On some class of sets with multi-sheeted envelope of holomorphy

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Abstract. *Let m be a smooth CR-manifold of dimension $2n-2$ contained in the unit sphere $\partial\mathbb{B}^n$ in $\mathbb{C}^n, n > 2$. We prove that if ∂M is the finite union of boundaries of $(n-1)$-dimensional analytic varieties and M is globally minimal, then the envelope of holomorphy of $\partial\mathbb{B}^n \setminus \bar{m}$ is multi-sheeted. In some cases we prove that the number of sheets is exactly two. It is also proved (for 4-dimensional CR-manifolds contained in $\partial\mathbb{B}^3$) that the property of being globally minimal (even being of finite type at each point) is generic.*

For general domains in $\mathbb{C}^n$ it is difficult to understand from the geometry of the domain whether its envelope of holomorphy is single-sheeted over $\mathbb{C}^n$. It has become a fruitful attempt to consider the mentioned problem for suitably small neighbourhoods (or one-sided neighbourhoods) of subsets of the unit sphere in $\mathbb{C}^n$ (more general, of subsets of strictly pseudoconvex boundaries in $\mathbb{C}^n$).

Bedford and Klingenberg [BK] (and later Kružilin [K]) proved that the envelope of holomorphy of generic two-spheres in strictly pseudoconvex boundaries in $\mathbb{C}^2$ (in the sense of the space of maximal ideals of the algebra of germs of analytic functions near the two-sphere) is a three-ball which is foliated by analytic discs, in particular, the envelope of holomorphy is single-sheeted.

The case of open subsets of strictly pseudoconvex boundaries in $\mathbb{C}^2$ was first considered by Stout [St1]. Let $G \subset \mathbb{C}^n$ be a strictly pseudoconvex domain and let $\omega \subset \partial G$ be an open subset of the boundary with connected components $\omega_1, \omega_2, \ldots$. A connected open subset U_j is called a one-sided neighbourhood of ω_j if U_j is the intersection of G with an open subset of $\mathbb{C}^n$ which intersects ∂G along ω_j. If $U_1, U_2, \ldots$ are disjoint one-sided neighbourhoods of $\omega_1, \omega_2, \ldots$ then $\cup_j U_j$ is called a one-sided neighbourhood of ω. Using local analytic extension from (neighbourhoods of) hypersurfaces near strictly pseudoconvex points, one can choose a one-sided neighbourhood U of ω such that any analytic function in an arbitrary one-sided neighbourhood of ω (and also any continuous CR-function on ω) has analytic extension to U. Call such sets U one-sided neighbourhoods of ω with *a priori* analytic continuation property. The envelope of holomorphy of such a set U (more precisely, the disjoint union of envelopes of holomorphy of its connected components) is called the essential envelope of holomorphy of ω and is denoted by $E(\omega)$. (It is easy to see that $E(\omega)$ does not depend on the choice of the set U with the above mentioned property.)

Lupacciolu-Stout, Alexander-Stout and Slodkowski (see e.g. the survey paper [St2]) proved that for any bounded strictly pseudoconvex domain $G \subset \mathbb{C}^2$ and any open subset ω of the boundary ∂G the essential envelope of holomorphy $E(\omega)$ is single-sheeted, moreover, it coincides with $G \setminus \hat{K}_{A(G)}$, where $K = \partial G \setminus \omega$ and $\hat{K}_{A(G)} = \{z \in \bar{G} : |f(z)| \leq \max_K |f| \text{ for each } f \in A(G)\}$ is the hull of K with respect to the algebra $A(G)$ of holomorphic functions in G which are continuous in the closure $\bar{G}$.

For dimensions larger than 2 the situation is different. Čirka and Stout [ČSt] were the first who constructed a strictly pseudoconvex domain G in $\mathbb{C}^4$ (the union of the unit ball with a tubular neighbourhood of a simple smooth arc which joins two distinct points in the unit sphere) and an open subset ω of ∂G with two-sheeted essential envelope of holomorphy $E(\omega)$.

In this paper we consider the case of "large" open subsets ω of strictly pseudoconvex boundaries, more precisely, of complements of subsets of codimension one submanifolds of such boundaries. Let $G \subset \mathbb{C}^n, n > 2$, be an arbitrary bounded strictly pseudoconvex domain with C^2 boundary and let $M \subset \partial G$ be a connected generic CR-manifold of class C^2 and of dimension $2n-2$ (hence of CR-dimension $n-2$). A subset A of M is called CR-invariant if it is the union of some CR-orbits of M. It is proved in [J2] that a compact subset K of M is removable (i.e. $E(\partial G \setminus K) = G$) if K does not contain non-empty CR-invariant subsets of M. Moreover, for any compact $K \subset M$ the equality $E(\partial G \setminus K) = E(\partial G \setminus \tilde{K})$ holds, where $\tilde{K}$ is the largest compact CR-invariant subset of M contained in K. The following problem was raised in [J2].

Problem. *For which compact CR-invariant subsets K of M the essential envelope of holomorphy $E(\partial G \setminus K)$ is multi-sheeted?*

It is natural to expect that this happens always if K contains a CR-orbit of M of full dimension. Our theorem 1 gives a partial answer to this problem.

Theorem 1. *Let M be an orientable generic CR-manifold of real dimension $2n-2$ and of class C^2 contained in the unit sphere $\partial\mathbb{B}^n$ of $\mathbb{C}^n, n > 2$. Let $K \subset M$ be a compact CR-invariant subset of M such that*

1. *$\partial K = \cup_{j=1}^{\ell} N_j$ is the disjoint union of finitely many connected compact maximally complex CR-manifolds N_j of dimension $2n-3$ and of class C^2.*
2. *K contains a CR-orbit m of M of dimension $2n-2$.*

   Then the envelope of holomorphy $E(\partial\mathbb{B}^n \setminus K)$ is multi-sheeted. Moreover, each point of m is a limit of points over which $E(\partial\mathbb{B}^n \setminus K)$ is multi-sheeted.

Recall that a CR-manifold is called globally minimal if any pair of its points can be joined by a piecewise CR-curve.

Remark 1. *Condition 2 of the theorem is satisfied in particular if the following condition 2′ holds.*

2′. The interior of K (with respect to M), int$K$, is globally minimal and, hence, it is itself a CR-orbit of M.

If the conditions of theorem 1 are satisfied with condition 2 replaced by the stronger condition 2′, then the whole set K is contained in the closure of the set of points over which $E(\partial\mathbb{B}^n \setminus K)$ is multi-sheeted.

Remark 2. *The proof of theorem 1 applies without modification for $\mathbb{B}^n$ replaced by any strictly pseudoconvex domain $G \subset \mathbb{C}^n, n > 2$, with C^2 boundary such that $H^2(G, \mathbb{Z}) = 0$.*

Remark 3. *Under the condition of theorem 1 the N_j are compact CR-orbits of M of real codimension 1. Each N_j is orientable, hence by the theorem of Harvey and Lawson [HaL], N_j bounds an analytic variety $X_j \subset \mathbb{B}^n$ of complex dimension $n-1$. We will use the condition that the N_j are of class C^2. If we require that M is e.g. of class $C^{2,\alpha}$ this is satisfied automatically.*

The relation of the theorem to the posed problem is the following. Suppose M is an orientable generic CR-manifold of real dimension $2n-2$ and of class $C^{2,\alpha}$ contained in a strictly pseudoconvex boundary in $\mathbb{C}^n, n > 2$. Let K be a CR-invariant compact subset of M which contains a CR-orbit m of M of full dimension. Then the boundary ∂m of m is the union of CR-orbits of codimension 1 in M, hence each such orbit either bounds an analytic variety of dimension $n-1$ or it is an exceptional orbit (see [J2] for more details). So the full answer to the problem is, in particular, related to the question whether non-empty compact exceptional minimal CR-invariant subsets of generic $(2n-2)$-dimensional CR-manifolds contained in strictly pseudoconvex boundaries in $\mathbb{C}^n, n > 2$, are always non-removable.

The natural question arises whether it is a generic property of CR-manifolds to be globally minimal. The following proposition gives the positive answer to this question in the situation which is interesting for us here. It states, in particular, that CR-invariant compact subsets of sufficiently smooth 4-dimensional CR-manifolds in strictly pseudoconvex boundaries in $\mathbb{C}^3$ which enjoy property 1 of theorem 1 generically also have property $2'$, moreover, generically for such a CR-manifold each point is of finite type and, hence, minimal in the sense of Tumanov. For simplicity the assertion is stated and proved only for dimension $n = 3$. We believe that the same methods allow to prove the corresponding result for arbitrary dimensions. We also believe that similar arguments but more extended calculation (and a clever induction) allows to prove, for example, that each generic CR-manifold of class C^∞ of arbitrary codimension in $\mathbb{C}^n$, ($n \geq 2$ arbitrary), has arbitrarily small deformations which are globally minimal, moreover, each point of which is of finite type.

Proposition. *Let G be a strictly pseudoconvex domain in $\mathbb{C}^3$ with C^∞-smooth boundary ∂G. Let M be an orientable C^4-smooth generic CR-submanifold of bG of real dimension 4 and of CR-dimension 1. Let K be a compact CR-invariant subset of M. Then there are arbitrarily small C^4-deformations $\tilde{M}$ of M in ∂G which fix the set $M \setminus intK$ (here $intK$ is the set of all interior points of K with respect to M) and such that the deformation $int\tilde{K}$ of the set $intK$ is C^∞-smooth and globally minimal. Moreover, it is of type 4, more precisely, the Lie brackets of order up to 3 of two real vector fields generating $T_p^{\mathbb{C}}(int\tilde{K})$ span the whole tangent space $T_p(int\tilde{K})$ for all $p \in int\tilde{K}$.*

The following two examples illustrate theorem 1 and proposition.

Example 1. Let $M = \{z \in \partial\mathbb{B}^3 : z_1 \in (-\frac{1}{2}, \frac{1}{2})\}$. It is not hard to check that M is a generic CR-manifold of real dimension 4 (here $n = 3$, $2n - 2 = 4$). The boundary $\partial M = \{-\frac{1}{2}\} \times \{|z_2|^2 + |z_3|^2 = 1 - \frac{1}{4}\} \cup \{\frac{1}{2}\} \times \{|z_2|^2 + |z_3|^2 = 1 - \frac{1}{4}\}$ satisfies condition 1 of the theorem. Hence there exist arbitrarily small (in C^4 topology) C^∞ deformations $\tilde{M}$ of M with fixed $\partial\tilde{M} = \partial M$ such that $\partial\mathbb{B}^3 \setminus \bar{\tilde{M}}$ has multi-sheeted essential envelope of holomorphy $E(\partial\mathbb{B}^3 \setminus \bar{\tilde{M}})$.

Example 2. Let $c < \frac{1}{\sqrt{3}^3}$ and close to $\frac{1}{\sqrt{3}^3}$. The set $N = \{z = (z_1, z_2, z_3) \in \partial\mathbb{B}^3 : z_1 z_2 z_3 = c\}$ is a smooth compact connected 3-dimensional maximally complex CR-manifold. Using Gromov's h-principle one can see that N bounds a smooth generic 4-dimensional CR-manifold M^* contained in $\partial\mathbb{B}^3$ (see also [JSh] where this fact is proved in an elementary way). By theorem 1 and proposition, for a smooth deformation $\tilde{M}$ of M^* with $\partial\tilde{M} = \partial M^*$ the envelope of holomorphy $E(\partial\mathbb{B}^3 \setminus \bar{\tilde{M}})$ is multi-sheeted. Note that in this case $\tilde{M}$ has connected boundary.

In fact, in both cases the essential envelope of holomorphy is exactly two-sheeted provided the manifold $\tilde{M}$ is close enough to a manifold which admits a foliation (maybe, with singularities) into maximally complex CR-manifolds of dimension $2n - 3$. We postpone the proof of this fact until the end of the paper.

Proof of theorem 1. 1. Here we recall a result from the theory of CR-manifolds which we will use in the proof. Let $m \subset \mathbb{C}^n$ be the manifold which appeared in condition 2 of theorem 1. Recall that a wedge W with edge m is a set of the form $W = \{x + y \in \mathbb{C}^n : x \in m,\ y \in \mathcal{K}(x)\}$ where $\mathcal{K}(x)$ is an open truncated cone in $\mathbb{C}^n$ with vertex x. The manifold m is a generic globally minimal CR-manifold of class C^2 and codimension one in a strictly pseudoconvex boundary in $\mathbb{C}^n$, hence there exists a wedge W with edge m such that any continuous CR-function on m has analytic continuation to W. Moreover, the following stability assertion holds.

Let m' be a manifold which is close enough to m in C^2 (i.e. m' is obtained by moving m along a C^2 small vector field defined near m). Then clearly m' is again a generic CR-manifold. Moreover, the same analytic continuation property takes place for a wedge W' with edge m' such that W' is close to W (in the sense that the parameters of corresponding truncated cones $\mathcal{K}(x)$ and $\mathcal{K}(x')$ at related points $x \in m, x' \in m'$, are uniformly close for vertices x ranging over compact subsets of m).

For the wedge extendibility result for globally minimal generic CR-manifolds of class $C^{2,\alpha}$ and of arbitrary codimension in $\mathbb{C}^n$ see [M], [J1]. In the case of class C^2 see [J3], where also the stability result is shown (see the proof of proposition 4.1 there). For the smoothness class $C^{2,\alpha}$ the stability result follows directly from known proofs of wedge-extendibility. Indeed, the main tool for proving wedge-extendibility is to attach analytic discs consecutively to the union of the CR-manifold with suitable sets which are covered by already constructed analytic discs. The analytic discs are contained in the set to which functions extend analytically and cover a wedge at final step. The construction of the mentioned analytic discs uses Bishop's equation for suitable parameters. These equations are solvable if the CR-manifold is of class $C^{2,\alpha}$. Moreover, they remain to be solvable after small perturbation and the solutions depend continuously on the perburbation. This implies the stability result.

2. Here we fix notation and introduce coordinates in a spherical layer around $\partial\mathbb{B}^n$ which will be convenient for the proof.

Each connected component N_j of ∂K bounds an $(n-1)$-dimensional analytic variety V_j, moreover, in a neighbourhood of $\partial\mathbb{B}^n$ each V_j is a C^2 manifold with C^2-boundary N_j. (It follows by locally presenting V_j as graph of a CR-function of class C^2 over a piece of a strictly pseudoconvex hypersurface in $\mathbb{C}^{n-1}$ and using analytic extension of the graphing function). By Forstnerič [F], T_pN_j is not contained in the complex tangent space $T_p^c\partial\mathbb{B}^n$ for each $p \in N_j$, i.e., $\bar{V}_j$ is transversal to $\partial\mathbb{B}^n$. Hence there exists a non-singular vector field v defined in a neighbourhood of the union $\cup_{j=1}^{\ell} N_j$ which is for each j tangental to $\bar{V}_j$ at points of $\bar{V}_j$ and is transversal to $r\partial\mathbb{B}^n$ for all r close to 1. Normalizing, we may assume that at each point p the equality $v(p) = n(p) + \tau(p)$ holds, where $n(p)$ is the unit inner normal to $|p|\partial\mathbb{B}^n = \partial(|p|\mathbb{B}^n)$ (the sphere around zero to which p belongs) and $\tau(p)$ is orthogonal to $n(p)$. Extend v to a vector field with the latter properties in a spherical layer $\{z \in \mathbb{C}^n : 1-\varepsilon < |z| < 1+\varepsilon\}$ for some $\varepsilon > 0$ and denote the extended vector field again by v. For each $t \in [0,\varepsilon)$ the flow Φ_t of v maps $\partial\mathbb{B}^n$ diffeomorphically onto $(1-t)\partial\mathbb{B}^n$, moreover, it maps each $N_j \subset \partial\mathbb{B}^n$ diffeomorphically onto $V_j \cap (1-t)\partial\mathbb{B}^n$ and maps

$\partial\mathbb{B}^n \setminus \cup_{j=1}^{\ell} N_j$ diffeomorphically onto $(1-t)\partial\mathbb{B}^n \setminus \cup_{j=1}^{\ell} V_j$. For $t \in (0,\varepsilon)$ denote $M_{1-t} \stackrel{\text{def}}{=} \Phi_t(M), K_{1-t} \stackrel{\text{def}}{=} \Phi_t(K), m_{1-t} \stackrel{\text{def}}{=} \Phi_t(m)$. In particular, $K_{1-t} \subset (1-t)\partial\mathbb{B}^n$, it is diffeomorphic to K and its boundary equals $(1-t)\partial\mathbb{B}^n \cap \cup_{j=1}^{\ell} V_j$. Define $E = \cup_{1-\varepsilon<r<1} M_r, e = \cup_{1-\varepsilon<r<1} \text{int} K_r, \bar{e} = \cup_{1-\varepsilon<r<1} K_r$. (Here $\text{int}K$ denotes the interior of K with respect to M.) Then E and e are orientable C^2 hypersurfaces in $\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n} \subset \mathbb{C}^n$, $\bar{e}$ is a hypersurface with boundary which is closed in $\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n}$. Denote $U' = (\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n}) \setminus \bar{e}$. U' is open and defines a one-sided neighbourhood of $\partial\mathbb{B}^n \setminus K$ (which is diffeomorphic to $(\partial\mathbb{B}^n \setminus K) \times (0,\varepsilon)$). It will be convenient to use the flow Φ_t and the set U' in the sequel.

3. Let U be a one-sided neighbourhood of $\partial\mathbb{B} \setminus K$ such that analytic functions in any one-sided neighbourhood of $\partial\mathbb{B}^n \setminus K$ extend to U. We may assume that $U \subset U'$, in particular, U does not intersect $\cup_{j=1}^{\ell} N_j$. The following lemma holds.

Lemma 1. *There exist manifolds m^+ and m^- contained in U which are arbitrarily close in C^2 to m with the same boundary $\partial m^+ = \partial m^- = \partial m$ and are situated on different sides of E in $\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n}$. There are wedges W^+ and W^- with edges m^+ and m^-, respectively, contained in $\mathbb{B}^n$ which are close to W if m^+ and m^- are close to m such that the following holds. Each analytic function in U has analytic extension from m^+ to W^+ and from m^- to W^-. In other words, W^- and W^+ embed naturally into the essential envelope of holomorphy $E(\partial\mathbb{B}^n \setminus K)$.*

Lemma 1 together with the following lemma imply the theorem.

Lemma 2. *There exists an analytic function $\mathcal{F}$ in $U' \supset U$ such that the following holds. There exist intersecting wedges W^+ and W^- as in lemma 1 with edges m^+ and m^- contained in U such that the analytic extensions of $\mathcal{F}$ to W^+ and to W^-, respectively, differ on a non-empty open subset of the intersection $W^+ \cap W^-$. More precisely, for each point $q \in m$ the wedges may be chosen so that the respective analytic continuations of $\mathcal{F}$ to intersecting open truncated cones $\mathcal{K}^+(q(m^+)) \subset W^+$ and $\mathcal{K}^-(q(m^-)) \subset W^-$ differ a.e. on their intersection. Moreover, the points $q(m^+) \in m^+$ and $q(m^-) \in m^-$ can be taken close to q for W^+ and W^- close to W.*

Proof of Lemma 1. Since M is orientable, there is a non-singular C^2 vector field ν in a neighbourhood of M which is transversal to M and tangential to $\partial\mathbb{B}^n$. Let φ be a small non-negative C^2 function in a neighbourhood of M which vanishes exactly on $M \setminus m$. Consider the flow of the vector fields $\varphi \cdot \nu$ and $-\varphi \cdot \nu$ and apply the time 1 mapping to m (and to $\bar{m}$). We obtain two CR-manifolds contained in $\partial\mathbb{B}^n \setminus \bar{m}$, which are C^2 close to m, have the same boundary as m and are situated locally on different sides of m in $\partial\mathbb{B}^n$. Move each of the two manifods slightly in the direction of the vector field v defined in part 2 of the proof, precisely, apply to each of the manifolds the time 1 mapping of the flow ψv for a small non-negative C^2 function ψ in a neighbourhood of M, which vanishes on $M \setminus m$ and is chosen in such a way that we obtain two manifolds m^- and m^+ contained in U. If φ and ψ are small, then m^- and m^+ are close to m. Moreover, $\partial m^- = \partial m^+ = \partial m$. The existence of the required wedges W^+, W^- follows from part 1 of the proof. By pseudoconvexity of $\mathbb{B}^n$ both are contained in $\mathbb{B}^n$. $\square$

Proof of Lemma 2. Observe first that for each j there exists an analytic function f_j in $\mathbb{B}^n$ with simple zero on V_j and no other zeros. (f_j is a minimal defining function in the terminology of [Č1].) It is characterized by the property that (locally) it divides any other analytic function which vanishes (locally) on V_j. An equivalent characterization is that the gradient of the function does not vanish at regular points of the variety (see [Č1]). Locally such a function exists since V_j is an irreducible analytic variety of pure dimension $n-1$. The global existence of the f_j follows from solvability of the second Cousin problem in $\mathbb{B}^n$.

Put $f = \prod_{j=1}^{\ell} f_j$. Let $p \in U'$ and let $\mathcal{F}_p$ be a holomorphic branch of $\sqrt{f}$ defined on a small ball around p (in other words, $\mathcal{F}_p$ is a functional element at p which defines a branch of $\sqrt{f}$). It is clear that $\mathcal{F}_p$ has unrestricted analytic continuation in $\mathbb{B}^n \setminus \cup_{j=1}^{\ell} V_j$ (i.e., it has analytic continuation along any curve there). We will show that $\mathcal{F}_p$ extends to an analytic function $\mathcal{F}$ in U' which satisfies the requirements of lemma 2.

Prove first single-valued analytic continuation of $\mathcal{F}_p$ to U'. Put $r = |p| \in (1-\varepsilon, 1)$. By the product structure of U' it is enough to prove that $\mathcal{F}_p$ extends to an analytic function $\mathcal{F}^r$ in a neighbourhood of $r\partial\mathbb{B}^n \setminus K_r$. Indeed, suppose this has been done. Consider an arbitrary point $z \in U'$. Then $z = \Phi_{1-|z|}(\zeta)$ for some $\zeta \in \partial\mathbb{B}^n \setminus K$. Consider the flow line $\Phi_t(\zeta), t \in (0,\varepsilon)$. It intersects $|z|\partial\mathbb{B}^n$ at z and $r\partial\mathbb{B}^n$ at some point η. Provide analytic continuation of $\mathcal{F}^r$ from a small ball around η along the flow line $t \to \Phi_t(\zeta)$. (The flow line does not meet $\cup_{j=1}^{\ell} V_j$, so this analytic continuation is possible.) We obtain an analytic function in a small ball around z (a functional element at z). By the properties of the flow Φ_t, the functional elements at points of U' obtained in the above described way constitute an analytic function in U' which coincides with $\mathcal{F}^r$ near $r\partial\mathbb{B}^n \setminus K_r$.

By a slight abuse of languge in the following we will call two functional elements at a point p equal, if they are equivalent, i.e. if they define the same function in a small enough ball around p. The existence of the analytic function $\mathcal{F}^r$ follows now directly from the following lemma.

Lemma 3. *Take an arbitrary point $z \in r\partial\mathbb{B}^n \setminus K_r$ and an arbitrary smooth simple closed curve $\gamma \subset r\partial\mathbb{B}^n \setminus K_r$ which starts at z. Let $\mathcal{F}_z$ be a functional element at z which defines some branch of $\sqrt{f}$ in some ball around z. Then analytic continuation of $\mathcal{F}_z$ along γ leads to the same functional element at z.*

Proof. The curve γ bounds a smoothly embedded disc $\Delta \subset r\partial\mathbb{B}^n$. Indeed, removing from $r\partial\mathbb{B}^n$ a point which is not contained in γ and choosing coordinates on the punctured sphere we may assume that γ is obtained by a smooth embedding of the unit circle $\mathbb{T} = \partial\mathbb{D} \subset \mathbb{C}$ into $\mathbb{R}^{2n-1}$. Extend this mapping to a smooth mapping of the closed disc $\bar{\mathbb{D}}$ into $\mathbb{R}^{2n-1}$ whose restriction to a small annulus attached to the circle $\mathbb{T}$ is an embedding. By Wintney's embedding theorem (see e.g. [Hi] Ch. 2) this mapping can be approximated by an embedding of the disc $\mathbb{D}$. (Here we use that $2n-1$ exceeds twice the real dimension of $\mathbb{D}$, since $n > 2$.) Moreover, (by a partition of the unit on $\mathbb{D}$) we may assume that this embedding coincides with the previous mapping near $\mathbb{T}$. This proves the existence of Δ. Endow Δ with the orientation induced from $\mathbb{D}$ by the embedding.

Let $\tilde{M}_r$ be a small neighbourhood of K_r on M_r which still does not meet $\gamma = \partial\Delta$. (Recall $\gamma \subset r\partial\mathbb{B}^n \setminus K_r$.) By the transversality theorem [Hi, Th. 3.2.1], there is a small smooth perturbation of Δ inside $r\partial\mathbb{B}^n$ which fixes $\partial\Delta = \gamma$ and is transversal to $\tilde{M}_r$ and to each of the $V_j \cap r\partial\mathbb{B}^n$. Denote this perturbation again by Δ. Then $\Delta \cap \cup_{j=1}^{\ell} V_j$ consists of isolated points and $\Delta \cap \tilde{M}_r$ consists of curves. Hence the compact set $\Delta \cap K_r$ is the finite union of disjoint compact (connected) curves, $\Delta \cap K_r = \cup_k s_k \cup \cup_{k'} c_{k'}$, where each $c_{k'}$ is a loop, i.e. a simple closed C^2 curve, and s_k is a closed segment, i.e. a simple C^2 curve with two boundary points belonging to $\cup_{j=1}^{\ell} V_j$. Note that Δ intersects the union of the V_j only at the endpoints of the segments s_k. In particular, the loops $c_{k'}$ do not meet $\cup_{j=1}^{\ell} V_j$. Note that $\mathcal{F}_z$ has analytic continuation along any curve in $\bar{\Delta} \setminus \cup_{j=1}^{\ell} V_j$, in particular, by the remark just made, $\mathcal{F}_z$ has analytic continuation along any curve in $\bar{\Delta} \setminus \cup_k s_k$.

To prove the lemma, it is enough to prove that $\mathcal{F}_z$ extends to a single-valued analytic function in a neighbourhood of $\bar{\Delta} \setminus \cup_k s_k$. For this it is enough, as in the classical situation of the monodromy theorem, to prove the following assertion. For each k let γ_k be a simple closed Jordan curve in Δ which surrounds exactly one of segments s_k (i.e. the corresponding s_k are contained in the connected component of $\bar{\Delta} \setminus \gamma_k$ which does not contain $\partial\Delta$, the other segments are contained in the other component). Let z_k be a point of γ_k. Then lemma 3 holds for γ and z replaced by γ_k and z_k.

Fix now k and prove the assertion. Since any branch of $\sqrt{f}$ extends to interior points of s_k from both sides in Δ, we may consider instead of a closed Jordan curve surrounding s_k the following curve γ_k which is the limit of such Jordan curves.

Denote the endpoints of s_k by a_k and b_k. Then $a_k \in V_{j(a_k)}$, $b_k \in V_{j(b_k)}$ for some indices $j(a_k)$ and $j(b_k)$. Consider a small circle in Δ around a_k and denote its intersection with s_k by $\hat{a}_k$. Analogously, consider a small circle in Δ around b_k and let $\hat{b}_k$ be its intersection point with s_k. Let $\hat{s}_k$ be the part of s_k between $\hat{a}_k$ and $\hat{b}_k$. Denote by $\hat{s}_k^+$ (the plus edge of $\hat{s}_k$) an identical copy of the curve $\hat{s}_k$ which is considered as limit of curves situated "on the right of s_k". On $\hat{s}_k^+$ we consider analytic continuation of the related branches of $\sqrt{f}$ "from the right of $\hat{s}_k$". Respectively, define $\hat{s}_k^-$ (the minus edge of $\hat{s}_k$) to be a copy of the curve $-\hat{s}_k$ (the curve $\hat{s}_k$ with inversed orientation) with analytic continuation of branches of $\sqrt{f}$ "from the left of $\hat{s}_k$" (i.e. from the right of $-\hat{s}_k$). To each point ζ in $\hat{s}_k$ corresponds a point ζ^+ in the plus edge $\hat{s}_k^+$ and a point ζ^- in the minus edge $\hat{s}_k^-$. Consider the point $z_k = \hat{a}_k^-$ (the point on the minus edge $\hat{s}_k^-$ which corresponds to $\hat{a}_k$). Let α_k surround a_k along the above-mentioned circle in Δ counterclockwise (with respect to orientaiton introduced from $\mathbb{D}$) starting from the point $\hat{a}_k^-$ and terminating at the point $\hat{a}_k^+$. Let β_k surround b_k along the other circle in Δ counterclockwise starting at $\hat{b}_k^+$ and terminating at $\hat{b}_k^-$. Let γ_k be the sum of the curves α_k, $\hat{s}_k^+, \beta_k, \hat{s}_k^-$ in the just mentioned order.

Consider a functional element at $\hat{a}_k$ which defines a branch of $\sqrt{f}$ and denote it by $\mathcal{F}_{\hat{a}_k^-}$. We will study now analytic continuation of $\mathcal{F}_{\hat{a}_k^-}$ along γ_k. Recall that $a_k \in V_{j(a_k)}$ and is close to $\partial\mathbb{B}^n$, so it is a regular point of $V_{j(a_k)}$. Hence there is a neighbourhood $U(a_k)$ of a_k in which f has the form $f = f_{j(a_k)} \cdot g_{j(a_k)}$, where $g_{j(a_k)}$ does not have roots in $U(a_k)$ and $V_{j(a_k)} \cap U(a_k)$ is the zero set of $f_{j(a_k)}$ in $U(a_k)$. Moreover, $df_{j(a_k)}$ does not vanish on this set and hence (since $f_{j(a_k)}$ is analytic) has real rank 2 at each point of this set.

Moreover, $V_{j(a_k)}$ and Δ are transversal, i.e. the kernel of $df_{j(a_k)}$ computed at the point a_k and the tangent space $T_{a_k}\Delta$ are transversal (as real linear subspaces of $T_{a_k}(r\partial\mathbb{B}^n)$). Hence, $df_{j(a_k)}$ computed at the point a_k and restricted to $T_{a_k}\Delta$ is a linear isomorphism. This implies that, provided $U(a_k)$ is small enough, the restriction of $f_{j(a_k)}$ on $\Delta \cap U(a_k)$ is a diffeomorphism. Moreover, $f_{j(a_k)}(a_k) = 0$. We may assume that the curve α_k is contained in $\Delta \cap U(a_k)$ together with the small disc on Δ bounded by α_k. $f_{j(a_k)}$ maps the small closed disc diffeomorphically onto an open set in $\mathbb{C}$ which contains 0 and is bounded by a simple closed Jordan curve (of class C^2). In particular, the curve $f_{j(a_k)} \circ \alpha_k$ surrounds 0 once (either counterclockwise or clockwise). This means that $f_{j(a_k)}$ changes argument by 2π or by -2π along α_k (running from $\hat{a}_k^-$ to $\hat{a}_k^+$ counterclockwise). Hence $\sqrt{f} = \sqrt{f_{j(a_k)}} \cdot \sqrt{g_{j(a_k)}}$ changes argument by $+\pi$ or $-\pi$ along α_k. We obtained two functional elements $\mathcal{F}_{\hat{a}_k^-}$ and $\mathcal{F}_{\hat{a}_k^+}$ at $\hat{a}_k$ such that $\mathcal{F}_{\hat{a}_k^+}$ is the analytic continuation of $\mathcal{F}_{\hat{a}_k^-}$ along α_k, and the two functional elements differ by the factor $e^{\pm i\pi} = -1$.

Provide analytic continuation of $\mathcal{F}_{\hat{a}_k^-}$ and $\mathcal{F}_{\hat{a}_k^+}$ along $\hat{s}_k^-$ in backward direction and along $\hat{s}_k^+$, respectively. For any point $\zeta \in \hat{s}_k$ we obtain functional elements $\mathcal{F}_{\zeta^+}$ and $\mathcal{F}_{\zeta^-}$ on the plus edge and on the minus edge, respectively, which differ by the factor -1 (i.e. by sign). Repeat the arguments done for the point a_k and the curve α_k, with respect to the point b_k and the curve β_k. We assume that β_k is contained in a sufficiently small neighbourhood of b_k, and obtain that analytic continuation of $\mathcal{F}_{\hat{b}_k^+}$ along β_k leads to a functional element at $\hat{b}_k^-$ which differ from $\mathcal{F}_{\hat{b}_k^+}$ by sign and, hence, equals $\mathcal{F}_{\hat{b}_k^-}$.

Summarizing, we proved that analytic continuation of $\mathcal{F}_{\hat{a}_k^-}$ along γ_k leads to the same functional element, which proves lemma 3. $\square$

For the proof of lemma 2 we have to show that analytic continuations of $\mathcal{F}$ to some wedges W^+ and W^-, found according to lemma 1, differ on some part of $W^+ \cap W^-$.

First, associate with the hypersurface $e = \cup_{1-\varepsilon<r<1} \mathrm{int} K_r$ its plus copy e^+ and its minus copy e^- (so that the edge m^+ of W^+ is on the side of e^+). We claim that for each $\zeta \in e$ analytic continuation of $\mathcal{F}$ from U' to the corresponding points $\zeta^+ \in e^+$ and $\zeta^- \in e^-$ give functional elements $\mathcal{F}_{\zeta^+}$ and $\mathcal{F}_{\zeta^-}$ at ζ which differ by sign. Indeed, let $z \in r\partial\mathbb{B}^n \cap V_j$ for some $r \in (1-\varepsilon, 1)$ and some j. As above, we choose a small smoooth disc Δ which contains z, is contained in $r\partial\mathbb{B}^n$ and is transversal to V_j and to M_r. Δ intersects K_r along a small curve with endpoint z in Δ. Let α be a curve in Δ which surrounds z once, starts at some point $\zeta_0^- \in m_r^- \cap \Delta$ and terminates at $\zeta_0^+ \in m_r^+ \cap \Delta$. Then $\mathcal{F}_{\zeta_0^+}$ is obtained from $\mathcal{F}_{\zeta_0^-}$ by analytic continuation along α and hence, as above, $\mathcal{F}_{\zeta_0^+}$ and $\mathcal{F}_{\zeta_0^-}$ differ by sign. Analytic continuation of $\mathcal{F}_{\zeta_0^+}$ and $\mathcal{F}_{\zeta_0^-}$, respectively, along curves in e gives the required claim.

Choose now manifolds m^+ and m^- contained in U on the plus side and the minus side of e, respectively, and close in C^2 to m. (Recall m is a globally minimal CR-orbit of M of dimension $2n-2$ contained in K.) Let $q \in m$ and let $\mathcal{K}(q)$ be a truncated cone with vertex q contained in the wedge W. Note that $\mathcal{K}(q) \subset \mathbb{B}^n$. Shrinking it we may assume that it is at positive distance from $\cup_{j=1}^{\ell} \bar{V}_j$ and contained in $\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n}$. We may also assume (choosing m^+ and m^- close enough to m) that there are points $q(m^+) \in m^+$ and $q(m^-) \in m^-$ close to q and truncated cones $\mathcal{K}(q(m^+)) \subset W^+$ and $\mathcal{K}(q(m^-)) \subset W^-$ so close to $\mathcal{K}(q)$ that the following holds: The two cones intersect, are contained in $\mathbb{B}^n \setminus (1-\varepsilon)\overline{\mathbb{B}^n}$ and do not meet $\cup_{j=1}^{\ell} \bar{V}_j$. Join $q(m^-)$ and $q(m^+)$ by a curve σ which consists of two straight line pieces, one of them contained in $\{q(m^-)\} \cup \mathcal{K}(q(m^-))$, the other one in $\{q(m^+)\} \cup \mathcal{K}(q(m^+))$. We assume that the curve intersects e transversally and exactly once. Denote the intersection point by p. Then $\sigma \setminus \{p\}$ is contained in U' where $\mathcal{F}$ is analytic. Let σ^+ be the component of $\sigma \setminus \{p\}$ which contains $q(m^+)$ and σ^- the other component. Then σ^+ is in U' on the plus side of e. Hence analytic continuation of $\mathcal{F}$ from m^+ along $-\sigma^+$ gives the functional element $\mathcal{F}_{p^+}$. Respectively, analytic continuation of $\mathcal{F}$ from m^- along σ^- gives the functional element $\mathcal{F}_{p^-}$ which differs from $\mathcal{F}_{p^+}$ by sign. Hence, analytic continuation of $\mathcal{F}$ from m^+ to $\mathcal{K}(q(m^+)) \subset W^+$ cannot coincide with analytic continuation of $\mathcal{F}$ from m^- to $\mathcal{K}(q(m^-)) \subset W^-$. Lemma 2 (and hence theorem 1) is proved. $\square$

Proof of proposition. First, we consider an arbitrary small C^4-deformation $\approx{M}$ of M in ∂G which fixes the set $M \setminus \mathrm{int} K$ such that the deformation $\mathcal{K}$ of the set $\mathrm{int} K$ is C^∞-smooth. Since the manifold M was assumed to be generic, and since the property of a manifold of being generic is stable under small deformations, we can assume that the manifold $\overset{\approx}{M}$ is also generic.

Consider the normal bundle N of $\mathcal{K}$ in ∂G. Denote by F the natural diffeomorphism of a small tubular neighbourhood of $\mathcal{K}$ in N onto an open subset of ∂G. The mapping F gives a correspondence of smooth sections of N close to the zero-section and smooth 4-manifolds which are small deformations of $\mathcal{K}$ in ∂G. Let $J^4(\mathcal{K}, N)$ be the set of all 4-jets of germs of sections of N. Denote by π the natural projection of $J^4(\mathcal{K}, N)$ onto $\mathcal{K}$. Let p be a point of $\mathcal{K}$ and let s be a germ of a smooth section of N defined in a neighbourhood of p which is close enough to the zero-section. Then $\mathcal{K}_s \overset{\text{def}}{=} F(s)$ is a germ of a smooth hypersurface in ∂G. Let Z be a (1,0)-vector field which is complex tangential to $\mathcal{K}_s$. Then the real vector fields $\mathrm{Re}Z, \mathrm{Im}Z, \mathrm{Im}[Z,\bar{Z}], \mathrm{Im}[Z,[Z,\bar{Z}]], \mathrm{Im}[Z,[Z,[Z,\bar{Z}]]]$ and $\mathrm{Im}[\bar{Z},[Z,[Z,\bar{Z}]]]$ are tangential to $\mathcal{K}_s$. Moreover, since G is strictly pseudoconvex, it follows that the first three vector fields are linearly independent (see also (7) below). Define a subset $\mathcal{N}$ of $J^4(\mathcal{K}, N)$ as follows: the fiber $\mathcal{N}_p = \mathcal{N} \cap \pi^{-1}(p)$ of the set $\mathcal{N}$ over the point p is defined to be the equivalence class of 4-jets of germs of sections s as described above, for which the vectors $\mathrm{Im}[Z,[Z,\bar{Z}]]$, $\mathrm{Im}[Z,[Z,[Z,\bar{Z}]]]$ and $\mathrm{Im}[\bar{Z},[Z,[Z,\bar{Z}]]]$ are linearly dependent on the vectors $\mathrm{Re}Z$, $\mathrm{Im}Z$ and $\mathrm{Im}[Z,\bar{Z}]$ at the point $F(s(p))$.

The following lemma (whose proof is based on direct calculations) is the main step in the proof of the proposition.

Lemma 4. *The set $\mathcal{N}$ is a stratified (in the sense of Whitney, see [GWPL], p.11) subset of $J^4(\mathcal{K}, N)$. Moreover, for each point $p \in \mathcal{K}$ the corresponding fiber $\mathcal{N}_p$ of $\mathcal{N}$ has real codimension 5 in the fiber of $J^4(\mathcal{K}, N)$ over p.*

Proof of lemma 4. Let p be an arbitrary point of $\mathcal{K}$, suppose s is a germ of a smooth section of N defined in a neighbourhood of p and $s(p)$ is the point of s that corresponds to p under the natural projection from N to $\mathcal{K}$. Consider a linear change of coordinates in $\mathbb{C}^3$ such that the point $F(s(p))$ is the origin in the new coordinates $z_1, z_2, w (w = u + iv)$ and such that in a neighbourhood of this point one has

$$G = \{v > H(z_1, z_2, u)\},$$

where H is a C^∞-smooth function with the properties

$$H(0) = H'_{z_1}(0) = H'_{\bar{z}_1}(0) = H'_{z_2}(0) = H'_{\bar{z}_2}(0) = H'_u(0) = 0. \tag{1}$$

If we choose the z_1-axis to be equal to $T_0^{\mathbb{C}}(\mathcal{K}_s)$, then, maybe after replacing z_2 by a suitable linear function of z_2 and w, one has in a neighbourhood of the origin that

$$\mathcal{K}_s = \{u = \Psi(z_1, z_2), v = H(z_1, z_2, u)\},$$

where Ψ is a C^∞-smooth function defined in a neighbourhood of the origin in $\mathbb{C}^2_{z_1,z_2}$ such that

$$\Psi(0) = \Psi'_{z_1}(0) = \Psi'_{\bar{z}_1}(0) = 0, \Psi'_{z_2}(0) = \Psi'_{\bar{z}_2}(0) = 1. \tag{2}$$

Direct calculation shows that the following (1,0)-vector field Z defined in a neighbourhood of the origin is complex tangential to $\mathcal{K}_s$: $Z = \partial_{z_1} + a\partial_{z_2} + b\partial_w$, where

$$a = -\frac{H'_{z_1} + (H'_u + i)\Psi'_{z_1}}{H'_{z_2} + (H'_u + i)\Psi'_{z_2}} \tag{3}$$

and

$$b = -2\frac{H'_{z_1}\Psi'_{z_2} - H'_{z_2}\Psi'_{z_1}}{H'_{z_2} + (H'_u + i)\Psi'_{z_2}}. \tag{4}$$

Note, that any vector field which is complex tangential to $\mathcal{K}_s$ is uniquely determined by its projection onto the (z_1, z_2)-plane. Therefore, we can consider the vector field $Z^* = \partial_{z_1} + a\partial_{z_2}$ instead of Z. Then the computation below of the Lie brackets of Z^* and $\bar{Z}^*$ will give us the (z_1, z_2)-components of the Lie brackets of Z and $\bar{Z}$ in the jet space $J^4(\mathcal{K}, N)$. For the first Lie bracket $[Z^*, \bar{Z}^*]$ we obtain

$$[Z^*, \bar{Z}^*] = -2i\mathrm{Im}(\bar{Z}^*(a)\partial_{z_2}) = -2i\mathrm{Im}(a'_{\bar{z}_1}\partial_{z_2} + a'_{\bar{z}_2}\bar{a}\partial_{z_2}). \tag{5}$$

To compute the coefficients explicitly, put

$$a = \alpha \cdot A$$

where α is the numerator and $-\frac{1}{A}$ the denominator of the fraction in (3). Then by (1) and (2)

$$\alpha(0) = 0, \quad A(0) = i.$$

Direct computation gives

$$\begin{aligned} a'_{\bar{z}_1} &= \alpha'_{\bar{z}_1}A + \alpha A'_{\bar{z}_1} \\ a'_{\bar{z}_2} &= \alpha'_{\bar{z}_2}A + \alpha A'_{\bar{z}_2} \\ \bar{Z}^*(a) &= a'_{\bar{z}_1} + a'_{\bar{z}_2}\bar{a} = \alpha'_{\bar{z}_1}A + \alpha A'_{\bar{z}_1} + \alpha'_{\bar{z}_2} \cdot \bar{\alpha} \cdot |A|^2 + |\alpha|^2 A'_{\bar{z}_2}\bar{A}. \end{aligned} \tag{6}$$

Here

$$\alpha'_{\bar{z}_1} = H''_{\bar{z}_1 z_1} + H''_{\bar{z}_1 u}\Psi'_{z_1} + (H'_u + i)\Psi''_{\bar{z}_1 z_1} = H''_{\bar{z}_1 z_1}(0) + i\Psi''_{\bar{z}_1 z_1}(0) + O(z).$$

All terms in (6) except the first one are of order $O(z)$, hence

$$\bar{Z}^*(a) = iH_{\bar{z}_1 z_1}(0) - \Psi''_{\bar{z}_1 z_1}(0) + O(z).$$

It follows from (5) that

$$\begin{aligned} [Z^*, \bar{Z}^*] = -2i\mathrm{Im}(\bar{Z}^*(a)\partial_{z_2}) = -i(H''_{\bar{z}_1 z_1}(0)\partial_{x_2} + \Psi''_{\bar{z}_1 z_1}(0)\partial_{y_2}) + \\ O(z)\partial_{x_2} + O(z)\partial_{y_2}. \end{aligned} \tag{7}$$

On the next step we compute the second Lie bracket and extract the derivatives of the highest (third) order

$$[Z^*,[Z^*,\bar{Z}^*]] = [\partial_{z_1} + a\partial_{z_2}, Z^*(\bar{a})\partial_{\bar{z}_2} - \bar{Z}^*(a)\partial_{z_2}] = (\partial_{z_1}Z^*(\bar{a}))\partial_{\bar{z}_2} - \\ - (\partial_{z_1}\bar{Z}^*(a))\partial_{z_2} + a(\partial_{z_2}Z^*(\bar{a}))\partial_{\bar{z}_2} - a(\partial_{z_2}\bar{Z}^*(a))\partial_{z_2} - Z^*(\bar{a})(\partial_{\bar{z}_2}a)\partial_{z_2} + \\ + \bar{Z}^*(a)(\partial_{z_2}a)\partial_{z_2}.$$

Since $\alpha(0) = 0$ and since the last two terms in the preceding expression contain only derivatives of H and Ψ up to second order, it follows from (1), (2) and (6) that

$$\begin{aligned}[Z^*,[Z^*,\bar{Z}^*]] = {} & -i(H'''_{z_1z_1\bar{z}_1}(0)\partial_{x_2} + \Psi'''_{z_1z_1\bar{z}_1}(0)\partial_{y_2}) + (\mathcal{A}_2\partial_{x_2} + \mathcal{B}_2\partial_{y_2}) + \\ & + O(z)\partial_{x_2} + O(z)\partial_{y_2},\end{aligned} \tag{8}$$

where $\mathcal{A}_2$ and $\mathcal{B}_2$ consist of terms containing only first and second order derivatives of H and Ψ.

Similarly,

$$\begin{aligned}[Z^*,[Z^*,[Z^*,\bar{Z}^*]]] = {} & -i(H^{(4)}_{z_1z_1z_1\bar{z}_1}(0)\partial_{x_2} + \Psi^{(4)}_{z_1z_1z_1\bar{z}_1}(0)\partial_{y_2}) + \\ & + (\mathcal{A}_3\partial_{x_2} + \mathcal{B}_3\partial_{y_2}) + O(z)\partial_{x_2} + O(z)\partial_{y_2}\end{aligned} \tag{9}$$

and

$$\begin{aligned}[\bar{Z}^*,[Z^*,[Z^*,\bar{Z}^*]]] = {} & -i(H^{(4)}_{z_1z_1\bar{z}_1\bar{z}_1}(0)\partial_{x_2} + \Psi^{(4)}_{z_1z_1\bar{z}_1\bar{z}_1}(0)\partial_{y_2}) + \\ & + (\mathcal{A}'_3\partial_{x_2} + \mathcal{B}'_3\partial_{y_2}) + O(z)\partial_{x_2} + O(z)\partial_{y_2},\end{aligned} \tag{10}$$

where $\mathcal{A}_3$, $\mathcal{B}_3$, $\mathcal{A}'_3$ and $\mathcal{B}'_3$ consist of terms containing only derivatives of H and Ψ up to the third order.

To finish the proof of lemma 4, let us first note that the real vector fields $U_1 = \mathrm{Re}Z$ and $V_1 = \mathrm{Im}Z$ are complex tangential to $\mathcal{K}_s$ and that the Lie brackets up to order 3 of these vector fields are equal (up to multiplication by non-zero constants) to the vector fields $V_2 = \mathrm{Im}[Z,\bar{Z}]$, $U_3 = \mathrm{Re}[Z,[Z,\bar{Z}]]$, $V_3 = \mathrm{Im}[Z,[Z,\bar{Z}]]$ and linear combinations of the vector fields $U_4 = \mathrm{Re}[Z,[Z,[Z,\bar{Z}]]]$, $V_4 = \mathrm{Im}[Z,[Z,[Z,\bar{Z}]]]$ and $W_4 = \mathrm{Im}[\bar{Z},[Z,[Z,\bar{Z}]]]$. Since, by our assumptions, the domain G is strictly pseudoconvex, it follows that $H''_{\bar{z}_1z_1}(0) > 0$. Hence (7) implies that the vector fields U_1, V_1 and V_2 are linearly independent. The subset $\mathcal{N}$ of the jet space $J^4(\mathcal{K}, N)$ is defined by the condition that each of the vectors U_3, V_3, U_4, V_4 and W_4 is a linear combination of U_1, V_1, V_2. The property of a vector to be a linear combination of some other vectors is an algebraic relation on the coordinates of the vectors. Formulas (3)-(10) above show that the coordinates of the vector fields $U_1, V_1, V_2, U_3, V_3, U_4, V_4$ and W_4 are rational functions of the derivatives of the functions H and Ψ. We conclude that $\mathcal{N}$ is defined in $J^4(\mathcal{K}, N)$ by algebraic equations and therefore is a stratified subset of $J^4(\mathcal{K}, N)$. Moreover, from formulas (3)–(10) one obtains 5 equations, each of them describing the condition for one of the vectors U_3, V_3, U_4, V_4 and W_4 to be a linear combination of the vectors U_1, V_1 and V_2. The mentioned formulas show that in different equations the appearing partial derivatives of H and Ψ of highest order are different. Hence, the 5 equations define $\mathcal{N}$ as a complete intersection and the codimension of $\mathcal{N}$ in $J^4(\mathcal{K}, N)$ is equal to 5. The proof of lemma 4 is completed. $\square$

To finish the proof of the proposition, let us consider a locally finite covering $\{B_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ by balls (in the intrinsic metric of $\mathcal{K}$) such that the radius of every ball B_n is less than half of the distance (also in the intrinsic metric of $\mathcal{K}$) from the center of the ball to the boundary $\partial\mathcal{K}$ of $\mathcal{K}$. Consider a sequence of balls $\{B_n'\}_{n=1}^{\infty}$ in $\mathcal{K}$, which is also a covering of $\mathcal{K}$, such that $B_n' \subset\subset B_n$ for all n. Since $\dim_{\mathbb{R}} \mathcal{K} = 4$, it follows from our lemma 4 and theorem 1.1 on p.127 in [GWPL], that there is a perturbation $\mathcal{K}_1$ of $\mathcal{K}$ in C^∞ which keeps $\mathcal{K}$ unchanged outside B_1 and such that the 4-jet of $\mathcal{K}_1$ over $\bar{B}_1'$ does not intersect the set $\mathcal{N}$. Then, again by our lemma 4 and theorem 1.1 in [GWPL], there is a perturbation $\mathcal{K}_2$ of $\mathcal{K}_1$ in C^∞ which keeps $\mathcal{K}_1$ unchanged outside B_2 and such that the 4-jet of $\mathcal{K}_2$ over $\bar{B}_2'$ does not intersect the set $\mathcal{N}$. Moreover, this perturbation of $\mathcal{K}_1$ can be chosen so small in C^4-metric that the 4-jet of $\mathcal{K}_2$ is still disjoint from $\mathcal{N}$ over $\bar{B}_1'$. In a similar way we can inductively construct a perturbation $\mathcal{K}_3$ of $\mathcal{K}_2, \mathcal{K}_4$ of $\mathcal{K}_3$ and so on. On the step number m our construction looks as follows. Let $\mathcal{K}_m$ be a perturbation of $\mathcal{K}_{m-1}$ in C^∞ (guaranteed by lemma 4 and theorem 1.1 of [GWPL]) which keeps $\mathcal{K}_{m-1}$ unchanged outside B_m and such that the 4-jet of $\mathcal{K}_m$ over $\bar{B}_m'$ is disjoint from $\mathcal{N}$. We required on step $m-1$ that the 4-jet of $\mathcal{K}_{m-1}$ is disjoint from $\mathcal{N}$ over $\cup_{i=1}^{m-1} \bar{B}_i'$. On step m we choose $\mathcal{K}_m$ close enough to $\mathcal{K}_{m-1}$ so that the 4-jet of $\mathcal{K}_m$ is still disjoint from $\mathcal{N}$ over $\cup_{i=1}^{m-1} \bar{B}_i'$ (and, therefore, over $\cup_{i=1}^{m} \bar{B}_i'$). Since the covering $\{B_n\}_{n=1}^{\infty}$ was chosen to be locally finite, it follows that for each n the pertubations $\mathcal{K}_m$ will stay the same over the ball B_n for all m big enough. Therefore, we have a well defined limit $\mathcal{K}^*$ of $\mathcal{K}_m$ as $m \to \infty$ which is C^∞-smooth and, by our construction, the 4-jet of $\mathcal{K}^*$ is disjoint from the set $\mathcal{N}$ over the whole of $\cup_{m=1}^{\infty} \bar{B}_m' = \mathcal{K}$. Then, by the definition of the set $\mathcal{N}$, we have that the set $\mathcal{K}^*$ is of type 4 and, hence, is globally minimal. Moreover, since the balls of our covering are getting smaller when we approach the boundary $\partial\mathcal{K}$ of $\mathcal{K}$, it follows that if we choose perturbations $\mathcal{K}_m$ is such a way, that the distance from the 4-jet of $\mathcal{K}_m$ to the 4-jet of $\mathcal{K}_{m-1}$ is decreasing to zero rapidly enough as $m \to \infty$, then the manifold $\tilde{M} = (M \setminus \mathcal{K}) \cup \mathcal{K}^*$ will still be C^4-smooth. This completes the proof of proposition. $\square$

Remark 4. *Proposition is proved under the assumption that the domain G has C^∞-smooth boundary. In this case the set $\mathcal{K}$ can be perturbed to a C^∞-smooth set $\mathcal{K}^*$ of type 4. One can easily adapt our arguments to the case of domains with C^4-smooth boundary. Then the perturbed set $\mathcal{K}^*$ will also be C^4-smooth.*

Theorem 1 can be generalized in the following way.

Theorem 2. *Let M be an orientable smooth manifold of real dimension 4 contained in the unit sphere $\partial\mathbb{B}^3$. Suppose M has smooth boundary ∂M which is* *the disjoint union of finitely many connected smooth compact maximally complex CR-manifolds of dimension 3. Then there exist arbitrarily small smooth deformations $\tilde{M}$ of M with fixed boundary $\partial\tilde{M} = \partial M$ such that $\tilde{M}$ has only finitely many complex tangencies $q_1, \ldots, q_N$ and the manifold $\tilde{M} \setminus \{q_1, \ldots, q_N\}$ is globally minimal.*

Moreover, the essential envelope of holomorphy $E(\partial\mathbb{B}^n \setminus \bar{\tilde{M}})$ is multi-sheeted. Each point of $\bar{\tilde{M}}$ is in the closure of points over which $E(\partial\mathbb{B} \setminus \bar{\tilde{M}})$ is multi-sheeted.

Proof. Let N be one of the boundary components of M. Then a small neighbourhood of N on M is a generic CR-manifold (although the whole manifold M may not be so). Indeed, by Forstnerič [F], for each $p \in N$ the tangent space T_pN is not contained in the complex tangent space $T_p^c\partial\mathbb{B}^3$. Hence, for $p \in N$ the tangent space $T_p\bar{M}$ is not contained in $T_p^c\partial\mathbb{B}^3$. By continuity this is true for all points $p' \in M$ close to N.

It follows from the jet transversality theorem (see e.g. [Hi] Ch. 3) that there is a small perturbation M^* of M which fixes M near ∂M such that the tangent space to M^* is not complex at all points of M^*, except maybe for a finitely many points $\{q_1, q_2, \ldots, q_N\}$. Indeed, consider for each point $p \in \partial\mathbb{B}^3$ the set of all oriented 4-planes T_p^4 such that $p \in T_p^4 \subset T_p(\partial\mathbb{B}^3)$. Each plane from this family is uniquely determined by the unit vector $n_p \in T_p(\partial\mathbb{B}^3)$ which is ortogonal to T_p^4 and such that the pair (T_p^4, n_p) defines the natural orientation on $T_p(\partial\mathbb{B}^3)$. Therefore, this family of 4-planes is diffeomorphic to the space of unit vectors in a 5-dimensional euclidean space, i.e. to S^4. Among these 4-planes there are two planes that correspond to the complex tangent space $T_p^{\mathbb{C}}(\partial\mathbb{B}^3)$ (one with positive orientation, the other one with negative). We obtain a set consisting of 2 points, i.e. a submanifold of S^4 which has codimension 4. This means that the jet transversality theorem is used to approximate the inclusion map of the manifold M into $\partial\mathbb{B}^3$ by a map whose 1-jet map is transversal to a codimension 4 subset of the 1-jet space. Since the dimension of M is also 4 and M is proved to be without complex tangencies near ∂M, we obtain the required M^*.

The proposition above gives a small perturbation $\tilde{M}$ of M^* which fixes ∂M and the complex tangencies such that $\tilde{M} \setminus \{q_1, \ldots, q_N\}$ is a globally minimal generic CR-manifold contained in $\partial\mathbb{B}^n$. This situation is enough for the proof of theorem 1 to go through. $\square$

Now we come back to the examples and prove that in the mentioned situations the envelope of holomorphy is exactly two-sheeted.

Claim 1. *Let M be as in example 1. If $\bar{\tilde{M}}$ is chosen as described there and close in C^4 norm to $\bar{M}$ then $E(\partial\mathbb{B}^3 \setminus \bar{\tilde{M}})$ is two-sheeted.*

Claim 2. *Let $c \in (0, \frac{1}{\sqrt{3}^3})$ and close to $\frac{1}{\sqrt{3}^3}$. Denote by M the set $M = \{z = (z_1, z_2, z_3) \in \partial\mathbb{B}^3 : z_1 \cdot z_2 \cdot z_3 \in (c, +\infty)\}$. Then M is a smooth 4-dimensional manifold. Its boundary $\partial M = N$ is a connected compact maximally complex CR-manifold. There exist arbitrarily small (in C^4) smooth deformations $\tilde{\bar{M}}$ of $\bar{M}$ with $\partial\tilde{M} = \partial M$ such that $\tilde{M}$ is a generic globally minimal CR-manifold. In this case $E(\partial\mathbb{B}^3 \setminus \tilde{\bar{M}})$ is two-sheeted. The whole set $\tilde{\bar{M}}$ is contained in the closure of the set of points over which $E(\partial\mathbb{B}^3 \setminus \tilde{\bar{M}})$ is two-sheeted.*

We will prove here claim 2. Claim 1 is proved in the same way but simpler in view of the simpler topology of the first example.

Proof of Claim 2. The set

$$\mathcal{H} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : Im(z_1 \cdot z_2 \cdot z_3) = 0, Re(z_1 \cdot z_2 \cdot z_3) > c\}$$

is a smooth (even real analytic) hypersurface which is foliated by the complex analytic hypersurfaces $\mathcal{H}_\alpha = \{z_1 \cdot z_2 \cdot z_3 = \alpha\}, \alpha \in (c, +\infty)$. If a point is contained in $\mathcal{H}$, then none of its coordinates vanishes. It is easy to see that $\mathcal{H}$ intersects $\partial\mathbb{B}^3$ transversally, so M is a smooth 4-dimensional manifold. M is foliated by the 3-dimensional compact maximally complex CR-manifolds $N_\alpha = \{z_1 \cdot z_2 \cdot z_3 = \alpha\} \cap \partial\mathbb{B}^3, \alpha \in (c, \frac{1}{\sqrt{3}^3})$. For $\alpha = \frac{1}{\sqrt{3}^3}$ the intersection $\mathcal{H}_\alpha \cap \partial\mathbb{B}^3$ degenerates. It is a 2-torus which consists of complex tangencies of M. For $\alpha > \frac{1}{\sqrt{3}^3}$, $\mathcal{H}_\alpha$ does not meet $\partial\mathbb{B}^3$. It is clear that $\partial M = N (= \mathcal{H}_c \cap \partial\mathbb{B}^3)$.

Choose a generic CR-manifold $M^* \subset \partial\mathbb{B}^3$ such that $\overline{M^*}$ is a C^4 small perturbation of $\bar{M}$ with $\partial M^* = \partial M$ (see [JSh]). Let $\tilde{\bar{M}}$ be C^4 close to $\overline{M^*}$ with $\partial\tilde{M} = \partial M$, such that $\tilde{M}$ is globally minimal (see proposition).

It will be convenient to use the following coordinates

$$(z_1, z_2, \zeta_3) = (z_1, z_2, z_1 z_2 z_3)$$

on the set

$$\Omega = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : Re(z_1 z_2 z_3) > c\}.$$

Ω is a domain of holomorphy. In these coordinates M has the form

$$M = \{(z_1, z_2, \xi_3 + i\eta_3) : \eta_3 = 0, \xi_3 > c, |z_1|^2 + |z_2|^2 + \frac{\xi_3^2}{|z_1|^2|z_2|^2} = 1\}.$$

The manifold $N = \partial M$ bounds the analytic manifold $V = \mathbb{B}^3 \cap \mathcal{H}_c$. The set $M \cup N \cup V$ is a closed piecewise smooth manifold. It bounds the following set $Q = \mathcal{H} \cap \mathbb{B}^3 \subset \mathcal{H}$,

$$\begin{aligned} Q &= \{(z_1, z_2, \xi_3 + i\eta_3) \in \Omega \cap \mathbb{B}^3 : \eta_3 = 0\} \cong \\ &\cong \{(z_1, z_2, \xi_3) \in \mathbb{C}^2 \times \mathbb{R} : \xi_3 > c, |z_1|^2 + |z_2|^2 + \frac{\xi_3^2}{|z_1|^2|z_2|^2} < 1\}. \end{aligned}$$

The manifold $\tilde{M}$ has the form

$$\tilde{M} = \{(z_1, z_2, \xi_3 + i\eta_3) : \eta_3 = \mathcal{E}_3(z_1, z_2, \xi_3), (z_1, z_2, \xi_3) \in \Gamma\},$$

where

$$\Gamma = \{(z_1, z_2, \xi_3) : \xi_3 > c, |z_1|^2 + |z_2|^2 + \frac{\xi_3^2}{|z_1|^2|z_2|^2} = 1 - \mathcal{E}_3^2(z_1, z_2, \xi_3)\}$$

and $\mathcal{E}_3$ is a small smooth real valued function. We consider Γ as a subset of $\mathcal{H}$ and $\tilde{M}$ as a graph over Γ. Note that $\Gamma \subset \bar{Q}$. (Points $(z_1, z_2, \xi_3) \in \Gamma$ for which $\mathcal{E}_3 = 0$ are contained in $M \subset \partial Q$, points for which $\mathcal{E}_3 \neq 0$ are contained in Q.) Moreover, $\partial\Gamma = \partial M$ and $\Gamma \cup N \cup V$ bounds an open subset $\mathcal{G}$ of Q.

Our plan is now the following. Let U be a one-sided neighbourhood of $\partial\mathbb{B}^3 \setminus \tilde{M}$ with a priori analytic extension. Note first that each analytic function u in U extends to a neighbourhood of $\mathbb{B}^3 \setminus (\Omega \cup V)$ (i.e. to the union of $\mathbb{B}^3 \setminus \bar{\Omega}$ with a neighbourhood of $(\mathbb{B}^3 \cap \partial\Omega) \setminus V$).

In fact for $c' + it \in \mathbb{C}$, $|c' + it| < \frac{1}{\sqrt{3}^3}$, the analytic hypersurface $\mathcal{H}_{c'+it} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 z_2 z_3 = c' + it\}$ intersects $\partial\mathbb{B}^3$ transversally. For those $c' + it$ the intersection $\mathcal{H}_{c'+it} \cap \mathbb{B}^3$ is a Stein manifold. If moreover, $c' < c$, or $t \neq 0$ and $|c - c'|$ is small compared with $|t|$, then the boundary of $\mathcal{H}_{c'+it} \cap \mathbb{B}^3$ is contained in $\partial\mathbb{B}^3 \setminus \bar{\tilde{M}}$. Hence, for such $c' + it$ each analytic function u in U extends to a neighbourhood of $\mathcal{H}_{c'+it} \cap \mathbb{B}^3$. The assertion follows by applying the Hartogs-Bochner theorem to $\mathbb{B}^3 \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 : Re(z_1 z_2 z_3) < c - \varepsilon\}$ for some small $\varepsilon > 0$. (The function u is analytic near $\mathbb{B}^3 \cap \{Re(z_1 z_2 z_3) = c - \varepsilon\}$ and in a one-sided neighbourhood of $\partial\mathbb{B}^3 \cap \{Re(z_1 z_2 z_3) \leq c - \varepsilon\}$.)

Note that $\tilde{M}$ divides $\partial\mathbb{B}^3 \cap \Omega$ into two connected components ω_+ and ω_-. Let U_+ and U_- be one-sided neighbourhoods of ω_+ and ω_-, respectively, with a priori analytic extension to U_+ and U_-, respectively. Assume ω_+ is on the side of $\tilde{M}$ where $\eta_3 > \mathcal{E}_3(z_1, z_2, \xi_3)$.

Denote by $C(\mathcal{G})$ the "cylinder over $\mathcal{G}$",

$$C(\mathcal{G}) = \{(z_1, z_2, \zeta_3) = (z_1, z_2, \xi_3 + i\eta_3) : \eta_3 \in \mathbb{R}, \xi_3 \in \mathcal{G}\}.$$

Claim 3. *$C(\mathcal{G})$ is pseudoconvex, provided the constant c in claim 2 is close to $\frac{1}{\sqrt{3}^3}$ and $\tilde{M}$ is close to M.*

Proof of claim 3. Denote by $C(Q)$ the cylinder over Q. Both domains, $C(\mathcal{G})$ and $C(Q)$, are invariant under translation in the η_3-direction and the projection parallel to the η_3-direction is equal to $\mathcal{G}$ and Q, respectively. It is enough to prove that the boundary points of $C(Q)$, for which $\xi_3 = \frac{1}{\sqrt{3}^3}$, are strictly pseudoconvex boundary points for $C(Q)$.

Indeed, if c is close to $\frac{1}{\sqrt{3}^3}$, then for all points of $\partial C(Q)$ which project to $M \subset \partial Q$ the ξ_3-coordinate is close to $\frac{1}{\sqrt{3}^3}$ and hence the corresponding point is a strictly pseudoconvex boundary point of $C(Q)$. Since the boundary part Γ of $\mathcal{G}$ is C^2 close to M, all points of $\partial C(\mathcal{G})$ which project to Γ are strictly pseudoconvex boundary points. The remaining boundary points of $\partial C(\mathcal{G})$ are contained in the real hyperplane $\{\xi_3 = c\}$ (in coordinates (z_1, z_2, ζ_3)) and hence they are pseudoconvex boundary points as well.

Prove now that all points of the set

$$E = \{(z_1, z_2, \xi_3) \in \partial C(Q), \xi_3 = \frac{1}{\sqrt{3}^3}\}$$

(and hence all their translates in η_3-direction) are strictly pseudoconvex boundary points. In (z_1, z_2, z_3)-coordinates E consists of those points $(z_1, z_2, z_3) \in \partial\mathbb{B}^3$ for which $|z_1| = |z_2| = |z_3| = \frac{1}{\sqrt{3}}$. The analytic hypersurface $\{z_1 \cdot z_2 \cdot z_3 = \frac{1}{\sqrt{3}^3}\}$ is tangent to $\partial\mathbb{B}^3$ at those points (see above or [JSh] for more details). In (z_1, z_2, ζ_3)-coordinates this means that the complex hyperplane $\{\zeta_3 = \frac{1}{\sqrt{3}^3}\}$ is tangent to $\partial\mathbb{B}^3$ at points of E. Hence $\partial\mathbb{B}^3$ and $\partial C(Q)$ have the same complex tangent space at points $p \in E$ (we use (z_1, z_2, ζ_3)-coordinates).

Each point $p \in E$ is strictly pseudoconvex for $\partial\mathbb{B}^3$ (in (z_1, z_2, ζ_3) coordinates). Since (near $p \in E$) $\partial\mathbb{B}^3$ and $\partial C(Q)$ do not differ in complex tangent directions (their intersections with $\{\eta_3 = 0\}$ coincide near $p \in E$) the Levi form of $\partial C(Q)$ at $p \in E$ is strictly positive in complex tangent directions. $\square$

Claim 4. *There exist two continuous functions h_+ and h_- on $\bar{\mathcal{G}}$ such that*

1. $h_+ \geq h_-$ on $\mathcal{G}$.

2. $h_+ = h_- = \mathcal{E}_3$ on Γ and $h_+ = h_- = 0$ on $N \cup V$.

(i.e. the graphs $g_+(\bar{\mathcal{G}})$ and $g_-(\bar{\mathcal{G}})$ of the functions h_+ and h_- over $\bar{\mathcal{G}}, g_\pm(\bar{\mathcal{G}}) = \{(z_1, z_2, \xi_3 + ih_\pm(z_1, z_2, \xi_3) : (z_1, z_2, \xi_3) \in \bar{\mathcal{G}}\}$, have common boundary which is equal to the union of $\bar{V}$ and $\tilde{M}$, $\tilde{M}$ being equal to the graph of $\mathcal{E}_3$ over Γ.)

3. The function u extends analytically to the relatively compact domain bounded by ω_+, by part of $\partial\Omega$ and by the graph $g_+(\bar{\mathcal{G}})$ of h_+ over $\bar{\mathcal{G}}$. (Denote this domain $\mathcal{B}_+$.)

Similarly, u extends analytically to the domain bounded by ω_-, by part of $\partial\Omega$ and by $g_-(\bar{\mathcal{G}})$. (Denote this domain $\mathcal{B}_-$.)

Suppose for a moment that claim 4 is proved. We will use the following result due to Trepreau [Tr1], see also [Č2].

Let C_+ be the "semi-cylinder" over $\mathcal{G}$

$$C_+ = \{(z_1, z_2, \xi_3 + i\eta_3) : (z_1, z_2, \xi_3) \in \mathcal{G}, \eta_3 > h_+\}.$$

Its envelope of holomorphy equals C_+^* which is a semi-cylinder of the same form with h_+ replaced by $h_+^*(h_+^* \leq h_+)$.

To apply the result, assume (changing h_+ slightly, if necessary) that u is analytic in $\mathcal{B}_+$ and in a neighbourhood of the graph of h_+ over $\mathcal{G}$. By an argument of Trepreau (see e.g. [Tr2]), u can be written as $u = u_+ + u_-$, where u_+ is analytic in C_+ and on the graph of h_+ and u_- is analytic in $C(\mathcal{G}) \setminus \bar{C}_+$ and on the graph of h_+. (The idea is to multiply u by a smooth cut-off function in C_+ which is 1 near the graph of h_+ and extend by zero to get a smooth function v in C_+. Let $(\bar\partial v)_+$ be equal to $\bar\partial v$ in C_+ and zero else. Since $C(\mathcal{G})$ is pseudoconvex, there is a function w in $C(\mathcal{G})$ such that $\bar\partial w = (\bar\partial v)_+$. Put $u_+ = v - w, u_- = w$.) Hence, u_+ extends to C_+^*, and, therefore, $u = u_+ + u_-$ extends analytically to $C_+^* \cap \mathbb{B}^3$.

In the same way, u extends analytically to $C_-^* \cap \mathbb{B}^3$ with C_-^* defined symmetrically. We obtained that u has analytic continuation to the following domains: a neighbourhood of $\mathbb{B}^3 \setminus (\Omega \cup V), \mathcal{B}_+, C_+^*$ and $\mathcal{B}_-, C_-^*$. (Note that $\mathcal{B}_+ \setminus C_+^*$ consists of points $(z_1, z_2, \zeta_3) \in \mathbb{B}^3$ for which $(z_1, z_2, \xi_3) \in Q \setminus \mathcal{G}$). Continuation of u coincides on the intersection of any pair of the domains above, except, maybe, C_+^* and C_-^*, which contain the overlapping wedges W^+ and W^-, respectively. The domains above cover a two-sheeted Riemann domain over $\mathbb{C}^3$ (all pairs of domains are glued along their intersection except C_+^* and C_-^*). Denote by π the natural projection to $\mathbb{C}^3$ defined on the suitably defined closure of the Riemann domain (see [O] or [GR]). Each boundary point of the Riemann domain is pseudoconvex. Indeed, this is clear for points which project to $\partial\mathbb{B}^3$, it is also clear for points which project to the graph of either h_+^* over the open set $\mathcal{G}$ or of h_-^* over $\mathcal{G}$. The remaining boundary part of the Riemann domain projects homeomorphically to the analytic hypersurface V. Note that there are points in $\mathbb{C}^3$ close to V which are covered twice by the Riemann domain. Changing coordinates in a neighbourhood of $\pi^{-1}(V)$ on the Riemann domain (e.g. by lifting $(z_1, z_2, \sqrt{\zeta_3 - c})$ to the Riemann domain) one can easily see that $\pi^{-1}(V)$ consists of pseudoconvex points (lying on a 2-dimensional analytic manifold contained in the boundary).

By theorems of Oka and Grauert, the Riemann domain is Stein (see [G], [O]). Hence it coincides with the envelope of holomorphy $E(\partial\mathbb{B}^3 \setminus \bar{\tilde{M}})$.

It remains to give the

Proof of claim 4. Let u be an analytic function in the one-sided neighbourhood U of $\partial\mathbb{B}^3 \setminus \tilde{M}$ with a priori analytic extension. Note, that parts of ω_+ and ω_- (the connected components of $\partial\mathbb{B}^3 \setminus \tilde{M}$) can be represented as graphs over $\mathcal{G}$. Recall that

$$\mathcal{G} = \{(z_1, z_2, \xi_3) : \xi_3 > c, |z_1|^2 + |z_2|^2 + \frac{\xi_3^2}{|z_1|^2|z_2|^2} < 1 - \mathcal{E}_3^2(z_1, z_2, \xi_3)\}.$$

For $(z_1, z_2, \xi_3) \in \bar{\mathcal{G}}$ we put $|\mathcal{E}_3(z_1, z_2, \xi_3)| = (1 - |z_1|^2 - |z_2|^2 - \frac{\xi_3^2}{|z_1|^2|z_2|^2})^{1/2}$. This function is a continuous extension of $|\mathcal{E}_3|$ from the part of $\partial\mathcal{G}$ where it was defined previously. Then

$$\partial\mathbb{B}^3 \cap C(\mathcal{G}) = \{(z_1, z_2, \xi_3 + i\eta_3) : (z_1, z_2, \xi_3) \in \mathcal{G}, \eta_3 = \pm|\mathcal{E}_3(z_1, z_2, \xi_3|\}.$$

The "upper" and "lower" component of $\partial\mathbb{B}^3 \cap C(\mathcal{G})$, which correspond respectively to the sign $+$ and to the sign $-$ in the last formula, are graphs of functions over $\mathcal{G}$ and both components are contained in $\partial\mathbb{B}^3 \setminus \tilde{M}$ (more precisely, in ω_+ and ω_- respectively).

The two functions $|\mathcal{E}_3|$ and $-|\mathcal{E}_3|$ do not coincide at points $p \in \partial\mathcal{G}$ where $\mathcal{E}_3 \neq 0$ (i.e., at points of $\partial\mathcal{G}$ which are contained in Q). In this case $\mathbb{B}^3 \cap \partial C(\mathcal{G})$ containes a segment of the form $\{p + i(0, 0, \eta_3) : \eta_3 \in (-|\mathcal{E}_3|, |\mathcal{E}_3|)\}$.

Our aim is to prove that u extends analytically to all these segments. More precisely, we want to prove the following. Let $\partial_j^+(\mathcal{G})$ be the components of $\partial\mathcal{G}$ where $\mathcal{E}_3$ is negative and $\partial_j^-(\mathcal{G})$, respectively, the components where $\mathcal{E}_3$ is positive. Denote by ∂_j^+ and ∂_j^- the corresponding components of $\mathbb{B}^3 \cap \partial C(\mathcal{G})$ and by $\mathcal{A}_j^+$ and $\mathcal{A}_j^-$ the components of $\mathbb{B}^3 \setminus \overline{C(\mathcal{G})}$ which are bounded by ∂_j^+ (∂_j^-, respectively) and a subset of $\partial\mathbb{B}^3$. Note, that the following part of the boundary of ∂_j^+

$$\{p + i(0, 0, \eta_3) : p \in \partial_j^+(\mathcal{G}), \eta_3 = \mathcal{E}_3(p) < 0\}$$

is contained in $\tilde{M}$, while the part

$$\{p + i(0, 0, \eta_3) : p \in \partial_j^+(\mathcal{G}), \eta_3 = |\mathcal{E}_3(p)| > 0\}$$

is contained in $\partial\mathbb{B}^3 \setminus \tilde{M}$, hence in ω^+. We assert that the function u (in fact, the function u_+), which was originally defined in a one-sided neighbourhood of ω_+, extends to an analytic function in $\cup_j \mathcal{A}_j^+$ and on a neighbourhood of $\cup_j \partial_j^+$. The analogous assertion holds for u_-.

Assume for a moment that this assertion is true and prove that this implies claim 4. In fact, join the "upper" component of $\partial\mathbb{B}^3 \cap C(\mathcal{G})$ with $\cup_j \partial_j^+$ and with the union of all common "edges"

$$\{p + i(0, 0, \eta_3) : p \in \partial_j^+(\mathcal{G}); \eta_3 = |\mathcal{E}_3(p)| > 0\} \subset \omega_+.$$

Join to this set a part of the hypersurface which bounds Ω, namely, the set $\{(z_1, z_2, \zeta_3) \in \overline{\mathbb{B}^3} : \xi_3 = c, \eta_3 > 0\}$. The union is a manifold with boundary $\tilde{M} \cup N \cup V$. The function u_+ is analytic in a one-sided neighbourhood of the upper part of $\partial\mathbb{B}^3 \cap C(\mathcal{G})$ and by our assumption it extends analytically to the rest of the manifold and to $\cup_j \mathcal{A}_j^+$.

Move all points of the manifold slightly to the side of $\mathbb{B}^3 \cap C(\mathcal{G})$ (fixing the boundary) so that we obtain a graph of a continuous function h_+ over $\bar{\mathcal{G}}$, the boundary of the graph being $\tilde{M} \cup N \cup V$, and the function u_+ extends analytically to the domain $\mathcal{B}_+$ bounded by ω_+, by part of $\partial\Omega$ and by the graph of h_+ over $\bar{\mathcal{G}}$.

The analogous construction gives the function h_- on $\bar{\mathcal{G}}$. It is clear from the construction that $h_+ > h_-$ on $\mathcal{G}$ and $h_+ = h_-$ on $\partial\mathcal{G}$.

Prove now the mentioned assertion that the function u_+ has analytic extension to $\cup_j \mathcal{A}_j^+$ and to a neighbourhood of $\cup_j \partial_j^+$. We will use here the fact that $\partial C(Q)$ is sufficiently close to $\partial C(\mathcal{G})$ (e.g. measured in C^2 metric for ∂Q and $\partial \mathcal{G}$ which form the bases for the cylinders $\partial C(Q)$ and $\partial C(\mathcal{G})$), and $C(Q)$ is strictly pseudoconvex. The latter implies that for each point $p \in \partial C(Q)$ there exists a ball b_p (in $\mathbb{C}^3$) with center p and a biholomorphic change of coordinates in b_p which makes $b_p \cap \partial C(Q)$ strictly convex from the side of $C(Q)$. If $\partial C(\mathcal{G})$ is close enough to $\partial C(Q)$ the same is true for each point $p \in \partial C(\mathcal{G})$ and a compactness argument (for the bases ∂Q and $\partial \mathcal{G}$) allows to take the radius of the balls b_p estimated from below by a constant $r > 0$ depending only on Q. Moreover, the Hessian of the defining function of $b_p \cap \partial C(\mathcal{G})$ in the mentioned coordinates has a uniform estimate. This implies in particular, the following. If we denote by h_p the tangent space $T_p \partial C(\mathcal{G})$ to $\partial C(\mathcal{G})$ at p in the chosen coordinates then $\partial C(\mathcal{G}) \cap b_p \setminus \{p\}$ is on one side of h_p and, moreover, $\partial C(\mathcal{G}) \cap \partial b_p$ has distance $\geq ar^2$ from $h_p \cap \partial b_p$ for some positive constant a depending only on Q.

Let $p \in \partial_j^+$ for some j (hence, $p \in C(\mathcal{G}) \cap \mathbb{B}^3 \subset \partial C(\mathcal{G}) \setminus \partial C(Q)$). If the C^2 distance of $\partial C(Q)$ and $\partial C(\mathcal{G})$ is small compared to the value ar^2, then $\partial C(Q) \cap \partial b_p$ is closer to $\partial C(\mathcal{G}) \cap \partial b_p$ than to $h_p \cap \partial b_p$ and hence $\partial C(Q) \cap \partial b_p$ is on the same side of h_p as $\partial C(\mathcal{G}) \cap \partial b_p$.

Denote by $\sqcap_p$ the closed halfspace (in b_p) which corresponds to h_p and does not contain points of $\partial C(\mathcal{G}) \setminus \{p\}$. The just mentioned arguments imply that $\partial C(Q) \cap \sqcap_p$ is compact (in other words $\partial C(Q) \cap \sqcap_p$ is separated away from ∂b_p). Moreover, no point in $\partial C(Q) \cap \sqcap_p$ is contained in $\partial C(\mathcal{G})$ (since $\partial C(\mathcal{G}) \setminus \{p\} \subset b_p \setminus \sqcap_p$ and $p \notin \partial C(Q)$).

Note that by the definition of Q the inclusion $\Omega \cap \overline{\mathbb{B}^3} \subset \overline{C(Q)}$ holds, hence $\partial \mathbb{B}^3 \setminus \overline{C(\mathcal{G})} \subset \overline{C(Q)} \setminus \overline{C(\mathcal{G})}$. It follows that $\partial \mathbb{B}^3 \cap \sqcap_p$ is compact (i.e. separated away from ∂b_p). Applying e.g. Kontinuitätssatz we obtain analytic continuation u_p of u to a neighbourhood of $\mathbb{B}^3 \cap \sqcap_p$. Replace $\sqcap_p$ by a relatively closed subset of $\sqcap_p$ which is convex (in Euclidean coordinates) and contains the point p. E.g. we may take $b_p \cap K_p$, where K_p is a closed convex cone with vertex p and symmetry axis being the outer normal to $C(\mathcal{G})$ at p.

Each point which is close to the strictly pseudoconvex part of $\partial C(\mathcal{G})$ and not contained in $C(\mathcal{G})$ is contained in $b_p \cap K_p$ for some p. In particular, this holds for points in $\mathbb{B}^3 \setminus C(\mathcal{G})$.

We need to show that if for some p and p' in a connected component $\mathcal{A}_j^+ \cup \partial_j^+$ the convex (and hence connected) set $b_p \cap K_p \cap K_{p'} \cap \mathbb{B}^3$ is non-empty, then $u'_p = u_p$ on this set. This holds if p and p' are close, since then the set contains points in $\mathbb{B}^3$ which are close to $\partial \mathbb{B}^3$. A connectedness argument gives the required equality for all $p, p' \in \mathcal{A}_j^+ \cup \partial_j^+$. We obtained a well-defined analytic function in a neighbourhood of $\mathcal{A}_j^+ \cup \partial_j^+$. $\square$

Note that here we gave an alternative proof of the fact that $E(\partial \mathbb{B}^3 \setminus \; \bar{\tilde{M}})$

is multi-sheeted. It avoided the construction of a holomorphic function in the one-sided neighbourhood U which separates W^+ and W^-. Instead we found a pseudoconvex Riemann domain over $\mathbb{C}^3$ which contains U and non-intersecting copies of W^+ and W^- with edges m^+ and m^- glued to U. We are not able to construct such Riemann domains in the general case.

Question 1. *Let $\mathcal{H}$ be a smooth Levi-flat hypersurface in $\mathbb{C}^n, n \geq 3$. Let M be a smooth $(2n-2)$-manifold contained in $\mathcal{H} \cap \partial\mathbb{B}^n$ such that its boundary ∂M is the finite union of disjoint connected compact maximally complex CR-manifolds of dimension $2n-3$. Let $\tilde{M}$ be a smooth generic globally minimal CR-manifold such that $\bar{\tilde{\tilde{M}}}$ is close (in C^4) to $\bar{M}$ and has the same boundary as M. Is $E(\partial\mathbb{B}^n \setminus \bar{\tilde{\tilde{M}}})$ two-sheeted?*

Question 2. *Can one extend the theorems to the case when $\mathbb{B}^n$ is replaced by any smoothly bounded strictly pseudoconvex domain G (not necessarily such that $H^2(G, \mathbb{Z}) = 0$)?*

Question 1 seems to be related to the following question asked by Tomassini [To].

Question 3. *Let $\mathcal{H}$ be a relatively closed Levi-flat hypersurface contained in $\mathbb{B}^n (n \geq 2)$. Does it have a Stein neighbourhood basis?*

Question 4. *Let K be as in theorem 1 or be equal to $\bar{\tilde{\tilde{M}}}$ in examples 1 or 2. What is the relation between the polynomial hull $\hat{K}$ of K and the closure of the set over which the essential envelope of holomorphy of $\mathbb{B}^n \setminus K$ is multi-sheeted?*

Note that the latter set is contained in $\hat{K}$ (see [ČSt]).

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