $\triangle$ There is a bijective correspondence (given by the function log) between multiplicative characters $G \longrightarrow \mathbb{R}_+$ and additive characters $G \longrightarrow \mathbb{R}$. We say that the set of multiplicative characters $G \longrightarrow \mathbb{R}_+$ is finite dimensional if the space Char (G) of additive characters $G \longrightarrow \mathbb{R}$ is finite dimensional.

Proposition 2.12. *Let $G = \mathbb{Z} \ltimes_A H$, where $H \in TG$. Suppose that the space Char (H) is finite dimensional and the matrix A determining the action of generator of $\mathbb{Z}$ on the space Char (H) has no eigenvalues on the unit circle other than 1. Then $G \in TG$.*

The arguments are similar to those given in the proof of Proposition 2.11.

Examples.

2.13. *Metabelian group $G = < a, b | a^{-1}ba = b^2 >$ is Tychonoff.*

Δ Indeed $G = \mathbb{Z} \ltimes H$, where $\mathbb{Z}$ is infinite cyclic group generated by element a and $H = Q_2$ is the group of rational numbers of the form $\frac{k}{2^n}$, $k, n \in \mathbb{Z}$ with operation of addition. The element a acts on H as multiplication by 2.

Any character φ on H is determined by value $\varphi(1)$ so the space of characters is 1-dimensional. We have

$$\varphi^{a^n}(g) = \varphi(2^n g) = [\varphi(g)]^{2^n}$$

and thus the orbit $\{\varphi^{a^n}(g)\}_{n=-\infty}^{+\infty}$ is bounded if and only if $\varphi(g) = 1$ and if this hold for any g then φ is trivial. Thus $G \in TG$. Δ

2.14. *Let $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$, $G = \mathbb{Z}^d wr \mathbb{Z}_k$ (wr means the wreath product). The group G can be defined as*

$$G = \mathbb{Z}^d \ltimes (\mathbb{Z}_k)^{\mathbb{Z}^d},$$

where the group $\mathbb{Z}^d$ acts on the space $(\mathbb{Z}_k)^{\mathbb{Z}^d}$ of $\mathbb{Z}_k$ - configurations on $\mathbb{Z}^d$ by shifts.

The group $\mathbb{Z}_k^{\mathbb{Z}^d}$ has only trivial character. Thus $G \in TG$.

2.15. *The group $\mathbb{Z} wr \mathbb{Z}$ is not Tychonoff, because it can be mapped onto infinite dihedral group.*

2.16. *The Tychonoff property may not be preserved when pass to subgroup:*

Let

$$H = < a, b | [a, b] = c, \quad [a, c] = [b, c] = 1 >$$

be nilpotent Heisenberg group and let automorphism $\varphi \in \mathrm{Aut} H$ be defined as

$$\varphi \begin{cases} a \longrightarrow ab \\ b \longrightarrow a\,. \end{cases}$$

Then φ induces automorphism of the group $\mathbb{Z}^2 = H/[H, H]$ determined by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

the eigenvalues of which $\lambda_{1,2} = (1 \pm \sqrt{5})/2$ do not belong to the unit circle.

The automorphism φ acts on the generator c of the center $Z(H)$, as

$$c = [a, b] \xrightarrow{\varphi} [b, a] = c^{-1}.$$

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Let

$$G = \mathbf{Z} \ltimes_{\varphi} H =$$
$$= < a, b, c, d \,|\, [a, b] = c, [a, c] = [b, c] = 1, \quad d^{-1}ad = ab, \quad d^{-1}bd = a >$$

By proposition 2.12 the group G is Tychonoff. At the same time G contains the subgroup

$$< c, d \,|\, d^{-1}cd = c^{-1} >$$

which can be mapped onto infinite dihedral group and so is not Tychonoff.

§3. Harmonic functions and Tychonoff property.

Let G be a countable group and $p(g)$ be probabilistic distribution on $G : p(g) \geq 0, \quad \sum_{g \in G} p(g) = 1$.

A function $f : G \longrightarrow \mathbb{R}$ is called μ - harmonic (μ is a real number) if $Pf = \mu f$, where P is the Markovian operator determined by the relation:

$$(Pf)(g) = \sum_{h \in G} p(h) f(gh).$$

The left shift of a μ - harmonic function is again a μ - harmonic function. If $\mu = 1$ then we get the standard notion of harmonic function.

A distribution $p(g)$ is called generating if its support

$$\mathrm{supp} p(g) = \{g \in G : p(g) \neq 0\}$$

generates G.

Proposition 3.1. *If given G there is a generating probabilistic distribution with finite support such that for any $\mu > 0$ every positive μ - harmonic function is constant on cosets of the commutator subgroup then G is Tychonoff.*

$\triangle$ From the condition it follows that every bounded harmonic function on G is constant and this implies the amenability of G [2].

Let $p(g)$ be a distribution on G for which every μ - harmonic function is constant on cosets of commutator subgroup when $\mu > 0$ and let G act by affine transformations on a cone K with compact base B determined by functional Φ.

We can define affine continuous mapping $T : K \longrightarrow K$,

$$Tx = \sum_{g \in G} p(g) gx$$

for which there is an invariant ray L_ξ, $\xi \in B$.

The function $f(g) = \Phi(g\xi)$ is positive and μ - harmonic. Indeed

$$(Pf)(g) = \sum_{h \in G} p(h) f(gh) = \sum_{h \in G} \Phi(gh\xi) p(h) =$$
$$= \Phi(g \sum_{h \in G} p(h) h\xi) = \Phi(gT\xi) = \mu \Phi(g\xi) = \mu f(g).$$

By our assumption, this function is constant on cosets of the commutator subgroup. In particular, $f(g) = 1$ if $g \in G' = [G, G]$. Thus G' - orbit of the point ξ belongs to the base B.

We can consider the action of G' on the convex closure of the orbit $\{g\xi\}_{g \in G'}$ and using the amenability of G' to claim the existence of G' - invariant point $\eta \in B$.

Now let $K' \subset K$ be nonempty convex closed cone of G' - fixed points. The cone K' is G - invariant. Thus the action of G'on K' induces the action of $G_{ab} = G/G'$ on K' by affine transformations. This action has an invariant ray which is G - invariant as well. $\triangle$

Definition 3.2. A group G is ZA - group if G has increasing transfinite central chain of normal subgroups

$$1 = G_1 < \cdots < G_\alpha < \cdots < G_\gamma = G \tag{2}$$

where

$$G_\lambda = \cup_{\alpha < \lambda} G_\alpha$$

if λ is a limit ordinal and for any α

$$G_{\alpha+1}/G_\alpha < Z(G/G_\alpha)$$

(as usual $Z(H)$ denotes the center of a group H).

The following statement is similar to the main result of [6] and our proof follows the one given in [6]. We observe only that the lemma 2 from [6] must be a little bit corrected either in the part of formulation or in the part of the proof.

Let us call G a superliouville group if for any generating probabilistic distribution $p(g)$ every positive μ - harmonic function is constant on cosets of the commutator subgroup.

Theorem 3.3. *Any countable ZA - group is superliouville.*

Corollary 3.4. *Any nilpotent group is Tychonoff and so any locally nilpotent group is Tychonoff as well.*

Remark 3.5. *By theorem of A. Malcev [5] any finitely generated ZA-group is nilpotent. Thus ZA is proper sub-class of the class of locally nilpotent groups.*

Proof of the theorem 3.3. Let p(g) be generating distribution on ZA - group G and let for some $\mu > 0$ the set of μ - harmonic functions be nonempty. We fix this μ and denote by K the convex cone of positive μ - subharmonic functions that is functions with the property $Pf \leq \mu f$.

Let $V(G)$ be the space of real valued functions on G endowed with the topology of pointwise convergence.

If $f_n \in K, \quad n \in N$ is a net and $f_n \to f$ then

$$Pf \leq \lim_n Pf_n \leq \mu \lim_n f_n = \mu f.$$

By our assumption, this function is constant on cosets of the commutator subgroup. In particular, $f(g) = 1$ if $g \in G' = [G, G]$. Thus G' - orbit of the point ξ belongs to the base B.

We can consider the action of G' on the convex closure of the orbit $\{g\xi\}_{g\in G'}$ and using the amenability of G' to claim the existence of G' - invariant point $\eta \in B$.

Now let $K' \subset K$ be nonempty convex closed cone of G' - fixed points. The cone K' is G - invariant. Thus the action of G'on K' induces the action of $G_{ab} = G/G'$ on K' by affine transformations. This action has an invariant ray which is G - invariant as well. $\triangle$

Definition 3.2. A group G is ZA - group if G has increasing transfinite central chain of normal subgroups

$$1 = G_1 < \cdots < G_\alpha < \cdots < G_\gamma = G \tag{2}$$

where

$$G_\lambda = \cup_{\alpha<\lambda} G_\alpha$$

if λ is a limit ordinal and for any α

$$G_{\alpha+1}/G_\alpha < Z(G/G_\alpha)$$

(as usual $Z(H)$ denotes the center of a group H).

The following statement is similar to the main result of [6] and our proof follows the one given in [6]. We observe only that the lemma 2 from [6] must be a little bit corrected either in the part of formulation or in the part of the proof.

Let us call G a superliouville group if for any generating probabilistic distribution $p(g)$ every positive μ - harmonic function is constant on cosets of the commutator subgroup.

Theorem 3.3. *Any countable ZA - group is superliouville.*

Corollary 3.4. *Any nilpotent group is Tychonoff and so any locally nilpotent group is Tychonoff as well.*

Remark 3.5. *By theorem of A. Malcev [5] any finitely generated ZA-group is nilpotent. Thus ZA is proper sub-class of the class of locally nilpotent groups.*

Proof of the theorem 3.3. Let p(g) be generating distribution on ZA - group G and let for some $\mu > 0$ the set of μ - harmonic functions be nonempty. We fix this μ and denote by K the convex cone of positive μ - subharmonic functions that is functions with the property $Pf \leq \mu f$.

Let $V(G)$ be the space of real valued functions on G endowed with the topology of pointwise convergence.

If $f_n \in K, \quad n \in N$ is a net and $f_n \to f$ then

$$Pf \leq \lim_n Pf_n \leq \mu \lim_n f_n = \mu f.$$

Thus the cone $K \subset V(G)$ is closed.

Let Φ be the functional on $V(G)$ determined by the relation $\Phi(f) = f(1)$. Then the base $B = \{f \in K : \Phi(f) = 1\}$ is a compact set becouse from $Pf \leq \mu f$ it is easy to deduce that

$$f(g) \leq \frac{\mu^n}{p(g_1)\cdots p(g_n)} \tag{3}$$

where the elements g_i, $i = 1, \cdots, n$ are selected in such a manner that $g = g_1 \cdots g_n$ and $p(g_i) > 0$, $i = 1, \cdots, n$.

From (3) it follows that all functions from B are majorized by the function from the right-hand side of (3), which gives the compactness of B.

We can introduce the partial ordering on the cone $K : x < y$, $x, y \in K$ if $y - x \in K$. Then the cone K is a lattice: for any $x, y \in K$ there is an infimum $z = \inf(x, y)$ that is the element such that $x - z, y - z \in K$ and if $x - z', y - z' \in K$ for some other $z' \in K$ then $z - z' \in K$

In our case z is determined by the relation

$$z(g) = \min\{x(g), y(g)\}.$$

The following statement follows from theorem of Choquet and Deny and is a part of a more general statement from [3] (theorem 6.2).

Proposition 3.6. *The set E of extremal points of B is a Borel set and any point $b \in B$ is a resultant of some unique probabilistic measure $d\nu$ defined on E i.e. b can be presented in the form*

$$b = \int_E x d\nu(x).$$

If f is μ - harmonic function then corresponding measure $d\nu$ is concentrated on μ - harmonic functions. Indeed, let f be μ - harmonic and

$$f = \int_E x d\nu_f(x).$$

Then

$$Pf \leq \int_E Px d\nu_f(x)$$

and

$$0 = Pf - \mu f \leq \int_E (Px - \mu x) d\nu_f(x).$$

But $Px - \mu x \leq 0$, so for any $g \in G$ the set

$$F_g = \{x \in E : (Px - \mu x)(g) = 0\}$$

has ν_f - measure 1 and thus

$$\nu_f(\bigcap_{g \in G} F_g) = 1$$

because G is countable.

Now we are going to characterize extremal μ - harmonic functions as characters of the group G.

Any such function will be denoted by $k(x)$.

Lemma 3.7. *If $z \in Z(G)$ then*

$$k(xz) = k(x)k(z). \tag{4}$$

Δ Since the left shift of μ - harmonic function is again μ - harmonic and z is an element of the center we get the relation

$$k(x) = p\frac{k(xz)}{k(z)} + q\frac{c(xz)}{c(z)},$$

where the function c is defined as

$$c(xz) = k(x) - bk(xz),$$

the number b is selected to satisfy the inequality

$$0 < b < \frac{k(x)}{k(xz)}$$

and $p = bk(z)$.

Now we observe that the functions

$$\frac{k(xz)}{k(z)}, \quad \frac{c(xz)}{c(z)}$$

belong to B and as k is extremal point we get that k coinsides with each of this functions that leads to (4). Δ

Lemma 3.8. *Let $x, y \in G$ and $z = [x, y] \in Z(G)$. Then $k(z) = 1$.*

Δ If $[x_1, y], [x_2, y] \in Z(G)$ then

$$[x_1 x_2, y] = [x_1, y][x_2, y] \in Z(G).$$

This shows that if $[x, y] \in Z(G)$ then

$$[x^n, y] = [x, y]^n \in Z(G).$$

Let K be closed cone generated by functions $k(g^n x)$, $n \in \mathbf{Z}$, the base B be defined as $B = \{f \in K : f(1) = 1\}$ and T_y be continuous map from K to K, which preserves B and is defined as

$$(T_y f)(x) = \frac{f(yx)}{f(y)}.$$

Every function $h \in K$ satisfies the relation $h(xz) = h(x)h(z)$, when $z = Z(G)$.
By a theorem of Tychonoff there is $h \in K$ such that $Th = h$ i.e. $h(x)h(y) = h(yx)$ for any $x \in G$.

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$$k(xz) = k(x)k(z). \tag{4}$$

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$\triangle$ If $[x_1, y], [x_2, y] \in Z(G)$ then

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By a theorem of Tychonoff there is $h \in K$ such that $Th = h$ i.e. $h(x)h(y) = h(yx)$ for any $x \in G$.

Besides this there is a constant $b > 0$ such that for any $n \in \mathbf{Z}$, the inequality

$$0 < b < \frac{h(x^n y)}{h(x^n)}$$

holds.
But $x^n y = yx^n[x^n y]$,

$$h(x^n y) = h(yx^n)k([x,y])^n = h(yx^n)k([x,y]),$$

and we get for any $n \in \mathbf{Z}$

$$k^n([x,y]) = \frac{h(x^n y)}{h(yx^n)} = \frac{h(x^n y)}{h(y)h(x^n)} > \frac{b}{h(y)}$$

that leads to the equality $k([x,y]) = 1$. Δ

Now let us finish the proof of the theorem. For this purpose we will prove by transfinite induction on α that $k(x)$ is constant on cosets of subgroup $[G, G_\alpha] < G$.

Let (2) be a central series of a group G. If $x \in G$, $y \in G_3$ then $[x,y] \in G_2 < Z(G)$ and by lemma 3.8 $k([x,y]) = 1$. So k is equal to 1 on the subgroup $[G, G_3]$ and by lemma 3.7 k is constant on cosets of subgroup $[G, G_3]$

Let us pass from (1) to the central series of "smaller" length:

$$1 < G_3/[G,G_3] < \cdots < G_\alpha/[G,G_3] < \cdots < G/[G,G_3]$$

After such factorization the distribution $p(g)$ on G will be projected on some distribution $p^{(3)}(g)$ on the group $G^{(3)} = G/[G,G_3]$ and the function k will be projected on positive μ - harmonic with respect to the distribution $p^{(3)}(g)$ function $k^{(3)}$ on the group $G^{(3)}$. Moreover, $k^{(3)}$ will be extremal point in the base of the corresponding cone of μ - harmonic function on $G^{(3)}$.

Let us suppose now, that for some ordinal λ every positive μ - harmonic function on G is constant on cosets of any subgroup $[G, G_\alpha]$ $\alpha < \lambda$. In case λ is limit ordinal this property can be extended on cosets of subgroup $[G, G_\lambda]$ as well.

If λ is not a limit ordinal and $\lambda = \mu + 1$ then let us consider the central series

(5) $$1 < G_\mu/[G,G_\mu] < G_\lambda/[G,G_\mu] < \cdots < G/[G,G_\mu].$$

Let $p^{(\mu)}(g)$ and $k^{(\mu)}$ be distribution and μ - harmonic function on $G^{(\mu)} = G/[G,G_\mu]$ that are projections of μ and k respectively.

If $x \in G^{(\mu)}$, $y \in G_\lambda/[G,G_\mu]$
then

$$[x,y] \in G_\mu/[G,G_\mu] < Z(G/[G,G_\mu])$$

and so the function $k^{(\mu)}$ is constant on cosets of subgroup

$$[G/[G,G_\mu], G_\lambda/[G,G_\mu]] < G/[G,G_\mu].$$

Thus we can pass from (5) to the series

$$1 < G_\lambda/[G,G_\lambda] < \cdots < G/[G,G_\lambda]$$

and to define on the group $G^{(\lambda)} = G/[G,G_\lambda]$ projections $p^{(\lambda)}(g)$ and $k^{(\lambda)}$ of $p(g)$, k respectively.
This gives the possibility to apply the inductive assumption and to prove the theorem. Δ

Remark 3.9. *As pointed out to me by B. Weiss, there is a direct proof shorter than the one given above of the fact that every nilpotent group is Tychonoff.*

Remark 3.10. *There are examples showing that a group, containing nilpotent subgroup of finite index can have extremal μ - harmonic functions that are not characters. Here is the simplest one.*

Let G be infinite dihedral group, given by the presentation (1) and the distribution p be uniform on the set $\{a, b\}$ of generators: $p(a) = p(b) = \frac{1}{2}$. The elements of G can be identified with words not containing the subwords aa, bb.

The Cayley graph of G looks like Cayley graph of infinite cyclic group

$$\overline{\cdots bab \quad ba \quad b \quad 1 \quad a \quad ab \quad aba \cdots}$$

and Markovian operator T, acts on functions on G analogously to the operator $\tilde{T}$ on the group $\mathbb{Z}$:

$$(\tilde{T}f)(n) = \frac{1}{2}(f(n-1) + f(n+1)).$$

We can apply theorem 3.3 (or classical results) to the last operator and deduce that extremal points of the base of the cone of positive solutions of the equation $\tilde{T}f = \mu f$, $\mu > 0$ exist if $\mu \geq 1$ and have the form $f_\xi(n) = \xi^n$ where ξ is some positive number satisfying the equation

$$\mu \xi^n = \frac{1}{2}(\xi^{n-1} + \xi^{n+1})$$

i.e.

$$\xi = \frac{\mu \pm \sqrt{\mu^2 - 1}}{2}$$

Respectively the function $f_\xi(g) = \xi^{\sigma(g)}$, where $\sigma(g)$ is the length of an element g taken with the sign + if irreducible form of g starts on a and taken with the sign − in opposite case, is extremal μ - harmonic function on G but is not a character.

Remark 3.11. *There are groups with Tychonoff property having nonconstant bounded harmonic functions. For instance, any group $G = \mathbb{Z}^d wr \mathbb{Z}_k$ is Tychonoff (see example 2.14) and has nonconstant bounded harmonic functions when $d \geq 3$ [4]. Thus the class TG does not coincide with the class of superliouville groups.*

§4. The Tychonoff property and indicability.

A group is called indicable if it can be mapped onto infinite cyclic group.

Theorem 4.1. *Any infinite, finitely generated Tychonoff group is indicable.*

Proof. Let G be such a group. The theorem will be proved if we construct an action of G without fixed points by affine transformations on convex cone K with compact base B. Indeed, then the action of G on any invariant ray L_ξ, $\xi \in B$:

$$g\xi = \varphi(g)\xi$$

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$$g\xi = \varphi(g)\xi$$

determines desired homomorphism $\varphi : G \longrightarrow \mathbf{R}_+$ with infinite cyclic image.

We are going to prove that the cone K of positive μ - harmonic functions satisfies the need property where μ is any fixed number greater than 1 and the distribution $p(g)$ has a finite support that generates G.
The group G acts on K by left shifts. This action is affine and has no nonzero fixed points. The same arguments which were given in the proof of theorem 3.3 show that the base

$$B = \{f \in K : f(1) = 1\}$$

is compact in the topology of pointwise convergence. Thus the only point remaining is to prove that the cone K is nonzero.

Let $p(g)$ be a distribution on G with finite support A that generates G. Let $p(n, x, y)$ be the probability of transmission from x to y in n steps in right random walk on G, determined by distribution $p(g)$: starting from x we can reach xa in one step with probability $p(a)$. The Markovian operator P corresponding to this random walk is determined by the relations:

$$(Pf)(x) = \sum_{y \in G} p(y)f(xy) = \sum_{y \in G} p(1, x, y)f(y).$$

For any λ, $|\lambda| < 1$ the series

$$g^\lambda(x, y) = \sum_{n=0}^{\infty} \lambda^n p(n, x, y)$$

converges and we can define the generalized Martin's kernels

$$k_y^\lambda(x) = \frac{g^\lambda(x, y)}{g^\lambda(1, y)}$$

and functions

$$\Pi_y^\lambda(x) = \sum_{i=0}^{\infty} \lambda^i f(i, x, y),$$

where $f(i, x, y)$ is the probability of the first getting from x into y on i - th step. It is clear that

$$p(n, x, y) = \sum_{i=0}^{n} f(i, x, y)p(n - i, y, y)$$

and so

$$g^\lambda(x, y) = \Pi_y^\lambda(x) g^\lambda(y, y)$$

because

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda^n p(n, x, y) &= \sum_{n=0}^{\infty} \lambda^n \sum_{i=0}^{n} f(i, x, y)p(n - i, y, y) = \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} \lambda^i f(i, x, y)\lambda^{n-i} p(n - i, y, y) = \\ &= \sum_{i=0}^{\infty} \lambda^i f(i, x, y) \sum_{n=i}^{\infty} \lambda^{n-i} p(n - i, y, y) = \Pi_y^\lambda(x) g^\lambda(y, y).\end{aligned}$$

Thus

$$k_y^\lambda(x) = \frac{\Pi_y^\lambda(x)}{\Pi_y^\lambda(1)}.$$

Lemma 4.2. *The following equality*

$$k_y^\lambda(x) - \lambda P k_y^\lambda(x) = \begin{cases} 0 & \text{if } x \neq y \\ \frac{1}{g^\lambda(1,y)} & \text{if } x = y \end{cases}$$

holds.

Δ In fact we have

$$\begin{aligned} Pk_y^\lambda(x) =& \frac{1}{g^\lambda(1,y)} \sum_{h\in G} p(1,x,h) g^\lambda(h,g) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{h\in G} p(1,x,h) \sum_{n=0}^{\infty} \lambda^n p(n,h,y) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{n=0}^{\infty} \lambda^n \sum_{h\in g} p(1,x,h) p(n,h,y) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{n=0}^{\infty} \lambda^n p(n+1,x,y) = \\ =& \frac{1}{g^\lambda(1,y)} [\frac{1}{\lambda} g^\lambda(x,y) - \frac{1}{\lambda} p(0,x,y)]. \Delta \end{aligned}$$

We claim that when x and λ are fixed the set of numbers $\{k_y^\lambda(x), \quad y \in G\}$ is bounded.

Really let A be the set of generators of G, C_G be the Cayley graph of a group G constructed with respect to the generating set A.

Let us fix for every element $x \in G$ a path l_x in C_G that joins 1 and x. Let $p(l_x)$ be the probability of the path l_x (the product of probabilities of transmission along links of this path) and let t_x be the length of l_x.

If $y \notin l_x$, then

$$f(i,1,y) > p(l_x) \cdot f(i - t_x, x, y).$$

Therefore

$$\frac{\lambda^{i-t_x} f(i-t_x,x,y)}{\lambda^i f(i,1,y)} \leq \frac{1}{\lambda^{t_x} p(l_x)}$$

and

$$\begin{aligned} k_y^\lambda(x) =& \frac{\Pi_y^\lambda(x)}{\Pi_y^\lambda(1)} = \frac{\sum_{i=0}^{\infty} \lambda^i f(i,x,y)}{\sum_{i=0}^{\infty} \lambda^i f(i,1,y)} \leq \\ \leq& \frac{\sum_{i=0}^{\infty} \lambda^i f(i,x,y)}{\sum_{i=0}^{\infty} \lambda^{i+t_x} f(i+t_x,1,y)} \leq \frac{1}{\lambda^{t_x} p(l_x)} \end{aligned}$$

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holds.
△ In fact we have

$$\begin{aligned} Pk_y^\lambda(x) =& \frac{1}{g^\lambda(1,y)} \sum_{h\in G} p(1,x,h) g^\lambda(h,g) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{h\in G} p(1,x,h) \sum_{n=0}^\infty \lambda^n p(n,h,y) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{n=0}^\infty \lambda^n \sum_{h\in g} p(1,x,h) p(n,h,y) = \\ =& \frac{1}{g^\lambda(1,y)} \sum_{n=0}^\infty \lambda^n p(n+1,x,y) = \\ =& \frac{1}{g^\lambda(1,y)} [\frac{1}{\lambda} g^\lambda(x,y) - \frac{1}{\lambda} p(0,x,y)].\triangle \end{aligned}$$

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$$f(i,1,y) > p(l_x) \cdot f(i - t_x, x, y).$$

Therefore

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and

$$\begin{aligned} k_y^\lambda(x) =& \frac{\Pi_y^\lambda(x)}{\Pi_y^\lambda(1)} = \frac{\sum_{i=0}^\infty \lambda^i f(i,x,y)}{\sum_{i=0}^\infty \lambda^i f(i,1,y)} \leq \\ \leq& \frac{\sum_{i=0}^\infty \lambda^i f(i,x,y)}{\sum_{i=0}^\infty \lambda^{i+t_x} f(i+t_x,1,y)} \leq \frac{1}{\lambda^{t_x} p(l_x)} \end{aligned}$$

Thus the set of functions $\{k_y^\lambda(x)\}_{y\in G}$ is majorized by the function $(\lambda^{l_x} p(l_x))^{-1}$.

Now we take any sequence $y_n \in G,\quad y_n \to \infty$ and extract a subsequence y_{n_k} such that the sequence $k^\lambda_{y_{n_k}}$ converges to some positive function $k^\lambda(x)$ which is $\frac{1}{\lambda}$ - harmonic. Really, because the distribution $p(g)$ has finite support

$$Pk^\lambda(x) = P\lim_{k\to\infty} k^\lambda_{y_{n_k}}(x) = \lim_{k\to\infty} Pk_{y_{n_k}}(x)$$

passing to the limit in the relation

$$\frac{1}{\lambda}k^\lambda_{y_{n_k}}(x) - Pk^\lambda_{y_{n_k}}(x) = \begin{cases} 0, & \text{if } x \neq y_{n_k} \\ \frac{1}{\lambda g^\lambda(1, y_{n_k})}, & \text{if } x = y_{n_k} \end{cases}$$

we get the relation

$$\frac{1}{\lambda}k^\lambda(x) = Pk^\lambda(x).$$

We have proved that on any infinite finitely generated group for any $\mu > 1$ there is a positive μ - harmonic function. The cone of such functions is nonzero and satisfies all necessary conditions. The theorem is proved.

§5. One fixed-point theorem for actions on cones.

Let a group G act by affine transformations on a cone K. We shall call such action bounded if the orbit of any point $\xi \in K$ is bounded. Thus the orbits can accumulate to zero, but not to infinity.

A finitely generated group G is called a group of subexponential growth if

$$\gamma = \lim_{n\to\infty} \sqrt[n]{\gamma(n)} = 1,$$

where $\gamma(n)$ is the growth function of the group G with respect to some finite system of generators ($\gamma(n)$ is equal to the number of elements of G that can be presented as a product of $\leq n$ of generators and its inverses).

Theorem 5.1. *Let a group G of subexponential growth act by affine transformations on nonzero convex cone K with compact base in locally convex topological vector space and this action is bounded.*

Then there is a G - fixed point $\xi \in K,\ \xi \neq 0$.

Proof. Let $p(g)$ be symmetric (that is $p(g) = p(g^{-1})$ for any $g \in G$) probabilistic distribution the support of which is finite and generates G and let P be the corresponding Markovian operator

$$(Pf)(g) = \sum_{h\in G} p(h) f(gh).$$

We can define continuous map $T : K \to K$, where

$$Tx = \sum_{h\in G} p(h)hx.$$

By Tychonoff theorem there is a T - invariant ray L_ξ, $\xi \in B$, that is $T\xi = \lambda\xi$ for some $\lambda > 0$.

We are going to prove that $\lambda = 1$. Let Φ be a functional, determining the base B of the cone K and let f be a function on G determined by the relation

$$f(g) = \Phi(g\xi).$$

Then f is λ - harmonic function:

$$\begin{aligned}(Pf)(g) = \sum_{h\in G} p(h)f(gh) = \sum_{h\in G} p(h)\Phi(gh\xi) = \Phi(\sum_{h\in G} p(h)gh\xi) = \\ =\Phi(g\sum_{h\in G} p(h)h\xi) = \Phi(g\lambda\xi) = \lambda\Phi(g\xi) = \lambda f(g).\end{aligned}$$

From the relation $(P^n f)(1) = \lambda^n f(1)$ we get the inequality $p(n,1,1) \leq \lambda^n$ where $p(n,1,1)$ is the probability of returning to the unit after n - steps in the right random walk on a group G.

It is well-known that any group of subexponential growth is amenable and by theorem of H.Kesten the spectral radius

$$r = \lim_{n\to\infty} \sup \sqrt[n]{p(n,1,1)}$$

is equal to 1 for a symmetric random walks on any amenable group. Thus $\lambda \geq 1$ in our case. But because the action of G on K is bounded, one can find a number $d > 0$ such that

$$\Phi(f)\cdot\lambda^n = \Phi(P^n f) \leq d$$

which leads to the equality $\lambda = 1$.

Thus the function $f(g)$ is bounded a harmonic function on the group G. By the theorem of Avez [1] the function f is constant and so the orbit O_ξ of the point ξ is a subset of the base B. The closed convex hull $\overline{O}_\xi$ of the orbit O_ξ will be a compact set on which G acts by affine transformations and this action has a fixed point because G is amenable.

This finishes the proof of the theorem.

Remark 5.2. *The statement of the theorem 5.1 holds for any group for which there is symmetric probabilistic distribution with finite support having the property that every positive bounded harmonic function is constant. There are some solvable groups of exponential growth having this property. The simplest example is the group of the form $\mathbb{Z} \ltimes_A \mathbb{Z}^d$ where not all eigenvalues of the matrix A lie on the unit circle.*

Acknowledgements

This paper was written during my stay at the Max Planck Mathematical Institute in Bonn and the Institute of Mathematics of the Hebrew University in Jerusalem. I would like to thank F. Hirzebruch and A. Lubotzky for giving me the opportunity for these fruitful visits.

I also thank B. Weiss, Y. Shalom and A. Starkov for their helpful remarks.

By Tychonoff theorem there is a T - invariant ray L_ξ, $\xi \in B$, that is $T\xi = \lambda\xi$ for some $\lambda > 0$.

We are going to prove that $\lambda = 1$. Let Φ be a functional, determining the base B of the cone K and let f be a function on G determined by the relation

$$f(g) = \Phi(g\xi).$$

Then f is λ - harmonic function:

$$(Pf)(g) = \sum_{h\in G} p(h)f(gh) = \sum_{h\in G} p(h)\Phi(gh\xi) = \Phi(\sum_{h\in G} p(h)gh\xi) =$$
$$= \Phi(g\sum_{h\in G} p(h)h\xi) = \Phi(g\lambda\xi) = \lambda\Phi(g\xi) = \lambda f(g).$$

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