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AUTOMORPHISM AND COHOMOLOGY II: COMPLETE INTERSECTIONS

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Abstract. We prove that the automorphism group of a general complete intersection X in $\mathbb{P}^n$ is trivial with a few well-understood exceptions. We also prove that the automorphism group of a complete intersection X acts on the cohomology of X faithfully with a few well-understood exceptions.

Contents

1. Introduction

In this paper we prove two results concerning automorphisms of complete intersections in projective spaces: one on their "generic triviality" and the other on the faithfulness of their actions on cohomology groups.

The earliest result in the first direction, to the best of our knowledge, is due to H. Matsumura and P. Monsky [11]:

Theorem 1.1 (Matsumura and Monsky)**.** *Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree d over a field k. Then*

- $\mathrm{Aut}(X)$ *is finite if* $n \geq 3$, $d \geq 3$ *and* $(n,d) \neq (3,4)$;
- $\mathrm{Aut}(X) = \{1\}$ *if X is generic,* $n \geq 3$ *and* $d \geq 3$ *except the case that* $(n,d) = (3,4)$ *and* $\mathrm{char}(k) > 0$.

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Naturally, one may ask whether this result generalizes to complete intersections. Let us first make the following definition.

Definition 1.2. Let k be a field and X be a closed subscheme of $\mathbb{P}_k^n$. The linear automorphism group of X with respect to the projective embedding $X \subset \mathbb{P}_k^n$, denoted by $\mathrm{Aut}_L(X)$ for $L = \mathcal{O}_X(1)$, is the subgroup of $\mathbf{PGL}_{n+1}(k)$ whose linear action on $\mathbb{P}_k^n$ takes X onto X itself.

The first part of Theorem 1.1 was generalized to complete intersections by O. Benoist [2]. He proved that $\mathrm{Aut}_L(X)$ is finite for a smooth complete intersection $X \subset \mathbb{P}^n$ with some well-known exceptions. His proof came down to the computation $H^0(T_X)$, which was achieved by an induction argument based on the spectral sequence associated to the Koszul complex.

The generalization of the second part of Theorem 1.1, i.e., the triviality of the automorphism group of a general complete intersection is going to be settled in this paper. Our theorem shows that, away from some exceptions, the generic number of automorphisms is 1. When $\mathrm{char}(k) = 0$, our result is optimal. When $\mathrm{char}(k) > 0$, there are only a very limited number of exceptions which we believe can be settled in a case-by-case manner if the need arises; but we will not pursue these cases in this paper.

Theorem 1.3. *Let X be a smooth complete intersection in $\mathbb{P}_k^n$ of type $(d_1, d_2, \ldots, d_c)$, i.e., $X = X_1 \cap X_2 \cap \ldots \cap X_c$ with $\deg X_i = d_i$. If $\dim X \geq 2$, $\deg X \geq 3$ and $2 \leq d_1 \leq d_2 \leq \ldots \leq d_c$, then*

(1) $\mathrm{Aut}_L(X) = \{1\}$ *when $k = \mathbb{C}$ and X is a general complete intersection of type $(d_1, d_2, \ldots, d_c) \neq (2, 2)$,*
(2) $\mathrm{Aut}_L(X) = (\mathbb{Z}/2\mathbb{Z})^n$ *for X general of type $(2, 2)$ and $\mathrm{char}(k) \neq 2$, and*
(3) $\mathrm{Aut}_L(X) = \{1\}$ *when X is a general complete intersection of type $(d_1, d_2, \ldots, d_c)$ except*
   - $d_1 = d_2 = \ldots = d_c = 2$, *or*
   - $(n, d_1, d_2, \ldots, d_c) = (5, 2, 3)$, $(6, 2, 3)$, $(6, 2, 4)$, $(6, 3, 3)$, $(7, 2, 2, 3)$, $(7, 2, 3)$, $(8, 2, 3)$, $(8, 3, 3)$, $(9, 2, 2, 3)$, $(10, 2, 3)$.

Remark 1.4. Retain the hypotheses and notations of Theorem 1.3 and suppose that X is one of the following smooth complete intersections in $\mathbb{CP}^n$:

- $\dim X \geq 3$, or
- $\dim X \geq 2$ and $\omega_X \neq \mathcal{O}_X$, or
- $\dim X = 2$, $\omega_X \cong \mathcal{O}_X$ and $\mathrm{rank}_{\mathbb{Z}} \mathrm{Pic}(X) = 1$, i.e., X is a K3 surface of Picard rank 1.

Then it is known that $\mathrm{Aut}(X) = \mathrm{Aut}_L(X)$. Thus, by Theorem 1.3, when X is general in its moduli, the automorphism group of X is trivial.

The triviality of $\mathrm{Aut}_L(X)$ for a general complete intersection $X \subset \mathbb{CP}^n$, to the best of our knowledge, was only known in some special cases:

- aforementioned hypersurface case [11];
- $3 \leq d_1 < d_2 \leq ... \leq d_c$ [8, Lemma 2.12];

- $(d_1, d_2, ..., d_c) = (2, 2, 2)$ [8, Lemma 2.13].

For a very general complete intersection X of Calabi-Yau or general type, we have a stronger statement: there are no non-trivial dominant rational self maps $\sigma : X \dashrightarrow X$ [3].

For the automorphisms and cohomology of a variety, the following question is fundamental.

Question 1.5. Is the natural action of the automorphism group of a variety on its cohomology faithful?

Our second result, which is derived from Theorem 1.3 above and the method in the papers [12] and [13] of the second author, concerns about the question. Historically speaking, this fundamental question is first explored for varieties of low dimension. For example, Burns, Rapoport, Shafarevich, Ogus . . . confirm this question for K3 surfaces. Few high-dimensional varieties are known to have a positive answer to this question. We answer this question for complete intersections completely.

Theorem 1.6. *Let k be an algebraically closed field and X be a smooth complete intersection in $\mathbb{P}^n_k$ of type $(d_1, d_2, \ldots, d_c)$. Suppose that $\deg X \geq 3$ and $2 \leq d_1 \leq d_2 \leq \ldots \leq d_c$.*

(1) *If $(d_1, d_2, \ldots, d_c) = (2, 2)$, $\mathrm{char}(k) \neq 2$ and $\dim(X) = m(> 1)$ is odd (resp. even), then the kernel of*
$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^m_{\text{ét}}(X, \mathbb{Q}_\ell))$$
is $(\mathbb{Z}/2\mathbb{Z})^{m+1}$ (resp. trivial) for X general, where ℓ is a prime different from $\mathrm{char}(k)$.

(2) *Suppose that X is not an elliptic curve. If $(d_1, d_2, \ldots, d_c) \neq (2, 2)$, then the map*
$$\mathrm{Aut}(X) \to \mathrm{Aut}(\mathrm{H}^m_{\text{ét}}(X, \mathbb{Q}_\ell)) \tag{1.6.1}$$
is injective.

Let us remark that the validity of item 1.6 (2) relies on certain infinitesimal Torelli results for complete intersections, i.e., the injectivity of the cup product map
$$\mathrm{H}^1(X, T_X) \to \bigoplus_{p+q=m} \mathrm{Hom}(\mathrm{H}^q(X, \Omega^p_X), \mathrm{H}^{q+1}(X, \Omega^{p-1}_X)). \tag{1.6.2}$$
The injectivity indeed holds true for all smooth complete intersections except some trivial cases when the characteristic of the ground field k is zero (result of [5]). In fact, Flenner's idea also generalizes to an algebraically closed field without assuming $k = \mathbb{C}$. We include a proof of this result in an appendix to the paper.

Let us briefly describe the structure of this paper.

As we find both illuminating, we present two different approaches to the generic triviality theorem. In Section 2, we prove the generic triviality of

$\mathrm{Aut}_L(X)$ by studying the action of $\mathrm{Aut}(X)$ on $H^1(T_X)$. This works only in $\mathrm{char}(k) = 0$ but we are able to settle the generic triviality of $\mathrm{Aut}_L(X)$ for all complete intersections over $\mathbb{C}$ and thus Theorem 1.3(1). We also take care of Theorem 1.3(2) in Section 2. In Sections 3 and 4, we prove the generic triviality of $\mathrm{Aut}_L(X)$ via a degeneration argument. This approach has the advantage of working in all characteristics but it does not work for intersections of quadrics. Thus we are able to obtain Theorem 1.3(3).

In Section 5, we use equivariant deformation theory to derived Theorem 1.6 from Theorem 1.3 when the varieties are over a field of characteristic zero. The last section addresses varieties over an arbitrary field, where some special cases are carefully studied and the lifting theory as developed in [13] is applied.

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The authors were informed that recently Ariyan Javanpeykar and Daniel Loughran [8], independent of what we have done here, are also able to prove the injectivity of the Betti analogue of (1.5.1) assuming the "generic triviality" hypothesis (i.e., assuming the validity of our Theorem 1.3(1)). In this paper, we settle the "generic triviality" issue, and bootstrap this to get the results that hold true even in positive characteristics by a systematical use of the lifting technique developed by the second author.

The paper of Ariyan Javanpeykar and Daniel Loughran cited above also discerns that the faithfulness of (1.5.1) has some interesting consequences in the realm of arithmetic of algebraic varieties. In fact, assuming (1.5.1), the curvature property of the period space implies that the moduli stack of these complete intersections is Brody hyperbolic. In particular, all subvarieties of this moduli stack are log-general type. This makes the moduli stack, when descends to a number field K, meet the hypothesis of the conjecture of Lang-Vojta. As argued as in Theorem 6.6 of [8], Lang-Vojta conjecture would then imply the Shafarevich conjecture for these complete intersections. We thank A. Javanpeykar and D. Loughran for some email exchange explaining their work.

2. Equivariant Kodaira Spencer Map

In this section, we will show that $\mathrm{Aut}_L(X) = \{1\}$ for $X \subset \mathbb{CP}^n$ general with the exception of X of type $(2, 2)$. This is achieved by studying the natural action $\mathrm{Aut}(X)$ on $H^1(T_X)$. We make the following simple observation on this action: if $\sigma \in \mathrm{Aut}(X)$ can be "deformed" as X deforms in a family, then σ preserves the Kodaira-Spencer classes of this family at X.

Proposition 2.1. *Let $\pi : X \to B$ be a flat family of projective varieties over a smooth variety B with π smooth and proper, where X and B are varieties over an algebraically closed field. Then for a very general point $b \in B$, the image of the Kodaira-Spencer map $\kappa : T_{B,b} \to H^1(T_{X_b})$ is invariant under* *the actions of* $\mathrm{Aut}(X_b)$. *If we fix a line bundle* L *of* X *relatively ample over* B, *then for a general point* $b \in B$, $\kappa(T_{B,b})$ *is invariant under the actions of* $\mathrm{Aut}_L(X_b)$.

Proof. Note that $\mathrm{Aut}(X_b)$ is a locally Noetherian scheme. At a very general point b, every $\sigma_b \in \mathrm{Aut}(X_b)$ can be "deformed" over B. That is, after shrinking B and a base change unramified over b, there exists a $\sigma \in \mathrm{Aut}(X/B)$ such that σ_b is the restriction of σ to b, where $\mathrm{Aut}(X/B)$ is the automorphism group of X preserving B. This gives us the commutative diagram

(2.1.1)
$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ & \searrow^{\pi} & \downarrow \pi \\ & & B \end{array}$$

which leads to the commuting exact sequences

(2.1.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{X_b} & \longrightarrow & T_X\big|_{X_b} & \longrightarrow & N_{X_b/X} & \longrightarrow & 0 \\ & & \downarrow \sigma_{b,*} & & \downarrow \sigma_* & & \| & & \\ 0 & \longrightarrow & T_{X_b} & \longrightarrow & T_X\big|_{X_b} & \longrightarrow & N_{X_b/X} & \longrightarrow & 0 \end{array}$$

where $N_{X_b/X} \cong \pi^* T_{B,b}$. Consequently, the resulting Kodaira-Spencer map commutes with the action of σ_b, i.e., the diagram

(2.1.3)
$$\begin{array}{ccc} T_{B,b} & \xrightarrow{\kappa} & H^1(T_{X_b}) \\ \| & & \downarrow \sigma_{b,*} \\ T_{B,b} & \xrightarrow{\kappa} & H^1(T_{X_b}) \end{array}$$

commutes. Therefore, the image of κ is invariant under the action of σ_b.

If we fix a polarization L, $\mathrm{Aut}_L(X_b)$ is a scheme. Therefore, at a general point b, every $\sigma_b \in \mathrm{Aut}_L(X_b)$ can be "deformed" over B. Then the above argument shows that $\kappa(T_{B,b})$ is invariant under the action of σ_b. □

The above proposition holds in all characteristics. In the rest of the section, with the exception of Proposition 2.5, we work over $k = \mathbb{C}$.

As a complete intersection $X \subset \mathbb{P}^n$ of type $(d_1, d_2, \ldots, d_n)$ deforms in $\mathbb{P}^n$, the image of the Kodaira-Spencer map is exactly the cokernel $\mathrm{coker}(J_X)$ of J_X given in the diagram

(2.1.4)
$$\begin{array}{ccccccc} & & & & H^0(\mathcal{O}_X(1))^{\oplus n+1} & \xrightarrow{J_X} & \bigoplus_{i=1}^{c} H^0(\mathcal{O}_X(d_i)) \\ & & & & \downarrow & & \| \\ 0 & \longrightarrow & H^0(T_X) & \longrightarrow & H^0(T_{\mathbb{P}^n}\big|_X) & \longrightarrow & H^0(N_{X/\mathbb{P}^n}) \end{array}$$

where $H^0(\mathcal{O}_X(1))^{\oplus n+1} \to H^0(X, T_{\mathbb{P}^n})$ is induced by the Euler sequence and $N_{X/\mathbb{P}^n}$ is the normal bundle of $X \subset \mathbb{P}^n$.

Thus, by Proposition 2.1, $\operatorname{coker}(J_X)$ is invariant under the action $\operatorname{Aut}_L(X)$, i.e., $\operatorname{Aut}_L(X)$ acts trivially on $\operatorname{coker}(J_X)$ for X general.

Since we have known that $\operatorname{Aut}_L(X)$ is finite, every $\sigma \in \operatorname{Aut}_L(X)$ has finite order. If $\operatorname{Aut}_L(X) \neq \{1\}$, then there exists $\sigma \in \operatorname{Aut}_L(X)$ of order p for some p prime. That is, $\sigma^p = 1$ and $\sigma \neq 1$. Since $\sigma \in \operatorname{Aut}(\mathbb{P}^n)$, it has a dual action on $\mathbb{P}H^0(\mathcal{O}(1))$ and hence on $\mathbb{P}H^0(\mathcal{O}(m))$ for all $m \in \mathbb{Z}^+$. A lift of σ to $H^0(\mathcal{O}(m))$ is diagonalizable and we may choose a lift such that

$$H^0(\mathcal{O}(m)) = \bigoplus E_{m,\xi} \tag{2.1.5}$$

where $E_{m,\xi}$ is the eigenspace of σ acting on $H^0(\mathcal{O}(m))$ corresponding to the eigenvalue ξ satisfying $\xi^p = 1$.

Since $\sigma \in \operatorname{Aut}_L(X)$, $\sigma \in \operatorname{Aut}_L(X_1 \cap X_2 \cap \ldots \cap X_l)$ for all $d_l < d_{l+1}$. Therefore, σ also acts on the subspace $H^0(I_{X_1 \cap X_2 \cap \ldots \cap X_l}(m)) \subset H^0(\mathcal{O}(m))$ and this action is also diagonalizable, where $I_{X_1 \cap X_2 \cap \ldots \cap X_l}$ is the ideal sheaf of $X_1 \cap X_2 \cap \ldots \cap X_l$ in $\mathbb{P}^n$. Thus, we can choose the defining equations $F_1, F_2, \ldots, F_c$ of $X_1, X_2, \ldots, X_c$ such that

$$\begin{bmatrix} \sigma(F_1) \\ \sigma(F_2) \\ \vdots \\ \sigma(F_c) \end{bmatrix} = \begin{bmatrix} \xi_1 & & & \\ & \xi_2 & & \\ & & \ddots & \\ & & & \xi_c \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_c \end{bmatrix} \tag{2.1.6}$$

for some $\xi_i^p = 1$. That is, $F_i \in E_{d_i,\xi_i}$ for $i = 1, 2, \ldots, c$.

Note that J_X is explicitly given by

$$J_X \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix} = \left[\frac{\partial F_i}{\partial z_j}\right]_{c \times (n+1)} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix} \tag{2.1.7}$$

for $(z_0, z_1, ..., z_n)$ the homogeneous coordinates of $\mathbb{P}^n$.

Under our choice of the defining equations of X, the action of σ on $\operatorname{coker}(J_X)$ is induced by

$$\begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_c \end{bmatrix}^{\sigma} = \begin{bmatrix} \xi_1^{-1} & & & \\ & \xi_2^{-1} & & \\ & & \ddots & \\ & & & \xi_c^{-1} \end{bmatrix} \begin{bmatrix} \sigma(G_1) \\ \sigma(G_2) \\ \vdots \\ \sigma(G_c) \end{bmatrix} \tag{2.1.8}$$

for $G_i \in H^0(\mathcal{O}(d_i))$. This comes from the observation that σ sends an infinitesimal deformation $\{F_i + t_i G_i = 0\}$ of X to

$$\{\sigma(F_i + t_i G_i) = 0\} = \{F_i + t_i \xi_i^{-1} \sigma(G_i) = 0\}.$$

The fact that σ acts trivially on $\operatorname{coker}(J_X)$ is equivalent to saying that

$$\begin{bmatrix} \sigma(G_1) - \xi_1 G_1 \\ \sigma(G_2) - \xi_2 G_2 \\ \vdots \\ \sigma(G_c) - \xi_c G_c \end{bmatrix} \in \operatorname{Im} J_X \tag{2.1.9}$$

for all $G_i \in H^0(\mathcal{O}(d_i))$. In other words,

$$\begin{aligned} &\operatorname{Im} J_X + (E_{X,d_1,\xi_1} \oplus E_{X,d_2,\xi_2} \oplus \ldots \oplus E_{X,d_c,\xi_c}) \\ &= H^0(\mathcal{O}_X(d_1)) \oplus H^0(\mathcal{O}_X(d_2)) \oplus \ldots \oplus H^0(\mathcal{O}_X(d_c)), \end{aligned} \tag{2.1.10}$$

where $\operatorname{Im}(J_X)$ is the image of J_X and $E_{X,d,\xi}$ is the restriction of $E_{d,\xi}$ to $H^0(\mathcal{O}_X(d))$. We will show that this cannot hold by a dimension count. That is, we are going to show that

$$\begin{aligned} &\dim (\operatorname{Im} J_X + (E_{X,d_1,\xi_1} \oplus E_{X,d_2,\xi_2} \oplus \ldots \oplus E_{X,d_c,\xi_c})) \\ &< h^0(\mathcal{O}_X(d_1)) + h^0(\mathcal{O}_X(d_2)) + ... + h^0(\mathcal{O}_X(d_c)). \end{aligned} \tag{2.1.11}$$

Let

$$a_j = \dim E_{1,\eta_j} \tag{2.1.12}$$

where $\eta_0, \eta_1, \ldots, \eta_{p-1}$ are the p-th roots of unit. Note that $\sum a_j = n+1$ and at least two a_j's are positive, i.e., $\mu \geq 2$ for

$$\mu = \#\{j : a_j > 0, 0 \leq j \leq p-1\} \tag{2.1.13}$$

since $\sigma \neq 1$. Our argument for (2.1.11) is based on the following inequalities:

Lemma 2.2. *Let X, J_X, σ, ξ_i, E_{X,d_i,ξ_i} and a_j be given as above. Then*

$$\begin{aligned} &\dim \operatorname{Im} J_X - \dim (\operatorname{Im} J_X \cap (E_{X,d_1,\xi_1} \oplus E_{X,d_2,\xi_2} \oplus \ldots \oplus E_{X,d_c,\xi_c})) \\ &\leq (n+1)^2 - \sum_{j=0}^{p-1} a_j^2. \end{aligned} \tag{2.2.1}$$

Proof of Lemma 2.2. We choose homogeneous coordinates $(z_0, z_1, ..., z_n)$ of $\mathbb{P}^n$ such that $\sigma(z_i) = \lambda_i z_i$ for each i, where $\lambda_i \in \{\eta_0, \eta_1, ..., \eta_{p-1}\}$. We may identify $H^0(\mathcal{O}_X(1))^{\oplus n+1}$ with the space of $(n+1) \times (n+1)$ matrices via

$$\big[b_{kl}\big]_{0 \leq k,l \leq n} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_n \end{bmatrix} \in H^0(\mathcal{O}_X(1))^{\oplus n+1}.$$

For each j, let

$$V_j = \big\{\big[b_{kl}\big] : b_{kl} = 0 \text{ for } \lambda_k \neq \eta_j \text{ or } \lambda_l \neq \eta_j\big\} \subset H^0(\mathcal{O}_X(1))^{\oplus n+1}$$

Then by (2.1.7), we see that

$$J_X(V_0 \oplus V_1 \oplus \ldots \oplus V_{p-1}) \subset E_{X,d_1,\xi_1} \oplus E_{X,d_2,\xi_2} \oplus \ldots \oplus E_{X,d_c,\xi_c}.$$

Combining with the fact that $\dim V_j = a_j^2$, we arrive at (2.2.1). □

Lemma 2.3. *Let σ, $E_{d,\xi}$ and a_j be given as above. Then*

$$\binom{n+d}{d} - \dim E_{d,\xi} > (n+1)^2 - \sum_{j=0}^{p-1} a_j^2 \tag{2.3.1}$$

if $d, n \geq 3$ and

$$\binom{n+2}{2} - \dim E_{2,\xi} \geq \frac{(n+1)^2}{2} - \frac{1}{2}\sum_{j=0}^{p-1} a_j^2 \tag{2.3.2}$$

for all $\xi \in \mathbb{C}$.

Proof of Lemma 2.3. WLOG, we may assume that $a_0 \geq a_1 \geq \ldots \geq a_{p-1}$. We let $V_j = E_{1,\eta_j}$ and write

$$\begin{aligned} H^0(\mathcal{O}(d)) = {} & \operatorname{Sym}^d V_0 \oplus \sum_{j\neq 0} \operatorname{Sym}^{d-1} V_0 \otimes V_j \\ & \oplus \operatorname{Sym}^d V_1 \oplus \sum_{j\neq 1} \operatorname{Sym}^{d-1} V_1 \otimes V_j \\ & \oplus ... \oplus \operatorname{Sym}^d V_{p-1} \oplus \sum_{j\neq p-1} \operatorname{Sym}^{d-1} V_{p-1} \otimes V_j \\ & \oplus \sum_{i<j<k} \operatorname{Sym}^{d-2} V_i \otimes V_j \otimes V_k \oplus ... \end{aligned} \tag{2.3.3}$$

when $d \geq 3$. Note that $\operatorname{Sym}^d V_0$, $\operatorname{Sym}^{d-1} V_0 \otimes V_1$, ..., $\operatorname{Sym}^{d-1} V_0 \otimes V_{p-1}$ have different weights under the action of σ. Therefore,

$$\begin{aligned} & \dim E_{d,\xi} \cap \left(\operatorname{Sym}^d V_0 \oplus \sum_{j\neq 0} \operatorname{Sym}^{d-1} V_0 \otimes V_j \right) \\ & \leq \max\Big(\dim \operatorname{Sym}^d V_0, \dim \operatorname{Sym}^{d-1} V_0 \otimes V_1, \\ & \qquad \ldots, \dim \operatorname{Sym}^{d-1} V_0 \otimes V_{p-1}\Big) \\ & = \max\left(\binom{a_0+d-1}{d}, \binom{a_0+d-2}{d-1}\binom{a_1}{1} \right). \end{aligned} \tag{2.3.4}$$

In other words,

$$\begin{aligned}
&\dim\left(\operatorname{Sym}^d V_0 \oplus \sum_{j\neq 0}\operatorname{Sym}^{d-1} V_0\otimes V_j\right)\\
&-\dim E_{d,\xi}\cap\left(\operatorname{Sym}^d V_0 \oplus \sum_{j\neq 0}\operatorname{Sym}^{d-1} V_0\otimes V_j\right)\\
&\geq \binom{a_0+d-1}{d}+\sum_{j\neq 0}\binom{a_0+d-2}{d-1}\binom{a_j}{1}\\
&-\max\left(\binom{a_0+d-1}{d},\binom{a_0+d-2}{d-1}\binom{a_1}{1}\right).
\end{aligned}\tag{2.3.5}$$

By the same argument, we have

$$\begin{aligned}
&\dim\left(\operatorname{Sym}^d V_i \oplus \sum_{j\neq i}\operatorname{Sym}^{d-1} V_i\otimes V_j\right)\\
&-\dim E_{d,\xi}\cap\left(\operatorname{Sym}^d V_i \oplus \sum_{j\neq i}\operatorname{Sym}^{d-1} V_i\otimes V_j\right)\\
&\geq \binom{a_i+d-1}{d}+\sum_{j\neq i, j\geq 1}\binom{a_i+d-2}{d-1}\binom{a_j}{1}
\end{aligned}\tag{2.3.6}$$

for all $i \geq 1$ and

$$\begin{aligned}
&\dim\left(\operatorname{Sym}^{d-2} V_i\otimes V_j\otimes\sum_{k>j} V_k\right)\\
&-\dim E_{d,\xi}\cap\left(\operatorname{Sym}^{d-2} V_i\otimes V_j\otimes\sum_{k>j} V_k\right)\\
&\geq \sum_{k>j+1}\binom{a_i+d-3}{d-2}\binom{a_j}{1}\binom{a_k}{1}
\end{aligned}\tag{2.3.7}$$

for all $i < j$. Combining (2.3.5), (2.3.6) and (2.3.7), we conclude

$$\begin{aligned}
\binom{n+d}{d}-\dim E_{d,\xi} \geq \sum_i\binom{a_i+d-1}{d}&+\sum_{\substack{i\neq j\\ j\geq 1}}\binom{a_i+d-2}{d-1}a_j\\
&-\max\left(\binom{a_0+d-1}{d},\binom{a_0+d-2}{d-1}a_1\right)\\
&+\sum_{i<j<k-1}\binom{a_i+d-3}{d-2}a_ja_k.
\end{aligned}\tag{2.3.8}$$

Note that the right hand side of (2.3.1) is simply $\sum_{i<j} 2a_i a_j$. So it suffices to verify the following:

$$\begin{aligned}
&\binom{a_i+d-2}{d-1}a_j + \binom{a_j+d-2}{d-1}a_i \geq 2a_i a_j \text{ for } 1 \leq i < j\\
&\binom{a_0+d-2}{d-1}a_j + \binom{a_j+d-1}{d} \geq 2a_0 a_j \text{ for } j \geq 2\\
&\min\left(\binom{a_0+d-1}{d}, \binom{a_0+d-2}{d-1}a_1\right) + \binom{a_1+d-1}{d} \geq 2a_0 a_1\\
&\sum_{i<j<k-1} \binom{a_i+d-3}{d-2}a_j a_k \geq 0.
\end{aligned} \tag{2.3.9}$$

Therefore, we conclude

$$\binom{n+d}{d} - \dim E_{d,\xi} \geq \sum_{i<j} 2a_i a_j = (n+1)^2 - \sum_{j=0}^{p-1} a_j^2. \tag{2.3.10}$$

And the equality in (2.3.10) holds only if all equalities hold in (2.3.5), (2.3.6), (2.3.7) and (2.3.9), which cannot happen. So (2.3.1) follows.

The proof of (2.3.2) is quite straightforward and we leave it to the readers. □

Lemma 2.4. *Let σ, $E_{d,\xi}$, a_j and μ be given as above. Then*

$$\begin{aligned}
\dim E_{d,\xi} \leq &\sum_{k=0}^{n+1-\mu} \frac{(\mu-1)^k}{\mu^{k+1}} \binom{n+d-k}{d}\\
&+ \frac{(\mu-1)^{n+1-\mu}}{\mu^{n+2-\mu}} \binom{d+\mu-1}{d+1}
\end{aligned} \tag{2.4.1}$$

for all $\xi \in \mathbb{C}$.

Proof of Lemma 2.4. We let $e(d, a_0, a_1, ..., a_{p-1})$ be the maximum of $\dim E_{d,\xi}$ for all $\xi \in \mathbb{C}$ and all σ such that $a_j = \dim E_{1,\eta_j}$.

We choose homogeneous coordinates $(z_0, z_1, \dots, z_n)$ of $\mathbb{P}^n$ such that $\sigma(z_i) = \lambda_i z_0$ and $\lambda_0, \lambda_1, ..., \lambda_{\mu-1}$ are distinct. We observe that for each monomial $z_0^{b_0} z_1^{b_1} ... z_n^{b_n}$ with $b_0 > 0$, $z_0^{b_0} z_1^{b_1} ... z_n^{b_n}(z_i/z_0)$ have different weights under the action of σ for $i = 0, 1, ..., \mu - 1$. This observation leads to the recursive inequality

$$e(d, a_0, a_1, \dots, a_{p-1}) \leq \frac{1}{\mu}\binom{n+d}{d} + \frac{\mu-1}{\mu} e(d, a_0 - 1, a_1, \dots, a_{p-1}) \tag{2.4.2}$$

where we assume that $a_0 = \max(a_j)$ and $\lambda_0 = \eta_0$.

It is easy to check that (2.4.1) follows from (2.4.2) by induction. □

Now we are ready to prove Theorem 1.3. By induction, we just have to deal with the following cases:

- $d_1 = d_2 = \cdots = d_c = d$ ($d \geq 3$ or $d, c \geq 2$)
- $2 = d_1 = d_2 < d_3 = \cdots = d_c = d$ and
- $2 = d_1 < d_2 = \cdots = d_c = d$.

We only need to prove (2.1.11) in these cases.

The case $d_1 = d_2 = \cdots = d_c = d$. By (2.2.1), we have

$$\begin{aligned} &\dim\left(\operatorname{Im} J_X + (E_{X,d,\xi_1} \oplus E_{X,d,\xi_2} \oplus \ldots \oplus E_{X,d,\xi_c})\right) \\ &\leq (n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{c} \dim E_{X,d,\xi_i}. \end{aligned} \tag{2.4.3}$$

Thus, in order to prove (2.1.11), it suffices to prove

$$(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{c} \dim E_{X,d,\xi_i} < ch^0(\mathcal{O}_X(d)) = c\binom{n+d}{d} - c^2. \tag{2.4.4}$$

If $\xi_{i_1}, \xi_{i_2}, \ldots \xi_{i_k}$ are distinct, we observe that

$$\dim E_{X,d,\xi_{i_1}} + \dim E_{X,d,\xi_{i_2}} + \ldots + \dim E_{X,d,\xi_{i_k}} \leq h^0(\mathcal{O}_X(d)). \tag{2.4.5}$$

We write $\{1, 2, ..., c\} = B_1 \sqcup B_2 \sqcup \ldots$ as a disjoint union of sets B_i such that $\xi_k = \xi_l$ if and only if $\{k, l\} \subset B_i$ for some i. Let $b_i = |B_i|$. Note that

$$\dim E_{d,\xi_k} - \dim E_{X,d,\xi_k} \geq b_i \text{ for } k \in B_i. \tag{2.4.6}$$

Suppose that $b_1 \geq b_2 \geq \ldots$. Combining (2.4.5) and (2.4.6), we derive

$$\begin{aligned} \sum_{i=1}^{c} \dim E_{X,d,\xi_i} &\leq b_2 h^0(\mathcal{O}_X(d)) + \sum_{k \in B} \dim E_{X,d,\xi_k} \\ &\leq b_2 h^0(\mathcal{O}_X(d)) + \sum_{k \in B} \dim E_{d,\xi_k} - b_1(b_1 - b_2) \end{aligned}$$

for all $B \subset B_1$ with $|B| = b_1 - b_2$. Note that $b_1 + b_2 \leq c$. So it comes down to verifying

$$(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + (b_1 - b_2) \dim E_{d,\xi} + 2b_1 b_2 < b_1 \binom{n+d}{d} \tag{2.4.7}$$

for all $\xi \in \mathbb{C}$, $b_1 \geq b_2 \in \mathbb{N}$ and $b_1 + b_2 = c$.

It is easy to check that (2.4.7) follows from (2.3.1) when $d \geq 3$ for all $n \geq c + 2 \geq 3$ and from (2.3.2) when $d = 2$ for all $n \geq c + 1 \geq 4$. This proves (2.4.7) and hence (2.4.4).

We are left with the only exceptional case $d = c = 2$, which is settled by the following proposition.

Proposition 2.5. *For a general smooth intersection X of two quadrics in $\mathbb{P}^n$, $\operatorname{Aut}(X) = (\mathbb{Z}/2\mathbb{Z})^n$ if $n \geq 4$ and $\operatorname{char}(k) \neq 2$.*

Proof. Let $W \subset \mathbb{P}^n \times \mathbb{P}^1$ be a pencil of quadrics whose base locus is X. Since $\mathrm{Aut}(X)$ acts $H^0(I_X(2))$, every $\sigma \in \mathrm{Aut}(X)$ induces an automorphism of W with diagram

$$\begin{array}{ccc} W & \xrightarrow{\sigma} & W \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{g} & \mathbb{P}^1 \end{array} \tag{2.5.1}$$

Let $D = \{b \in \mathbb{P}^1 : W_b \text{ singular}\}$ be the discriminant locus of the pencil W. Clearly, $g(D) = D$.

We can make everything explicit. Let Q_A and Q_B be two members of the pencil whose defining equations are given by two symmetric matrices A and B. Obviously, we may take $A = I$ and assume that a member of pencil is given by $B - tI$. Then D consists of exactly the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_n$ of B. By [15, Proposition 2.1], the set $D = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ is a set of $n+1 \geq 5$ general points on $\mathbb{P}^1$. Therefore, $g \in \mathrm{Aut}(\mathbb{P}^1)$ sending D to D must be the identity map and σ preserves the fiberation $W/\mathbb{P}^1$, i.e., it induces an automorphism of W_b for each b. In particular, σ maps W_b to W_b for each $b \in D$. Every W_b has exactly one node, i.e., an ordinary double point p_i for $b = \lambda_i$, which is given by the eigenvector of B corresponding to λ_i. Clearly, σ fixes $p_0, p_1, \ldots, p_n$.

The matrix B can always be orthogonally diagonalized (this follows from [15, Proposition 2.1]). Therefore, W is given by

$$(\lambda_0 z_0^2 + \lambda_1 z_1^2 + \cdots + \lambda_n z_n^2) - t(z_0^2 + z_1^2 + \cdots + z_n^2) = 0 \tag{2.5.2}$$

after some action of $\mathrm{Aut}(\mathbb{P}^n)$. Since σ fixes $p_0, p_1, \ldots, p_n$, it has to be a $(k^*)^{n+1}$ action on $(z_0, z_1, \ldots, z_n)$. And since it preserves the fibers of $W/\mathbb{P}^1$, it must be

$$\sigma(z_0, z_1, \ldots, z_n) = (\pm z_0, \pm z_1, \ldots, \pm z_n) \tag{2.5.3}$$

and we then conclude that $\mathrm{Aut}(X) = (\mathbb{Z}/2\mathbb{Z})^{n+1}/\{\pm 1\} = (\mathbb{Z}/2\mathbb{Z})^n$. □

Example 2.6 (A (2,2)-complete intersection with an extra automorphism). Let k be an algebraically closed field. Assume that the characteristic of k is not equal to 2. Let $n > 0$ be an integer prime to the characteristic of k. Let η be a primitive nth root of unit. Consider the quadric hypersurfaces

$$D = \{z \in \mathbb{P}^{n-1} : \textstyle\sum_{i=0}^{n-1} z_i^2 = 0\} \text{ and } E = \{z \in \mathbb{P}^{n-1} : \textstyle\sum_{i=1}^{n-1} \eta^i z_i^2 = 0\}$$

and the linear transformation σ on $\mathbb{P}^{n-1}$ given by

$$z_i \mapsto \eta^{1/2} z_{i+1},$$

where the subscripts are understood as integers modulo n. Then σ sends

$$\sum_{i \in \mathbb{Z}/n} z_i^2 \mapsto \sum_{i \in \mathbb{Z}/n} \eta \cdot z_{i+1}^2 \text{ and } \sum_{i \in \mathbb{Z}/n} \eta^i z_i^2 \mapsto \sum_{i \in \mathbb{Z}/n} \eta^i \cdot \eta \cdot z_{i+1}^2.$$

It follows that σ is simultaneous automorphisms of D and E and hence an automorphism of the (2,2)-complete intersection $X = D \cap E$. It is elementary to check that X is nonsingular. This automorphism is not induced from the standard automorphism of X obtained by "flipping the factors".

The case $2 = d_1 = d_2 < d_3 = \cdots = d_c = d$. By (2.2.1), it suffices to prove

$$\begin{aligned}(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \dim E_{X,2,\xi_1} + \dim E_{X,2,\xi_2} + \sum_{i=3}^{c} \dim E_{X,d,\xi_i} \\ < 2h^0(\mathcal{O}_X(2)) + (c-2)h^0(\mathcal{O}_X(d))\end{aligned} \tag{2.6.1}$$

where

$$h^0(\mathcal{O}_X(d)) = \binom{n+d}{n} - 2\binom{n+d-2}{n} + \binom{n+d-4}{n} - (c-2).$$

Applying the same argument as before to E_{X,d,ξ_i} for $i \ge 3$, we can further reduce (2.6.1) to

$$\begin{aligned}(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{2} \dim E_{X,2,\xi_i} + (b_1 - b_2)\dim E_{d,\xi} + 2b_1b_2 \\ < 2h^0(\mathcal{O}_X(2)) + b_1\left(\binom{n+d}{n} - 2\binom{n+d-2}{n} + \binom{n+d-4}{n}\right)\end{aligned} \tag{2.6.2}$$

for all $\xi \in \mathbb{C}$, $b_1 \ge b_2 \in \mathbb{N}$ and $b_1 + b_2 = c - 2$.

For $E_{X,2,\xi_1}$ and $E_{X,2,\xi_2}$, it follows from (2.3.2) that

$$\begin{aligned}&(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{2} \dim E_{X,2,\xi_i} \\ &\le (n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{2} \dim E_{2,\xi_i} - 2 \\ &\le 2\binom{n+2}{2} - 2 = 2h^0(\mathcal{O}_X(2)) + 2.\end{aligned} \tag{2.6.3}$$

Suppose that $\mu \ge 3$. We have

$$\binom{n+d}{d} - \dim E_{d,\xi} \ge \frac{2}{3}\binom{n+d-1}{d-1} + \frac{4}{9}\binom{n+d-2}{d-1} \tag{2.6.4}$$

for $n \ge 3$ by (2.4.1). It is easy to check that

$$\frac{2}{3}\binom{n+d-1}{d-1} + \frac{4}{9}\binom{n+d-2}{d-1} > 2\binom{n+d-2}{n} + 2 \tag{2.6.5}$$

for all $d \ge 3$ and $n \ge 4$. Thus, (2.6.2) follows from (2.6.3), (2.6.4) and (2.6.5) for all $d \ge 3$ and $n \ge c+1 \ge 4$.

Suppose that $\mu = 2$. If $\xi_1 \ne \xi_2$,

$$\dim E_{X,2,\xi_1} + \dim E_{X,2,\xi_2} \le h^0(\mathcal{O}_X(2)) \tag{2.6.6}$$

and hence

$$\begin{aligned}(n+1)^2 &- \sum_{j=0}^{p-1} a_j^2 + \sum_{i=1}^{2} \dim E_{X,2,\xi_i}\\ &\le (n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + h^0(\mathcal{O}_X(2))\\ &\le \frac{(n+1)^2}{2} + h^0(\mathcal{O}_X(2)) = 2h^0(\mathcal{O}_X(2))\end{aligned} \tag{2.6.7}$$

for $n \ge 3$. On the other hand, if $\xi_1 = \xi_2$,

$$\dim E_{X,2,\xi_1} + \dim E_{X,2,\xi_2} \le \dim E_{2,\xi_1} + \dim E_{2,\xi_2} - 4 \tag{2.6.8}$$

and (2.6.7) still holds by (2.3.2).

By (2.4.1), we have

$$\begin{aligned}\binom{n+d}{d} - \dim E_{d,\xi} \ge \frac{1}{2}\binom{n+d-1}{d-1} + \frac{1}{4}\binom{n+d-2}{d-1}\\ + \frac{1}{8}\binom{n+d-3}{d-1}\end{aligned} \tag{2.6.9}$$

for $\mu \ge 2$ and $n \ge 3$. It is easy to check that

$$\frac{1}{2}\binom{n+d-1}{d-1} + \frac{1}{4}\binom{n+d-2}{d-1} + \frac{1}{8}\binom{n+d-3}{d-1} > 2\binom{n+d-2}{n} \tag{2.6.10}$$

for all $d \ge 3$ and $n \ge 4$. Thus, (2.6.2) follows from (2.6.7), (2.6.9) and (2.6.10) for all $d \ge 3$ and $n \ge c+1 \ge 4$.

The case $2 = d_1 < d_2 = d_3 = \cdots = d_c = d$. By (2.2.1), it suffices to prove

$$\begin{aligned}(n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \dim E_{X,2,\xi_1} + \sum_{i=2}^{c} \dim E_{X,d,\xi_i}\\ < h^0(\mathcal{O}_X(2)) + (c-1)h^0(\mathcal{O}_X(d))\end{aligned} \tag{2.6.11}$$

where

$$h^0(\mathcal{O}_X(d)) = \binom{n+d}{n} - \binom{n+d-2}{n} - (c-1).$$

By (2.3.2), we have

$$\begin{aligned}(n+1)^2 &- \sum_{j=0}^{p-1} a_j^2 + \dim E_{X,2,\xi_1} - h^0(\mathcal{O}_X(2))\\ &\le (n+1)^2 - \sum_{j=0}^{p-1} a_j^2 + \dim E_{2,\xi_1} - 1 - \left(\binom{n+2}{2} - 1\right)\\ &\le \frac{(n+1)^2}{2} - \frac{1}{2}\sum_{j=0}^{p-1} a_j^2.\end{aligned} \tag{2.6.12}$$

So (2.6.11) holds if

$$\frac{(n+1)^2}{2} - \frac{1}{2}\sum_{j=0}^{p-1} a_j^2 + \sum_{i=2}^{c} \dim E_{X,d,\xi_i} < (c-1)h^0(\mathcal{O}_X(d)). \tag{2.6.13}$$

Applying the same argument as before to E_{X,d,ξ_i} for $i \geq 2$, we can further reduce (2.6.13) to

$$\begin{aligned} &\frac{(n+1)^2}{2} - \frac{1}{2}\sum_{j=0}^{p-1} a_j^2 + (b_1 - b_2)\dim E_{d,\xi} + 2b_1b_2 \\ &< b_1\left(\binom{n+d}{n} - \binom{n+d-2}{n}\right) \end{aligned} \tag{2.6.14}$$

for all $\xi \in \mathbb{C}$, $b_1 \geq b_2 \in \mathbb{N}$ and $b_1 + b_2 = c - 1$.

By (2.6.9) and (2.6.10), we have

$$\begin{aligned} &\left(b_1 - b_2 - \frac{1}{2}\right)\dim E_{d,\xi} + 2b_1b_2 \\ &< \left(b_1 - \frac{1}{2}\right)\binom{n+d}{n} - b_1\binom{n+d-2}{n} \end{aligned} \tag{2.6.15}$$

for all $d \geq 3$ and $n \geq c + 2 \geq 4$. Combining (2.6.15) with (2.3.1), we obtain (2.6.14). This finishes the proof of Theorem 1.3(1).

3. Resolution of Relative Birational Maps

3.1. **Triviality of** $\mathrm{Aut}(X)$ **via Degeneration.** A naive idea for proving the generic triviality of automorphism is to use the degeneration method specializing a complete intersection to a special one, and then verify that the specialization does not admit any automorphism. However this method relies on the separatedness of the moduli space (i.e., one has to deal with the issue on when automorphisms specialize along with the complete intersection).

O. Benoist's paper [2] addresses this issue, but restricted to the realm of smooth complete intersections, where an automorpism-free example is hard to come up with; on the other hand, deducing separatedness in general requires some restrictions to singularities and may well fit into another article. To solve our problem, we only pursue this analysis in a simple degeneration situation: we only consider degenerations to double point singularities, and verify that in this process the automorphisms do specialize with the variety as well. With the presence of the singularities it is then easy to prove the triviality of the automorphism group.

Having said our idea and motivation, the key question is thus the following:

Question 3.1. Let B be a smooth curve over an algebraically closed field k, 0 be a closed point of B and $B^* = B\backslash\{0\}$. Under what conditions can $f \in \mathrm{Aut}_L(X^*/B^*)$ be extended to $f \in \mathrm{Aut}_L(X/B)$ for a flat family $X \subset \mathbb{P}_k^n \times B$ of closed subschemes in $\mathbb{P}^n$ over B? Here $X^* = X\backslash X_0$ and

$\mathrm{Aut}_L(X/B)$ and $\mathrm{Aut}_L(X^*/B^*)$ are the automorphism groups of X and X^* preserving $L = \mathcal{O}(1)$ and the bases B and B^*, respectively.

Let t be the local coordinate of a formal neighborhood of B at 0. Assuming that X_t is linearly nondegenerate, we may regard f as a birational self-map $f : \mathbb{P}^n \times B \dashrightarrow \mathbb{P}^n \times B$ preserving the base B, which is also an automorphism over B^*. Namely, we have the diagram

$$\begin{array}{ccc} \mathbb{P}^n \times B^* & \xrightarrow{\sim} & \mathbb{P}^n \times B^* \\ \downarrow \subset & & \downarrow \subset \\ \mathbb{P}^n \times B & \overset{f}{\dashrightarrow} & \mathbb{P}^n \times B \\ \downarrow & & \downarrow \\ B & = & B. \end{array} \tag{3.1.1}$$

We want to show that if f sends X_t to itself for $t \neq 0$ and some $X \subset \mathbb{P}^n \times B$, f extends to an automorphism over $t = 0$, i.e., f is regular. Let us first figure out how to resolve the indeterminacy of f.

Let $P = \mathbb{P}^n \times B$. The indeterminacy of f can be resolved by a sequence of blowups over linear subspaces

$$\Lambda_{f,1} \supsetneq \Lambda_{f,2} \supsetneq \ldots \supsetneq \Lambda_{f,l} \tag{3.1.2}$$

of $P_0 = \mathbb{P}^n$. To see it, we choose homogeneous coordinates $(z_0, z_1, \ldots, z_n)$ of $\mathbb{P}^n$ such that $f^* H^0(\mathcal{O}_P(1))$ is spanned by

$$\begin{gathered} t^{\mu_0} z_0, t^{\mu_0} z_1, \ldots, t^{\mu_0} z_{a_1}, \\ t^{\mu_1} z_{a_1+1}, t^{\mu_1} z_{a_1+2}, \ldots, t^{\mu_1} z_{a_2}, \\ \ldots \\ t^{\mu_{l-1}} z_{a_{l-1}+1}, t^{\mu_{l-1}} z_{a_{l-1}+2}, \ldots, t^{\mu_{l-1}} z_{a_l}, \\ t^{\mu_l} z_{a_l+1}, t^{\mu_l} z_{a_l+2}, \ldots, t^{\mu_l} z_n \end{gathered} \tag{3.1.3}$$

for $0 = \mu_0 < \mu_1 < \cdots < \mu_{l-1} < \mu_l$ and $0 \le a_1 < a_2 < \cdots < a_{l-1} < a_l < n$. That is, f is simply given by

$$\begin{aligned} f(z_0, z_1, ..., z_n) = (&t^{\mu_0} z_0, t^{\mu_0} z_1, \ldots, t^{\mu_0} z_{a_1}, \\ &t^{\mu_1} z_{a_1+1}, t^{\mu_1} z_{a_1+2}, \ldots, t^{\mu_1} z_{a_2}, \ldots, \\ &t^{\mu_{l-1}} z_{a_{l-1}+1}, t^{\mu_{l-1}} z_{a_{l-1}+2}, \ldots, t^{\mu_{l-1}} z_{a_l}, \\ &t^{\mu_l} z_{a_l+1}, t^{\mu_l} z_{a_l+2}, \ldots, t^{\mu_l} z_n). \end{aligned} \tag{3.1.4}$$

Then the indeterminacy of f is resolved by subsequently blowing up P along the subschemes

$$
\begin{aligned}
z_0 = z_1 = \cdots = z_{a_1} = t^{\mu_1-\mu_0} &= 0,\\
\frac{z_0}{t^{\mu_1-\mu_0}} = \frac{z_1}{t^{\mu_1-\mu_0}} = \cdots = \frac{z_{a_1}}{t^{\mu_1-\mu_0}} = z_{a_1+1} = \cdots = z_{a_2} = t^{\mu_2-\mu_1} &= 0,\\
\vdots &= \vdots\\
\frac{z_0}{t^{\mu_{l-1}-\mu_0}} = \cdots = z_{a_{l-1}+1} = \cdots = z_{a_l} = t^{\mu_l-\mu_{l-1}} &= 0,
\end{aligned}
\tag{3.1.5}
$$

with $\Lambda_{f,i}$ in (3.1.2) given by

$$\Lambda_{f,i} = \{z_0 = z_1 = \cdots = z_{a_i} = 0\} \text{ for } i = 1, 2, \ldots, l. \tag{3.1.6}$$

Let $\rho_f : \widehat{P}_f \to P$ be the resulting resolution of $f : P \dashrightarrow P$ such that $f \circ \rho_f$ is regular.

Let us also consider the inverse $g = f^{-1}$ of f:

$$
\begin{aligned}
g(z_0, z_1, ..., z_n) = (&t^{-\mu_0} z_0, t^{-\mu_0} z_1, \ldots, t^{-\mu_0} z_{a_1},\\
&t^{-\mu_1} z_{a_1+1}, t^{-\mu_1} z_{a_1+2}, \ldots, t^{-\mu_1} z_{a_2}, \ldots,\\
&t^{-\mu_{l-1}} z_{a_{l-1}+1}, t^{-\mu_{l-1}} z_{a_{l-1}+2}, \ldots, t^{-\mu_{l-1}} z_{a_l},\\
&t^{-\mu_l} z_{a_l+1}, t^{-\mu_l} z_{a_l+2}, \ldots, t^{-\mu_l} z_n).
\end{aligned}
\tag{3.1.7}
$$

Similarly, the indeterminacy of g can be resolved by a sequence of blowups over linear subspaces

$$\Lambda_{g,1} \supsetneq \Lambda_{g,2} \supsetneq \ldots \supsetneq \Lambda_{g,l} \tag{3.1.8}$$

with $\Lambda_{g,i}$ given by

$$\Lambda_{g,i} = \{z_n = z_{n-1} = \cdots = z_{a_{l+1-i}+1} = 0\} \text{ for } i = 1, 2, \ldots, l. \tag{3.1.9}$$

Let $\rho_g : \widehat{P}_g \to P$ be the resulting resolution of $g : P \dashrightarrow P$ such that $g \circ \rho_g$ is regular. Clearly,

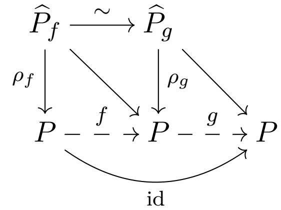
$$\dim \Lambda_{f,i} = n - a_i - 1,\ \dim \Lambda_{g,i} = a_{l+1-i}, \tag{3.1.10}$$

$$\dim \Lambda_{f,i} + \dim \Lambda_{g,l+1-i} = n - 1 \text{ and } \Lambda_{f,i} \cap \Lambda_{g,l+1-i} = \emptyset \tag{3.1.11}$$

for $0 \le i \le l+1$, where we set $\Lambda_{f,0} = \Lambda_{g,0} = P_0$ and $\Lambda_{f,l+1} = \Lambda_{g,l+1} = \emptyset$.

Indeed, it is not hard to see that $\rho_g^{-1} \circ (f \circ \rho_f) : \widehat{P}_f \dashrightarrow \widehat{P}_g$ is regular and we have the diagram

(3.1.12)



Furthermore, the maps $f \circ \rho_f$ and $g \circ \rho_g$ can be decribed as follows. Let

$$\begin{aligned} \rho_f^* P_0 &= F_0 + F_1 + \cdots + F_l \\ \rho_g^* P_0 &= G_0 + G_1 + \cdots + G_l \end{aligned} \tag{3.1.13}$$

where F_0 and G_0 are the proper transforms of P_0 under ρ_f and ρ_g and F_i and G_i are the exceptional divisors satisfying $\rho_f(F_i) = \Lambda_{f,i}$ and $\rho_g(G_i) = \Lambda_{g,i}$ for $i = 1, 2, \ldots, l$, respectively. Then

- $\sum F_i$ and $\sum G_i$ has simple normal crossings, where $F_i \cap F_j \neq \emptyset$ and $G_i \cap G_j \neq \emptyset$ if and only if $|i - j| \leq 1$.
- The map $\rho_g^{-1} \circ (f \circ \rho_f) : \widehat{P}_f \to \widehat{P}_g$ induces isomorphisms

$$F_i \xrightarrow[\sim]{\rho_g^{-1} \circ (f \circ \rho_f)} G_{l-i} \tag{3.1.14}$$

  and hence

$$f \circ \rho_f(F_i) = \Lambda_{g,l-i} \text{ and } g \circ \rho_g(G_i) = \Lambda_{f,l-i} \tag{3.1.15}$$

  for $i = 0, 1, \ldots, l$.
- We have the commutative diagram

$$\begin{array}{ccc} F_i & \xrightarrow{f \circ \rho_f} & \Lambda_{g,l-i} \\ \Big\downarrow {\scriptstyle \rho_f} & & \Big\downarrow {\scriptstyle \pi_{\Lambda_{g,l-i+1}, \Lambda_{f,i}}} \\ \Lambda_{f,i} & \xrightarrow{\pi_{\Lambda_{f,i+1}, \Lambda_{g,l-i}}} & \Lambda_{f,i} \cap \Lambda_{g,l-i} \end{array} \tag{3.1.16}$$

  where we use the notation $\pi_{\Lambda,\Lambda'}$ to denote the linear projection

$$\pi_{\Lambda,\Lambda'} : \mathbb{P}^n \xrightarrow{|I_\Lambda(1)|} \Lambda' \subset \mathbb{P}^n \tag{3.1.17}$$

  of $\mathbb{P}^n$ through a linear subspace $\Lambda \subset \mathbb{P}^n$ to a linear subspace $\Lambda' \subset \mathbb{P}^n$ satisfying $\dim \Lambda + \dim \Lambda' = n - 1$ and $\Lambda \cap \Lambda' = \emptyset$. We simplify the notation and let $\pi_\Lambda = \pi_{\Lambda,\Lambda'}$ if Λ' is irrelevant or obvious by context. So we write $\pi_{\Lambda_{f,i+1}} = \pi_{\Lambda_{f,i+1}, \Lambda_{g,l-i}}$ and $\pi_{\Lambda_{g,l-i+1}} = \pi_{\Lambda_{g,l-i+1}, \Lambda_{f,i}}$ in (3.1.16). The fiber of $\pi_{\Lambda_{g,l-i+1}}$ over a point $q \in \Lambda_{f,i} \cap \Lambda_{g,l-i}$ is the linear subspace spanned by q and $\Lambda_{g,l-i+1}$, i.e.,

$$\pi^{-1}_{\Lambda_{g,l-i+1}}(q) = C_{q, \Lambda_{g,l-i+1}} \tag{3.1.18}$$

  for $q \in \Lambda_{f,i} \cap \Lambda_{g,l-i}$, where we use $C_{A,B}$ to denote the chord variety

$$C_{A,B} = \overline{\bigcup_{\substack{a \in A \\ b \in B \\ a \neq b}} l_{ab}} \tag{3.1.19}$$

  of two subschemes A and B in $\mathbb{P}^n$, i.e., the closure of the union of all lines l_{ab} joining $a \in A$ and $b \in B$ with $a \neq b$.

- In addition, $f\circ\rho_f$ maps a fiber $\rho_f^{-1}(p)\cap F_i$ of $\rho_f : F_i \to \Lambda_{f,i}$ over $p\in\Lambda_{f,i}\backslash\Lambda_{f,i+1}$ isomorphically to the fiber $\pi^{-1}_{\Lambda_{g,l-i+1}}(q)$:

$$\rho_f^{-1}(p)\cap F_i \xrightarrow[\sim]{f\circ\rho_f} C_{q,\Lambda_{g,l-i+1}} \cong \mathbb{P}^{a_i+1} \tag{3.1.20}$$

for all $p\in\Lambda_{f,i}\backslash\Lambda_{f,i+1}$ and $q=\pi_{\Lambda_{f,i+1}}(p)$. Note that

$$\dim(\rho_f^{-1}(p)\cap F_i) = \dim F_i - \dim\Lambda_{f,i} = \dim C_{q,\Lambda_{g,l-i+1}} \tag{3.1.21}$$

by (3.1.11). In summary, $f\circ\rho_f : F_i\to\Lambda_{g,l-i}$ is characterized by the diagram (3.1.16) and it maps the fibers of $F_i/\Lambda_{f,i}$ isomorphically to those of $\Lambda_{g,l-i}/(\Lambda_{f,i}\cap\Lambda_{g,l-i})$ over the open set $\Lambda_{f,i}\backslash\Lambda_{f,i+1}$.
- Similarly, the map $g\circ\rho_g : G_i\to\Lambda_{f,l-i}$ can be characterized in the same way by the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{g\circ\rho_g} & \Lambda_{f,l-i} \\ {\scriptstyle\rho_g}\downarrow & & \downarrow{\scriptstyle\pi_{\Lambda_{f,l-i+1},\Lambda_{g,i}}} \\ \Lambda_{g,i} & \xrightarrow{\pi_{\Lambda_{g,i+1},\Lambda_{f,l-i}}} & \Lambda_{g,i}\cap\Lambda_{f,l-i} \end{array} \tag{3.1.22}$$

Our main purpose of the above discussion is to estimate $\dim f\circ\rho_f(V)$ of an irreducible variety $V\subset F_i$. Suppose that $\rho_f(V)\not\subset\Lambda_{f,i+1}$. By (3.1.20), $f\circ\rho_f$ maps a fiber $V\cap\rho_f^{-1}(p)$ of $V/\Lambda_{f,i}$ over $p\in\Lambda_{f,i}\backslash\Lambda_{f,i+1}$ isomorphically onto its image. So $\dim f\circ\rho_f(V)$ has at least a lower bound

$$\dim f\circ\rho_f(V)\ge\dim V-\dim\rho_f(V). \tag{3.1.23}$$

If $f\circ\rho_f(V)\not\subset\Lambda_{g,l-i+1}$, it has a better lower bound

$$\begin{aligned}\dim f\circ\rho_f(V) &\ge \dim\pi_{\Lambda_{f,i+1}}(\rho_f(V)\backslash\Lambda_{f,i+1})+\dim V-\dim\rho_f(V)\\ &=\dim V-\dim\left((\rho_f(V)\backslash\Lambda_{f,i+1})\cap C_{q,\Lambda_{f,i+1}}\right)\end{aligned} \tag{3.1.24}$$

for a general point $q\in\rho_f(V)$, where we use the fact that the fibers of the linear projection π_Λ through Λ are $C_{q,\Lambda}$.

3.2. **Extensibility of $f\in\mathrm{Aut}_L(X^*/B^*)$.** Based on the above description of how $f\in\mathrm{Aut}(\mathbb{P}^n\times B^*/B^*)$ is resolved as a birational map $f\in\mathrm{Bir}(\mathbb{P}^n\times B/B)$, we can prove a criterion on the extensibility of $f\in\mathrm{Aut}_L(X^*/B^*)$.

Let us first introduce some more notations. Let Λ be a linear subspace in $\mathbb{P}^n$ and D be a local complete intersection scheme in $\mathbb{P}^n$ containing Λ. We denote the map between the normal sheaves of Λ and D by $\xi_{D,\Lambda}$ in

$$\mathcal{O}_\Lambda(1)^{\oplus r} = N_\Lambda \xrightarrow{\xi_{D,\Lambda}} N_D\Big|_\Lambda \tag{3.1.25}$$

where $r=\mathrm{codim}_{\mathbb{P}^n}\Lambda$ and N_Λ and N_D are the normal sheaves of Λ and D in $\mathbb{P}^n$, respectively. Let us consider the condition

$$\xi_{D,\Lambda}\circ\alpha\ne0 \text{ for all } \alpha\in\mathrm{Hom}(\mathcal{O}_\Lambda(1),N_\Lambda) \text{ and } \alpha\ne0. \tag{3.1.26}$$

In particular, when $D = \{\Phi(z_0, z_1, \ldots, z_n) = 0\}$ is a hypersurface defined by a homogeneous polynomial Φ and $\Lambda = \{z_0 = z_1 = \ldots = z_{r-1} = 0\}$, the map $\xi_{D,\Lambda}$ is explicitly given by

$$\xi_{D,\Lambda}(s_0, s_1, \ldots, s_{r-1}) = \sum_{i=0}^{r-1} s_i \frac{\partial \Phi}{\partial z_i} \tag{3.1.27}$$

and the condition (3.1.26) is equivalent to saying that

$$\frac{\partial \Phi}{\partial z_0}, \frac{\partial \Phi}{\partial z_1}, \ldots, \frac{\partial \Phi}{\partial z_{r-1}} \text{ are linearly independent in } H^0(\mathcal{O}_\Lambda(d-1)) \tag{3.1.28}$$

for $d = \deg D$.

Indeed, for a local complete intersection $X \subset \mathbb{P}^n$ containing a linear subspace $\Lambda \subset \mathbb{P}^n$,

$$\xi_{X,\Lambda} \circ \alpha \neq 0 \text{ for all } \alpha \in \operatorname{Hom}(\mathcal{O}_\Lambda(1), N_\Lambda) \text{ and } \alpha \neq 0 \tag{3.1.29}$$

if and only if there exists a hypersurface $D \in |I_X(d)|$ for some $d \in \mathbb{Z}^+$ satisfying (3.1.26), where I_X is the ideal sheaf of X in $\mathbb{P}^n$.

We are ready to state our criterion. Its hypotheses are quite technical and seemingly awkward but they will be made more understandable by its proof.

Proposition 3.2. *Let B be a smooth curve over an algebraically closed field k, 0 be a closed point of B and $B^* = B\backslash\{0\}$. Let X and $Y \subset \mathbb{P}^n_k \times B$ be two flat families of closed subschemes of pure dimension $0 < m < n$ in $\mathbb{P}^n$ over B and let $f \in \operatorname{Aut}(\mathbb{P}^n \times B^*/B^*)$ be an automorphism inducing isomorphisms $f_t : X_t \xrightarrow{\sim} Y_t$ for $t \neq 0$. Suppose that when Z is one of X_0 and Y_0,*

$$\begin{gathered}\dim(Z \cap \Lambda) \le \max\left(\frac{m}{2}, m + \dim\Lambda - n\right)\\ \text{for all linear subspaces } \Lambda \subset \mathbb{P}^n,\end{gathered} \tag{3.2.1}$$

$$\begin{aligned}\dim \pi_\Lambda((Z\backslash\Lambda) \cap \Lambda') \ge{}& \dim(Z \cap \Lambda')\\ &- \min\Bigg(\max\left(\frac{m-1}{2}, m + \dim\Lambda - n + 1\right),\\ &\qquad \max\left(\dim\Lambda, n - 1 - \dim\Lambda'\right)\Bigg)\\ &\text{for all pairs of linear subspaces } \Lambda \subsetneq \Lambda' \subset \mathbb{P}^n \text{ of } \dim\Lambda' > n - \frac{m}{2},\end{aligned} \tag{3.2.2}$$

and

$$\begin{gathered}\text{for all pairs of linear subspaces } \Lambda \text{ and } \Lambda' \subset \mathbb{P}^n \text{ satisfying}\\ \dim\Lambda = n - \dim\Lambda' \ge \frac{m}{2} - 1 \text{ either } \Lambda \not\subset X_0 \text{ or } \Lambda \subset X_0,\\ \text{there exists a hypersurface } D \in |I_{X_0}(d)| \text{ in } \mathbb{P}^n \text{ containing } X_0\\ \text{with the property (3.1.26) and } Y_0 \cap \Lambda' \text{ is not contained}\\ \text{in the union of two proper linear subspaces in } \Lambda',\end{gathered} \tag{3.2.3}$$

where π_Λ is the linear projection of $\mathbb{P}^n$ through Λ defined in (3.1.17) *and I_{X_0} is the ideal sheaf of X_0 in $\mathbb{P}^n$. Then f extends to an automorphism $f \in \mathrm{Aut}(\mathbb{P}^n \times B/B)$ inducing an isomorphism $f : X \xrightarrow{\sim} Y$. Here all intersections are set theoretic intersections.*

Proof. Suppose that f is not regular. We resolve the indeterminacies of f and $g = f^{-1}$ as in 3.1 with the same notations. Then $l > 0$.

Let $\widehat{X} \subset \widehat{P}_f$ be the proper transform of X under ρ_f and $\widehat{Y} \subset \widehat{P}_g$ be the proper transform of Y under ρ_g. Since $\rho_g^{-1} \circ (f \circ \rho_f)$ is an isomorphism between $\widehat{P}_f$ and $\widehat{P}_g$, it induces an isomorphism between $\widehat{X}$ and $\widehat{Y}$. Therefore, by (3.1.14) and (3.1.15), we conclude that

$$\begin{aligned} f \circ \rho_f(\widehat{X} \cap F_i) = \rho_g(\widehat{Y} \cap G_{l-i}) = Y_0 \cap \Lambda_{g,l-i} \text{ and} \\ g \circ \rho_g(\widehat{Y} \cap G_i) = \rho_f(\widehat{X} \cap F_{l-i}) = X_0 \cap \Lambda_{f,l-i} \end{aligned} \tag{3.2.4}$$

for $i = 0, 1, ..., l$. Note that

$$X_0 \cap \Lambda_{f,l} \neq \emptyset \text{ and } Y_0 \cap \Lambda_{g,l} \neq \emptyset \tag{3.2.5}$$

since $\widehat{X} \cap F_0 \neq \emptyset$ and $\widehat{Y} \cap G_0 \neq \emptyset$ as X_0 and Y_0 are not linearly degenerate by (3.2.1).

The basic idea of the proof is to apply (3.1.23) and (3.1.24) to give a lower bound for $\dim f \circ \rho_f(\widehat{X} \cap F_i)$ and then compare it with the upper bound on $\dim(Y_0 \cap \Lambda_{g,l-i})$ given by (3.2.1).

For an irreducible component V of $\widehat{X} \cap F_i$ satisfying $\rho_f(V) \not\subset \Lambda_{f,i+1}$,

$$\begin{aligned} \dim f \circ \rho_f(V) &\geq m - \dim(X_0 \cap \Lambda_{f,i}) \\ &\geq m - \min\left(\dim \Lambda_{f,i}, \max\left(\frac{m}{2}, m + \dim \Lambda_{f,i} - n\right)\right) \\ &= \max\Big(m + \dim \Lambda_{g,l-i+1} - n + 1, \\ &\qquad \min\left(\frac{m}{2}, \dim \Lambda_{g,l-i+1} + 1\right)\Big) \end{aligned} \tag{3.2.6}$$

by (3.1.23) and (3.2.1). If $f \circ \rho_f(V) \subset \Lambda_{g,l-i+1}$, then

$$\begin{aligned} \dim f \circ \rho_f(V) &\leq \dim(Y_0 \cap \Lambda_{g,l-i+1}) \\ &\leq \min\left(\dim \Lambda_{g,l-i+1}, \max\left(\frac{m}{2}, m + \dim \Lambda_{g,l-i+1} - n\right)\right) \end{aligned} \tag{3.2.7}$$

by (3.2.1). Clearly, (3.2.6) and (3.2.7) hold if and only if

$$\begin{aligned} \dim \Lambda_{g,l-i+1} \geq \frac{m}{2} = \dim(Y_0 \cap \Lambda_{g,l-i+1}) = \dim f \circ \rho_f(V) \\ \geq m + \dim \Lambda_{g,l-i+1} - n + 1. \end{aligned} \tag{3.2.8}$$

In conclusion,

$$\begin{aligned} \rho_f(V) \not\subset \Lambda_{f,i+1} \Rightarrow f \circ \rho_f(V) \not\subset \Lambda_{g,l-i+1} \\ \text{if } 2 \nmid m \text{ or } \dim \Lambda_{f,i} \geq n - \frac{m}{2} \text{ or } \dim \Lambda_{f,i} \leq \frac{m}{2} - 1 \end{aligned} \tag{3.2.9}$$

for all integers i and irreducible components $V \subset \widehat{X} \cap F_i$, where we set $\Lambda_{f,i} = \Lambda_{g,i} = \mathbb{P}^n$ for $i \le 0$ and $\Lambda_{f,i} = \Lambda_{g,i} = \emptyset$ for $i > l$. Similarly,

$$\begin{gathered} \rho_g(W) \not\subset \Lambda_{g,i+1} \Rightarrow g \circ \rho_g(W) \not\subset \Lambda_{f,l-i+1} \\ \text{if } 2 \nmid m \text{ or } \dim \Lambda_{g,i} \ge n - \frac{m}{2} \text{ or } \dim \Lambda_{g,i} \le \frac{m}{2} - 1 \end{gathered} \tag{3.2.10}$$

for all integers i and irreducible components $W \subset \widehat{Y} \cap G_i$.

Let i be an integer such that

$$\dim \Lambda_{f,i} > n - \frac{m}{2}. \tag{3.2.11}$$

Note that $\rho_f(V) \not\subset \Lambda_{f,i+1}$ by (3.2.1) and $f \circ \rho_f(V) \not\subset \Lambda_{g,l-i+1}$ by (3.2.9) for all irreducible components $V \subset \widehat{X} \cap \Lambda_{f,i}$.

Then by (3.1.24), (3.2.1) and (3.2.2), we have

$$\begin{aligned} &\min\left(\dim \Lambda_{g,l-i}, \max\left(\frac{m}{2}, m + \dim \Lambda_{g,l-i} - n\right)\right) \\ &\ge \dim(Y_0 \cap \Lambda_{g,l-i}) = \dim f \circ \rho_f(\widehat{X} \cap F_i) \\ &\ge m - \dim(X_0 \cap \Lambda_{f,i}) + \dim \pi_{\Lambda_{f,i+1}}((X_0 \backslash \Lambda_{f,i+1}) \cap \Lambda_{f,i}) \\ &\ge m - \min\Bigg(\max\left(\frac{m-1}{2}, m + \dim \Lambda_{f,i+1} - n + 1\right), \\ &\qquad \max\left(\dim \Lambda_{f,i+1}, n - 1 - \dim \Lambda_{f,i}\right)\Bigg) \\ &= \max\Bigg(\min\left(\frac{m+1}{2}, \dim \Lambda_{g,l-i}\right), \\ &\qquad \min\left(m + \dim \Lambda_{g,l-i} - n + 1, m - \dim \Lambda_{g,l-i+1}\right)\Bigg). \end{aligned} \tag{3.2.12}$$

This leads to two cases: either

$$m + \dim \Lambda_{g,l-i} - n \ge \max\left(\frac{m+1}{2}, m - \dim \Lambda_{g,l-i+1}\right) \text{ or} \tag{3.2.13}$$

$$\frac{m}{2} \ge \dim(Y_0 \cap \Lambda_{g,l-i}) = \dim \Lambda_{g,l-i}. \tag{3.2.14}$$

Suppose that (3.2.13) holds. Then

$$\dim \Lambda_{g,l-i} + \dim \Lambda_{g,l-i+1} \ge n \text{ and } \dim \Lambda_{g,l-i} > n - \frac{m}{2}. \tag{3.2.15}$$

Hence we may apply the same argument to estimate $\dim g \circ \rho_g(\widehat{Y} \cap G_{l-i})$ and obtain

$$\begin{aligned}
&\min\left(\dim \Lambda_{f,i}, \max\left(\frac{m}{2}, m + \dim \Lambda_{f,i} - n\right)\right)\\
&\geq \dim(X_0 \cap \Lambda_{f,i}) = \dim g \circ \rho_g(\widehat{Y} \cap G_{l-i})\\
&\geq \max\left(\min\left(\frac{m+1}{2}, \dim \Lambda_{f,i}\right),\right.\\
&\qquad \left.\min\left(m + \dim \Lambda_{f,i} - n + 1, m - \dim \Lambda_{f,i+1}\right)\right).
\end{aligned} \tag{3.2.16}$$

And since $\dim \Lambda_{f,i} > n - m/2$, we must have

$$m + \dim \Lambda_{f,i} - n \geq m - \dim \Lambda_{f,i+1} \Rightarrow \dim \Lambda_{f,i} + \dim \Lambda_{f,i+1} \geq n. \tag{3.2.17}$$

On the other hand,

$$(\dim \Lambda_{f,i} + \dim \Lambda_{f,i+1}) + (\dim \Lambda_{g,l-i} + \dim \Lambda_{g,l-i+1}) = 2(n-1). \tag{3.2.18}$$

So (3.2.15) and (3.2.17) cannot hold simultaneously. Consequently, we must have (3.2.14), i.e., $\Lambda_{g,l-i}$ is a linear subspace contained in Y_0 of dimension no more than $m/2$. In summary, we conclude that

$$\dim \Lambda_{g,l-i} \leq \frac{m}{2} \text{ and } \Lambda_{g,l-i} \subset Y_0 \text{ if } \dim \Lambda_{f,i} > n - \frac{m}{2}. \tag{3.2.19}$$

Applying the same argument to g, we obtain

$$\dim \Lambda_{f,l-i} \leq \frac{m}{2} \text{ and } \Lambda_{f,l-i} \subset X_0 \text{ if } \dim \Lambda_{g,i} > n - \frac{m}{2}. \tag{3.2.20}$$

We may start to apply (3.2.20) to $i = 0$ since $\dim \Lambda_{g,0} = n$. For every integer i satisfying the conclusion of (3.2.20), if $\dim \Lambda_{f,l-i} < m/2 - 1$, then $\dim \Lambda_{g,i+1} > n - m/2$ and we can continue to apply (3.2.20) to $i+1$ to obtain

$$\dim \Lambda_{f,l-i-1} \leq \frac{m}{2} \text{ and } \Lambda_{f,l-i-1} \subset X_0. \tag{3.2.21}$$

This process has to stop for some i since $\dim \Lambda_{f,0} = n$. Therefore, there exists $i \in \mathbb{Z}$ such that

$$\frac{m}{2} \geq \dim \Lambda_{f,l-i} \geq \frac{m}{2} - 1 \text{ and } \Lambda_{f,l-i} \subset X_0. \tag{3.2.22}$$

Note that

$$\Lambda_{f,l-i} = \{z_0 = z_1 = \cdots = z_{a_{l-i}} = 0\} \tag{3.2.23}$$

by (3.1.6) with $\dim \Lambda_{f,l-i} = n - a_{l-i} - 1$.

By (3.2.3), there exists a hypersurface $D = \{\Phi(z_0, z_1, ..., z_n) = 0\} \subset \mathbb{P}^n$ containing X_0 as a scheme and satisfying (3.1.28). Therefore,

$$\left.\frac{\partial \Phi}{\partial z_{a_{l-i}}}\right|_{\Lambda_{f,l-i}} \not\equiv 0. \tag{3.2.24}$$

Thus, the fiber $\rho_f^{-1}(p) \cap \widehat{X} \cap F_{l-i}$ over a general point $p \in \Lambda_{f,l-i}$ is linearly degenerate in $\rho_f^{-1}(p) \cap F_{l-i}$, i.e.,

$$\begin{aligned} \rho_f^{-1}(p) \cap \widehat{X} \cap F_{l-i} \subset \Gamma_p \cong \mathbb{P}^{a_{l-i}} \\ \subset \rho_f^{-1}(p) \cap F_{l-i} \cong \mathbb{P}^{a_{l-i}+1} \end{aligned} \tag{3.2.25}$$

for a hyperplane Γ_p in $\mathbb{P}^{a_{l-i}+1}$.

By (3.1.20), $f \circ \rho_f$ induces an isomorphism

$$\rho_f^{-1}(p) \cap F_{l-i} \xrightarrow[\sim]{f\circ\rho_f} C_{q,\Lambda_{g,i+1}} \cong \mathbb{P}^{a_{l-i}+1} \tag{3.2.26}$$

for all $p \in \Lambda_{f,l-i} \backslash \Lambda_{f,l-i+1}$ and $q = \pi_{\Lambda_{f,l-i+1}}(p)$. Consequently, $f \circ \rho_f(\Gamma_p)$ is a hyperplane in $C_{q,\Lambda_{g,i+1}}$.

Let V and W be irreducible components of $\widehat{X} \cap F_{l-i}$ and $\widehat{Y} \cap G_i$, respectively, such that

$$W = \rho_g^{-1} \circ f \circ \rho_f(V) \text{ and } V = \rho_f^{-1} \circ g \circ \rho_g(W). \tag{3.2.27}$$

Note that

$$\dim \Lambda_{g,i} \geq \dim \Lambda_{g,i+1} + 1 = n - \dim \Lambda_{f,i} \geq n - \frac{m}{2}. \tag{3.2.28}$$

By (3.2.3), $Y_0 \cap \Lambda_{g,i}$ is not contained in a union of two proper linear subspaces in $\Lambda_{g,i}$. Hence $Y_0 \cap \Lambda_{g,i} \not\subset \Lambda_{g,i+1}$.

Let W be an irreducible component of $\widehat{Y} \cap G_i$ satisfying $\rho_g(W) \not\subset \Lambda_{g,i+1}$. Similar to (3.1.23), we have

$$\begin{aligned} \dim g \circ \rho_g(W) &\geq \dim W - \dim \rho_g(W) = m - \dim(Y_0 \cap \Lambda_{g,i}) \\ &= n - \dim \Lambda_{g,i} = \dim \Lambda_{f,l-i+1} + 1. \end{aligned} \tag{3.2.29}$$

Therefore, $\dim g \circ \rho_g(W) \not\subset \Lambda_{f,l-i+1}$. Then, similar to (3.1.24), we have

$$\begin{aligned} \dim g \circ \rho_g(W) &\geq \dim \pi_{\Lambda_{g,i+1}}(\rho_g(W) \backslash \Lambda_{g,i+1}) \\ &\quad + \dim W - \dim \rho_g(W) \\ &= \dim W - \dim\left((\rho_g(W) \backslash \Lambda_{g,i+1}) \cap C_{q,\Lambda_{g,i+1}}\right) \end{aligned} \tag{3.2.30}$$

for a general point $q \in \rho_g(W)$. Combining with (3.2.1), we have

$$\begin{aligned} \dim \rho_f(V) &= \dim g \circ \rho_g(W) \\ &\geq m - \dim(\rho_g(W) \cap C_{q,\Lambda_{g,i+1}}) \\ &\geq m - \max\left(\frac{m}{2}, m + \dim \Lambda_{g,i+1} + 1 - n\right) \\ &= \min\left(\frac{m}{2}, \dim \Lambda_{f,l-i}\right) = \dim \Lambda_{f,l-i} \end{aligned} \tag{3.2.31}$$

for a general point $q \in \rho_g(W)$. Therefore, $\rho_f(V) = \Lambda_{f,l-i}$ and hence

$$\Gamma_p \supset \rho_f^{-1}(p) \cap V \neq \emptyset \tag{3.2.32}$$

by (3.2.25) for a general point $p \in \Lambda_{f,l-i}$. Thus,

$$\rho_g(W) \cap C_{q,\Lambda_{g,i+1}} = f \circ \rho_f(\rho_f^{-1}(p) \cap V) \subset f \circ \rho_f(\Gamma_p) \tag{3.2.33}$$

for $q = \pi_{\Lambda_{f,l-i+1}}(p)$, where $f \circ \rho_f(\Gamma_p)$ is a proper linear subspace in $C_{q,\Lambda_{g,i+1}}$, as pointed out above. Consequently,

$$Z \cap C_{q,\Lambda_{g,i+1}} \subset f \circ \rho_f(\Gamma_p) \tag{3.2.34}$$

for every irreducible component Z of $Y_0 \cap \Lambda_{g,i}$ satisfying $Z \not\subset \Lambda_{g,i+1}$, a general point $p \in \Lambda_{f,l-i}$ and $q = \pi_{\Lambda_{f,l-i+1}}(p)$.

Let $\Lambda_q' = C_{q,\Lambda_{g,i+1}}$. Then it follows from (3.2.34) that

$$Y_0 \cap \Lambda_q' \subset \Lambda_{g,i+1} \cup f \circ \rho_f(\Gamma_p) \tag{3.2.35}$$

for $p \in \Lambda_{f,l-i}$ general and $q = \pi_{\Lambda_{f,l-i+1}}(p)$. This contradicts (3.2.3) because

$$\dim \Lambda_q' = \dim \Lambda_{g,i+1} + 1 = n - \dim \Lambda_{f,l-i}. \tag{3.2.36}$$

□

Remark 3.3. The condition in (3.2.3) on $Y_0 \cap \Lambda'$ is implied by (3.2.1) when $\dim \Lambda < m/2$: for $\dim \Lambda' > n - m/2$, no components of $Y_0 \cap \Lambda'$ are linearly degenerate in Λ'. So that part of the hypothesis is only relevant in the case that $\dim \Lambda = m/2$.

4. Complete intersections with nodes

4.1. **Singular locus of $X \cap \Lambda$.** To show the triviality of $\mathrm{Aut}_L(X)$ for a general complete intersection X, it suffices to produce a special X_0 satisfying the hypotheses of Proposition 3.2 and $\mathrm{Aut}_L(X_0) = \{1\}$. Here we consider a complete intersection $X_0 \subset \mathbb{P}_k^n$ with $n+1$ nodes (ordinary double points) in general position. An automorphism in $\mathrm{Aut}_L(X_0)$ permutes these $n+1$ points and hence lies in the subgroup $(k^*)^n \ltimes \Sigma_{n+1}$ of $\mathrm{Aut}(\mathbb{P}_k^n)$. Then it is easy to show that $\mathrm{Aut}_L(X) = \{1\}$ for a general deformation of X_0. The trickier part is the verification of the hypotheses (3.2.1), (3.2.2) and (3.2.3). The key to prove these statements for a complete intersection X in $\mathbb{P}^n$ is to bound the dimension of the singular locus of $X \cap \Lambda$ for a linear subspace $\Lambda \subset \mathbb{P}^n$.

Throughout this section, we work over an algebraically closed field k.

Although we are only interested in the singular locus of complete intersections in $\mathbb{P}^n$, we can study the problem in a more general setting. Let $X_1, X_2, \ldots, X_r$ be r local complete intersection subschemes of a smooth projective variety Y and $X = X_1 \cap X_2 \cap \ldots \cap X_c$. We do not assume that $X_1, X_2, \ldots, X_r$ meet properly. Let us consider the natural map

$$T_Y\Big|_X \xrightarrow{\psi_X} N_{X_1}\Big|_X \oplus N_{X_2}\Big|_X \oplus \ldots \oplus N_{X_c}\Big|_X \tag{4.0.1}$$

where $N_{X_i} = (I_{X_i}/I_{X_i}^2)^\vee$ are normal sheaves of X_i in Y. We let

$$\Psi_{X_1,X_2,\ldots,X_c,r} = \{p \in X : \mathrm{rank}(\psi_{X,p}) = c - r\} \tag{4.0.2}$$

be the r-th strata of the locus along which ψ_X drops rank, or equivalently, X_i fail to meet transversely in Y. And we write

$$\Psi_{X_1,X_2,\ldots,X_c} = \bigcup_{r\geq 1} \Psi_{X_1,X_2,\ldots,X_c,r}. \tag{4.0.3}$$

When there is no ambiguity, we simply write

$$\Psi_{X,r} = \Psi_{X_1,X_2,\ldots,X_c,r} \text{ and } \Psi_X = \Psi_{X_1,X_2,\ldots,X_c}. \tag{4.0.4}$$

Note that Ψ_X is the singular locus X_{sing} of X and $\Psi_{X,r}$ is independent of the choices of X_i if $X_1, X_2, ..., X_c$ meet properly in Y.

When X_i are hypersurfaces in $Y = \mathbb{P}^n$, $\Psi_{X_1,X_2,\ldots,X_c,r}$ is given by

$$\Psi_{X_1,X_2,\ldots,X_c,r} = \left\{p \in X : \text{rank} \left[\frac{\partial F_i}{\partial z_j}(p)\right]_{c\times(n+1)} = c - r\right\} \tag{4.0.5}$$

where $F_i(z_0, z_1, \ldots, z_n)$ are the homogeneous defining polynomials of X_i.

To describe $\Psi_{X,r}$ in general, we let

$$\begin{aligned} P_X &= \mathbb{P}(N_{X_1}\Big|_X \oplus N_{X_2}\Big|_X \oplus \ldots \oplus N_{X_c}\Big|_X)^\vee \\ &= \text{Proj}\left(\text{Sym}^\bullet(N_{X_1}\Big|_X \oplus N_{X_2}\Big|_X \oplus \ldots \oplus N_{X_c}\Big|_X)\right) \end{aligned} \tag{4.0.6}$$

and let $U_X \subset P_X$ be the image of the map

$$\text{Proj}\left(\text{Sym}^\bullet \operatorname{coker}(\psi_X)\right) \longrightarrow P_X \tag{4.0.7}$$

induced by $N_{X_1}\big|_X \oplus N_{X_2}\big|_X \oplus \ldots \oplus N_{X_c}\big|_X \to \operatorname{coker}(\psi_X)$. Then

$$\Psi_X = \pi(U_X) \tag{4.0.8}$$

under the projection $\pi : P_X \to X$. Indeed, the fiber of $U_X \to X$ over a point $p \in \Psi_{X,r}$ is simply

$$U_X \cap \pi^{-1}(p) \cong \mathbb{P}^{r-1}. \tag{4.0.9}$$

Another way to define U_X is defining it as the vanishing locus of the section

$$\overline{\psi}_X \in \text{Hom}(\pi^* T_Y\big|_X, \mathcal{O}_{P_X}(1)) = H^0(\pi^*\Omega_Y\big|_X \otimes \mathcal{O}_{P_X}(1)), \tag{4.0.10}$$

which is the lift of $\psi_X \in \text{Hom}(T_Y\big|_X, N_{X_1}\big|_X \oplus N_{X_2}\big|_X \oplus \ldots \oplus N_{X_c}\big|_X)$.

In the special case that $X_i = \{F_i(z) = 0\}$ are hypersurfaces of degree d_i in $Y = \mathbb{P}^n$, we have

$$P_X = \mathbb{P}\left(\mathcal{O}_X(-d_1) \oplus \mathcal{O}_X(-d_2) \oplus \cdots \oplus \mathcal{O}_X(-d_c)\right). \tag{4.0.11}$$

We may replace T_Y by $\mathcal{O}(1)^{\oplus n+1}$ via the Euler sequence. Then

$$\overline{\psi}_X \in H^0(\pi^*\mathcal{O}_X(-1) \otimes \mathcal{O}_{P_X}(1))^{\oplus n+1} \tag{4.0.12}$$

and U_X is cut out by $n+1$ sections in $|\pi^*\mathcal{O}_X(-1)\otimes\mathcal{O}_{P_X}(1)|$. Actually, it is very explicitly given by

$$\begin{aligned} U_X &= \{G_0(y,z) = G_1(y,z) = \cdots = G_n(y,z) = 0\} \\ &= G_0 \cap G_1 \cap \ldots \cap G_n \end{aligned}$$

(4.0.13) $$G_j(y,z) = \sum_{i=1}^{c} y_i \frac{\partial F_i}{\partial z_j} \in H^0(\pi^*\mathcal{O}_X(-1)\otimes\mathcal{O}_{P_X}(1))$$

$$\text{and } \overline{\psi}_X = G_0(y,z)dz_0 + G_1(y,z)dz_1 + \cdots + G_n(y,z)dz_n,$$

where $G_j = \{G_j(y,z) = 0\}$ for $j = 0, 1, \ldots, n$.

Back to the general case, we let $\Lambda \subset Y$ be another local complete intersection in Y and write

(4.0.14) $$\Psi_{X\cap\Lambda,r} = \Psi_{X_1,X_2,\ldots,X_c,\Lambda,r}$$

with $P_{X\cap\Lambda}$ and $U_{X\cap\Lambda}$ defined accordingly. To order to compare $\Psi_{X,r}$ and $\Psi_{X\cap\Lambda,r}$, we let $\rho_{X,\Lambda} : P_{X\cap\Lambda} \dashrightarrow P_X$ be the rational map induced by

(4.0.15) $$\bigoplus_{i=1}^{c} N_{X_i}\Big|_{X\cap\Lambda} \longrightarrow \bigoplus_{i=1}^{c} N_{X_i}\Big|_{X\cap\Lambda} \oplus N_\Lambda\Big|_{X\cap\Lambda}.$$

Suppose that Λ is smooth. Then $\rho_{X,\Lambda}$ is regular along $U_{X\cap\Lambda}$ and let us consider its image

(4.0.16) $$V_{X,\Lambda} = \rho_{X,\Lambda}(U_{X\cap\Lambda}).$$

It is not hard to see that

(4.0.17) $$\Psi_{X\cap\Lambda} = \pi(V_{X,\Lambda}) \text{ and } V_{X,\Lambda}\cap\pi^{-1}(p) \cong \mathbb{P}^{r-1}$$

for $p \in \Psi_{X\cap\Lambda,r}$. More importantly, $V_{X,\Lambda}$ is the vanishing locus of

(4.0.18) $$\overline{\zeta}_{X,\Lambda} \in \mathrm{Hom}(\pi^* T_\Lambda\Big|_{X\cap\Lambda}, \mathcal{O}_{P_X}(1)) = H^0(\pi^*\Omega_\Lambda\Big|_{X\cap\Lambda}\otimes\mathcal{O}_{P_X}(1))$$

which is the lift of the map $\zeta_{X,\Lambda}$ given by

(4.0.19) $$\begin{array}{ccccccccc} 0 & \longrightarrow & T_\Lambda\Big|_{X\cap\Lambda} & \longrightarrow & T_Y\Big|_{X\cap\Lambda} & \longrightarrow & N_\Lambda\Big|_{X\cap\Lambda} & \longrightarrow & 0 \\ & & & \searrow^{\zeta_{X,\Lambda}} & \downarrow{\scriptstyle\psi_X} & & & & \\ & & & & \oplus N_{X_i}\Big|_{X\cap\Lambda} & & & & \end{array}$$

where the top row is, of course, exact. By (4.0.19), we see that $\zeta_{X,\Lambda}$ drops rank wherever ψ_X does. So the vanishing locus of $\overline{\zeta}_{X,\Lambda}$ contains that of $\overline{\psi}_X$ over $X\cap\Lambda$. That is,

(4.0.20) $$U_X \cap \pi^{-1}(X\cap\Lambda) \subset V_{X,\Lambda}.$$

Furthermore, if the top row of (4.0.19) splits, $\overline{\psi}_X - \overline{\zeta}_{X,\Lambda}$ is a section in

(4.0.21) $$\overline{\psi}_X - \overline{\zeta}_{X,\Lambda} \in H^0(\pi^* N_\Lambda^\vee\Big|_{X\cap\Lambda}\otimes\mathcal{O}_{P_X}(1))$$

and its vanishing locus cuts out U_X on $V_{X,\Lambda}$, i.e.,

$$U_X \cap \pi^{-1}(X \cap \Lambda) = V_{X,\Lambda} \cap \{\overline{\psi}_X - \overline{\zeta}_{X,\Lambda} = 0\}. \tag{4.0.22}$$

When $X_i = \{F_i(z) = 0\}$ are hypersurfaces of degree d_i in $Y = \mathbb{P}^n$ and Λ is a linear subspace of $\mathbb{P}^n$, we may choose

$$\Lambda = \{z_0 = z_1 = \cdots = z_{l-1} = 0\} \tag{4.0.23}$$

for simplicity. Then

$$\Psi_{X\cap\Lambda,r} = \Big\{p \in X \cap \Lambda : \operatorname{rank}\left[\frac{\partial F_i}{\partial z_j}(p)\right]_{\substack{1\le i\le c\\ l\le j\le n}} = c - r\Big\}, \tag{4.0.24}$$

$$\begin{aligned} V_{X,\Lambda} &= \{G_l(y,z) = G_{l+1}(y,z) = \cdots = G_n(y,z) = 0\}\\ &= G_l \cap G_{l+1} \cap \ldots \cap G_n,\\ \overline{\zeta}_{X,\Lambda} &= G_l(y,z)dz_l + G_{l+1}(y,z)dz_{l+1} + \cdots + G_n(y,z)dz_n, \end{aligned} \tag{4.0.25}$$

and

$$\begin{aligned} U_X \cap \pi^{-1}(X \cap \Lambda) &= V_{X,\Lambda} \cap \{\overline{\psi}_X - \overline{\zeta}_{X,\Lambda} = 0\}\\ &= V_{X,\Lambda} \cap G_0 \cap G_1 \cap ... \cap G_{l-1} \end{aligned} \tag{4.0.26}$$

with G_j given in (4.0.13).

Lemma 4.1. *Let $X = X_1 \cap X_2 \cap \ldots \cap X_c$ be a complete intersection in $\mathbb{P}^n_k$ of degrees $d_i = \deg X_i \ge 2$ for $i = 1, 2, ..., c$. Then*

- *For all linear subspaces $\Lambda \subset \mathbb{P}^n$, $r \in \mathbb{Z}^+$ and irreducible components W of $\overline{\Psi}_{X\cap\Lambda,r}$,*

$$\min(\dim W, \dim W + r + \dim\Lambda - n - 1) \le \max(-1, \dim(\Psi_X \cap W)) \tag{4.1.1}$$

and hence

$$\dim \Psi_{X\cap\Lambda} \le n - \dim\Lambda + \max(-1, \dim(\Psi_X \cap \Lambda)), \tag{4.1.2}$$

where we set $\dim\emptyset = -\infty$ and $\overline{\Psi}_{X\cap\Lambda,r}$ is the closure of $\Psi_{X\cap\Lambda,r}$.

- *For all linear subspaces $\Lambda \subset \mathbb{P}^n$,*

$$\dim(X\cap\Lambda) \le \max\left(\frac{n-c}{2}, \frac{n-c+1+\dim(\Psi_X\cap\Lambda)}{2}, \dim\Lambda - c\right). \tag{4.1.3}$$

In particular, if $\dim X = n - c$ and X has at worst isolated singularities, then

$$\dim(X\cap\Lambda) \le \begin{cases} \max\left(\dfrac{\dim X}{2}, \dim\Lambda - c\right) & \text{if } \Lambda \cap X_{sing} = \emptyset\\ \max\left(\dfrac{\dim X + 1}{2}, \dim\Lambda - c\right) & \text{otherwise.}\end{cases} \tag{4.1.4}$$

Proof. Without loss of generality, we may assume that Λ is given by (4.0.23) for some choice of the homogeneous coordinates $(z_0, z_1, \ldots, z_n)$ of $\mathbb{P}^n$ with $\dim\Lambda = n - l$. Let $F_i \in H^0(\mathcal{O}(d_i))$ be the defining polynomials of X_i and let

P_X, $U = U_X$ and $V = V_{X,\Lambda}$ be defined as above in (4.0.6), (4.0.11), (4.0.13), (4.0.16) and (4.0.25).

For an irreducible component W of $\overline{\Psi}_{X\cap\Lambda,r}$, the general fibers of V over W are $\mathbb{P}^{r-1}$ by (4.0.17). Therefore,

$$\dim\left(V \cap \pi^{-1}(W)\right) \geq \dim W + r - 1. \tag{4.1.5}$$

By (4.0.26), $V \cap R = U \cap \pi^{-1}(X \cap \Lambda)$ and hence

$$V \cap \pi^{-1}(W) \cap R \subset U \cap \pi^{-1}(W) \tag{4.1.6}$$

for $R = G_0 \cap G_1 \cap \ldots \cap G_{l-1}$. And since $\pi(U) = \Psi_X$ by (4.0.8),

$$\pi\left(V \cap \pi^{-1}(W) \cap R\right) \subset \Psi_X \cap W. \tag{4.1.7}$$

Since $d_i \geq 2$, $\pi^*\mathcal{O}_X(-1) \otimes \mathcal{O}_{P_X}(1)$ is ample and hence each G_j is an ample divisor in P_X. Therefore,

$$\dim\left(V \cap \pi^{-1}(W) \cap R\right) \geq \dim W + r - 1 - l \tag{4.1.8}$$

by (4.1.5) and

$$\dim \pi\left(V \cap \pi^{-1}(W) \cap R\right) \geq \min\left(\dim W, \dim W + r - 1 - l\right). \tag{4.1.9}$$

Then (4.1.1) follows from (4.1.7) and (4.1.9).

Obviously, $\dim(X \cap \Lambda) = \dim \Lambda - c$ if

$$\dim \Lambda - c \geq \dim \Psi_{X\cap\Lambda}. \tag{4.1.10}$$

Otherwise, $\dim(X \cap \Lambda) > \dim \Lambda - c$ and a general point $p \in X \cap \Lambda$ lies in $\Psi_{X\cap\Lambda,0}$, where $X_1, X_2, ..., X_r, \Lambda$ meet transversely, which is a contradiction.

By (4.1.2), (4.1.10) holds when

$$\dim \Lambda \geq \left\lceil \max\left(\frac{n+c-1}{2}, \frac{n+c+\dim(\Psi_X \cap \Lambda)}{2}\right)\right\rceil \tag{4.1.11}$$

and (4.1.3) follows.

Finally, when $\dim X = n - c$ and X has at worst isolated singularities, $\dim \Psi_X = \dim X_{\text{sing}} \leq 0$. So we have (4.1.4). □

The condition $d_i \geq 2$ is crucial in Lemma 4.1. It ensures that the divisors G_j are ample and so we have (4.1.8) and (4.1.9). On the other hand, the lemma trivially fails when one of $d_i = 1$.

We see that (4.1.4) is just short of giving us (3.2.1), although the bound is optimal for complete intersections with isolated singularities. So it needs a little improvement for complete intersections with nodes.

4.2. **Complete intersections in $\mathbb{P}^n$ with $n+1$ nodes.**

Lemma 4.2. *Let $X = X_1 \cap X_2 \cap \ldots \cap X_c$ be a complete intersection in $\mathbb{P}^n_k$ of degrees $d_i = \deg X_i \ge 2$ for $i = 1, 2, ..., c$. Suppose that*

- *$d_1 \le d_2 \le \ldots \le d_c$, $d_c \ge 3$,*
- *X is smooth of dimension $n - c$ outside of a linearly non-degenerate set of $n+1$ points $p_0, p_1, \ldots, p_n$,*
- *X has $n+1$ nodes at $p_0, p_1, \ldots, p_n$,*
- *$X_1, X_2, \ldots, X_{c-1}$ meet transversely everywhere and*
- *X does not contain the line $\overline{p_i p_j}$ for all $0 \le i < j \le n$.*

Then for all linear subspaces $\Lambda \subsetneq \mathbb{P}^n$, $r \in \mathbb{Z}^+$ and irreducible components W of $\overline{\Psi}_{X\cap\Lambda,r}$,

$$\text{either } W \subset \{p_0, p_1, \ldots, p_n\} \text{ or } \dim W \le n - \dim\Lambda - r \tag{4.2.1}$$

and hence

$$\dim \Psi_{X\cap\Lambda} \le n - \dim\Lambda - 1 \tag{4.2.2}$$

$$\dim(X \cap \Lambda) \le \max\left(\frac{\dim X}{2}, \dim\Lambda - c\right). \tag{4.2.3}$$

In addition, if $X_1, X_2, \ldots, X_a$ meet transversely everywhere for $a \in \mathbb{Z}$ satisfying $d_a = 2 < d_{a+1}$, then $X \cap \Lambda$ is not contained in a union of two proper linear subspaces in Λ for all linear subspaces $\Lambda \subset \mathbb{P}^n$ of $2\dim\Lambda = n + c$.

Proof. Without loss of generality, we may choose X_c such that X_c has $n+1$ double points at $p_0, p_1, \ldots, p_n$.

We use the same setup of the proof of Lemma 4.1. Let $Y = \mathbb{P}^2$ and $f : \widehat{Y} \to Y$ be the blowup of Y at $p_0, p_1, \ldots, p_n$.

Let $\widehat{X}_i$ be the proper transforms of X_i under f for $i = 1, 2, \ldots, c$. By our hypotheses on X_i, we see that

$$\begin{aligned} \widehat{X}_i &\in \left|\mathcal{O}_{\widehat{Y}}(d_i f^*L - E_0 - E_1 - \cdots - E_n)\right| \\ &= \left|\mathcal{O}_{\widehat{Y}}(d_i f^*L - E)\right| \text{ for } i < c \\ \widehat{X}_c &\in \left|\mathcal{O}_{\widehat{Y}}(d_c f^*L - 2E)\right| \end{aligned} \tag{4.2.4}$$

and $\widehat{X} = \widehat{X}_1 \cap \widehat{X}_2 \cap \ldots \cap \widehat{X}_c$ is smooth, where $L = \mathcal{O}_Y(1)$, E_i are the exceptional divisors of f over p_i for $i = 0, 1, \ldots, n$ and $E = \sum E_i$. Let

$$\widehat{P}_X = \mathbb{P}\left(\mathcal{O}_{\widehat{Y}}(-\widehat{X}_1) \oplus \mathcal{O}_{\widehat{Y}}(-\widehat{X}_2) \oplus \ldots \oplus \mathcal{O}_{\widehat{Y}}(-\widehat{X}_c)\right) \tag{4.2.5}$$

and $\widehat{f} : \widehat{P}_X \dashrightarrow P_X$ be the rational map in

$$\begin{array}{ccc} \widehat{P}_X & \stackrel{\widehat{f}}{\dashrightarrow} & P_X \\ \downarrow{\scriptstyle \widehat{\pi}} & & \downarrow{\scriptstyle \pi} \\ \widehat{X} & \xrightarrow{f} & X \end{array} \tag{4.2.6}$$

Let $\widehat{V}$ be the proper transform of V under $\widehat{f}$. Note that $\Psi_{\widehat{X}} = U_{\widehat{X}} = \emptyset$ since $\widehat{X}$ is smooth. Thus,

$$\widehat{V} \cap \widehat{R} \subset U_{\widehat{X}} = \emptyset \text{ for } \widehat{R} = \widehat{G}_0 \cap \widehat{G}_1 \cap \ldots \widehat{G}_{l-1}, \tag{4.2.7}$$

where $\widehat{G}_j \in \left|\widehat{\pi}^* \mathcal{O}_{\widehat{Y}}(-f^*L + E) \otimes \mathcal{O}_{\widehat{P}_X}(1)\right|$ are the proper transforms of G_j for $j = 0, 1, \ldots, n$.

Let W be an irreducible component of $\overline{\Psi}_{X\cap\Lambda,r}$ and $\widehat{W} \subset \widehat{Y}$ be its proper transform under f. By (4.2.7), we necessarily have

$$\widehat{V} \cap \widehat{\pi}^{-1}(\widehat{W}) \cap \widehat{R} = \emptyset. \tag{4.2.8}$$

Note that the divisor $(d_c - 1)f^*L - E$ is ample for $d_c > 3$ and big and numerically effective (nef) for $d_c = 3$; $((d_c - 1)f^*L - E).\Gamma = 0$ for an irreducible curve $\Gamma \subset \widehat{Y}$ only if Γ is the proper transform of one of lines $\overline{p_i p_j}$. Therefore, the divisors $\widehat{G}_j$ are big and nef on $\widehat{P}_X$ and $\widehat{G}_j.\Gamma = 0$ for an irreducible curve $\Gamma \subset \widehat{P}_X$ only if $\widehat{\pi}(\Gamma) \subset E$ or $f \circ \widehat{\pi}(\Gamma)$ is one of lines $\overline{p_i p_j}$. Consequently, $\widehat{G}_j$ are big and nef on every subvariety $Z \subset \widehat{P}_X$ satisfying that $f \circ \widehat{\pi}(Z) \neq p_i, \overline{p_i p_j}$ and hence

$$\widehat{G}_0 \widehat{G}_1 \ldots \widehat{G}_{l-1} Z > 0 \tag{4.2.9}$$

for all such subvarieties Z of $\dim Z = l$.

If W is one of p_i, there is nothing to prove. Suppose that $W \neq p_i$ for $i = 0, 1, \ldots, n$. Since X does not contain the lines $\overline{p_i p_j}$, $W \neq \overline{p_i p_j}$ for all $0 \le i < j \le n$. Therefore,

$$l > \dim\left(\widehat{V} \cap \widehat{\pi}^{-1}(\widehat{W})\right) \ge \dim W + r - 1 \tag{4.2.10}$$

by (4.2.8) and (4.2.9). This proves (4.2.1) and (4.2.2) follows. Using the same argument for (4.1.3), we obtain (4.2.3).

When $2\dim\Lambda = n + c$, X and Λ meet properly and

$$\dim \Psi_{X\cap\Lambda} \le n - \dim\Lambda - 1 < \dim(X \cap \Lambda) = \frac{n-c}{2} \tag{4.2.11}$$

by (4.2.2) and (4.2.3). So $X \cap \Lambda$ is reduced.

Suppose that $X \cap \Lambda$ is contained in a union $\Lambda_1 \cup \Lambda_2$ for two proper linear subspaces Λ_i in Λ for $i = 1, 2$. With loss of generality, we may assume that $\dim\Lambda_i = \dim\Lambda - 1$ for $i = 1, 2$. Since $X \cap \Lambda$ is reduced, $\Lambda_1 \cup \Lambda_2$ contains $X \cap \Lambda$ as a scheme. That is,

$$\Lambda_1 \cup \Lambda_2 \in |I_{X\cap\Lambda}(2)| \tag{4.2.12}$$

where $I_{X\cap\Lambda}$ is the ideal sheaf of $X \cap \Lambda$ in Λ. Since $X \cap \Lambda$ is the complete intersection of X_i in Λ, we see that

$$\Lambda_1 \cup \Lambda_2 \in |I_{X_1\cap X_2\cap\ldots\cap X_a\cap\Lambda}(2)|. \tag{4.2.13}$$

So $\Lambda_1 \cup \Lambda_2$ is a member of the linear system spanned by $X_1, X_2, \ldots, X_a$ in Λ. Therefore, $\Lambda_1 \cap \Lambda_2 \cap X_2 \cap \ldots \cap X_a \subset \Psi_{X_1 \cap X_2 \cap \ldots \cap X_a \cap \Lambda}$ and hence

$$\begin{aligned} \dim \Psi_{X_1 \cap X_2 \cap \ldots \cap X_a \cap \Lambda} &\geq \dim(\Lambda_1 \cap \Lambda_2 \cap X_2 \cap \ldots \cap X_a) \\ &= \frac{n+c}{2} - a - 1. \end{aligned} \tag{4.2.14}$$

On the other hand, since $X_1 \cap X_2 \cap \ldots \cap X_a$ is smooth,

$$\dim \Psi_{X_1 \cap X_2 \cap \ldots \cap X_a \cap \Lambda} \leq n - \dim \Lambda - 1 = \frac{n-c}{2} - 1 \tag{4.2.15}$$

by (4.1.2), which contradicts (4.2.14) as $a < c$. □

Of course, the above lemma takes care of (3.2.1) and part of (3.2.3) in Proposition 3.2. Next, let us verify (3.2.2).

Lemma 4.3. *Let $X = X_1 \cap X_2 \cap ... \cap X_c$ be a complete intersection in $\mathbb{P}^n_k$ with the properties in Lemma 4.2. Then*

$$\begin{aligned} \dim \pi_\Lambda((X \backslash \Lambda) \cap \Lambda') \geq \min\Big(& 2\dim \Lambda' - n - c, \\ & \dim \Lambda' - \dim \Lambda - 1\Big) \end{aligned} \tag{4.3.1}$$

for all linear subspaces $\Lambda \subsetneq \Lambda' \subset \mathbb{P}^n$ satisfying $2\dim \Lambda' > n + c$.

Furthermore,

$$\begin{aligned} \dim \pi_\Lambda((X \backslash \Lambda) \cap \Lambda') \geq \min\Big(& \dim \Lambda' - \dim \Lambda - c, \\ & 2\dim \Lambda' + 1 - n - c\Big) \end{aligned} \tag{4.3.2}$$

for all linear subspaces $\Lambda \subsetneq \Lambda' \subset \mathbb{P}^n$ satisfying $2\dim \Lambda' > n + c$ if one of the following is true:

- *$c \geq 3$;*
- *$c = 2$ and $d_1 \geq 3$;*
- *$c = 2$, $d_1 = 2$ and*

$$(X.\Lambda'.A)_p < 2d_2 \tag{4.3.3}$$

  for all $p \in \{p_0, p_1, \ldots, p_n\}$, all hyperplanes $\Lambda' \subset \mathbb{P}^n$ passing through p and a general linear subspace $A \subset \mathbb{P}^n$ with $p \in A$ and $\dim A = 3$, where $(V_1.V_2.\ldots.V_m)_p$ is the intersection multiplicity of $V_1, V_2, \ldots, V_m$ at a point p;
- *$c = 1$ and*

$$C_{\Lambda, X\cap\Lambda'} \not\subset X \tag{4.3.4}$$

  for all linear subspaces $\Lambda \subsetneq \Lambda'$ of $\mathbb{P}^n$ satisfying $\dim \Lambda + \dim \Lambda' = n - 1$ and $\dim \Lambda' - \dim \Lambda > 2$, where $C_{A,B}$ is the chord variety defined in (3.1.19).

Proof. Let Λ^* be a general linear subspace in Λ' satisfying

$$\dim\Lambda + \dim\Lambda^* = \dim\Lambda' - 1 \text{ and } \Lambda^*\cap\Lambda = \emptyset. \tag{4.3.5}$$

We may consider the restriction π_Λ to Λ' as the linear projection $\Lambda' \dashrightarrow \Lambda^*$ through Λ. Let $\Gamma \subset \Lambda^*$ be the closure of the reduced image $\pi_\Lambda((X\backslash\Lambda)\cap\Lambda')$ of $(X\backslash\Lambda)\cap\Lambda'$ under π_Λ. Then

$$X\cap\Lambda' \subset C_{\Gamma,\Lambda}. \tag{4.3.6}$$

By (4.2.2) and (4.2.3), $X\cap\Lambda'$ is reduced of expected dimension $\dim\Lambda' - c$ and a general fiber of $\pi_\Lambda : (X\backslash\Lambda)\cap\Lambda' \to \Gamma$ is $(X\backslash\Lambda)\cap C_{q,\Lambda}$ with dimension

$$\dim\left((X\backslash\Lambda)\cap C_{q,\Lambda}\right) = \dim(X\cap\Lambda') - \dim\Gamma = \dim\Lambda' - c - \dim\Gamma \tag{4.3.7}$$

for $q\in\Gamma$ general.

If $\Gamma = \Lambda^*$, then (4.3.1) follows. Otherwise, we let $L\subset\Lambda^*$ be a hyperplane in Λ^* tangent to Γ at a general point $q\in\Gamma$. Then $\Sigma = C_{L,\Lambda}$ is tangent to $C_{\Gamma,\Lambda}$ along $C_{q,\Lambda}$, where Σ is the linear subspace spanned by L and Λ and hence of $\dim\Sigma = \dim\Lambda' - 1$. Therefore, Σ and X fail to meet transversely along $(X\backslash\Lambda)\cap C_{q,\Lambda}$, i.e.,

$$\Psi_{X\cap\Sigma} \supset (X\backslash\Lambda)\cap C_{q,\Lambda} \tag{4.3.8}$$

and consequently,

$$\dim\Psi_{X\cap\Sigma} \ge \dim((X\backslash\Lambda)\cap C_{q,\Lambda}) = \dim\Lambda' - c - \dim\Gamma \tag{4.3.9}$$

by (4.3.7). Combining with (4.2.2), we obtain

$$\dim\Lambda' - c - \dim\Gamma \le \dim\Psi_{X\cap\Sigma} \le n - \dim\Sigma - 1 = n - \dim\Lambda' \tag{4.3.10}$$

and (4.3.1) follows.

To prove (4.3.2), we observe that

$$\begin{aligned}\dim\pi_\Lambda((X\backslash\Lambda)\cap\Lambda') &= \dim(X\cap\Lambda') - \dim((X\backslash\Lambda)\cap C_{q,\Lambda})\\ &\ge \dim\Lambda' - c - \dim C_{q,\Lambda}\\ &= \dim\Lambda' - \dim\Lambda - c - 1\end{aligned} \tag{4.3.11}$$

for a general point $q\in X\cap\Lambda'$. So it suffices to prove (4.3.2) when

$$\dim\Lambda' - \dim\Lambda - c \le 2\dim\Lambda' + 1 - n - c \tag{4.3.12}$$

i.e., $\dim\Lambda + \dim\Lambda' \ge n-1$. Moreover, when $\dim\Lambda + \dim\Lambda' \ge n$,

$$\begin{aligned}\dim\pi_\Lambda((X\backslash\Lambda)\cap\Lambda') \ge \min\Big(&2\dim\Lambda' - n - c,\\ &\dim\Lambda' - \dim\Lambda - 1\Big)\\ \ge \dim\Lambda' - &\dim\Lambda - c\end{aligned} \tag{4.3.13}$$

by (4.3.1). Thus, we have reduced (4.3.2) to

$$\begin{gathered}\dim\pi_\Lambda((X\backslash\Lambda)\cap\Lambda') \ge \dim\Lambda' - \dim\Lambda - c\\ \text{for } \Lambda\subsetneq\Lambda',\ \dim\Lambda + \dim\Lambda' = n-1 \text{ and } \dim\Lambda' > \frac{n+c}{2}.\end{gathered} \tag{4.3.14}$$

Suppose that (4.3.14) fails. Then the equality in (4.3.11) holds, which happens if and only if $C_{q,\Lambda} \subset X$ for $q \in X \cap \Lambda'$ general. That is, $C_{q,\Lambda} \subset X$ for all $q \in X \cap \Lambda'$ and hence

$$C_{X\cap\Lambda',\Lambda} \subset X \text{ and } X \cap \Lambda' = C_{\Lambda,\Gamma} \tag{4.3.15}$$

for $\Gamma = X \cap \Lambda^*$. This settles the case $c = 1$ by (4.3.4). Suppose that $c \geq 2$.

Since $X \cap \Lambda'$ is reduced, $X \cap \Lambda' = C_{\Lambda,\Gamma}$ as schemes. Therefore, $X \cap \Lambda'$ is a complete intersection

$$X \cap \Lambda' = \bigcap_{i=1}^{c} (X_i \cap \Lambda') = \bigcap_{i=1}^{c} C_{\Lambda,X_i^*} \tag{4.3.16}$$

of c cones over $X_i^* = X_i \cap \Lambda^*$ with vertices Λ in Λ'. It is then easy to see

$$(X.\Lambda'.A)_p = d_1 d_2 \ldots d_c \tag{4.3.17}$$

for all $p \in \Lambda$ and a general linear subspace $A \subset \mathbb{P}^n$ with $p \in A$ and $\dim A = n + c - \dim \Lambda'$.

It is also easy to see from (4.3.16) that

$$\Lambda \subset \Psi_{X\cap\Lambda',c} \tag{4.3.18}$$

and hence either $\Lambda \subset \{p_0, p_1, \ldots, p_n\}$ or

$$\dim \Lambda \leq \dim \Psi_{X\cap\Lambda',c} \leq n - \dim \Lambda' - c \tag{4.3.19}$$

by (4.2.1), which contradicts the assumption $\dim \Lambda + \dim \Lambda' = n - 1$.

So $\Lambda = p \in \{p_0, p_1, \ldots, p_n\}$ and $\dim \Lambda' = n - 1$. If Λ' meets one of X_i transversely at p, then one of $X_i \cap \Lambda'$ is smooth at p, while all C_{p,X_i^*} are singular at p; it is impossible by (4.3.16). So Λ' has to be tangent to all of $X_1, X_2, \ldots, X_{c-1}$ at p. And since $X_1, X_2, \ldots, X_{c-1}$ meet transversely at p, this is only possible when $c = 2$. Since X has a node at p, $X_2 \cap \Lambda'$ has a node at p. Comparing the two sides of (4.3.16), we see that one of C_{p,X_i^*} must have a double point at p. Namely, $d_1 = \deg X_1 = 2$. Then we have (4.3.3), which contradicts (4.3.17). This proves (4.3.2). □

Clearly, (4.3.1) and (4.3.2) imply (3.2.2). Of course, we still need to take care of the hypotheses on X in Lemma 4.2 and 4.3 and also (3.2.3). This is achieved by taking X generic.

Lemma 4.4. *Let $\{p_0, p_1, \ldots, p_n\}$ be a linearly non-degenerate set of $n + 1$ points in $\mathbb{P}^n_k$ and $c, d_1, d_2, ..., d_c$ be integers satisfying $n \geq c + 2 \geq 3$, $d_c \geq 3$ and $d_c \geq d_{c-1} \geq ... \geq d_1 \geq 2$. Suppose that $X_1, X_2, \ldots, X_c$ are general members of*

$$\begin{aligned} &X_i \in |\mathcal{O}(d_i) \otimes \mathcal{O}(-p_0 - p_1 - \cdots - p_n)| \text{ for } i = 1, 2, \ldots, c-1 \\ &X_c \in |\mathcal{O}(d_c) \otimes \mathcal{O}(-2p_0 - 2p_1 - \cdots - 2p_n)| \end{aligned} \tag{4.4.1}$$

in $\mathbb{P}^n$ and $X = X_1 \cap X_2 \cap \ldots \cap X_c$. Then

- *$X_1, X_2, \ldots, X_i$ meet transversely everywhere for all $i < c$.*
- *X is smooth of $\dim X = n - c$ outside of the nodes at $p_0, p_1, \ldots, p_n$.*
- *The lines $\overline{p_i p_j} \not\subset X$ for all $0 \leq i < j \leq n$ if $d_c \geq 4$ or $c \geq 2$.*

- *X satisfies* (4.3.3) *when $c = 2$ and $d_1 = 2$.*
- *X satisfies* (4.3.4) *when $c = 1$ and $(n, d_1) \neq (4, 3), (5, 3)$.*
- *X satisfies* (3.2.3) *except*
  - *$n - c = 2$, or*
  - *$3 \leq n - c \leq 4$ and $2 \sum d_i \leq 4n - c - 3$, or*
  - *$(n, d_1, d_2, \ldots, d_c) = (6, 3)$, $(7, 3)$, $(8, 3)$, $(9, 3)$, $(7, 4)$, $(7, 2, 3)$, $(8, 2, 3)$, $(8, 3, 3)$, $(9, 2, 2, 3)$, $(10, 2, 3)$, $(11, 3)$.*

Proof. It follows from Bertini's theorem on hyperplane sections, which holds in all characteristics, that $X_1, X_2, \ldots, X_i$ meet transversely everywhere for $i = 1, 2, \ldots, c - 1$.

Without loss of generality, we let

$$\begin{aligned} p_0 &= (1, 0, 0, ..., 0) \\ p_1 &= (0, 1, 0, ..., 0) \\ \vdots &= \qquad \vdots \\ p_n &= (0, 0, 0, ..., 1) \end{aligned} \tag{4.4.2}$$

under the homogeneous coordinates $(z_0, z_1, ..., z_n)$ of $\mathbb{P}^n$. Then

$$\begin{aligned} H^0(\mathcal{O}(d) \otimes \mathcal{O}(-\sum_{i=0}^n mp_i)) = \text{Span}\Bigg\{ \prod_{i=0}^n z_i^{m_i} : \sum_{i=0}^n m_i = d, \\ 0 \leq m_i \leq d - m \Bigg\}. \end{aligned} \tag{4.4.3}$$

It is easy to see from (4.4.3) that $\overline{p_i p_j} \not\subset X_c$ for X_c general and $d_c \geq 4$ and $\overline{p_i p_j} \not\subset X_1$ for X_1 general and $c \geq 2$. So $\overline{p_i p_j} \not\subset X$ for all $0 \leq i < j \leq n$ if $d_c \geq 4$ or $c \geq 2$.

When $n \geq 3$, $d \geq 3$ and $m = 2$, the linear series (4.4.3) gives a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^N$, which is an isomorphism onto its image outside of the union of 2-planes Λ_{pqr} spanned by three distinct points p, q, r in $\{p_0, p_1, \ldots, p_n\}$. So by Bertini's theorem, X_c is smooth outside of the union of Λ_{pqr}. On the other hand, it is easy to check that a general member of (4.4.3) is smooth on Λ_{pqr} outside of p, q, r. Therefore, X_c is smooth outside of $p_0, p_1, \ldots, p_n$.

When $n \geq 2$, $d \geq 2$ and $m = 1$, the linear series (4.4.3) gives a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^N$, which is an isomorphism onto its image outside of the union of lines $\overline{p_i p_j}$ for $0 \leq i < j \leq n$. So by Bertini's theorem, X is smooth outside of $\overline{p_i p_j}$. More precisely, when $d \geq 3$ and $m = 1$, the linear series (4.4.3) gives a rational map that is an isomorphism onto its image outside of $p_0, p_1, \ldots, p_n$. When $d \geq 2$ and $m = 1$, a general member of (4.4.3) meets each $\overline{p_i p_j}$ only at p_i and p_j. Therefore, X is smooth outside of $p_0, p_1, \ldots, p_n$. And it is easy to see from (4.4.3) that X has nodes at $p_0, p_1, \ldots, p_n$.

The existence of such X also shows that the locus of such complete intersections with $n + 1$ moving nodes in general position has codimension $n + 1$ in the total space of complete intersections of type $(d_1, d_2, \ldots, d_c)$.

When $c = 2$ and $d_1 = 2$, (4.3.3) follows easily from (4.4.3).

To see (4.3.4), we do some simple dimension counting. We construct the correspondence

$$\begin{aligned}\Sigma = \Big\{(X, \Lambda', \Lambda) : \Lambda \subsetneq \Lambda' \text{ and } C_{\Lambda, X\cap\Lambda'} \subset X\Big\} \\ \subset |\mathcal{O}(d)| \times \mathbb{G}(n-l-1, n) \times \mathbb{G}(l, n)\end{aligned} \tag{4.4.4}$$

where $d = d_c$, $n \ge 2l + 4$ and $\mathbb{G}(a, n)$ is the Grassmannian of a-dimensional linear subspaces of $\mathbb{P}^n$. Its dimension $\dim \Sigma$ can be easily computed by projecting it to $\mathbb{G}(n-l-1, n) \times \mathbb{G}(l, n)$:

$$\begin{aligned}\dim \Sigma &= (2n - 3l - 1)(l + 1) + \dim |\mathcal{O}(d)| \\ &\quad - \left(\binom{n+d-l-1}{d} - \binom{n+d-2l-2}{d}\right) \\ &< \dim |\mathcal{O}(d)| - (n+1)\end{aligned} \tag{4.4.5}$$

for $d \ge 4$ and $n \ge 2l + 4$ or $d = 3$ and $n \ge \max(6, 2l + 4)$. So the projection of Σ to $|\mathcal{O}(d)|$ has codimension bigger than $n + 1$, while the subvariety parameterizing hypersurfaces in $|\mathcal{O}(d)|$ with $n + 1$ nodes has codimension $n + 1$. So $X = X_c$ satisfies (4.3.4).

Similarly, to see (3.2.3), we construct the correspondence

$$\begin{aligned}\Sigma = \Big\{(X_1, X_2, ..., X_c, \Lambda) : \Lambda \subset X = X_1 \cap X_2 \cap \ldots \cap X_c \\ X \text{ and } \Lambda \text{ fail (3.1.29)}\Big\} \\ \subset \prod_{i=1}^{c} |\mathcal{O}(d_i)| \times \mathbb{G}(l, n).\end{aligned} \tag{4.4.6}$$

Again, $\dim \Sigma$ can be computed by projecting Σ to $\mathbb{G}(l, n)$. If the codimension of the projection of Σ to $\prod |\mathcal{O}(d_i)|$ is larger than $n + 1$ for $2l \ge n - c - 2$, then X satisfies (3.2.3).

Note that (3.1.29) fails if

$$\xi_{X,\Lambda} \circ \alpha = 0 \text{ for some } \alpha \in \operatorname{Hom}(\mathcal{O}_\Lambda(1), N_\Lambda) \text{ and } \alpha \ne 0. \tag{4.4.7}$$

If $\Lambda = \{z_0 = z_1 = \ldots = z_{n-l-1} = 0\}$, then it translates to

$$\begin{aligned}\left[\frac{\partial F_i}{\partial z_0}\right]_{1\le i\le c}, \left[\frac{\partial F_i}{\partial z_1}\right]_{1\le i\le c}, \ldots, \left[\frac{\partial F_i}{\partial z_{n-l-1}}\right]_{1\le i\le c} \\ \text{are linearly dependent in } \bigoplus_{i=1}^{c} H^0(\mathcal{O}_\Lambda(d_i - 1)),\end{aligned} \tag{4.4.8}$$

where $X_i = \{F_i(z_0, z_1, \ldots, z_n) = 0\}$ for $i = 0, 1, \ldots, n$. Therefore,

$$\begin{aligned} \dim \prod_{i=1}^{c} |\mathcal{O}(d_i)| - \dim \Sigma = \sum_{i=1}^{c} \binom{d_i + l}{d_i} - (n-l)(l+1) \\ + \max\left(0, \sum_{i=1}^{c} \binom{d_i + l - 1}{l} + l + 1 - n\right). \end{aligned} \tag{4.4.9}$$

In the range $n - c \geq 2l \geq n - c - 2$, the right hand side of (4.4.9) is greater than $n + 1$ except

- $l = 0$ and $n - c = 2$, or
- $l = 1$, $2 \leq n - c \leq 4$ and $2 \sum d_i \leq 4n - c - 3$, or
- $l = 2$ and $(n, d_1, d_2, \ldots, d_c) = (5, 3)$, $(6, 3)$, $(7, 3)$, $(7, 4)$, $(7, 2, 3)$, $(8, 2, 3)$, $(8, 3, 3)$, $(9, 2, 2, 3)$, or
- $l = 3$ and $(n, d_1, d_2, \ldots, d_c) = (8, 3), (9, 3), (10, 2, 3)$, or
- $l = 4$ and $(n, d_1, d_2, \ldots, d_c) = (11, 3)$.

This shows that X satisfies (3.2.3) with the above exceptions. □

4.3. **Completion of the proof of Theorem 1.3(3).** We also need the following lemma in linear algebra.

Lemma 4.5. *Let $V \cong k^{n+1}$ be a vector space of dimension $n + 1$ over an algebraically closed field k and let $(k^*)^n$ act on V by*

$$(a_1, a_2, ..., a_n) \sum_{i=0}^{n} b_i \mathbf{e}_i = \sum_{i=0}^{n} a_i b_i \mathbf{e}_i \tag{4.5.1}$$

for a fixed basis $\{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_n\}$ of V, where $a_0 = 1$. Let $G(m, V)$ be the Grassmannian of m-dimensional subspaces of V. Then the induced action of $(k^*)^n$ on $G(m, V)$ has generically trivial stabilizers for all m. That is, for a general $\Lambda \in G(m, V)$,

$$\operatorname{stab}(\Lambda) = \{\xi \in (k^*)^n : \xi(\Lambda) = \Lambda\} = \{1\}. \tag{4.5.2}$$

Proof. Let us consider the Plücker embedding $\pi : G(m, V) \to \mathbb{P}(\wedge^m V)$. For a general $\Lambda \in G(m, V)$, it is easy to see that $\lambda_I \neq 0$ for all I in

$$\pi(\Lambda) = \sum_{\substack{I \subset \{0,1,\ldots,n\} \\ |I|=m}} \lambda_I \left(\bigwedge_{i \in I} \mathbf{e}_i \right). \tag{4.5.3}$$

For $\xi = (a_1, a_2, ..., a_n) \in (k^*)^n$, we have

$$\pi(\xi(\Lambda)) = \sum_{\substack{I \subset \{0,1,\ldots,n\} \\ |I|=m}} a_I \lambda_I \left(\bigwedge_{i \in I} \mathbf{e}_i \right) \text{ for } a_I = \prod_{i \in I} a_i. \tag{4.5.4}$$

Therefore, $\xi \in \operatorname{stab}(\Lambda)$ if and only if $a_I = a_J$ for all $I, J \subset \{0, 1, ..., n\}$ with $|I| = |J| = m$, which implies that $a_1 = a_2 = ... = a_n = 1$. □

Now we are ready to prove Theorem 1.3(3).

Proof of Theorem 1.3(3). Let $p_0, p_1, \ldots, p_n$ be $n+1$ general points in $\mathbb{P}^n$ and let $X_1, X_2, ..., X_c$ be general members of (4.4.1).

Since $X_{\text{sing}} = \{p_0, p_1, \ldots, p_n\}$, $\text{Aut}_L(X)$ is a subgroup of $\text{Aut}(\mathbb{P}^n)$ preserving $\{p_0, p_1, \ldots, p_n\}$. Namely, we have

$$\begin{aligned}\text{Aut}_L(X) &\subset \{\xi \in \text{Aut}(\mathbb{P}^n) : \xi(p_i) \in \{p_0, p_1, \ldots, p_n\} \text{ for } 0 \le i \le n\}\\ &= (k^*)^n \ltimes \Sigma_{n+1}\end{aligned} \tag{4.5.5}$$

with an obvious group homomorphism

$$\text{Aut}_L(X) \xrightarrow{\varphi} \Sigma_{n+1} \tag{4.5.6}$$

where Σ_{n+1} is the symmetric group acting on $\{p_0, p_1, \ldots, p_n\}$ and $\varphi(\xi)$ sends p_i to $\xi(p_i)$ for $i = 0, 1, \ldots, n$.

We claim that $\ker(\varphi) = \{1\}$. Clearly, $\ker(\varphi)$ is a subgroup of $(k^*)^n$, which acts on $\mathbb{P}^n$ simply by

$$\xi(z_0, z_1, ..., z_n) = (z_0, a_1 z_1, \ldots, a_n z_n) \text{ for } \xi = (a_1, \ldots, a_n). \tag{4.5.7}$$

This action can be lifted to a dual action on $H^0(\mathcal{O}(1))$ by

$$\xi(z_i) = a_i^{-1} z_i \text{ for } \xi = (a_1, ..., a_n) \text{ and } i = 0, 1, ..., n \tag{4.5.8}$$

where $a_0 = 1$. Thus, we have an induced action of $\ker(\varphi)$ on $H^0(\mathcal{O}(d))$ for all $d \in \mathbb{N}$. In particular, it preserves the subspace

$$V_{n,d,m} = H^0(\mathbb{P}^n, \mathcal{O}(d) \otimes \mathcal{O}(-mp_0 - mp_1 - \ldots - mp_n)) \tag{4.5.9}$$

for all $d, m \in \mathbb{N}$. Furthermore, since $\ker(\varphi) \subset \text{Aut}_L(X)$, it preserves the subspace $H^0(I_X(d))$.

When $d = d_1 = d_2 = ... = d_l < d_{l+1}$, $\ker(\varphi)$ preserves

$$H^0(I_X(d)) = \text{Span}\{F_1, F_2, ..., F_l\} \tag{4.5.10}$$

where $F_i \in H^0(\mathcal{O}(d_i))$ are the defining polynomials of X_i. Therefore,

$$\text{Span}\{\xi(F_1), \xi(F_2), ..., \xi(F_l)\} = \text{Span}\{F_1, F_2, ..., F_l\} \tag{4.5.11}$$

for $F_i \in V_{n,d,1}$ general and all $\xi \in \ker(\varphi)$. Applying Lemma 4.5 to $V = V_{n,d,1}$, we see that $\ker(\varphi)$ acts trivially on $\mathbb{P}V_{n,d,1}$. By (4.4.3) and (4.5.8), we conclude that $\ker(\varphi)$ is trivial.

When $d = d_1 = d_2 = ... = d_c$, the same argument as above shows that

$$\text{Span}\{\xi(F_1), ..., \xi(F_{c-1}), \xi(F_c)\} = \text{Span}\{F_1, ..., F_{c-1}, F_c\} \tag{4.5.12}$$

for all $\xi \in \ker(\varphi)$. Note that $F_1, ..., F_{c-1} \in V_{n,d,1}$ and $F_c \in V_{n,d,2}$.

Let $\pi : H^0(\mathcal{O}(d)) \to V_{n,d,2}$ be the projection given by the decomposition of $H^0(\mathcal{O}(d))$ into monomials in $z_0, z_1, ..., z_n$. Then $\xi \circ \pi = \pi \circ \xi$ for all $\xi \in \ker(\varphi)$. Therefore,

$$\text{Span}\{\xi(G_1), ..., \xi(G_{c-1}), \xi(F_c)\} = \text{Span}\{G_1, ..., G_{c-1}, F_c\} \tag{4.5.13}$$

for all $\xi \in \ker(\varphi)$ and $G_i = \pi(F_i)$. Clearly, for $F_i \in V_{n,d,1}$ general, $G_i = \pi(F_i)$ is general in $V_{n,d,2}$. Again, applying Lemma 4.5 to $V = V_{n,d,2}$, we see that

$\ker(\varphi)$ acts trivially on $\mathbb{P}V_{n,d,2}$. By (4.4.3) and (4.5.8), we again conclude that $\ker(\varphi)$ is trivial.

In conclusion, we have proved that $\ker(\varphi) = \{1\}$ and hence $\mathrm{Aut}_L(X)$ is a subgroup of Σ_{n+1}.

Let $\mathcal{X} \subset \mathbb{P}^n \times B$ be a flat family $\mathcal{X} \subset \mathbb{P}^n \times B$ of complete intersections of type $(d_1, d_2, \ldots, d_c)$ over a smooth quasi-projective curve B with $\mathcal{X}_0 = X$ for $0 \in B$. Here we exclude the hypersurface cases settled by Matsumura and Monsky's Theorem 1.1 and the exceptional cases listed in Theorem 1.3(3). In the remaining cases, either X satisfies the hypotheses of Proposition 3.2 by Lemma 4.2, 4.3 and 4.4 or $K_{\mathcal{X}/B}$ is nef. When $K_{\mathcal{X}/B}$ is nef, a birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}$ over B is an isomorphism in codimension 2. Therefore, every $f \in \mathrm{Aut}_L(\mathcal{X}^*/B^*)$ induces an automorphism on $H^0(\mathcal{O}_{\mathcal{X}}(1))$ and is hence an automorphism of $\mathcal{X}/B$ itself. Therefore, in all remaining cases, we conclude that every $f \in \mathrm{Aut}_L(\mathcal{X}^*/B^*)$ can be extended to $f \in \mathrm{Aut}_L(\mathcal{X}/B)$. Consequently, $\mathrm{Aut}_L(\mathcal{X}_b)$ is a subgroup of Σ_{n+1} for $b \in B$ general. To further show $\mathrm{Aut}_L(\mathcal{X}_b) = \{1\}$, we use the following argument.

Let $\mathcal{X} \subset \mathbb{P}^n \times B$ be a complete family of complete intersections in $\mathbb{P}^n$ of type $(d_1, d_2, ..., d_c)$, let $T_i \subset B$ be the locus of $X = X_1 \cap X_2 \cap ... \cap X_c$ with

$$\begin{aligned} &X_i \in |\mathcal{O}(d_i) \otimes \mathcal{O}(-p_0 - p_1 - \cdots - p_n)| \text{ for } i = 1, 2, \ldots, c-1 \\ &X_c \in |\mathcal{O}(d_c) \otimes \mathcal{O}(-2p_i - \sum_{j \neq i} p_j)| \end{aligned} \tag{4.5.14}$$

for $i = 0, 1, ..., n$ and let $S \subset B$ be the locus of $X = X_1 \cap X_2 \cap ... \cap X_c$ given by (4.4.1). Clearly, $S = T_0 \cap T_1 \cap ... \cap T_n$ and $\mathcal{X}_t$ has only one node at p_i for $t \in T_i$ general by Bertini's theorem as in the proof of Lemma 4.4.

Suppose that there exists $\xi_b \neq 1 \in \mathrm{Aut}_L(\mathcal{X}_b)$ for $b \in B$ general. Then ξ_b can be extended to $\xi \in \mathrm{Aut}_L(\mathcal{X}_U/U)$ over an open set $U \subset B$ after a finite base change such that $U \cap S \neq \emptyset$, where $\mathcal{X}_U = \mathcal{X} \times_B U$.

Let $s \in U \cap S$ be a general point of S. Clearly, $\xi_s \neq 1$. And since $\ker(\varphi) = 1$, we see that ξ_s acts non-trivially on the set $\{p_0, p_1, ..., p_n\}$, i.e., $\xi_s(p_i) \neq p_i$ for some $0 \leq i \leq n$. On the other hand, for a general point $t \in U \cap T_i$, $\mathcal{X}_t$ has only one node at p_i and hence $\xi_t(p_i) = p_i$. This is a contradiction since $s \in U \cap T_i$.

Therefore, $\mathrm{Aut}_L(\mathcal{X}_b) = \{1\}$ for $b \in B$ general. This finishes the proof of Theorem 1.3(3). □

The main missing case for generic triviality of $\mathrm{Aut}_L(X)$ is intersections of quadrics in positive characteristic.

5. Equivariant Deformations

In this section, we use equivariant deformation theory to show that the automorphism group of a complete intersection (except some cases) acts on the cohomology groups faithfully over complex numbers.

Let us first recall equivariant deformation theory from [7, 6] and [19]. We refer to [1] for a more elementary introduction of equivariant deformation theory for smooth varieties. We freely use the notations and results from [19]. Suppose Y is a scheme over a field k with a group action

$$G \times Y \to Y.$$

Let $\mathbb{L}^G_{Y/k}$ be the equivariant cotangent complex with respect to the group action. The complex $\mathbb{L}^G_{Y/k}$ is a complex of G-$\mathcal{O}_Y$-module, moreover, the underlying complex of $\mathbb{L}^G_{Y/k}$ is the ordinary contangent complex $\mathbb{L}_{Y/k}$. There is a forget map F which is the identity on the underlying complex

$$\mathrm{F} : \mathbb{L}^G_{Y/k} \xrightarrow{\cong} \mathbb{L}_{Y/k} .$$

It gives rise to a natural forgetful map

$$\mathrm{F} : \mathbb{E}\mathrm{xt}^q_{G-\mathcal{O}_Y}(\mathbb{L}^G_{Y/k}, \mathcal{O}_Y) \to \mathbb{E}\mathrm{xt}^q_{\mathcal{O}_Y}(\mathbb{L}_{Y/k}, \mathcal{O}_Y).$$

In the following, we assume $k = \mathbb{C}$. Recall that there is a natural forgetful map from the equivariant deformation space Def(Y,G) to the ordinary deformation space Def(Y):

$$\mathrm{Def}(Y, G) \to \mathrm{Def}(Y).$$

Note that there is a spectral sequence as follows

$$E_2^{p,q} = H^p(G, \mathbb{E}\mathrm{xt}^q_Y(A, B)) \Rightarrow \mathbb{E}\mathrm{xt}^{p+q}_G(A, B) \tag{5.0.1}$$

where A, B are two complexes of G-$\mathcal{O}_Y$-modules, cf. [19, Page 1163]. In particular, if G is a finite group, then the spectral sequence (5.0.1) degenerates at the E_2 page for $A = \mathbb{L}^G_Y$ and $B = \mathcal{O}_Y$. Therefore, we have

$$\begin{array}{ccc} \mathbb{E}\mathrm{xt}^q_Y(\mathbb{L}_Y, \mathcal{O}_Y)^G & = & \mathbb{E}\mathrm{xt}^q_G(\mathbb{L}^G_Y, \mathcal{O}_Y) \\ & \searrow & \downarrow \mathrm{F} \\ & & \mathbb{E}\mathrm{xt}^q_Y(\mathbb{L}_Y, \mathcal{O}_Y) \end{array}.$$

The map F is injective. Moreover, it follows from equivariant deformation theory that the map

$$\mathbb{E}\mathrm{xt}^2_G(\mathbb{L}^G_Y, \mathcal{O}_Y) \overset{\mathrm{F}}{\hookrightarrow} \mathbb{E}\mathrm{xt}^2_Y(\mathbb{L}_Y, \mathcal{O}_Y)$$

associates to the obstruction $\mathrm{obs}(Y_n, G)$ the obstruction $\mathrm{obs}(Y_n)$ with respect to the small extension $0 \to \mathbb{C} \to R' \to R \to 0$ of an Artin $\mathbb{C}$-algebra R, where Y_n is a deformation of Y over R with G action. So if Y is unobstructed ($\mathrm{obs}(Y_n) = 0$), then (Y, G) is unobstructed. By the Kodaira vanishing theorem, we have that

$$\mathbb{E}\mathrm{xt}^2(\mathbb{L}_Y, \mathcal{O}_Y) = H^2(Y, T_Y) = H^{l-2}(Y, \Omega^1_Y \otimes K_Y) = 0$$

if Y is a Fano variety over complex numbers $\mathbb{C}$ and of dimension l. If Y is a proper Calabi-Yau algebraic manifold, then Y is unobstructed by [9] and [18]. Therefore, in both cases, the equivariant deformation of (Y, G) is unobstructed, i.e., the deformation space $\mathrm{Def}(Y, G)$ is smooth.

Suppose that the finite group G is generated by an automorphism g of Y such that $H^*(g) = \mathrm{Id}$ on $H^*(Y, \mathbb{Q})$. Then we have the following diagram

$$\begin{array}{ccc} H^1(Y, T_Y)^G & \longrightarrow & \mathrm{Hom}(H^q(Y, \Omega_Y^p), H^{q+1}(Y, \Omega_Y^{p-1}))^G \\ \big\downarrow & & \big\| \\ H^1(Y, T_Y) & \longrightarrow & \mathrm{Hom}(H^q(Y, \Omega_Y^p), H^{q+1}(Y, \Omega_Y^{p-1})) \end{array}$$

where $p + q = \dim(Y)$. By the infinitesimal Torelli theorem [5], the bottom arrow is injective for some p and q if Y is a complete intersection in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$ with $(n, d_1, \ldots, d_c) \neq (3, 3)$ or $(n, 2, 2)$. In particular, we have $H^1(Y, T_Y)^G = H^1(Y, T_Y)$.

Note that the tangent space of $\mathrm{Def}(Y, G)$ is given by

$$\mathbb{E}\mathrm{xt}^1_Y(\mathbb{L}_Y^G, \mathcal{O}_Y) = H^1(Y, T_Y)^G.$$

It follows that

$$\mathrm{Def}(Y, G) = \mathrm{Def}(Y) \tag{5.0.2}$$

if Y is a Fano or Calabi-Yau complete intersection in $\mathbb{P}^n$ of type $(d_1, \ldots, d_c)$ with $(n, d_1, \ldots, d_c) \neq (3, 3)$ or $(n, 2, 2)$.

Proposition 5.1. *Let X be a smooth complete intersection in $\mathbb{CP}^n$ of type $(d_1, d_2, \ldots, d_c)$ for $(d_1, d_2, \ldots, d_c) \neq (2, 2)$. If X is not a cubic surface or an elliptic curve, then the map*

$$\mathrm{Aut}(X) \to \mathrm{Aut}(H^m(X, \mathbb{Q}))$$

is injective where $m = n - c$.

Proof. The proposition is well-known for smooth algebraic curves of genus at least two and K3 surfaces. If X is a surface with nontrivial canonical bundle, or the dimension of X is at least 3, then we have

$$\mathrm{Aut}(X) = \mathrm{Aut}_L(X).$$

Suppose g is a linear automorphism of X acting trivially on $H^m(X, \mathbb{Q})$. If X is a Fano or Calabi-Yau complete intersection, then we know that g is extendable to any nearby deformation Y of X by (5.0.2). It follows from Theorem 1.3 that g is a specialization of the identity maps. Therefore, we conclude that g is the identity.

Note that the induced action of g on the ring

$$S_X = \bigoplus_{n \geq 0} \mathrm{Sym}^d(H^0(X, \omega_X))$$

is trivial. If X is a complete intersection of general type, then it follows that g induces the identity on the scheme $X (\subseteq \mathrm{Proj}(S_X))$, thanks to the fact that X has very ample canonical bundle. □

Remark 5.2. By using deformation theory and the Riemann–Hilbert correspondence, we have an alternative way to show this theorem, see [13, Theorem 1.4 and Corollary 3.2],

6. Automorphism and Cohomology

Lemma 6.1. *Let X be a smooth cubic surface over an algebraically closed field k. If g is an automorphism of X such that $g^* = \mathrm{Id}$ on $H^2_{\text{ét}}(X, \mathbb{Q}_l)$ for a prime $l \neq \mathrm{char}(k)$, then $g = \mathrm{Id}$.*

Proof. It is clear that X is the blow-up of $\mathbb{P}^2$ at general six points $p_1, \ldots, p_6$. Denote by $\mathrm{Bl} : X \to \mathbb{P}^2$ the blow-up. It is clear that $H^2(X)$ has a basis $\{\mathrm{Bl}^*(\mathcal{O}_{\mathbb{P}^2}(1)), l_1, \ldots, l_6\}$ where l_i is the exceptional line corresponding to the point p_i. By the hypothesis, the automorphism g fixes the lines $l_1, \ldots, l_6$. It induces an automorphism $\widehat{g} : \mathbb{P}^2 \to \mathbb{P}^2$ such that $\widehat{g} \circ \mathrm{Bl} = \mathrm{Bl} \circ g$. It follows that $\widehat{g}(p_i) = p_i$. Since the six points p_i are in general position, we conclude $\widehat{g} = \mathrm{Id}$. Therefore, we have $g = \mathrm{Id}$. □

Intersection of quadrics. In the following, we deal with the automorphism groups of intersections of quadrics. We freely use the notations following [15]. We work over an algebraically closed field k and assume the characteristic of k is not equal to 2.

Let X be a smooth complete intersection of type $(2, 2)$ in $\mathbb{P}^n_k$. Suppose the dimension of X is m. By [15, Proposition 2.1], the complete intersection X is projectively equivalent to $Q_1 \cap Q_2$ with

$$Q_1 = V(\sum_{i=0}^{n} X_i^2 = 0),\ Q_2 = V(\sum_{i=0}^{n} \lambda_i X_i^2 = 0) \tag{6.1.1}$$

where $\{\lambda_i\}$ are distinct elements in k. We assume the intersection X of two quadrics is general. By Theorem 1.3, the automorphisms of X are given by $(\pm X_0, \pm X_1, \ldots, \pm X_n)$. In other words, we have

$$\mathrm{Aut}(X) \cong (\mathbb{Z}/2\mathbb{Z})^{n+1}/\{\pm 1\} \cong (\mathbb{Z}/2\mathbb{Z})^n = G.$$

Let us denote the automorphism $(X_0, \ldots, X_{i-1}, -X_i, X_{i+1} \ldots, X_n)$ by σ_i (cf. [15, Page 37]). Suppose $\mathrm{Aut}_{tr}(X)$ is the kernel of

$$\mathrm{Aut}(X) \to \mathrm{Aut}(H^m_{\text{ét}}(X, \mathbb{Q}_l)).$$

We have the following lemma.

Lemma 6.2. *If m is even, then $\mathrm{Aut}_{tr}(X)$ is trivial.*

Proof. Let $g \in \mathrm{Aut}_{tr}(X)$. By [15, Chapter 3, Definition on Page 48], we have a commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\widehat{g}} & A(X) \\ \big\downarrow & & \big\downarrow \\ H^m(X) & \xrightarrow{g^*=\mathrm{Id}} & H^m(X) \end{array}$$

We refer to [15, Chapter 3, Page 48] for the precise definition of $A(X)$. It follows from the remark [15, Chapter 3, Page 49] that the map

$$A(X) \to H^m(X)$$

is injective. In particular, we have $\widehat{g} = \mathrm{Id}$. By [15, Proposition 3.18], we have an injection

$$G \hookrightarrow \mathrm{Aut}(\Sigma) \cong \mathrm{Aut}(A(X), \cdot, \eta) \hookrightarrow \mathrm{Aut}(A(X)).$$

We refer to [15, Chapter 3 Page 41] for the precise definition of Σ. Therefore, we have $g = \mathrm{Id}$. ☐

Lemma 6.3. *If m is odd and at least 3, then $\mathrm{Aut}_{tr}(X)$ is $(\mathbb{Z}/2\mathbb{Z})^{m+1}$. Suppose an automorphism g of X is given by $\sigma_{i_1} \circ \ldots \circ \sigma_{i_s}$. Then $g \in \mathrm{Aut}_{tr}(X)$ if and only if s is even.*

Proof. Suppose $m = 2e + 1$. Let S be the variety parameterizing e-planes in X. By [15, Chapter 4, Page 55], the variety S is nonsingular of dimension $e + 1$. The obvious incidence subvariety $T = \{(s, x)|x \in s\} \subset S \times X$

$$\begin{array}{ccc} & T & \\ {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} \\ S & & X \end{array}$$

induces an isomorphism $H^1(S) \cong H^{2e+1}(X)$ with respect to the action of $g \in \mathrm{Aut}_L(X)$, cf. [15, Theorem 4.14]. Namely, the map g induces $\widehat{g} : S \to S$ and a commutative diagram

$$\begin{array}{ccc} H^1(S) & \xrightarrow{\cong} & H^{2e+1}(X) \\ \downarrow{\scriptstyle \widehat{g}^*} & & \downarrow{\scriptstyle g^*} \\ H^1(S) & \xrightarrow{\cong} & H^{2e+1}(X). \end{array}$$

We claim that if $\widehat{g}$ has a fixed point and $\widehat{g}^* = \mathrm{Id}$, then $g = \mathrm{Id}$ on X. In fact, by [15, Theorem 4.8], we know S is an abelian variety. From the injection

$$\mathrm{End}(S) \hookrightarrow \mathrm{End}(\mathrm{H}^1_{\text{ét}}(S)),$$

it follows that $\widehat{g} = \mathrm{Id}$. On the other hand, a general point $p \in X$ is the intersection of e-planes in X which contain p, cf. Lemma 6.4. In other words, we have $p = \cap P_i$ where P_i goes through e-planes in X containing p. It follows that

$$g(p) = g(\cap_{p \in P_i} P_i) = \cap_{p \in P_i} \widehat{g}(P_i) = \cap_{p \in P_i} P_i = p$$

for a general point $p \in X$. We conclude that $g = \mathrm{Id}$.

We claim that $\sigma_i^* = -\mathrm{Id}$ on $H^m(X) \cong H^1(S)$. In fact, since the automorphism σ_i fixes $\{X_i = 0\} \cap X$ (cf. (6.1.1)) which is a smooth complete intersection of type $(2, 2)$ of dimension $2e$, it is clear that σ_i fixes all the e-planes in $\{X_i = 0\} \cap X$. Let s be a e-plane in $\{X_i = 0\} \cap X$. It follows that $\widehat{\sigma_i}([s]) = [s]$. Let C be the hyperelliptic curve (cf. [15, Page 56]) given by:

$$z^2 = \Pi_{i=0}^n (x - \lambda_i y).$$

For the point $[s] \in S$, by [15, Proposition 4.2], we have a map

$$\Psi : C \to S$$

whose image is the closure of $C_s^{1'} = \{t \in S | \dim(s \cap t) = n-1\}$. Therefore, the automorphism $\widehat{\sigma_i} : S \to S$ maps $C_s^{1'}$ to $C_s^{1'}$. It induces an automorphism

$$g_i : C \to C$$

such that $\widehat{\sigma_i} \circ \Psi = \Psi \circ g_i$.

$$\begin{array}{ccc} C & \xrightarrow{\Psi} & S \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle \widehat{\sigma_i}} \\ C & \xrightarrow{\Psi} & S \end{array}$$

By [15, Proposition 4.2, Corollary 4.5, Lemma 4.7 and Theorem 4.8], we conclude that the morphism Ψ induces an isomorphism

$$\Psi^* : \mathrm{H}^1_{\text{ét}}(C, \mathbb{Q}_l) \to \mathrm{H}^1_{\text{ét}}(S, \mathbb{Q}_l).$$

By the main theorem of [14], there are some λ_i such that

$$\mathrm{Aut}(C) = \{1, \iota\}$$

where ι is the natural involution. In particular, it follows from the previous claim that $(\widehat{\sigma_i})^* \neq \mathrm{Id}$. We conclude $g_i \neq \mathrm{Id}$. Therefore, we have $g_i = \iota$. It follows that $\widehat{\sigma_i}^* = -\mathrm{Id}$ on $H^1(S) = H^m(X)$. So we show the claim holds for one particular intersection $X = X_{\{\lambda_i\}}$ of two quadrics which is determined by $\{\lambda_i\}_{i=0}^n$. In general, we consider the smooth family of complete intersections of two quadrics

$$\begin{array}{ccc} X_{\{\lambda_i\}} & \subseteq & \widehat{X} = \{\sum X_i^2 = 0, \sum \lambda_i X_i^2 = 0\} \\ \downarrow & & \downarrow \\ \{\lambda_i\} & \in & B \end{array}$$

Obviously, the automorphism σ_i acts on $\widehat{X}$ fiberwisely. Therefore, for any smooth complete intersection X of two quadrics, we have $\sigma_i^* = -\mathrm{Id}$ on $H^m(X)$. We prove the claim.

Hence, for an automorphism $g = \sigma_{i_1} \circ \ldots \circ \sigma_{i_s}$ of X, we conclude that $g \in \mathrm{Aut}_{tr}(X)$ if and only if s is even. □

Lemma 6.4. *Suppose that X is a smooth complete intersection of type $(2,2)$. If $\dim(X) = m = 2e+1$ is odd, then a general point $p \in X$ is the intersection of two e-planes in X.*

Proof. Suppose $T_{X,p}$ is the projective tangent space to X at p. By a remark in [15, Page 65], the intersection $T_{X,p} \cap X$ is a cone of a nonsingular intersection Y of two quadrics in $T_{X,p}$ if p is a general point of X. It is clear that $\dim(Y) = m-3$ is even. By [15, Theorem 3.8], there are two $(e-1)$-planes s and t in Y such that $s \cap t$ is empty. Note that the cones $\mathrm{Cone}(s)$ and $\mathrm{Cone}(t)$ with vertex p are two e-planes in X containing p. It is clear that the intersection $\mathrm{Cone}(s) \cap \mathrm{Cone}(t)$ is the point p. We show the lemma. □

Proof of Theorem 1.6.

Proof. According to the lemmas in this section and Proposition 5.1, it suffices to show that the theorem holds for the complete intersection X over an algebraically closed field k of positive characteristic when X is not a cubic surface or a complete intersection of type $(2,2)$.

In fact, it follows from [4, Lemma 1.14] that the theorem holds if the dimension of X is one and $(n, d_1, \ldots, d_c) \neq (3,2,2)$ (in this case, X is an elliptic curve). If the dimension of X is two, then we have

$$\mathrm{Aut}(X) = \mathrm{Aut}_L(X)$$

for X being Fano or of general type. If X is a K3 surface, then the theorem follows from [16, Theorem 3.4.2] if we assume $\mathrm{char}(k) \neq 2$. However, we do not need this assumption for a complete intersection K3 surface. In fact, if an automorphism g of X is fixing $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_l)$, then g is fixing the natural polarization $\mathcal{O}_X(1)$, i.e., $g \in \mathrm{Aut}_L(X)$ since the map

$$\mathrm{Pic(X)} \hookrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{Q}_l)$$

is injective. Therefore, we are reduced to proving the action of $\mathrm{Aut}_L(X)$ on the cohomology is faithful.

Suppose that $f : X \to X$ is a linear automorphism. Then it is of finite order, cf. [2]. We can apply the same method as [12, Lemma 2.2] to show that $\mathrm{H}^{\mathrm{m}}_{\text{ét}}(f) = \mathrm{Id}$ if and only if $\mathrm{H}^{\mathrm{m}}_{\text{cris}}(f) = \mathrm{Id}$.

Let $g : \overline{X} \to W(k)$ be a lifting of X over the Witt ring $W(k)$ and π be the uniformizer of $W(k)$. Assume $\mathrm{H}^{\mathrm{m}}_{\text{ét}}(f) = \mathrm{Id}$. If the infinitesmal Torelli Theorem holds for X, i.e., the cup product

$$\Psi_0 : \mathrm{H}^1(X, T_X) \to \bigoplus_{p+q=m} \mathrm{Hom}(\mathrm{H}^q(X, \Omega^p_X), \mathrm{H}^{q+1}(X, \Omega^{p-1}_X)) \tag{6.4.1}$$

is injective, cf. Appendix A, then, by [13, Theorem 1.7 and Corollary 5.4], such an automorphism f can be lifted from X over k to a lifting $\overline{X}$ of X over the Witt ring $W(k) = W$.

For the sake of completeness, we sketch the proof of [13, Theorem 1.7] here. We have smooth morphisms

$$g_n : X_n \to W_n$$

where $g_n = g|_{W_n}$ is the restriction of g to $W_n = W/(\pi^{n+1})$. It follows from the injection (6.4.1) that we have an injective cup product

$$\begin{array}{c} R^1 g_{0*}(f_0^* T_{X_0/k}) \otimes_k (\pi^{n+1}) \\ \Big\downarrow \widehat{\Psi}_n \\ \bigoplus_{p+q=m} \mathrm{Hom}(R^q g_{0*}(\Omega^p_{X_0/k}), R^{q+1} g_{0*}(\Omega^{p-1}_{X_0/k}) \otimes_k (\pi^{n+1})) \end{array}$$

where we use the fact that the Hodge spectral sequence of X_n/W_n degenerates at E_1 and are locally free so that the Hodge and de Rham cohomology sheaves commute with base change ([17, Exp IX]).

On the other hand, the map $f_n : X_n/W_{n+1} \to X_n/W_{n+1}$ induces a map $H^{p+q}_{cris}(f_n)$ as follows

$$\begin{array}{ccc} H^{p+q}_{cris}(X_n/W_{n+1}) & \longrightarrow & H^{p+q}_{cris}(X_n/W_{n+1}) \\ \| & & \| \\ H^{p+q}_{DR}(X_{n+1}/W_{n+1}) & \xrightarrow{H^{p+q}_{cris}(f_n)} & H^{p+q}_{DR}(X_{n+1}/W_{n+1}). \end{array}$$

Therefore, the map $H^{p+q}_{cris}(f_n) \otimes W_n (= \mathrm{Id})$ can be identified with $H^{p+q}_{DR}(f_n)(= \mathrm{Id})$ and hence it preserves the Hodge filtrations. The map $H^{p+q}_{cris}(f_n) \otimes W_n$ induces a diagram

$$\begin{array}{ccc} F^p_{Hodge} H^{p+q}_{DR}(X_{n+1}/W_{n+1}) & \longrightarrow & gr^{p-1}_F H^{p+q}_{DR}(X_{n+1}/W_{n+1}) \bigotimes_{W_{n+1}} (\pi^{n+1}) \\ & \searrow & \| \\ & & H^{q+1}(X_n, \Omega^{p-1}_{X_n/W_n}) \bigotimes_{W_n} (\pi^{n+1}) \end{array}$$

by the fact that (π^{n+1}) is a square zero ideal in W_{n+1}. In particular, we have the map $\rho(f_n)_q$ as follows.

$$\begin{array}{ccc} F^p_{Hodge} H^{p+q}_{DR}(X_n/W_n) & \longrightarrow & H^{q+1}(X_n, \Omega^{p-1}_{X_n/W_n}) \bigotimes_{W_n} (\pi^{n+1}) \\ {\scriptstyle proj} \downarrow & \nearrow {\scriptstyle \rho(f_n)_q} & \\ H^p(X_n, \Omega^q_{X_n/W_n}) & & \end{array}$$

From [13, Proposition 4.1], it follows that

$$\bigoplus_q \rho(f_n)_q = \pm \widehat{\Psi}_n(\mathrm{obs}(f_n))$$

where $\mathrm{obs}(f_n)$ is the obstruction element in

$$H^1(X_n, f_n^* T_{X_n/W_n}) \otimes (\pi^{n+1}) = H^1(X_0, f_0^* T_{X_0/k}) \otimes_k (\pi^{n+1}).$$

Since $H^m_{cris}(f_0) = \mathrm{Id}$ preserves the Hodge filtrations, we conclude that $\rho(f_n)$ is zero. It follows from the injectivity of $\widehat{\Psi}_n$ that $\mathrm{obs}(f_n)$ is zero. Hence, we have a formal automorphism $\varprojlim f_n$ on the formal scheme $\varprojlim X_n$.

By the Grothendieck existence theorem, the formal automorphism comes from an automorphism $\overline{f} : \overline{X}/W \to \overline{X}/W$. In other words, we can lift f over k to $\overline{f}$ over $W(k)$. Let $\overline{f}_K$ be the lifting over the fraction field $K = \mathrm{f.f}(W(k))$ of $W(k)$. By Proposition 5.1, we conclude that $\overline{f}_K$ is the identity. Consequently, the specialization f of $\overline{f}_K$ is also the identity. We prove the theorem. □

Appendix A. The infinitesimal Torelli theorem in arbitrary field

We give an exposition on infinitesimal Torelli theorem of complete intersections, a precise statement is given in Theorem A.1 and Theorem A.9. We present these notes because we need to apply a positive characteristic version of Theorem A.9 to deduce the faithfulness of automorphism actions on cohomology groups of certain complete intersections.

Flenner shows the infinitesimal Torelli theorem holds for certian complex manifolds [5]. We find that the infinitesimal Torelli theorem is algebraic and one can show it for algebraic varieties over an arbitrary field as long as one takes a bit care with the duality between symmetric powers. The proof we exposed in §A.5 is different from that of Flenner's (we find the spectral sequence argument makes the book-keeping less painful).

A.1. Divided powers. Recall that the rth *divided power* of a locally free sheaf E on a scheme, notation $D_r(E)$, is defined to be the dual

$$D_r(E) = \mathrm{Sym}^r(E^*)^*. \tag{A.1.1}$$

If $e_1, \ldots, e_n$ is a local frame of E, then we write

$$e_1^{(i_1)} \cdots e_n^{(i_n)}$$

to be the element dual to the basis elements

$$(e_1^*)^{i_1} \cdots (e_n^*)^{i_n}.$$

If $u = \sum u_i e_i$, one defines

$$u^{(r)} = \sum_{p_1 + \cdots p_n = r} u_1^{p_1} \cdots u_n^{p_n} e_1^{(p_1)} \cdots e_n^{(p_n)}. \tag{A.1.2}$$

This definition does not depend on the choice of basis. There is a natural pairing between divided power and symmetric power:

$$\mathrm{Sym}^{p+s}(E) \otimes D_s(E^*) \to \mathrm{Sym}^p(E), \tag{A.1.3}$$

dual to the algebra structure on $D_*(E^*)$.

The bialgebra structure on the symmetric algebra $\mathrm{Sym}^*(E)$ defines a bialgebra structure on $D_*(E)$.

Situation A.2. Let X be a smooth, proper scheme pure dimension n over a field k. Assume that there exists a short exact sequence of locally free sheaves

$$0 \to \mathcal{G} \to \mathcal{F} \to \Omega^1_X \to 0.$$

Let p be a positive integer no larger than n. We make the following two hypotheses:

(i) the map

$$\mathrm{H}^0(X, D_{n-p}(\mathcal{G}^*) \otimes \omega_X) \otimes \mathrm{H}^0(X, D_{p-1}(\mathcal{G}^*) \otimes \omega_X) \to \mathrm{H}^0(X, D_{n-1}(\mathcal{G}^*) \otimes \omega_X^2)$$

is surjective;

(ii) for all $0 \le j \le n-2$, we have

$$\mathrm{H}^{j+1}(X, \mathrm{Sym}^j(\mathcal{G}) \otimes \bigwedge^{n-1-j}(\mathcal{F}) \otimes \omega_X^{-1}) = 0.$$

We have a long exact sequence of locally free sheaves

$$0 \to \mathrm{Sym}^{p-1}(\mathcal{G}) \to \cdots \to \mathcal{G} \otimes \bigwedge^{p-2}(\mathcal{F}) \to \bigwedge^{p-1}(\mathcal{F}) \to \Omega_X^{p-1} \to 0. \tag{A.2.1}$$

Dualizing (A.2.1), replacing p by $p+1$, we get

$$(\Omega_X^p)^* \to \bigwedge^p(\mathcal{F}^*) \to \mathcal{G}^* \otimes \bigwedge^{p-1}(\mathcal{F}^*) \to \cdots \to D_p(\mathcal{G}^*) \to 0. \tag{A.2.2}$$

Since the pairing

$$\Omega_X^p \otimes \Omega_X^{n-p} \to \omega_X$$

is a perfect pairing, tensoring ω_X to (A.2.2) and change p by $n-p$ yields

$$0 \to \Omega_X^p \to \bigwedge^{n-p}(\mathcal{F}^*) \otimes \omega_X \to \cdots \to D_{n-p}(\mathcal{G}^*) \otimes \omega_X \to 0. \tag{A.2.3}$$

When $p = 1$, dualize (A.2.3) yields

$$0 \to \omega_X^* \otimes \mathrm{Sym}^{n-1}(\mathcal{G}) \to \cdots \to \omega_X^* \otimes \bigwedge^{n-1} \mathcal{F} \to T_X \to 0. \tag{A.2.4}$$

A.3. Three resolutions. Following Flenner, we introduce three complexes T, A and B of locally free sheaves that resolve the sheaves T_X, Ω_X^p and Ω_X^{p-1} respectively. Let notations be as in Situation A.2. Define

$$T_\bullet = \{\omega_X^* \otimes \mathrm{Sym}^{n-1}(\mathcal{G}) \to \cdots \to \omega_X^* \otimes \bigwedge^{n-1} \mathcal{F}\} \tag{A.3.1}$$

placed at cohomological degree $-(n-1), \dots, -1, 0$;

$$A^\bullet = \{\bigwedge^{n-p}(\mathcal{F}^*) \otimes \omega_X \to \cdots \to D_{n-p}(\mathcal{G}^*) \otimes \omega_X\} \tag{A.3.2}$$

placed at cohomological degree $0, 1, \dots, n-p$; and

$$B_\bullet = \{\mathrm{Sym}^{p-1}(\mathcal{G}) \to \cdots \to \mathcal{G} \otimes \bigwedge^{p-2}(\mathcal{F}) \to \bigwedge^{p-1}(\mathcal{F})\} \tag{A.3.3}$$

placed at cohomological degree $-(p-1), \dots, -1, 0$. Then we have quasi-isomorphisms

$$T \to T_X[0], \quad \Omega_X^p[0] \to A, \quad B \to \Omega_X^{p-q}[0].$$

The contraction

$$T_X \otimes \Omega_X^p \to \Omega_X^{p-1}$$

thus induces a morphism

$$\langle \cdot, \cdot \rangle : T \otimes A \to B, \tag{A.3.4}$$

in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{Coh}}(X)$. The morphism $\langle ., . \rangle$ has an explicit description: By mulitilinear algebra, there are natural contractions

$$\bigwedge^{s+t}(\mathcal{E}) \otimes \bigwedge^s(\mathcal{E}^*) \to \bigwedge^t(\mathcal{E}), \quad \mathrm{Sym}^{s+t}(\mathcal{E}) \otimes D_s(\mathcal{E}^*) \to \mathrm{Sym}^t(\mathcal{E}).$$

Note that the hypothesis A.2(ii) says precisely that

$$\mathrm{H}^{j+1}(X, T_j) = 0, \quad 0 \le j \le n-2.$$

Remark A.4. The map $u_{ij} : T_i \otimes A^j \to B_{i-j}$ is precisely induced by the above contractions of the summands of the source; and u is the totalization of u_{ij} when i and j vary. Although one would put faith on it as "what else could it be", the verification is a bit annoying. Flenner did this in his paper, but since we rarely use this description, let's not reproduce the lengthy proof.

A.5. Spectral sequences. We shall prove the infinitesimal Torelli theorem by an induction argument with the help of certain spectral sequences. There are spectral sequences

$$^{T}E_1^{-i,j} = \mathrm{H}^j(X, T_i), {}^{A}E_1^{i,j} = \mathrm{H}^j(X, A^i), {}^{B}E_1^{-i,j} = \mathrm{H}^j(X, B_i)$$

abutting to $\mathrm{H}^*(X, T_X)$, $\mathrm{H}^*(X, \Omega_X^p)$ and $\mathrm{H}^*(X, \Omega_X^{p-1})$ respectively. The map (A.3.4) thus induces a morphism

$$\langle \cdot, \cdot \rangle_m : {}^{T}E_m^{-i,j} \otimes {}^{A}E_m^{s,t} \to {}^{B}E_m^{s-i,j+t}.$$

on the level of spectral sequences (i.e., compatible wit differentials). This follows from the construction of the spectral sequences (since the naïve filtration, which is used to define these spectral sequences is preserved under the tensor product).

Ultimately we are interested in the map

$$\langle \cdot, \cdot \rangle : \mathrm{H}^1(X, T_X) \otimes \mathrm{H}^{n-p}(X, \Omega_X^p) \to \mathrm{H}^{n-p+1}(X, \Omega_X^{p-1}). \tag{A.5.1}$$

The product structure on the spectral sequences eventually computes the tensor product of assoicated graded vector spaces for the infinitesimal Torelli map.

We wish to prove that (A.5.1) is injective on the first factor, in the sense that if an element $\xi \in \mathrm{H}^1(X, T_X)$ induces the zero linear function, i.e., $\langle \xi, \cdot \rangle = 0$, then ξ must be zero. The relevant terms in ${}^{T}E_m^{-i,j}$ are those with $j - i = 1$.

Hypothesis A.2(ii) says that ${}^{T}E_1^{-i,j} = 0$ for all $j - i = 1$ and $i \neq -(n-1)$. Since $(-(n-1), n)$ is the left highest edge of the spectral sequence, we infer that $\mathrm{H}^1(X, T_X) = {}^{T}E_\infty^{-(n-1),n}$ and is the intersection

$$\bigcap_m \mathrm{Ker}(d_m^{-(n-1),n}) \subset {}^{T}E_1^{-(n-1),n}.$$

We shall denote by V_{n-1-m} the subspace $\mathrm{Ker}(d_m^{-(n-1),n})$ of $V_{n-1} = {}^{T}E_1^{-(n-1),n}$. Then $V_m \subset V_{m-1}$ and $\mathrm{H}^1(X, T_X)$ is V_0. It turns out Flenner's condition (i) of Hypothesis A.2 puts some nondeneneracy on the first stage of the spectral sequences.

Lemma A.6. *Let notations be as above. The pairing*

$$\langle \cdot, \cdot \rangle : V_0 \otimes {}^{A}E_1^{n-p,0} \to {}^{B}E_1^{-(p-1),n}$$

is injective on the first factor.

Proof. By the definition of the spectral sequence, the pairing is

$$\mathrm{H}^n(X, \omega_X^* \otimes \mathrm{Sym}^{n-1}(\mathcal{G})) \otimes \mathrm{H}^0(X, D_{n-p}(\mathcal{G}^*) \otimes \omega_X) \to \mathrm{H}^n(X, \mathrm{Sym}^{p-1}(\mathcal{G})).$$

But saying the injectivity on the first factor amounts to saying, thanks to the Serre duality, that the pairing in Situation A.2 (i) is surjective. We win. □

Lemma A.7. *For all $m > 0$, we have*

$$V_{n-m} \otimes {}^A E_m^{n-p,0} \to {}^B E_m^{-(p-1),n}$$

is injective on the first factor.

Proof. We have the following diagram

$$\begin{array}{ccccc} V_{n-m+1} & \otimes & {}^A E_{m-1}^{n-p,0} & \longrightarrow & {}^B E_{m-1}^{-(p-1),n} \\ \uparrow & & \downarrow\!\!\!\downarrow & & \uparrow \\ V_{n-m} & \otimes & {}^A E_{m}^{n-p,0} & \longrightarrow & {}^B E_{m}^{-(p-1),n} \end{array}$$

The right vertical arrow in injective since the coordinate $(-(p-1), n)$ of the spot forces so (it's the left-most highest spot for ${}^B E$). By the definition of V_{n-m}, an element v of it satisfies the property

$$\langle v, d_{m-1} a \rangle = 0.$$

Since ${}^A E_m^{n-p,0}$ is defined to be the quotient ${}^A E_{m-1}^{n-p,0} / \mathrm{Im}(d_{m-1})$, we see the second horizontal pairing is indeed injective on the first fact, by induction on m. □

Theorem A.1. *Let notations be as in Situation A.2. The pairing*

$$\mathrm{H}^1(X, T_X) \otimes \mathrm{H}^{n-p}(X, \Omega_X^p) \to \mathrm{H}^{n-p+1}(X, \Omega_X^{p-1})$$

is injective on the first factor.

Proof. Let ξ be an element in $E_\infty^{-(n-1),n} = V_0 = \mathrm{H}^1(X, T_X)$ that pairs to zero with the second factor. Then it kills all elements ${}^A E_\infty^{n-p,0}$ (the right lowest edge of the spectral sequence) since this is a subspace of $\mathrm{H}^{n-p}(X, \Omega_X^p)$. By Lemma A.7, ξ is zero. This completes the proof. □

A.8. Applications to complete intersections. Flenner's theorem has several applications. Most notably is the validity of the infinitesimal Torelli for complete intersections in projective spaces. Using Theorem A.1, Flenner, in his original paper, proves the infinitesimal Torelli theorem for complete intersections by verifying the his conditions (i) and (ii) for them. The proof is about linear algebra of polynomials, and only uses the Bott vanishing theorem for the projective space as extra input, hence does not depend on the ambient field. The only possible issue is the duality between symmetric power and divided power, but this is readily resolved by noticing the following fact: If $L_1, \ldots, L_r$ are line bundles on a variety X, then there is an isomorphic of *bialgebra*

$$D_*(L_1 \oplus \cdots \oplus L_r) \cong \bigoplus L_1^{i_1} \otimes \cdots \otimes L_r^{i_r}.$$

This is true, since the dual algebra of D_* is the symmetric algebra, and the symmetric algebra version of this isomorphism is well-known.

Proposition A.9. *Let k be any algebraically closed field. Let X be a complete intersection of type $(d_1, \dots, d_c)$ in a projective space $\mathbb{P}^{n+c}$. Then the conditions* (i), (ii) *of Hypothesis A.2 are verified for the exact sequence*

$$0 \to \bigoplus_{i=1}^{c} \mathcal{O}_X(-d_i) \to \Omega^1_{\mathbb{P}^{n+c}}|_X \to \Omega^1_X \to 0.$$

except the cases listed below. Hence the infinitesimal Torelli theorem holds for all complete intersections except for the ones listed below.

There are a few cases that (i) and (ii) do not verify, these exceptions are:

(1) (2,2) complete intersections,
(2) even dimensional (2,3) complete intersections,
(3) even dimensional (2,2,2) complete intersections,
(4) cubic fourfolds,
(5) quartic K3 surfaces, and
(6) quadrics.

As is well-known, the infinitesimal Torelli fails for quadrics and even dimensional (2,2) complete intersections.

We need the infinitesimal Torelli to apply our theorem to complete intersections in projective space of a field of positive characteristic. Since cubic fourfolds has been dealt with in the paper of the second-named author, for that application we shall still verify the infinitesimal Torelli holds true for (2,2,2) and (2,3) complete intersections of even dimension. Flenner, in his original paper, already proves these cases by other means (modifying his scheme of proofs). But we decide to provide an exposition to these facts for the sake of completeness. Both cases are proven similarly. Flenner treated the case (2,2,2); so we shall illustrate our method by dealing with (2,3) complete intersections in $\mathbb{P}^{2p+2}$.

Recall that the Bott vanishing theorem asserts that the cohomology groups

$$\mathrm{H}^b(\mathbb{P}^r, \Omega^a_{\mathbb{P}^r}(\ell))$$

vanish except in the following cases:

- $b = a$, $\ell = 0$,
- $b = 0$, $\ell > a$, or
- $b = r$, $\ell < a - r$.

For a smooth (2,3) complete intersection X in $\mathbb{P} = \mathbb{P}^{2p+2}$, there might be one extra term in ${}^T E_1$ that is nonzero, namely ${}^T E_1^{-(p-1),p}$. This term is

$$\mathrm{H}^p(X, \Omega^p_{\mathbb{P}}|_X \otimes \mathrm{Sym}^{p-1}(\mathcal{O}_X(-2) \otimes \mathcal{O}_X(-3)) \otimes \mathcal{O}_X(2p-2)).$$

By binomial theorem, this cohomology group can be decomposed into a direct sum

$$\bigoplus_{i=0}^{p-1} \mathrm{H}^p(X, \Omega^p_{\mathbb{P}}(-i)|_X)^{\oplus \binom{p-1}{i}}$$

By the Bott vanishing theorem and the spectral sequence associated to the Koszul complex, one sees that the only nontrivial summand in the decomposition is

$$k \cong \mathrm{H}^p(X, \Omega^p_{\mathbb{P}}|_X).$$

If this 1-dimensional space is killed by some differential of ${}^TE$, then it won't contribute anything to the eventual infinitesimal Torelli map, and we already win. If the contrary holds, and suppose $\xi \neq 0$ kills all the space $\mathrm{H}^p(X, \Omega^p_X)$. Let's derive a contradiction. Then each element $\eta \in \mathrm{H}^p(X, \Omega^p_X) = \mathrm{H}^0(X, \Omega^p_X[p])$ induces a homomorphism of spectral sequences

$$\langle \cdot, \eta \rangle : {}^TE^{a,b}_m \to {}^BE^{a,b+p}_m.$$

At the E_1-stage, by identifying ${}^BE_1^{-(p-1),2p}$ with $\mathrm{H}^{2p}(X, \omega_X)$ and ${}^TE_1^{-(p-1),p}$ with $\mathrm{H}^p(X, \Omega^p_{\mathbb{P}}|_X)$, this map is identified with the cup-product map

$$\cup\, \eta : \mathrm{H}^p(X, \Omega^p_{\mathbb{P}}|_X) \to \mathrm{H}^{2p}(X, \omega_X). \tag{A.9.1}$$

Since ${}^TE_1^{-(p-1),p} = {}^TE_\infty^{-(p-1),p}$, the image of the above map actually falls in ${}^BE_\infty^{-(p-1),2p} \subset \mathrm{H}^{2p}(X, \omega_X)$. The map (A.9.1) is not always zero as the pairing

$$\mathrm{H}^p(X, \Omega^p_{\mathbb{P}}|_X) \otimes \mathrm{H}^p(X, \Omega_X) \to \mathrm{H}^{2p}(X, \omega_X).$$

is not identically zero (which also yields $\mathrm{H}^{2p}(X, \omega_X)$ survives in ${}^BE_\infty$).

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