UNITARY REPRESENTATIONS OF GENERAL LINEAR GROUP OVER REAL AND COMPLEX FIELD

by

Marko Tadić

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Department of Mathematics
University of Zagreb
41001 Zagreb
Yugoslavia

MPI/SFB 85- 22

Introduction

Let G be a reductive Lie group. Fundamental question of harmonic analysis on the group G is to describe the unitary dual $\hat{G}$ of G. The unitary dual of G is the set of all equivalence classes of irreducible unitary representations of G. Harish-Chandra created theory of non-unitary representations mainly in order to study unitary representations. That ideas lead to the study of the unitary dual of G as the set of all unitarizable classes in the non-unitary dual $\tilde{G}$ of G. As the problem of non-unitary dual has been solved ([15] and [13]), it remains to solve the problem of unitarizability of classes in $\tilde{G}$.

This paper solves the unitarizability problem for groups $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. This problem is very closely related to the same problem for groups $SL(n,\mathbb{R})$ and $SL(n,\mathbb{C})$.

The history of the problem of unitary dual in the mentioned case is long, and this author will give only a few remarks about the history. The case of $n = 2$ was solved in 1947 by V. Bargmann, I.M. Gelfand and M.A. Naimark (see [1] and [7]). I.M. Gelfand and M.A. Naimark in 1950 gave a list of unitary representations of $SL(n,\mathbb{C})$ for which they presumed that exaust the whole unitary dual of $SL(n,\mathbb{C})$ ([6]). E.M. Stein showed in 1967 that the list of I.M. Gelfand and M.A. Naimark can not be complete, by constructing a new unitary representations ([22]). Since 1967 there was no construction of new elements of $SL(n,\mathbb{C})^\wedge$. Note that representations constructed by I.M. Gelfand, M.A. Naimark and E.M. Stein have a simple description.

In the case of $GL(n,\mathbb{R})$ it is possible to make similar constructions of unitary representations to that of I.M. Gelfand, M.A. Naimark and E.M. Stein. B. Speh obtained in [20] a new family of irreducible unitary representations of $GL(n,\mathbb{R})$.

It could be interesting to mention that the unitarizability problem for $GL(n)$ over non-archimedean local fields is solved in [23] and [27] (see also [25] and [24]).

Now we shall describe the main result of this paper. Let F be either $\mathbb{R}$ or $\mathbb{C}$. A character $g \longmapsto |\det g|_F$, $GL(n,F) \longrightarrow \mathbb{R}^\times$ is denoted by ν. Here $| \, |_F$ denotes the normalized absolute value on F. Set

$$\mathrm{Irr}^u = \bigcup_{n\geq 0} GL(n,F)^\wedge$$

Let D^u be the set of all sequare integrable classes in $\bigcup_{n\geq 1}\hat{G}_n$. Note that $D^u = GL(1,F)^\wedge$ if $F = \mathbb{C}$, and $D^u \subseteq GL(1,F)^\wedge \cup GL(2,F)^\wedge$ if $F = \mathbb{R}$.

Let n be a positive integer and $\delta \in D^u$. Suppose that $\delta \in GL(k,F)^\wedge$. The representation of $GL(nk,F)$ parabolically induced by

$$\nu^{\frac{n-1}{2}}\delta \otimes \nu^{\frac{n-1}{2}-1}\delta \otimes \ldots \otimes \nu^{-\frac{n-1}{2}}\delta$$

from suitable (upper) standard parabolic subgroup, has a unique irreducible quotient which will be denoted by $u(\delta,n)$. If $0 < \alpha < 1/2$, then we define $\pi(u(\delta,n),\alpha)$ to be parabolically induced representation of $GL(2nk,F)$ by

$$\nu^{\alpha}u(\delta,n) \otimes \nu^{-\alpha}u(\delta,n)$$

from suitable maximal standard parabolic subgroup.

If $\delta \in \hat{G}_1$, then $u(\delta,n): G_n \to \mathbb{C}$, $g \to \delta(\det g)$.

Now we can state our main result.

Theorem A: Let B be the set of all $u(\delta,n)$ and $\pi(u(\delta,n),\alpha)$ where $\delta \in D^u$, $0 < \alpha < 1/2$ and n is a postive integer.

(i) Let k be a non-negative integer and $\sigma_1,\ldots,\sigma_k \in B$. Denote by $\Theta((\sigma_1,\ldots,\sigma_k))$ the representation parabolically induced by $\sigma_1 \otimes\ldots\otimes \sigma_k$ from a suitable parabolic subgroup. Then

$$\Theta((\sigma_1,\ldots,\sigma_k)) \in \overline{\mathrm{Irr}^u}\ .$$

(ii) Suppose that $\Theta((\sigma_1,\ldots,\sigma_k)) = \Theta((\tau_1,\ldots,\tau_m))$ $(\sigma_i,\tau_j \in B)$. Then $k = m$ and there exists a permutation p of $\{1,2,\ldots,k\}$ such that

$$\sigma_i = \tau_{p(i)} \quad , \quad 1 \le i \le k \quad .$$

(iii) If $\pi \in \mathrm{Irr}^u$, then there exist $k \ge 0$ and $\sigma_1, \ldots, \sigma_k \in B$ such that

$$\pi = \Theta((\sigma_1, \ldots, \sigma_k)).$$

The rest of this paper is just the proof of Theorem A.

Note that by [23] and [27] the above theorem holds also for $GL(n)$ over non-archimedean fields. In that case the set D^u is much more complicated and bigger, and it is not completely classified yet.

After Theorem A, we can see that essentially all irreducible unitary representations of $GL(n,\mathbb{C})$ (or $SL(n,\mathbb{C})$) were known in 1967. Since 1967 only the completeness argument was missing.

The similar situation was with $GL(n,\mathbb{R})$. After B. Speh construction in [20] (were she proved Zuckerman conjecture for $GL(n,\mathbb{R})$) essentially all irreducible unitary representations of $GL(n,\mathbb{R})$ were known (it was possible to construct them by complementary series, induction and twisting by character form existing ones). In this case also, only the completeness argument was missing. The representations constructed by B. Speh correspond to representations $u(\delta,n)$ with δ square integrable representations of $GL(2,\mathbb{R})$.

The completeness argument presented in this paper is the archimedean version of the completeness argument for non-archimedean case, contained in our paper [23] from January, 1984.

It is interesting to mention that, as this author knows, there was missing the conjecture about completeness. The introduction of the paper [30], which was written in 1983, maybe the best indicate the situation about ideas on unitarizability. D. Vogan concluded there that no one knows what unitary dual should look like for semisimple Lie groups, and that one should try to guess as much as possible about unitary dual, to narrow and direct the search for its complete structure. See also Problem 4.8. and Table 4.9. of [30].

While in the case of archimedean $GL(n)$ only the completeness argument was missing, in the case of non-archimedean $GL(n)$ the construction of unitary representation was missing as well as the completeness argument (note that there the non-unitary dual is not classified completely yet). In the non-archimedean case the only completely settled group with respect to unitarizability was $SL(2)$ (1963, I.M. Gelfand and M.I. Graev) and closely related groups like $GL(2)$. Since in the non-archimedean case for each n, $GL(n)$ has a square integrable representations, one has to construct here unitary representations $u(\delta,m)$ for square integrable representations δ of $GL(n)$ with $n > 2$.

Recall once more that the solution of the unitarizability problem for $GL(n,\mathbb{C})$ and $GL(n,\mathbb{R})$ expressed in Theorem A, holds over any local field and note that the differences in the proofs for archimedean and non-archimedean case are not in the general ideas of the proofs but only in the technical details which comes from the nature of fields. Thus we can consider this paper as an archimedean part of the general solution of the unitarizability problem for GL(n) - groups over local fields (and completion to the papers [23]-[27]). For the case of inner forms of GL(n)-groups see Remark 2.3 .

Unitarizability of the representations constructed by B. Speh in [20] was proved by global methods. T.J. Enright, R. Parthasarathy, N.R. Wallach and J.A. Wolf proved it using cohomological methods. In the non-archimedean case we used one new method based on a simple ideas (see [23] and [24]), which is neither cohomological, nor global. This method can be applied also to $GL(n,\mathbb{R})$.

Now we shall describe briefly the content of this paper. In the first section the notation used in this paper is introduced. Some basic results from representation theory of $GL(n,F)$ are formulated in the way we use them in the following sections. We are using the ideas of A.V. Zelevinsky presented in [28]. The second section is short, but it is the most important part of our paper. At first we formulate five technical statements, (U0),(U1),...,(U4), and then we show that (U0)-(U4) implies (i),(ii) and (iii) of Theorem A. In this way Theorem A is reduced to a few technical statements. In the third section we check (U0)-(U4) when $F = \mathbb{C}$. A conjecture on ends of complementary series in the case $F = \mathbb{C}$, is formulated in the fourth section. This conjecture is related to a description of the natural topology of $GL(n,\mathbb{C})^\wedge$ by use of general results of D. Miličić about the topology of the dual space of C*-algebras with bounded trace ([16]). In the fifth and sixth section we are checking (U0)-(U4) in the case $F = \mathbb{R}$. In the seventh section we formulate a conjecture on ends of complementary series in the case $F = \mathbb{R}$.

The results about non-unitary dual which we are using in this paper are standard and well known facts of representation theory of reductive Lie groups. The notation which is using in this paper is adapted to the case of the general linear group. We have used in our paper the result of A.A. Kirillov paper [12]. As this author knows, there is no written paper at the present time where all details of the proof are supplied. Also we have used a result of H. Jacquet, proof of which has not appeared as this author knows. Details of proofs of these two results (which does not belong to this author) we leave for another paper. The Kirillov result is essentially a fact about distributions, while the Jacquet result in essentially an application of Mackey theory. Note that the techniques we are using in this paper are of quite different nature from the techniques involved in the Kirillov and Jacquet result.

I want to thank to Prof. H. Kraljević for the conversations

on the representation theory of the reductive Lie groups.

At the end, it is a pleasure to me to thank Max-Planck-Institut für Mathematik for the hospitality and support during which this work was finished.

1. Algebra of representations

The field of complex numbers is denoted by $\mathbb{C}$, the subfield of reals by $\mathbb{R}$, the subring of integers by $\mathbb{Z}$. The set of non-negative integers in $\mathbb{Z}$ is denoted by $\mathbb{Z}_+$, and the subset of positive ones is denoted by $\mathbb{N}$.

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Set

$$G_n = GL(n,F), \quad n \in \mathbb{Z}_+ .$$

Let K_n be a maximal compact subgroup of G_n. We can take K_n to be $U(n)$ if $F = \mathbb{C}$ and $K_n = O(n)$ if $F = \mathbb{R}$. The groups G_n are regarded as real Lie groups. Let q_n be the Lie algebra of G_n.

The category of all Harish-Chandra modules of G_n of finite length is denoted by $HC(G_n)$. Let R_n be the Grothendieck group of $HC(G_n)$. The set of all equivalence classes of irreducible Harish-Chandra modules of G_n is denoted by $\tilde{G}_n$. We shall identify an irreducible Harish-Chandra module with its class. The set of all unitarizable classes in $\tilde{G}_n$ is denoted by $\hat{G}_n$. The set $\tilde{G}_n$ will be identified with a subset of R_n in a natural way. In that case, R_n is a free $\mathbb{Z}$-module over $\tilde{G}_n$. We have a natural map

$$HC(G_n) \longrightarrow R_n,$$
$$\pi \longrightarrow \pi^{ss}$$

which we shall call semi-simplification.

Let $\pi \in R_n$. We say that π contains $\sigma_0 \in \tilde{G}_n$ if there exist $\sigma_0,\ldots,\sigma_k \in \tilde{G}_n, k \in \mathbb{Z}_+$, such that

$$\pi = \sigma_0 + \ldots + \sigma_k .$$

We say that π contains σ_0 with multiplicity one if $\sigma_i \neq \sigma_0$, $i = 1,\ldots,k$.

Suppose that π is a continuous representation of G_n on a Hilbert space. If the representation $\pi|K_n$ decomposes with finite multiplicities, then we say that π is an admissible representation. Let π be an admissible representation of G_n. Then the representation of $(\mathfrak{g}_n, K_n)$ on the space of K_n-finite vectors is called a Harish-Chandra module of π. If π is a finite length representation, then the semi-simplification of the Harish-Chandra module of π is denoted by π^{ss}.

Set $$R = \bigoplus_{n \geq 0} R_n .$$

Then R is a graded commutative group. We shall define the structure of a graded ring on R. For it, it is enough to define a $\mathbb{Z}$-linear mapping

$$\cdot \; R_{n_1} \times R_{n_2} \longrightarrow R_{n_1+n_2} , \quad n_1, n_2 \in \mathbb{Z}_+ ,$$

in fact, it is enough to define $\sigma_1 \times \sigma_2$ for $\sigma_i \in \tilde{G}_{n_i}, i = 1,2.$

Let $P = MN$ be the standard parabolic subgroup of $G_{n_1+n_2}$ given by

$$P = \{g = (g_{ij}) \in G_{n_1+n_2} \; ; \; g_{ij} = 0 \quad \text{if} \quad i > n_1 \quad \text{and} \; j \leq n_1\}$$

$$M = \{g = (g_{ij}) \in P; \; g_{ij} = 0 \text{ if } \; i \leq n_1 \quad \text{and} \quad j > n_1\} \; .$$

The unipotent radical of P is denoted by N. Let $\sigma_i \in \text{Alg}\, G_{n_i}$, $i = 1,2$. The tensor product $\tau = \sigma_1 \otimes \sigma_2$ is $(\mathfrak{g}_{n_1} \times \mathfrak{g}_{n_2}, K_{n_1} \times K_{n_2})$-module. Since M is naturally isomorphic to $G_{n_1} \times G_{n_2}, \mathfrak{g}_{n_1} \times \mathfrak{g}_2$ is considered as Lie algebra of M and $K_{n_1} \times K_{n_2}$ is considered as a maximal compact subgroup in M. Let τ^* be an admissible representation of M whose Harish-Chandra module is isomorphic to τ. We extend τ^* to a representation of P defining action of N to be trivial. Let $\pi^* = \text{Ind}\,(\tau^* | P, G_{n_1+n_2})$ be the induced representation of $G_{n_1+n_2}$ from P by τ^*. Here Ind denotes normalized induction (for definition one can see, for example, Definition 4.1.11 of [29]). The representation π^* is admissible. We shall denote by $\sigma_1 \times \sigma_2$ the Harish-Chandra module of π. Proposition 4.1.12. of [29] implies that $\sigma_1 \times \sigma_2$ is of the finite length. Therefore $(\sigma_1 \times \sigma_2)^{ss} \in R_{n_1+n_2}$.

Let $s,t \in R$. Then we can write

$$s = \sum_{\sigma \in \tilde{G}_{n_1}} a_\sigma \sigma \quad , \; a_\sigma \in \mathbf{Z}$$

$$t = \sum_{\tau \in \tilde{G}_{n_2}} b_\tau \tau \quad , \; b_\tau \in \mathbf{Z},$$

where $a_\sigma \neq 0$ only for finitely many σ and $b_\tau \neq 0$ only for finitely many τ. The above expressions are unique. Now we define

$$s \times t = \sum_{\substack{\sigma \in \mathfrak{S}_{n_1} \\ \tau \in \mathfrak{S}_{n_2}}} (s_\sigma b_\tau)\ (\sigma \times \tau)^{ss} .$$

Note that we could introduce R_n as the group of virtual characters. Then the multiplication in R correspond to inducing of characters. The formula for the character of the induced representation can be found in [18].

For $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Irr}$ we have the following equalities in R

$$\sigma_1 \times \sigma_2 = \sigma_2 \times \sigma_1 ,$$

$$\sigma_1 \times (\sigma_2 \times \sigma_3) = (\sigma_1 \times \sigma_2) \times \sigma_3 .$$

The first relation is a consequence of Proposition 4.1.20 of [29] about induction from associated parabolic subgroups. The induction by stages implies the second relation (Proposition 4.1.18 of [29]). Thus we have:

1.1. <u>Proposition:</u> The induction functor induces on

$$R = \bigoplus_{n \geq 0} R_n$$

a structure of a commutative associative graded $\mathbb{Z}$-algebra.

For a finer information about the ring R we need to consider a classification of irreducible (q_n, K_n)-modules since $\tilde{G}_n$ is $\mathbb{Z}$-basis of R_n.

Set
$$\mathrm{Irr} = \bigcup_{n=0}^{\infty} \tilde{G}_n ,$$
$$\mathrm{Irr}^u = \bigcup_{n=0}^{\infty} \hat{G}_n .$$

Clearly, Irr is a basis of $\mathbb{Z}$-module R.

The set of all equivalence classes of irreducible unitary representations of G_n (on Hilbert spaces) which are square integrable is denoted by $D^u(G_n)$. Set

$$D^u = \bigcup_{n=1}^{\infty} D^u(G_n).$$

Let $|\,|_F$ be the normalized absolute value on F (recall that $|\,|_{\mathbb{C}}$ is the square of usual absolute value of complex numbers). Define $\nu : G_n \longrightarrow \mathbb{R}$ by

$$\nu(g) = |\det g|_F .$$

Set
$$D(G_n) = \{\nu^{\alpha}\pi \ ; \ \alpha \in \mathbb{R}, \pi \in D^u(G_n)\} ,$$
$$D = \bigcup_{n=0}^{\infty} D(G_n).$$

If $\delta \in D$ then $e(\delta) \in \mathbb{R}$ and $\delta^u \in D^u$ are uniquely determined by the relation

$$\delta = \nu^{e(\delta)} \delta^u .$$

Note that $\pi \to \pi^{ss}$, $D \to \mathrm{Irr}$ is an injection and we identify D with the subset of Irr in this way.

Let X be a set. A function $f: X \longrightarrow \mathbb{Z}_+$ with the finite support is called a finite multiset in X. The set of all finite multisets in X is denoted by $M(X)$. The set $M(X)$ is an additive semigroup in a natural way. Let $f \in M(X)$. Suppose that $\{x_1, \dots, x_n\}$ is the support of f. Then we shall write f in the following way

$$f = (\underbrace{x_1, \dots, x_1}_{f(x_1)\text{-times}}, \underbrace{x_2, \dots, x_2}_{f(x_2)\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{f(x_n)\text{-times}})$$

If $f \in M(X)$, then we write

$$\mathrm{card}\, f = \sum_{x \in X} f(x).$$

We call card f the cardinal number of the multiset f.

The number $f(x)$, $f \in M(X)$, $x \in X$, will be called a multiplicity of x in f.

A subgroup of G_n is called a standard parabolic subgroup if it contains the subgroup of all upper triangular matrices of G_n.

Let $a \in M(D)$. Choose $\delta_i \in D(G_{n_i})$, $i = 1,\ldots,k$, so that $a = (\delta_1,\ldots,\delta_k)$ and $e(\delta_1) \geq e(\delta_2) \geq \ldots \geq e(\delta_k)$. Let P be the standard parabolic subgroup whose Levi factor is naturally isomorphic to

$$G_{n_1} \times G_{n_2} \times \cdots \times G_{n_k} \quad .$$

Now

$$\operatorname{Ind}(\delta_1 \otimes \delta_2 \otimes \cdots \otimes \delta_{n_k} \mid P, G_{n_1+\cdots+n_k})$$

has a unique irreducible quotient representation whose Harish-Chandra module will be denoted by $L(a)$. Set

$$\lambda(a) = \delta_1 \times \cdots \times \delta_k \in R.$$

Then $\lambda(a)$ contains $L(a)$.

The mapping

$$a \longmapsto L(a)$$
$$M(D) \longrightarrow \operatorname{Irr}$$

is a bijection. This is a version of Langlands classification of non-unitary dual of $GL(n)$'s. The way we present Langlands classification for GL(n)'S should be equal to the way of R.P. Langlands in [14], essentially equal to the way of H. Jacquet in [11].

If $\pi \in HC(G_n)$, then $\tilde{\pi}$ denotes the contragradient of π and $\overline{\pi}$ the conjugated module to π. We denote $\tilde{\overline{\pi}}$ by π^+ and call a Hermitian contragradient π. If π is isomorphic to π^+, then π is called a Hermitian representation.

For $a = (\delta_1, \ldots, \delta_n) \in M(D)$, $\alpha \in \mathbb{R}$, set

$$\begin{aligned}
\tilde{a} &= (\tilde{\delta}_1, \ldots, \tilde{\delta}_n) \\
\overline{a} &= (\overline{\delta}_1, \ldots, \overline{\delta}_n) \\
a^+ &= (\delta_1^+, \ldots, \delta_n^+) \\
\nu^\alpha a &= (\nu^\alpha \delta_1, \ldots, \nu^\alpha \delta_n).
\end{aligned}$$

If $\delta \in D$, then

$$\begin{aligned}
\delta &= \nu^{e(\delta)} \delta^u, \\
\overline{\delta} &= \nu^{e(\delta)} (\overline{\delta^u}) \\
\tilde{\delta} &= \nu^{-e(\delta)} (\delta^u)^\sim \\
\delta^+ &= \nu^{-e(\delta)} \delta^u \\
\nu^\alpha \delta &= \nu^{e(\delta)+\alpha} \delta .
\end{aligned}$$

1.2. <u>Proposition</u>: For $a \in M(D)$, $a \in \mathbb{R}$, we have

$$\begin{aligned}
\overline{L(a)} &= L(\overline{a}) \\
L(a)^+ &= L(a^+) \\
L(a)^\sim &= L(\tilde{a}) \\
\nu^\alpha L(a) &= L(\nu^\alpha a).
\end{aligned}$$

<u>Proof</u>: The first relation is obvious and it implies that the second and the third relation are equivalent.

The second relation is proved in the proof of Theorem 7 in [13] (see also [11],[5] and [19]).

The fourth relation can be proved directly by constructing of intertwining operator between induced representations, which restricts to an equivalence between $\nu^{\alpha}L(a)$ and $L(\nu^{\alpha}a)$.

1.3. <u>Proposition</u>: The ring R is a $\mathbb{Z}$-polynomial ring over indeterminates D. This means that

$$\{\lambda(a); a \in M(D)\}$$

is a $\mathbb{Z}$-basis of R.

<u>Proof:</u> This is well known fact because $\lambda(a)$, $a \in M(D)$ correspond to the standard characters which form a basis of the group of all virtual characters (for a fixed reductive Lie group). In fact, the proposition can be proved easily directly using [9] and properties of Langlands classification Lemma 3.4. and 5.5 of this paper also implies the proposition.

1.4. <u>Corollary:</u> (i) The ring R is factorial.

(ii) If $\delta \in D$, then δ is prime.

(iii) Let $\pi \in R$ be a homogenius element of the graded ring R, $\pi \neq 0$. Suppose that

$$\pi = \sigma_1 \times \sigma_2 \ , \sigma_1, \sigma_2 \in R.$$

Then δ_1 and δ_2 are homogenious elements .

(iv) The group of invertible elements in R is

$$\{L(\emptyset) , - L(\emptyset)\}.$$

Note that $L(\emptyset)$ is identity in R.

Proof: Proposition 1.3 implies (i) and (ii). Proposition 1.3 implies that R is an integral ring. This implies (iii). From (iii) we obtain (iv) directly.

1.5. Remark: The mappings

$$\pi \longrightarrow \bar{\pi}$$
$$\pi \longrightarrow \tilde{\pi}$$
$$\pi \longrightarrow \pi^+$$
$$\pi \longrightarrow \nu^\alpha \pi$$

induce automorphisms of graded ring R

$$- \sim + \nu^\alpha : R \longrightarrow R.$$

The first three automorphisms are involutive. Each of these automorphisms can be described by a permutation of indeterminates D. We say that $f \in R$ is Hermitian if $f = f^+$.

We shall need one more general fact about classification L: if $a,b \in M(D)$, then $L(a) \times L(b)$ contains $L(a+b)$.

This will be proved later.

2. Formal approach to unitary dual of general linear group

In this section F is either $\mathbb{R}$ of $\mathbb{C}$. For $\delta \in D$ and $n \in \mathbf{N}$ set

$$a(\delta,n) = (\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-1}{2}-1}\delta, \ldots, \nu^{-\frac{n-1}{2}}\delta),$$

$$u(\delta,n) = L(a(\delta,n)).$$

If $n = 0$, then we set $a(\delta,0) = \emptyset$ and $u(\delta,0) = L(\emptyset)$. The first section implies

$$\nu^{\alpha}u(\delta,n) = u(\nu^{\alpha}\delta,n) \quad , \quad \alpha \in \mathbb{R} .$$

Note that $a(\delta,n) = a(\delta,n)^{+}$ if $\delta \in D^{u}$. If $\sigma \in \mathrm{Irr}$ and $\alpha \in \mathbb{R}$, then we define

$$\pi(\sigma,\alpha) = (\nu^{\alpha}\sigma) \times (\nu^{-\alpha}\sigma^{+}) \in R.$$

Clearly, $\pi(\sigma,\alpha)$ are Hermitian elements of R. Note that $\pi(\sigma,\alpha) = \pi(\sigma,-\alpha)$ if σ is Hermitian.

In this section we shall describe the unitary dual of general linear group assuming that some facts hold. We shall deal with that facts in the following sections. Note that for the unitary duals of general linear groups it is enough to describe Irr^{u}.

We consider the following statements

(U0) If $\sigma,\tau \in \mathrm{Irr}^u$ then $\sigma \times \tau \in \mathrm{Irr}^u$.

(U1) If $\delta \in D^u$ and $n \in \mathbf{N}$, then $u(\delta,n) \in \mathrm{Irr}^u$.

(U2) If $\delta \in D^u$, $n \in \mathbf{N}$ and $0 < \alpha < 1/2$, then

$$\pi(u(\delta,n),\alpha) \in \mathrm{Irr}^u.$$

(U3) If $\delta \in D$ and $n \in \mathbf{N}$, then $u(\delta,n)$ is a prime element of the factorial ring R.

(U4) If $a,b \in M(D)$, then $L(a) \times L(b)$ contains $L(a+b)$.

Note that (U4) is a general statement about nonunitary dual and (U0) is a general statement about unitary dual.

The fact that $\pi \in \mathrm{Irr}$ is a prime element of R implies that π is not induced from proper parabolic subgroup. Therefore the property "π is a prime in R" is a strengthen of the property "π is not induced from proper parabolic subgroup". In proving (U3), we usualy prove a stronger fact than (U3).

<u>2.1.Proposition</u>: Suppose that(U0)-(U4) holds. Then Irr^u is a multiplicative semigroup and it is a free abelian semigroup over the basis

$$B = \{u(\delta,n), \pi(u(\delta,n),\alpha);\delta \in D^u, n \in \mathbf{N},\ 0 < \alpha < 1/2\}.$$

In other words:

(i) If $\pi_1,\ldots,\pi_i \in B$ then

$$\pi_1 \times \pi_2 \times \cdots \times \pi_i \in \mathrm{Irr}^u$$

(ii) If $\pi \in \mathrm{Irr}^u$, then there exist $\pi_1,\ldots,\pi_j \in \mathrm{Irr}^u$

unique up to a permuation, so that

$$\pi = \pi_1 \times \cdots \times \pi_j .$$

Proof: By (U0) Irr^u is a multiplicative semigroup. The statements (U1) and (U2) implies $B \subseteq \mathrm{Irr}^u$. Therefore (i) holds.

If $\pi_1 \times \cdots \times \pi_i = \sigma_1 \times \cdots \times \sigma_j$, $\pi_1, \ldots, \pi_i, \sigma_1, \ldots, \sigma_j \in B$ then (U3) implies that $i = j$ and that the sequences $\pi_1, \ldots, \pi_i$ and $\sigma_1, \ldots, \sigma_j$ differs up to a permutation.

It remains to prove the existance of presentation in (ii). Let $\pi \in \mathrm{Irr}^u$. Choose $a \in M(D)$ so that $\pi = L(a)$. Since π is unitarizable, π is Hermitian i.e. $\pi = \pi^+$. By Proposition 1.4 we have $a = a^+$. Recall that for $\delta = \nu^{e(\delta)}\delta^u \in D$ we have $\delta^+ = \nu^{-e(\delta)}\delta^u$. Therefore we can find $\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_m \in D^u$ and positive numbers $\alpha_1, \ldots, \alpha_m$ so that

$$a = (\gamma_1, \ldots, \gamma_m, \nu^{\alpha_1}\delta_1, \nu^{-\alpha_1}\delta_1, \nu^{\alpha_2}\delta_2, \nu^{-\alpha_2}\delta_2, \ldots, \nu^{\alpha_m}\delta_m, \nu^{-\alpha_m}\delta_m)$$

(cases $m = 0$ or $n = 0$ are possible). After a change of numeration, we can assume that $\alpha_1, \ldots, \alpha_u \in (1/2)\mathbb{Z}$ and $\alpha_{u+1}, \ldots, \alpha_m \notin (1/2)\mathbb{Z}$ for some $0 \leq u \leq m$. Now we can introduce $\sigma_1, \ldots, \sigma_v \in D^u$ and positive numbers $\beta_1, \ldots, \beta_v$ so that

$$a = (\gamma_1, \ldots, \gamma_n, \nu^{\alpha_1}\delta_1, \nu^{-\alpha_1}\delta_1, \ldots, \nu^{\alpha_u}\delta_u, \nu^{-\alpha_u}\delta_u, \nu^{\beta_1}\sigma_1, \nu^{-\beta_1}\sigma_1, \ldots$$
$$\ldots, \nu^{\beta_v}\sigma_v, \nu^{-\beta_n}\sigma_v).$$

Recall that we have now

$$\gamma_1,\ldots,\gamma_n,\delta_1,\ldots,\delta_u,\sigma_1,\ldots,\sigma_v \in D^u \quad \text{and}$$

$$\alpha_1,\ldots,\alpha_u,\beta_1,\ldots,\beta_v$$

positive numbers such that

$$\alpha_1,\ldots,\alpha_u \in (1/2)\mathbb{Z}$$

$$\beta_1,\ldots,\beta_v \notin (1/2)\mathbb{Z}.$$

The case of $n = 0$ or $u = 0$ or $v = 0$ is possible. Take $r_1,\ldots,r_v \in \mathbb{R}$ and $m_1,\ldots,m_v \in (1/2)\mathbb{Z}$ so that

$$\beta_i = r_i + m_i \ , \ i = 1,\ldots,v$$

and $0 < r_1,\ldots,r_v < 1/2$. Now $m_1,\ldots,m_v \geq 0$.

One can see directly that

$$(\nu^{\alpha_i}\delta_i,\nu^{-\alpha_i}\delta_i) + a(\delta_i,2\alpha_i-1) = a(\delta_i,2\alpha_i+1), \ i = 1,\ldots,u$$

$$(\nu^{m_j}\delta_j) + \nu^{-1/2}a(\sigma_j,2m_j) = a(\sigma_j,2m_j+1), \quad i = 1,\ldots,v$$

The second relation implies

$$(\nu^{\beta_j}\delta_j,\nu^{-\beta_j}\sigma_j) + \nu^{r_j-1/2}a(\sigma_j,2m_j) + \nu^{1/2-r_j}\, a(\sigma_j,2m_j) =$$

$$=(\nu^{r_j+m_j}\sigma_j,\nu^{-r_j-m_j}\sigma_j) + \nu^{r_j-1/2}a(\sigma_j,2m_j) + \nu^{1/2-r_j}a(\sigma_j,2m_j) =$$

$$= \nu^{r_j}a(\sigma_j,2m_j+1) + \nu^{-r_j}a(\sigma_j,2m_j+1),$$

$j = 1,\ldots,v.$

In the rest of the proof we shall use the following property. Let $a_1,a_2 \in M(D)$. Suppose that $L(a_1),L(a_2)$ are unitarizable. Then (U0) and (U4) implies

$$L(a_1) \times L(a_2) = L(a_1 + a_2).$$

By induction we obtain that

$$L(a_1) \times L(a_2) \times \ldots \times L(a_k) = L(a_1 + a_2 + \ldots + a_k)$$

if $a_1,\ldots,a_k \in M(D)$ so that $L(a_1),\ldots,L(a_k) \in \mathrm{Irr}^u$.

Now we shall finish the proof. We compute

$$\pi \times u(\delta_1,2\alpha_1-1) \times \ldots \times u(\delta_u,2\alpha_u-1)\times \pi(u(\sigma_1,2m_1),r_1-1/2)\times \ldots \times(u(\sigma_v,2m_v), r_v-1/2) =$$

$$= L(\gamma_1,\ldots,\gamma_m,\nu^{\alpha_1}\delta_1,\nu^{-\alpha_1}\delta_1,\ldots,\nu^{\alpha_u}\delta_u,\nu^{-\alpha_u}\delta_u,\nu^{\beta_1}\sigma_1,\nu^{-\beta_1}\sigma_1,\ldots, \nu^{\beta_v}\sigma_v,\nu^{-\beta_v}\sigma_v) \times$$

$$\times L(a(\delta_1,2\alpha_1 - 1)) \times \ldots \times L(a(\delta_u,2\alpha_u - 1)) \times$$

$$\times\ L(\nu^{r_1-1/2}a(\sigma_1,2m_1)+\nu^{1/2-r_1}a(\sigma_1,2m_1))\times\ldots\times L(\nu^{r_v-1/2}a(\sigma_v,2m_v)+$$

$$+\ \nu^{1/2-r_v}a(\sigma_v,2m_v))$$

$$= L((\gamma_1,\ldots,\gamma_n)+[(\nu^{\alpha_1}\delta_1,\nu^{-\alpha_1}\delta_1)+a(\delta_1,2\alpha_1-1)]+\ldots+[(\nu^{\alpha_u}\delta_u,\nu^{-\alpha_u}\delta_u)+$$

$$a(\delta_u,2\alpha_u-1)]+[(\nu^{-\beta_1}\sigma_1,\nu^{\beta_1}\sigma_1)+\nu^{r_1-1/2}a(\sigma_1,2m_1)+\nu^{1/2-r_1}a(\sigma_1,2m_1)]+$$

$$\ldots+[(\nu^{\beta_v}\sigma_v,\nu^{-\beta_v}\sigma_v)+\nu^{r_v-1/2}a(\sigma_v,2m_v)+\nu^{1/2-r_v}a(\sigma_v,2m_v)]) =$$

$$= L((\gamma_1,\ldots,\gamma_n)+a(\delta_1,2\alpha_1+1)+\ldots+a(\delta_u,2\alpha_u+1)+\nu^{r_1}a(\sigma_1,2m_1+1)$$

$$+\ \nu^{-r_1}a(\sigma_1,2m_j+1)+\cdots+\nu^{r_v}a(\sigma_v,2m_v+1)+\nu^{-r_v}a(\sigma_v,2m_v+1)) =$$

$$= L(\gamma_1)\times\cdots\times L(\gamma_n)\times L(a(\delta_1,2\alpha_1+1))\times\cdots\times L(a(\delta_u,2\alpha_u+1))\ \times$$

$$\times\ \pi(L(a(\sigma_1,2m_1+1),r_1)\times\cdots\times\pi(L(a(\sigma_v,2m_v+1),r_v)) =$$

$$= u(\gamma_1,1)\times\cdots\times u(\gamma_n,1)\times u(\delta_1,2\alpha_1+1)\times\cdots\times u(\delta_u,2\alpha_u+1)\ \times$$

$$\times\ \pi(u(\sigma_1,2m_1+1),r_1)\times\cdots\times\pi(u(\sigma_v,2m_v+1),r_v).$$

Thus π divides

$$u(\gamma_1,1)\times\cdots\times u(\gamma_n,1)\times u(\delta_1,2\alpha_1+1)\times\cdots\times u(\delta_u,2\alpha_u+1)\ \times$$

$$\times\ \pi(u(\sigma_1,2m_1+1),r_1)\times\cdots\times\pi(u(\sigma_v,2m_v+1),r_v).$$

Now (U3) implies that π is a product of some

$$u(\gamma_i,1),u(\delta_j,2\alpha_j+1),\ \nu^{r_k}u(\sigma_k,2m_k+1),\nu^{-r_k}u(\sigma_k,2m_k+1).$$

The fact that π is Hermitian implies that π is a product of some

$$u(\gamma_i,1),u(\delta_j,2\alpha_j+1),\ \pi(u(\sigma_k,2m_k+1),r_k).$$

Thus we proved existance of presentation of π into a product of elements of B. This concludes the proof.

2.2 Corollary: The mapping Θ defined by

$$\Theta:(\pi_1,\dots,\pi_n) \longmapsto \pi_1\times\dots\times\pi_n$$

$$M(B) \longrightarrow \mathrm{Irr}^u$$

is an isomorphism of semigroups if (U0)-(U4) holds.

2.3 Remark: The results of the first section hold for $GL(n)$ over any local field. Proposition 2.1 together with the proof presented here, hold also over any local field (see [23]). Proposition 2.1 with slight modifications holds in a more general situation. For example, suppose that A is a local non-archimedean division algebra. It is proved in [3] that the parabolically induced representation of $GL(n,A)$ by an irreducible square-integrable representation is irreducible. Therefore the same classification presented in the first section, together with all properties mentioned there, holds for non-unitary dual of $GL(n,A)$. In this situation we need to modify definitions of $u(\delta,n)$ and $\pi(u(\delta,n),\alpha)$. Let ν be the absolute value of the reduced norm of the central simple algebra of all $n\times n$-matrices with coefficients in A. The same letter denotes the restriction of ν to $GL(n,A)$. Let δ be an irreducible square integrable representation of $GL(n,A)$. Choose $e_\delta > 0$ such that $\nu^{(1/2)e_\delta}\delta \times \nu^{-(1/2)e_\delta}\delta$ is reducible and that e_δ is a minimal number with that property. Set

$$\nu_\delta = \nu^{e_\delta} \quad .$$

Now we define

$$u(\delta,n) = L(\nu_\delta^{-\frac{n-1}{2}}\delta, \nu_\delta^{1-\frac{n-1}{2}}\delta, \ldots, \nu_\delta^{\frac{n-1}{2}}\delta) ,$$

$$\pi(u(\delta,n),\alpha) = \nu_\delta^{\alpha}u(\delta,n)\times\nu_\delta^{-\alpha}u(\delta,n) \quad .$$

With this modifications Proposition 2.1., together with the proof, holds in this situation. In fact, Proposition 2.1. holds also without these modifications, but then it is easy to see that the statement (U3) and also (U1), (U) do not hold. Therefore, Proposition 2.1. is not interesting if we do not make necessary modifications.

In this way if, we prove (UO)-(U4) for $GL(n,A)$, we shall have solution of unitarizability problem for $GL(n,A)$.

Note that the constant e_δ are related the correspondence obtained in [3] between irreducible square-integrable representations of GL(n,A) and of GL(m,F), where F is a central field of A and m a suitable integer.

We can say that all e_δ are 1 if GL(n,A) is a split group, i.e. if A is a field.

3. Complex general linear group

In the preceeding section we have shown that (U0)-(U4) implies a description of the unitary dual of $GL(n,F)$. In this section we shall show that (U0)-(U4) holds if $F = \mathbb{C}$.

In this section we take F to be $\mathbb{C}$.

It is well known that here

$$D = G_1^{\sim} = (\mathbb{C}^{\times})^{\sim} \quad ,$$
$$D^u = G_1^{\wedge} = (\mathbb{C}^{\times})^{\wedge} \quad .$$

Let $\delta \in D^u$. It means that δ is a unitary character of $\mathbb{C}^{\times}$. Then $u(\delta,n)$ is just

$$g \longrightarrow \delta(\det g), \quad G_n \longrightarrow \mathbb{C}^{\times} \ .$$

Since δ is a unitary character, $u(\delta,n)$ is an one-dimensional unitary representation of G_n. Thus, (U1) holds.

Let $\delta \in D^u, n \in \mathbb{N}$ and $0 < \alpha < 1/2$. The representation $\pi(u(\delta,n),\alpha)$ restricted to $SL(2n,\mathbb{C})$ is irreducible and unitarizable by [22]. Since G_{2n} is a product of $SL(2n,\mathbb{C})$ and its own center, we have $\pi(u(\delta,n),\alpha) \in \mathrm{Irr}$. The representation $\pi(u(\delta,n),\alpha)$ is Hermitian, so its central character is unitary. Therefore $\pi(u(\delta,n),\alpha)$ is unitarizable. Note that the fact

$$\pi(u(\delta,n),\alpha) \in \mathrm{Irr}^u$$

is also proved in [17]. We have seen that (U2) holds.

Now we shall introduce a parametrisation of D. If $\delta \in D$, then there exist a unique $n \in \mathbb{Z}$ and a unique $\beta \in \mathbb{C}$ so that

$$\delta(z) = |z|^{2\beta}\left(\frac{z}{|z|}\right)^{n} = |z|_{\mathbb{C}}^{\beta}\left(\frac{z}{|z|}\right)^{n}, \quad z \in \mathbb{C}.$$

In this case we shall write

$$\delta = \delta(\beta, n).$$

Here $| \, |$ denotes the usual absolute value on $\mathbb{C}$ and we have

$$| \, |^2 = | \, |_{\mathbb{C}}.$$

Note that $\delta(\beta,n)$ is a unitary character if and only if the real part of β is zero. The mapping

$$\mathbb{C} \times \mathbb{Z} \longrightarrow (\mathbb{C}^{x})^{\sim}$$

is an isomorphism of topological groups.

Note that

$$\delta(\beta,u)^{+} = \delta(-\bar{\beta},n)$$

and $e(\delta(\beta,n)) = \operatorname{Re} \beta$.

For a given β and n there exist a unique $x, y \in \mathbb{C}$ so that

$$x + y = 2\beta$$

$$x - y = n.$$

Then we shall write

$$\delta(\beta,n) = \gamma(x,y).$$

In this way we obtain another parametrisation of D by the set

$$\{(x,y) \in \mathbb{C}^2 \ ; \ x - y \in \mathbb{Z}\} .$$

Note that

$$\gamma(x_1,y_1)\gamma(x_2,y_2) = \gamma(x_1 + x_2, y_1 + y_2).$$

By definition

$$\gamma(x,y)(z) = |z|^{x+y} \cdot \left(\frac{z}{|z|}\right)^{x-y} .$$

Note that $\gamma(x,y)^+ = \gamma(-\bar{y},-\bar{x})$ and $e(\gamma(x,y)) = (1/2) \operatorname{Re}(x + y)$.

We shall say that $\delta_1, \delta_2 \in D$ are linked if and only if

$$\delta_1 \times \delta_2 \notin \mathrm{Irr}.$$

By [8], δ_1 and δ_2 are linked if and only if there exist $p, q \in \mathbb{Z}$, $pq > 0$, so that

$$(\delta_1 \delta_2^{-1})(z) = z^p \bar{z}^q.$$

Note that

$$\gamma(p,q)(z) = z^p \bar{z}^q.$$

If the above relation holds then we have in R

$$\delta_1 \times \delta_2 = L((\delta_1, \delta_2)) + \nu_1 \times \nu_2$$

where $\nu_1, \nu_2 \in D$ are defined by.

$$\nu_1(z) = (\bar{z})^{-q} \delta_1(z), \; \nu_2(z) = (\bar{z})^q \delta_2(z).$$

We shall write

$$(\delta_1, \delta_2) \longmapsto (\nu_1, \nu_2).$$

In that case $\nu_1 \times \nu_2 \in \mathrm{Irr}$.

Let $\delta_i = \gamma(x_i, y_i) \in D$, $i = 1,2$. Then δ_1 and δ_2 are linked if and only if

$$x_1 - x_2 \in \mathbb{Z},\ y_1 - y_2 \in \mathbb{Z} \quad \text{and} \quad (x_1 - x_2)(y_1 - y_2) > 0.$$

Note that $x_1 - x_2 \in \mathbb{Z}$ implies $y_1 - y_2 \in \mathbb{Z}$ since $x_1 - y_1, x_2 - y_2 \in \mathbb{Z}$. If δ_1 and δ_2 are linked then

$$(\gamma(x_1,y_1),\gamma(x_2,y_2)) \longmapsfrom (\gamma(x_1,y_2),\gamma(x_2,y_1)).$$

Let $\delta_i = \delta(\beta_i, n_i) \in D$, $i = 1,2$. Set

$$\delta = \delta_1\delta_2^{-1} = \delta(\beta, n)$$

where $\beta = \beta_1 - \beta_2$, $n = n_1 - n_2$. Now δ_1 and δ_2 are linked if and only if

$$2\beta \in \mathbb{Z}, 2\beta - n \in 2\mathbb{Z} \quad \text{and} \quad 2|\beta| > |n|.$$

Set

$$\nu_1 = \delta\left(\frac{\beta_1+\beta_2}{2} + \frac{n_1-n_2}{4},\ \beta_1 - \beta_2 + \frac{n_1+n_2}{2}\right)$$

$$\nu_2 = \delta\left(\frac{\beta_1+\beta_2}{2} - \frac{n_1-n_2}{4},\ -(\beta_1 - \beta_2) + \frac{n_1+n_2}{2}\right)$$

Then

$$(\delta(\beta_1,n_1),\delta(\beta_2,n_2)) \longmapsfrom (\nu_1,\nu_2).$$

Let $(\delta_1,\ldots,\delta_n) \in M(D)$. Suppose that δ_i and δ_j are linked for some $1 \leq i < j \leq n$. Choose $\nu_i,\nu_j \in D$ so that

$$(\delta_i,\delta_j) \vdash (\nu_i,\nu_j).$$

Then we shall write

$$(\delta_1,\ldots,\delta_{i-1},\delta_i,\delta_{i+1},\ldots,\delta_{j-1},\delta_j,\delta_{j+1},\ldots,\delta_n) \vdash$$

$$\vdash (\delta_1,\ldots,\delta_{i-1},\nu_i,\delta_{i+1},\ldots,\delta_{j-1},\nu_j,\delta_{j+1},\ldots,\delta_n).$$

Let $a,b \in M(D)$. Then we write

$$a < b$$

if there exist $c_1,\ldots,c_k \in M(D)$, $k \geq 1$, so that

$$b \vdash c_1,\ c_1 \vdash c_2,\ldots,c_{k-1} \vdash c_k,\ c_k = a.$$

For $a,b \in M(D)$ we write $a \leq b$ if $a = b$ or $a < b$.

We shall show that $\leq$ is a partial ordering on $M(D)$ and we shall examine some properties of this ordering.

Let $a \in M(D)$. We say that

$$a = (\delta_1,\ldots,\delta_n)$$

is a standard presentation of a if

$$e(\delta_1) \geq e(\delta_2) \geq \ldots \geq e(\delta_n).$$

Then we define

$$e(a) = (e(\delta_1), e(\delta_2), \ldots, e(\delta_n)) \in \mathbb{R}^n .$$

Let $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $y = (y_1,\ldots,y_m) \in \mathbb{R}^m$. Then we write

$$x \leq y$$

if and only if $n = m$ and the following relations hold

$$\begin{aligned} x_1 &\leq y_1 \\ x_1 + x_2 &\leq y_1 + y_2 \\ &\vdots \\ x_1+\ldots+x_n &= y_1+\ldots+y_n \end{aligned}$$

It is obvious that $\leq$ is a partial ordering on $\cup \mathbb{R}^n (n \geq 0)$.

If $x = (x_1,\ldots,x_n) \in \mathbb{R}$ then we define

$$\text{Tr } x = x_1+\ldots+x_n.$$

For $a \in M(D)$ we define $\text{Tr } a$ to be $\text{Tr } e(a)$.

We have now one simple technical lemma on the notation which we have just introduced.

3.1 Lemma: (i) Let $a,b \in M(D)$ and $a \leq b$. Then the graduations of $L(a)$ and $L(b)$ in R are the same

(ii) Fix $a \in M(D)$. The set of all $b \in M(D)$ so that

$$a \leq b \quad \text{or} \quad b \leq a$$

is finite.

(iii) Suppose that $a \leq b$ for $a,b \in M(D)$. Then $e(a) \leq e(b)$ and $\mathrm{Tr}\ a = \mathrm{Tr}\ b$. We have $a < b$ if and only if $e(a) < e(b)$.

(iv) The relation $a \leq b$ on $M(D)$ is a partial ordering.

(v) Let $a_i, b_i \in M(D)$, $i = 1,2$. Suppose that $a_i \leq b_i$, $i = 1,2$. Then

$$a_1 + a_2 \leq b_1 + b_2.$$

We have $a_1 + a_2 = b_1 + b_2$ if and only if

$$a_i = b_i, \quad i = 1,2.$$

Proof: The definition of $\leq$ on $M(D)$ implies (i).
Let $a = (\gamma(x_1,y_1),\ldots,\gamma(x_n,y_n))$. Suppose that

$$b = (\gamma(x_1^*,y_1^*),\ldots, (x_n^*,y_n^*)) \in M(D)$$

so that $a \le b$ or $b \le a$. Then

$$x_i{}^* \in \{x_1,\dots,x_n\}$$
$$y_i{}^* \in \{y_1,\dots,y_n\} \ .$$

This implies (ii).

Let $a,b \in M(D)$ and $a \dashv b$. Then direct checking implies

$$e(a) < e(b)$$

and

$$Tr(a) = Tr(b) .$$

This implies (iii).

The statement (iii) implies (iv).

Let $a_i,b_i \in M(D)$, $a_i \le b_i$, $i = 1,2$. Then the definition of $\le$ on $M(D)$ implies

$$a_1 + a_2 \le b_1 + b_2 \ .$$

If $a_i = b_i$, $i = 1,2$, then clearly $a_1 + a_2 = b_1 + b_2$. Suppose now $a_1 + a_2 = b_1 + b_2$. Let $a_1 < b_1$. Then there exists $c \in M(D)$ so that

$$a_1 \le c \dashv b_1$$

and thus

$$a_1 + a_2 \le c + a_2 \dashv b_1 + a_2 \le b_2 + a_2 .$$

Therefore $a_1 + a_2 < b_2 + a_2$ and this contradicts to

$a_1 + a_2 = b_1 + b_2$ since $\leq$ is a partial ordering on $M(D)$.

3.2 Lemma: Let $a, b \in M(D)$. Suppose that

$$b \leq a.$$

Then $\lambda(a)$ contains $L(b)$.

Proof: We shall prove the lemma by the induction with respect to the partial ordering on $M(D)$. This is possible by (ii) of Lemma 3.1.

Let c be an element in $M(D)$. Then by definition of $L(c)$, $\lambda(c)$ contains $L(c)$.

Suppose that a is an minimal element of $M(D)$. Then $b = a$ and by the above remark, $\lambda(a)$ contains $L(b) = L(a)$.

Let $a^* \in M(D)$ be arbitrary. We suppose that the statement of the lemma holds for all $a \in M(D)$ such that

$$a < a^* .$$

Let $b \in M(D)$, $b \leq a^*$. If $b = a^*$, then $\lambda(a^*)$ contains $L(b) = L(a^*)$. Thus we need to consider only the case of $b < a^*$. By the definition of $<$, there exists $c \in M(D)$ so that

$$b \leq c \dashv a^*.$$

One can see directly, using commutativity and associativity of R, that each $\pi \in \mathrm{Irr}$ which is contained in $\lambda(c)$, need to be contained in $\lambda(a^*)$. Now the inductive assumption implies the lemma.

3.3. <u>Proposition</u>: If $a, b \in M(D)$, then $\lambda(a)$ contains $L(b)$ if and only if $b \leq a$.

Proof: If $b \leq a$ then $\lambda(a)$ contains $L(b)$ by the preceeding lemma.

Suppose that $\lambda(a)$ contains $L(b)$. Applying Example 3.16 of [21] (or Corollary 3.15 of [21]) we know that either $L(a) = L(b)$ i.e. $a = b$, or $L(b)$ is in the kernel of some factor of the long intertwining operator. In our situation it means that either $L(a) = L(b)$ i.e. $a = b$ or there exist $c \longrightarrow\!\!| \; a$ such that $\lambda(c)$ contains $L(b)$, because such kernels of factors of the long intertwining operator have form $\lambda(c)$ for $c \longrightarrow\!\!| \; a$ (see Lemma 3.8 of [21]).

We obtain the proposition by induction on a, with respect to the ordering of $M(D)$.

3.4 <u>Lemma</u>: Fix $a \in M(D)$.

(i) There exist $m_b^a \in \mathbb{N}$, $b \leq a$ such that

$$\lambda(a) = \sum_{b \leq a} m_b^a \, L(b)$$

holds in R.

(ii) We have $m_b^a = 1$ i.e.

$$(a) = L(a) + \sum_{b<a} m_b^a L(b).$$

(iii) There exist $m(a,b) \in \mathbb{Z}$, $b \leq a$ such that

$$L(a) = \sum_{b \leq a} m(a,b)\lambda(b).$$

(iv) We have $m(a,a) = 1$ i.e.

$$L(a) = (a) + \sum_{b<a} m(a,b)\lambda(b).$$

(v) Let $c \in M(D)$ so that $c < a$. Suppose that there does not exist $b \in M(D)$ so that

$$c < b < a.$$

Then $m(a,c) \neq 0$.

(vi) Let $c \in M(D)$ so that $c < a$. Suppose that for $d \in M(D)$ so that

$$d \dashv a$$

we have $e(c) \nmid e(d)$. Then $m(a,c) \neq 0$.

<u>Proof</u>: Proposition 3.3. implies (i). The fact that $L(a)$ has multiplicity one in $\lambda(a)$ implies (ii) (see [28]).

We shall show (iii),(iv) and (v) simultanously by the induction on a in $M(D)$. If $a \in M(D)$ is minimal, then

$L(a) = \lambda(a)$ by (ii). Therefore (iii) and (iv) hold. Note that there is no c such that $c < a$. Therefore (v) also holds.

Let $a \in M(D)$ be arbitrary. By (ii) and the inductive assumption we have

$$(*) \qquad L(a) = \lambda(a) - \sum_{b<a} m_b^a L(b) = \lambda(a) - \sum_{b<a} m_b^a(\lambda(b) + \sum_{d<b} m(b,d)\lambda(d)).$$

After a gathering of the terms in the above presentation of $L(a)$, we obtain (iv) and also (iii). Suppose that $m(a,b) = 0$. Then (*) implies that there exist $b' \in M(D)$ so that

$$b < b' < a.$$

This proves (v).

Suppose $c \in M(D)$ is like in (vi). Let $m(a,c) = 0$. By (v), there exist $b \in M(D)$ so that

$$c < b \leq d \longrightarrow a.$$

Then $e(c) < e(d)$ by (iii) of Lemma 3.1. We obtained a contradiction. This proves (vi).

The following proposition is just (U4).

3.5. <u>Proposition</u>: If $a,b \in M(D)$, then $L(a) \times L(b)$ contains $L(a+b)$. The multiplicity is one.

<u>Proof</u>: We compute in R

$$L(a) \times L(b) = (\lambda(a) + \sum_{c<a} m(a,c)\lambda(c))(\lambda(b) + \sum_{d<b} m(b,d)\lambda(d))$$

$$= \lambda(a)\times\lambda(b) + \sum_{d<b} m(b,d)\lambda(a)\times\lambda(d) + \sum_{c<a} m(a,c)\lambda(c)\times\lambda(b)$$

$$+ \sum_{\substack{c<a\\ d<b}} m(a,c)m(b,d)\lambda(c)\times\lambda(d) = L(a+b) + \sum_{u<a+b} m_u^{a+b}L(u) +$$

$$+ \sum_{d<b} m(b,d)\Big(\sum_{u\le a+d} m_u^{a+d}\, L(u)\Big) + \sum_{c<a} m(a,c)\ \Big(\sum_{u\ge c+b} m_u^{c+b}L(u)\Big) +$$

$$+ \sum_{\substack{c<a\\ d<b}} m(a,c)m(b,d)\Big(\sum_{u\le c+d} m_u^{c+d}L(u)\Big).$$

By (v) of Lemma 3.1 we have that $L(a)\times L(b)$ contains $L(a+b)$.

Now we return to $u(\delta,n), \delta\in D$. Fix $\delta\in D, \delta = \delta(\beta,k) = \gamma(x,y)$. Then

$$a(\delta,n) = (\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-1}{2}-1}\delta,\ldots,\nu^{-\frac{n-1}{2}}\delta) =$$

$$= (\delta(\beta+\tfrac{n-1}{2},k),\delta(\beta+\tfrac{n-1}{2}-1,k),\delta(\beta+\tfrac{n-1}{2}-2,k),\ldots,\delta(\beta-\tfrac{(n-1)}{2},k)).$$

$$= (\gamma(x+\tfrac{n-1}{2},\ y+\tfrac{n-1}{2}),\gamma(x+\tfrac{n-1}{2}-1,y+\tfrac{n-1}{2}-1),\ldots,\gamma(x-\tfrac{n-1}{2},y-\tfrac{n-1}{2})).$$

In the rest of this section we shall consider R as a polynomial algebra over $\mathbb{Z}$ in indeterminates D.

3.6 Lemma: Let $\delta^*\in D$. Then the degree of the polynomial $u(\delta,n)$ in the indetermine δ^* is either zero or one.

Proof: We have

$$u(\delta,n) = \sum_{a \le a(\delta,n)} m(a(\delta,n),a)\lambda(a).$$

Let $a_0 = (\gamma(x_1,y_1),\ldots,\gamma(x_n,y_n)) \le a(\delta,n)$. By the formula for $a \dashv b$ in terms of γ-coordinates we see that

$$\{x_1,\ldots,x_n\} = \{x + \frac{n-1}{2}, x + \frac{n-1}{2} - 1, \ldots, x - \frac{n-1}{2}\}.$$

Therefore $x_1,\ldots,x_n$ are all different and thus $\gamma(x_1,y_1),\ldots$ $\ldots, \gamma(x_n,y_n)$ are all different. This implies the lemma.

The following proposition is just (U3).

3.7 <u>Proposition</u>: In the factorial ring $R, u(\delta,n)$ is a prime element.

<u>Proof</u>: Set

$$X_1 = \gamma(x + \frac{n-1}{2}, y + \frac{n-1}{2})$$

$$X_2 = \gamma(x + \frac{n-1}{2} - 1, y + \frac{n-1}{2} - 1)$$

$$X_i = \gamma(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - i + 1)$$

$$X_n = \gamma(x - \frac{n-1}{2}, y - \frac{n-1}{2}).$$

Let $1 \le i < j \le n$. Then

$$(X_i, X_j) \longmapsfrom (Y_i(i,j), Y_j(i,j))$$

where $Y_i(i,j) = \gamma(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - j + 1)$

$$Y_j(i,j) = \gamma(x + \frac{n-1}{2} - j + 1, y + \frac{n-1}{2} - i + 1).$$

Put $a_0 = (X_1, X_2, \ldots, X_n)$,

$$a_{i,j} \doteqdot (X_1, \ldots, X_{i-1}, Y_i(i,j), X_{i+1}, \ldots, X_{j-1}, Y_j(i,j), X_{j+1}, \ldots, X_n).$$

Clearly $a_0 = a(\delta, n)$ and

$$a_{i,j} < a_0, \ 1 \leq i < j \leq n.$$

We have

$$e(a_0) = (\frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, -\frac{n-1}{2}) + (\mathrm{Re}(\frac{x+y}{2}), \ldots, \mathrm{Re}(\frac{x+y}{2}))$$

and

(*) $$e(Y_i(i,j), Y_j(i,j)) = (\frac{n-1}{2} - \frac{i+j}{2} + 1, \frac{n-1}{2} - \frac{i+j}{2} + 1)$$

$$+ (\mathrm{Re}\,\frac{x+y}{2}, \mathrm{Re}\,\frac{x+y}{2}).$$

Fix $1 \leq p \leq n - 1$. Take some $d \in M(D)$ so that

$$d \dashv a_0 \quad \text{and} \quad e(a_{p,p+1}) < e(d).$$

Choose $1 \leq i < j \leq n$ so that $d = a_{i,j}$. Thus

$$e(a_{p,p+1}) < e(a_{i,j}).$$

The definition of the ordering on $\mathbb{R}^n$ and (*) implies $p \leq i$. Using the fact that

$$\mathrm{Tr}\ a_{p,p+1} = \mathrm{Tr}\ a_{i,j}$$

we obtain that

$$j \leq p+1.$$

Therefore $\{i,j\} = \{p,p+1\}$ i.e. $i = p,\ j = p+1$.
Thus $a_{p,p+1} = a_{i,j}$ and $e(a_{p,p+1}) = e(d)$.

We proved that for $d \in M(D)$ so that $d \dashv a_0$ we have $e(a_{p,p+1}) \nmid e(d)$. By (vi) of Lemma 3.4 we have

$$m(a_0,a_{p,p+1}) \neq 0.$$

We shall now prove the proposition. Assume that $u(\delta,n)$ is not prime. It is enough to consider only the case of $n \geq 2$, because $u(\delta,1) = \delta$ is prime ((ii) of Corollary 1.4). There exist $P,Q \in R$ so that

$$u(\lambda,n) = P \times Q$$

and that neither P nor Q is invertible, i.e.

$$P \neq \pm 1,\ Q \neq \pm 1$$

(here $1 = L(\emptyset)$). By (iii) of Corollary 1.4., P and Q are homogenious elements of the graded ring R. Since the graduation of each indeterminate in D is one, P and Q are homogeneous polynomials.

By (iii) of Lemma 3.4 we have

$$u(\delta,n) = \lambda(a_0) + \sum_{a<a_0} m(a_0,a)\lambda(a).$$

Since the coefficient by $\lambda(a_0)$ is 1, we have that the degrees of the homogeneous polynomials P and Q are greater than zero. Write

$$P = \sum_{a \in M(D)} m(P,a)\lambda(a),$$

$$Q = \sum_{a \in M(D)} m(Q,a)\lambda(a).$$

Here $m(P,a) \neq 0$ only for finitely many $a \in M(D)$, and $m(Q,a) \neq 0$ only for finitely many $a \in M(D)$. We have

$$u(\delta,n) = \lambda(a_0) + \sum_{a<a_0} m(a_0,a)\lambda(a) =$$

$$= X_1 \times X_2 \times \dots \times X_n + \sum_{a<a_0} m(a_0,a)\lambda(a) =$$

$$= \Big(\sum_{a \in M(D)} m(P,a)\lambda(a)\Big)\Big(\sum_{b \in M(D)} m(Q,b)\lambda(b)\Big).$$

The above relation and the definition of multiplication of polynomials implies that there exist $a^\vee, b^\vee \in M(D)$ so that

$$a^\vee + b^\vee = a_0,$$
$$m(P,a^\vee) \neq 0$$
$$m(Q,b^\vee) \neq 0.$$

Since P and Q are of positive degree,

$$a^\vee \neq \emptyset \quad \text{and} \quad b^\vee \neq 0.$$

There exist a partition

$$\{\sigma(1),\ldots,\sigma(u)\} \cup \{\tau(1),\ldots,\tau(v)\} = \{1,2,\ldots,n\} ,$$

$$u > 0, \; v > 0, \quad u + v = n,$$

so that

$$a^{\vee} = (X_{\sigma(1)},\ldots,X_{\sigma(u)})$$

$$b^{\vee} = (X_{\tau(1)},\ldots,X_{\tau(v)})$$

Note that the degree of P is u, and of Q is v.

Since $a^{\vee} \neq \emptyset$ and $b^{\vee} \neq \emptyset$, we can find $1 \leq r \leq n-1$ and $1 \leq i \leq u$, $1 \leq j \leq v$ so that

$$\{r, r+1\} = \{\sigma(i), \tau(j)\} .$$

Without lost of generality we can assume

$$i = u, \; j = v.$$

Now we consider

$$u(\delta,n) = X_1 \times \ldots \times X_n + m(a_0, a_{r,r+1}) X_1 \times \ldots \times X_{r-1} \times Y_r(r,r+1) \times$$

$$\times Y_{r+1}(r,r+1) \times X_{r+2} \times \ldots \times X_n + \sum_{\substack{a<a_0 \\ a \neq a_{r,r+1}}} m(a_0, a)\lambda(a) = P \times Q =$$

$$= \Big(\sum_{a \in M(D)} m(P,a)\lambda(a)\Big) \Big(\sum_{b \in M(D)} m(Q,b)\lambda(b)\Big).$$

Since $m(a_0,a_{r,r+1}) \neq 0$, there exist $c^{\vee},d^{\vee} \in M(D)$ such that

$$c^{\vee} + d^{\vee} = a_{r,r+1}$$

$$m(P,c^{\vee}) \neq 0,$$

$$m(Q,d^{\vee}) \neq 0,$$

$$\text{card } c^{\vee} = u$$

$$\text{card } d^{\vee} = v.$$

Without lost of generality we can suppose that $Y_r(r,r+1)$ is in the support of $c^{\vee}$. Now we consider two possible cases.

Suppose that $Y_{r+1}(r,r+1)$ is also in the support of $c^{\vee}$. Then there exist a partition

$$\{1,2,\ldots,r-1,r+2,\ldots,n\} =$$

$$= \{\pi(1),\ldots,\pi(u-2)\} \cup \{\rho(1),\ldots,\rho(v)\}$$

so that

$$c^{\vee} = (X_{\pi(1)},\ldots,X_{\pi(u-2)}) + (Y_r(r,r+1),Y_{r+1}(r,r+1))$$

$$d^{\vee} = (X_{\rho(1)},\ldots,X_{\rho(v)})$$

Suppose that

$$\{\rho(1),\ldots,\rho(v)\} \subseteq \{\tau(1),\ldots,\tau(v)\} .$$

Then we have an equality. This can not hold because the right hand side has a non-trivial intersection with $\{r,r+1\}$ since $\{r,r+1\} = \{\sigma(u),\tau(v)\}$, while the left hand side have the trivial intersection with $\{r, r+1\}$ by definition of $\rho(1), \ldots, \rho(v)$. Thus

$$\{\rho(1), \ldots, \rho(v)\} \cap \{\sigma(1), \ldots, \sigma(u)\} \neq \emptyset.$$

Choose $1 \leq p \leq u$ and $1 \leq q \leq v$ so that

$$\sigma(p) = \rho(q).$$

Set $$s = \sigma(p) = \rho(q).$$

Now by construction the degree of the polynomial P in the indetermine $X_{\sigma(p)} = X_s$ is greater then or equal to 1. Also the degree of the polynomial Q in the indeterminate $X_{\rho(q)} = X_s$ is greater then or equal to 1. Thus the degree of $u(\delta, n)$ in the indeterminate X_s is greater than or equal to 2. This contradicts to Lemma 3.6. Therefore, $Y_{r+1}(r, r+1)$ is not in the support of c^v.

We have obtained that $Y_{r+1}(r, r+1)$ is in the support of d^v. Thus there exist a partition

$$\{1,2,\ldots,r-1,r+2,\ldots,n\} = \{\pi(1),\ldots,\pi(u-1)\} \cup \{\rho(1),\ldots,\rho(v-1)\}$$

so that

$$c^v = (X_{\pi(1)},\ldots,X_{\pi(u-1)},Y_r(r,r+1)),$$
$$d^v = (X_{\rho(1)},\ldots,X_{\rho(v-1)},Y_{r+1}(r,r+1)).$$

Suppose that $\{\pi(1),\ldots,\pi(u-1)\} \not\subseteq \{\sigma(1),\ldots,\sigma(u)\}$. Then

$$\{\pi(1),\ldots,\pi(u-1)\} \not\subseteq (\{\sigma(1),\ldots,\sigma(u)\} \setminus \sigma(u)).$$

since $\sigma(u) \notin \{\pi(1),\ldots,\pi(u-1)\}$. Thus

$$\{\pi(1),\ldots,\pi(u-1)\} \cap (\{\tau(1),\ldots,\tau(v)\} \cup \{\sigma(u)\}) \neq \emptyset .$$

This implies

$$\{\pi(1),\ldots,\pi(u-1)\} \cap \{\tau(1),\ldots,\tau(v-1)\} \neq \emptyset.$$

Let $\pi(p) = \rho(q) = s$ with $1 \leq p \leq u-1$, $1 \leq q \leq v-1$. For the same reason as above, the degree of $u(\delta,n)$ in the indetermine X_s is greater then or equal to 2. This contradicts to Lemma 3.6. Thus

$$\{\pi(1),\ldots,\tau(u-1)\} \subseteq \{\sigma(1),\ldots,\sigma(u)\} .$$

For the same reason as above, the degree of $u(\delta,n)$ in the indeterminate X_s is greater then or equal to 2. This contradicts to Lemma 3.6. Thus

$$\{\pi(1),\ldots,\pi(u-1)\} \subseteq \{\sigma(1),\ldots,\sigma(u)\} .$$

For the same reason

$$\{\rho(1),\ldots,\rho(v-1)\} \subseteq \{\tau(1),\ldots,\tau(v)\}$$

Now we have

$$\{\pi(1),\ldots,\pi(u-1)\} = \{\sigma(1),\ldots,\sigma(u-1)\}$$

$$\{\rho(1,\ldots,\rho(v-1)\} = \{\tau(1),\ldots,\tau(v-1)\}.$$

Without lost of generality we can suppose that

$$\pi(1) = \sigma(i)\ ,\quad i = 1,\ldots,u-1,$$

$$\rho(i) = \tau(i),\quad i = 1,\ldots,v-1.$$

Thus
$$a^{\vee} = (X_{\sigma(1)},\ldots,X_{\sigma(u)})$$
$$b^{\vee} = (X_{\tau(1)},\ldots,X_{\tau(v)})$$
$$c^{\vee} = (X_{\sigma(1)},\ldots,X_{\sigma(u-1)},Y_r(r,r+1))$$
$$d^{\vee} = (X_{\tau(1)},\ldots,X_{\tau(v-1)},Y_{r+1}(r,r+1)).$$

Set $T = \{X_{\sigma(1)},\ldots,X_{\sigma(u)},X_{\tau(1)},\ldots,X_{\tau(v-1)},Y_{r+1}(r,r+1)\}$
We can consider the degree of $f \in R$ with respect to the indeterminates in T (in fact, we have an isomorphisms $R \cong \mathbb{Z}[D] \cong (\mathbb{Z}[D\setminus T])[T])$. This degree we shall denote by $\deg_T$. We have shown that

$$\deg_T P \geq u \quad \text{since} \quad \deg_T \lambda(a^{\vee}) = u,$$
$$\deg_T Q \geq v \quad \text{since} \quad \deg_T \lambda(d^{\vee}) = v.$$

Thus $\deg_T u(\delta,n) = \deg_T P + \deg_T Q \geq u + v = n$. Since $\deg_T u(\delta,n)$ is less than or equal to the total degree of $u(\delta,n)$ which is n, we have

$$\deg_T u(\delta,n) = n.$$

Therefore, there exist $a \leq a_0$ so that

$$\deg_T \lambda(a) = n.$$

Now

$$\lambda(a) = \lambda(b) \times X_{\sigma(1)}^{s(1)} \times \dots \times X_{\sigma(u)}^{s(u)} \times X_{\tau(1)}^{t(1)} \times \dots \times X_{\tau(v-1)}^{t(v-1)} \times Y_{r+1}(r,r+1)^{t_0}$$

where $s(1),\dots,s(u)$, $t(1),\dots,t(v-1),t_0 \in \mathbb{Z}_+$, and $\deg_T \lambda(b) = 0$. Lemma 3.6. implies

$$0 \leq s(1),\dots,s(u),t(1),\dots,t(v-1),t_0 \leq 1.$$

Since $\deg_T \lambda(a) = n$ we have

$$s(1) = s(2) = \dots = s(u) = t(1) = \dots = t(v-1) = t_0 = 1.$$

The fact that the degree of $\lambda(a)$ is n implies $b = \emptyset$.

By the above construction we have obtained that

$$a = (X_{\sigma(1)},\dots,X_{\sigma(u)},X_{\tau(1)},\dots,X_{\tau(v-1)},Y_{r+1}(r,r+1)) \leq a_0.$$

By (iii) of Lemma 3.1 we have

$$\mathrm{Tr}(a) = \mathrm{Tr}(a_0).$$

The calculation of $e(a_0)$ implies

$$\mathrm{Tr}(a_0) = n \cdot \mathrm{Re}\left(\frac{x+y}{2}\right).$$

For the multiset $(X_{\sigma(1)},\dots,X_{\sigma(u)},X_{\tau(1)},\dots,X_{\tau(v-1)})$ we have the two following possibilities.

The first possibility is

$$(X_{\sigma(1)},\dots,X_{\sigma(u)},X_{\tau(1)},\dots,X_{\tau(v-1)}) =$$

$$= (X_1,\dots,X_{r-1},X_{r+1},\dots,X_n).$$

Then $Tr(a) = Tr(a_0) - TrX_r + TrY_{r+1}(r,r+1) =$

$$= n\ \mathrm{Re}\ (\tfrac{x+y}{2}) - (\tfrac{1}{2})\mathrm{Re}\ (x+y+(n-1)-2r+2)$$

$$+ \tfrac{1}{2}\ \mathrm{Re}(x+y+(n-1)-2r+1) = n\ \mathrm{Re}(\tfrac{x+y}{2}) - \tfrac{1}{2}\ .$$

We obtained

$$Tr(a) = n\ \mathrm{Re}\ (\tfrac{x+y}{2}) - \tfrac{1}{2}\ .$$

The second possibility is

$$(X_{\sigma(1)},\ldots,X_{\sigma(u)},X_{\tau(1)},\ldots,X_{\tau(v-1)}) =$$

$$= (X_1,\ldots,X_r,X_{r+2},\ldots,X_n).$$

Then

$$Tr(a) = Tr(a_0) - Tr\ X_{r+1} + Tr\ Y_{r+1}(r,r+1) =$$

$$= n\ \mathrm{Re}(\tfrac{x+y}{2}) + \tfrac{1}{2}\ .$$

In the both cases we obtained

$$Tr\ (a) \neq Tr\ (a_0).$$

This contradiction concludes the proof of the proposition.

Now let for a moment F be either $\mathbb{C}$ or $\mathbb{R}$. Let P_n be the subgroup of $G_n = GL(n,F)$ consisting of all matrices with the bottom row equal to $(0,\ldots,0,1)$. Let π be a topologically irreducible unitary representation of G_n (in some Hilbert space), $n \in \mathbb{N}$. By [12], the restriction of π to P_n remains topological irreducible. This result, by

H. Jacquet, implies the following one (see introduction of [2]). If σ is a topologically irreducible unitary representation of a Levi factor M of a parabolic subgroup of G_n, then the parabolically induced representation of G_n by σ from P is topologically irreducible (the induction we consider is normalized).

Passing to Harish-Chandra modules we obtain (UO) for $F = \mathbb{C}$ or $\mathbb{R}$.

At the end, Proposition 2.1 together with the results of this section implies Theorem A for the case $F = \mathbb{C}$.

4. A conjecture on $GL(n,\mathbb{C})$

In this section we assume $F = \mathbb{C}$. Note that

$$(\beta,k) \longmapsto \delta(\beta,k)\ ,\ k \in \mathbb{Z}, \beta \in \mathbb{C},\ \mathrm{Re}\,\beta = 0,$$

is a parametrization of D^u.

4.1. Conjecture: Let $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $\beta \in \mathbb{C}$ with $\mathrm{Re}\,\beta = 0$. Then the relation

$$\nu^{1/2}u(\delta(\beta,k),n) \times \nu^{-1/2}u(\delta(\beta,k),n) =$$

$$= u(\delta(\beta,k),n+1) \times u(\delta(\beta,k),n-1) +$$

$$+ u(\delta(\beta,k-1),n) \times u(\delta(\beta,k+1),n)$$

holds in R.

Suppose that $n = 1$. Then the above formula is

$$\nu^{1/2}\delta(\beta,k) \times \nu^{-1/2}\delta(\beta,k) =$$

$$= u(\delta(\beta,k),2) + \delta(\beta,k-1) \times \delta(\beta,k+1)$$

i.e.

$$\delta(\beta+1/2,k) \times \delta(\beta-1/2,k) =$$

$$= L(\delta(\beta+1/2,k),\ \delta(\beta-1/2,k)) + \delta(\beta,k-1) \times \delta(\beta,k+1).$$

The third section implies that this formula holds. Thus, Conjecture 4.1 holds if $u = 1$.

By an old result of I.M. Gelfand and M.A. Naimark, representations

$$u(\delta(\beta,k),n+1) \times u(\delta(\beta,k),n-1),$$

$$u(\delta(\beta,k-1),n) \times u(\delta(\beta,k+1),n)$$

are irreducible (they are degenerate unitary principal series). By Proposition 3.5. we have that

$$\nu^{1/2}u(\delta(\beta,k),n) \times \nu^{-1/2}u(\delta(\beta,k),n) =$$

$$\nu^{1/2}L(a(\delta(\beta,k),n)) \times \nu^{-1/2}L(a(\delta(\beta,k),n) =$$

$$= L(\nu^{1/2}a(\delta(\beta,k),n)) \times L(\nu^{-1/2}a(\delta(\beta,k),n))$$

contains

$$L(\nu^{1/2}a(\delta(\beta,k),n) + \nu^{-1/2}a(\delta(\beta,k),n))$$

with multiplicity one. Since

$$\nu^{1/2}a(\delta(\beta,k),n) + \nu^{-1/2}a(\delta(\beta,k),n) =$$

$$= a(\delta(\beta,k),n+1) + a(\delta(\beta,k),n-1)$$

we obtain:

4.2. <u>Proposition</u>: The representation

$$\nu^{1/2}u(\delta(\beta,k),n) \times \nu^{-1/2}u(\delta(\beta,k),n)$$

contains the irreducible representation

$$u(\delta(\beta,k),n+1) \times u(\delta(\beta,k),n-1)$$

with multiplicity one.

5. Real general linear group I

In this section we assume that $F = \mathbb{R}$. First we shall parametrize D. The signum characters of $\mathbb{R}^{\times}$ is denoted by sgn. Clearly $\mathrm{sgn} \in D^{u}$ and $\tilde{G}_1 \subseteq D$.

Let $\delta_1, \delta_2 \in \tilde{G}_1$. If $\delta_1 \times \delta_2 \in \mathrm{Irr}$ then

$$\delta_1 \times \delta_2 = L((\delta_1, \delta_2)).$$

By [8], $\delta_1 \times \delta_2 \notin \mathrm{Irr}$ if and only if there exist $p \in \mathbb{Z}, p \neq 0$, so that

$$\delta_1(t)\delta_2(t)^{-1} = t^{p} \,\mathrm{sgn}\, t \ , \quad t \in \mathbb{R}$$

Suppose that $\delta_1 \times \delta_2 \notin \mathrm{Irr}$. Then there exist $\gamma(\delta_1, \delta_2) \in D$ so that

$$\delta_1 \times \delta_2 = L((\delta_1, \delta_2)) + \gamma(\delta_1, \delta_2).$$

The mapping

$$(\delta_1, \delta_2) \longrightarrow \gamma(\delta_1, \delta_2)$$

$$\tilde{G}_1 \times \tilde{G}_1 \longrightarrow D \setminus \tilde{G}_1$$

is surjective. We have $\gamma(\delta_1, \delta_2) = \gamma(\delta_1', \delta_2')$ if and only if one of the following conditions is fulfilled

1) $\delta_1 = \delta_1'$ and $\delta_2 = \delta_2'$;
2) $\delta_1 = \delta_2'$ and $\delta_2 = \delta_1'$;

3) $\delta_1 = \delta_1' \operatorname{sgn}$ and $\delta_2 = \delta_2' \operatorname{sgn}$;

4) $\delta_1 = \delta_2' \operatorname{sgn}$ and $\delta_2 = \delta_1' \operatorname{sgn}$.

The relation $\delta_1 \times \delta_2 = L((\delta_1,\delta_2)) + \gamma(\delta_1,\delta_2)$ implies

$$\gamma(\delta_1,\delta_2)^+ = \gamma(\delta_1^+,\delta_2^+).$$

We have also

$$e(\gamma(\delta_1,\delta_2)) = 1/2(e(\delta_1) + e(\delta_2)).$$

Thus

$$\gamma(\delta_1,\delta_2) \in D^u \iff 1/2(e(\delta_1) + e(\delta_2)) = 0.$$

Let $\gamma(\delta_1,\delta_2) \in D^u$. By the definition we have

$$\delta_1\delta_2^{-1} = t^p \operatorname{sgn} t.$$

Let $\delta_i(t) = |t|^{\alpha_i}(\operatorname{sgn} t)^{m_i}$, $\alpha_i \in \mathbb{C}$, $m_i \in \{0,1\}$, $i = 1,2$.
Set $\delta_1^*(t) = |t|^{\alpha_1}$. Then

$$\gamma(\delta_1,\delta_2) = \gamma(\delta_1^*,\delta_2^*)$$

where $\delta_2^* = \delta_2(\operatorname{sgn})^{m_1}$. We have

$$|t|^{\alpha_1-\alpha_2}(\operatorname{sgn} t)^{m_1-m_2} = t^p \operatorname{sgn} t =$$

$$= |t|^p(\operatorname{sgn} t)^{p+1}.$$

Thus $\alpha_1 - \alpha_2 = p$ and $(\operatorname{sgn})^{m_1-m_2} = (\operatorname{sgn})^{p+1}$.

Now

$$\delta_2^*(t) = |t|^{\alpha_2} \operatorname{sgn}(t)^{m_2} \cdot \operatorname{sgn}(t)^{m_1} =$$

$$= |t|^{\alpha_2} \operatorname{sgn}(t)^{m_1-m_2} = |t|^{\alpha_2} \operatorname{sgn}(t)^{\alpha_1-\alpha_2+1} .$$

Let $x,y \in \mathbb{C}$ so that $x - y \in \mathbb{Z}\setminus\{0\}$. Set $\delta_1(t) = |t|^x$, $\delta_2(t) = |t|^y \operatorname{sgn}(t)^{x-y+1}$. Then $\delta_1\delta_2^{-1}(t) = t^{x-y} \operatorname{sgn} t$ and we define $\gamma(x,y)$ by

$$\gamma(x,y) = \gamma(\delta_1,\delta_2).$$

By definition $\gamma(x,y) = \gamma(y,x)$. If $\gamma(x,y) = \gamma(x',y')$ then $x = x'$ and $y = y'$ or $x = y'$ and $y = x'$. Now we have

$$\gamma(x,y)^+ = \gamma(-\bar{x}, -\bar{y}),$$

$$e(\gamma(x,y)) = \operatorname{Re} \frac{x+y}{2} .$$

Let $x,y \in \mathbb{C}$, $x - y \in \mathbb{Z}\setminus\{0\}$. Set

$$2\beta = x + y$$

$$n = x - y$$

Then $n \in \mathbb{Z} \setminus \{0\}$ and $\beta \in \mathbb{C}$. Clearly, x and y are uniquely determined by β and n. Set

$$\delta(\beta,n) = \gamma(x,y).$$

We have $\delta(\beta,n) = \delta(\beta,-n)$. If $\delta(\beta,n) = \delta(\beta',n')$ then $\beta = \beta'$ and $n = \pm n'$. Thus

$$\mathbb{C} \times \mathbb{N} \longrightarrow D\setminus\tilde{G}_1$$
$$(\beta,n) \longmapsto \delta(\beta,n)$$

is a bijection. We have directly

$$\delta(\beta,n)^+ = \delta(-\bar{\beta},n),$$
$$e(\delta(\beta,n)) = \mathrm{Re}\ \beta,$$
$$\delta(\tfrac{x+y}{2}, x-y) = \gamma(x,y),$$
$$\delta(\beta,n) = \gamma(\beta+\tfrac{n}{2}, \beta-\tfrac{n}{2}).$$

Note that for $\alpha \in \mathbb{R}$ we hve

$$\nu^\alpha\gamma(x,y) = \gamma(x+\alpha, y+\alpha)$$

$$\nu^\alpha\delta(\beta,n) = \delta(\beta+\alpha,n).$$

If $x \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, then we set

$$\gamma_\varepsilon(x)(t) = |t|^x (\mathrm{sgn}\ t)^\varepsilon \quad , \quad t \in G_1 .$$

Now $(x,\varepsilon) \longrightarrow \gamma_\varepsilon(x)$, $\mathbb{C}\times(\mathbb{Z}/2\mathbb{Z}) \longrightarrow \tilde{G}_1$ is a bijection which parametrizes $\tilde{G}_1$. Clearly

$$e(\gamma_\varepsilon(x)) = \mathrm{Re}\ x.$$

Recall that in this notation we have

5.1. Proposition: We have

$$\gamma_\varepsilon(x)\times\gamma_{\varepsilon'}(x') \notin \mathrm{Irr}$$

if and only if $x - x' \in \mathbb{Z}\setminus\{0\}$ and $x - x' + 1 = \varepsilon - \varepsilon'$ in $\mathbb{Z}/2\mathbb{Z}$. If the above representation reduces then we have

$$\gamma_\varepsilon(x) \times \gamma_{\varepsilon'}(x') = L(\gamma_\varepsilon(x),\gamma_{\varepsilon'}(x')) + \gamma(x,x').$$

Now we shall describe the infinitesimal character of $L(a)$ when $a \in M(D)$, Let $\delta \in D$. Suppose that $\delta \in \tilde{G}_1$. Then $\delta = \gamma_\varepsilon(x)$ for some $x \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Set

$$\chi(\delta) = (x) \in M(\mathbb{C}).$$

If $\delta \notin \tilde{G}_1$, then $\delta = \gamma(x,y)$ for some $x,y \in \mathbb{C}$. Set

$$\chi(\delta) = (x,y) \in M(\mathbb{C}) .$$

Let $a = (\delta_1,\ldots,\delta_n) \in M(D)$. Set

$$\chi(a) = \chi(\delta_1)+\ldots+ \chi(\delta_n) \in M(\mathbb{C}).$$

For $\delta \in D$ we have defined the graduation $\mathrm{gr}\,\delta$ of δ since δ is in a graded ring $R(\mathrm{gr}\delta \in \{1,2\})$. Let $a = (\delta_1,\ldots,\delta_n) \in M(D)$. We define

$$gr(a) = gr\delta_1 + \ldots + gr\delta_n .$$

With this definition we have

$$gr(a) = \text{card}\ \chi(a).$$

Let $\mathfrak{a}_n \subseteq \mathfrak{g}_n$ be the Lie algebra of the subgroup A_n of all diagonal elements in G_n. Let $\mathfrak{a}_n^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ be complexifications of these two algebras. The universal enveloping algebras of $\mathfrak{a}_n^{\mathbb{C}}$ and $\mathfrak{g}_n^{\mathbb{C}}$ are denoted by $U(\mathfrak{a}_n^{\mathbb{C}})$ and $U(\mathfrak{g}_n^{\mathbb{C}})$. The center of the algebra $U(\mathfrak{g}_n^{\mathbb{C}})$ is denoted by $Z(\mathfrak{g}_n^{\mathbb{C}})$. We consider the Harish-Chandra homomorphism

$$\xi : Z(\mathfrak{g}_n^{\mathbb{C}}) \longrightarrow U(\mathfrak{a}_n^{\mathbb{C}}) .$$

Let $(\mathfrak{a}_n^{\mathbb{C}})^*$ be the space of all complex linear functionals on $\mathfrak{a}_n^{\mathbb{C}}$. For $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$ let

$$\xi_\lambda : Z(\mathfrak{g}_n^{\mathbb{C}}) \longrightarrow \mathbb{C}$$

be the composition of ξ with evaluation at λ.

Let A_n^0 be the connected component of A_n containing identity and M the torsion subgroup of A_n. The normalizer of A_n in K_n is denoted by M'. Set $W = M'/M$. Now W acts on $\mathfrak{g}_n^{\mathbb{C}}$ and $\mathfrak{a}_n^{\mathbb{C}}$. As it is well known, every homomorphism of $Z(\mathfrak{g}_n^{\mathbb{C}})$ into $\mathbb{C}$ is obtained as ξ_λ for some $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$. Also $\xi_\lambda = \xi_\mu$ if and only if $W\lambda = W\mu$.

We identify $\mathfrak{g}_n^{\mathbb{C}}$ in a natural way with the Lie algebra of all complex $n \times n$ matrices. Then $\mathfrak{a}_n^{\mathbb{C}}$ is the subalgebra of all diagonal matrices in $\mathfrak{g}_n^{\mathbb{C}}$. Now W acts on $\mathfrak{a}_n^{\mathbb{C}}$ by permutations of diagonal elements and W is isomorphic to the permutation group of the order n.

Let $\lambda \in (\mathfrak{a}_n^{\mathbb{C}})^*$. Then there exist $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ so that

$$\lambda : \mathrm{diag}\,(x_1,\dots,x_n) \longmapsto \lambda_1 x_1 + \dots + \lambda_n x_n$$

Define $\lambda^{\vee} : A_n^0 \longrightarrow \mathbb{C}^{\times}$ by

$$\mathrm{diag}\,(a_1,\dots,a_n) \longmapsto a_1^{\lambda_1} \dots a_n^{\lambda_n}.$$

The mapping $\lambda \longmapsto \lambda^{\vee}$ is a group isomorphism of $(\mathfrak{a}_n^{\mathbb{C}})^*$ onto the group $\tilde{A}_n^0$ of all continuous homomorphisms of A_n^0, into $\mathbb{C}^{\times}$. Note that $\lambda \longmapsto (\lambda_1,\dots,\lambda_n)$ is an isomorphism of $(\mathfrak{a}_n^{\mathbb{C}})^*$ onto $\mathbb{C}^n$. In this relisation W acts by permutations of coordinates. Therefore

$$(\mathfrak{a}_n^{\mathbb{C}})^*/W$$

can be identified with the set of all multisets $(\lambda_1,\dots,\lambda_n)$ in $\mathbb{C}$ of cardinal number n.

Now let $\gamma_{\varepsilon_1}(x_1),\dots,\gamma_{\varepsilon_n}(x_n) \in \tilde{G}_1$. Set

$$a = (\gamma_{\varepsilon_1}(x_1),\dots,\gamma_{\varepsilon_n}(x_n)).$$

Let $\mu \in (\mathfrak{a}_n^{\mathbb{C}})^*$ correspond to $(x_1,\dots,x_n)$ under the above identification. Then by Lemma 4.1.8. of [26], $\lambda(a)$ has infinitesimal character which equals to ξ_μ.

Let $a \in M(D)$, $\mathrm{gr}\, a = n$. Set

$$\chi(a) = (x_1,\dots,x_n).$$

Then by Proposition 5.1, there exist $\varepsilon_1,\dots,\varepsilon_n \in \mathbb{Z}/2\mathbb{Z}$ so that each $\pi \in \mathrm{Irr}$ which is contained in $\lambda(a)$, is contained in

$$\lambda((\gamma_{\varepsilon_1}(x_1),\dots,\gamma_{\varepsilon_n}(x_n))).$$

Therefore the infinitesimal character of $\lambda(a)$ is ξ_μ , where μ is as above. Thus we have proved:

5.2. Lemma: Let $a,b \in M(D)$ with $\mathrm{gr}\, a = \mathrm{gr}\, b$. Then $L(a)$ and $L(b)$ have the same infinitesimal character if and only if

$$\chi(a) = \chi(b).$$

Now we shall describe a neccessary conditions for $a,b \in M(D)$ that $L(b)$ is contained in $\lambda(a)$.

Let $a = (\delta_1,\dots,\delta_m) \in M(D)$. We say that $(\delta_1,\dots,\delta_m)$ is in a standard order if

$$e(\delta_1) \geq e(\delta_2) \geq \dots \geq e(\delta_m).$$

Set $\underline{e}(\delta) = (e(\delta)) \in M(\mathbb{R})$ if $\delta \in G_1$ and $\underline{e}(\delta) = (e(\delta),\ e(\delta)) \in M(\mathbb{R})$ if $\delta \in D \setminus \tilde{G}_1$. Let $a = (\delta_1,\dots,\delta_m) \in M(D)$, $\mathrm{gr}\, a = n$. Suppose that

$$\underline{e}(\delta_1)+\dots+\underline{e}(\delta_m) = (x_1,\dots,x_n)$$

where $x_1 \geq \dots \geq x_n$. Then such $(x_1,\dots,x_n) \in \mathbb{R}^n$ is uniquely determined by a. We define $\underline{e}(a)$ by

$$\underline{e}(a) = (x_1,\ldots,x_n) \in \mathbb{R}^n .$$

The number $x_1+\ldots+x_n$ is denoted by Tr a.

We define a partial order on $\mathbb{R}^n$ as before by

$$(x_1,\ldots,x_n) \leq (y_1,\ldots,y_n)$$

if and only if

$$x_1 \leq y_1$$

$$x_1 + x_2 \leq y_1 + y_2$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$x_1 + \ldots x_n \leq y_1 + \ldots + y_n.$$

This is a partial order on $\mathbb{R}^n$.

For $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ we set $\operatorname{Tr} x = x_1+\ldots+x_n$. If $a \in M(D)$, then we set $\operatorname{Tr} a = \operatorname{Tr} \underline{e}(a)$.

5.3. <u>Lemma</u>: Let $a,b \in M(D)$. Suppose that $L(b)$ is contained in $\lambda(a)$. Then:

(i) $\operatorname{gr}(a) = \operatorname{gr}(b)$;

(ii) $\chi(a) = \chi(b)$;

(iii) $\operatorname{Tr} a = \operatorname{Tr} b$

(iv) $\underline{e}(b) \leq \underline{e}(a)$;

(v) We have $a \neq b$ if and only if

$$\underline{e}(b) \underset{\neq}{<} \underline{e}(a).$$

Proof: The first statement is obvious. Since $L(b)$ and $\lambda(a)$ have the same infinitesimal character, (ii) is a consequence of Lemma 5.2.

Let $c = (\delta_1,\dots,\delta_m) \in M(D)$. After a prenumeration we can suppose that

$$\delta_i = \gamma(x_i,y_i) \ , \ 1 \leq i < k$$

$$\delta_i = \gamma_{\varepsilon_i}(z_i) \ , \ k \leq i \leq m$$

for some $1 \leq k \leq m+1$, $x_i, y_i, z_i \in \mathbb{C}$, $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z}$. Then $\chi(c) = (x_1,y_1,\dots,x_{k-1},z_k,\dots,z_m)$. Now we have

$$\mathrm{Tr}\ c = \mathrm{Re}(x_1+y_1+x_2+y_2+\dots+x_{k-1}+y_{k-1}+z_k+\dots+z_m).$$

Thus $\mathrm{Tr}\ c$ depends only on $\chi(c)$. Now (ii) implies (iii).

Let $\quad \mathrm{gr}\ (a) = n.$ Set

$$\beta_1 = (1,0,\dots,0) - \frac{1}{n}(1,1,\dots,1)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\beta_i = (\underbrace{1,1,,\dots,1}_{i\text{-times}},0,\dots,0) - \frac{i}{n}(1,\dots,1)$$

$$\dots\dots\dots\dots\dots\dots$$

$$\beta_{n-1} = (1,\dots,1,0) - \frac{n-1}{n}(1,\dots,1).$$

We consider on $\mathbb{R}^n$ the following form

$$\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle = \sum_{i=1}^{n} x_i y_i.$$

Now by Proposition 4.13 of [4] we have

$$\langle\beta_i,\underline{e}(b)\rangle \leq \langle\beta_i,\underline{e}(a)\rangle \quad , i = 1,\ldots,n-1.$$

Also, if all $n-1$ above inequalities are equalities then $a = b$, by the same proposition. Note that

$$((x_1,\ldots,x_n),\beta_i) = x_1+\ldots+x_i - \frac{i}{n}\,\mathrm{Tr}(x_1,\ldots,x_n).$$

Using (iii) we obtain (iv) and (v).

5.4. Lemma: (i) Let $a \in m(D)$. The set of all $b \in M(D)$ so that $\chi(a) = \chi(b)$, is finite.

(ii) Let $a_i,b_i \in M(D)$, $i = 1,2$. Suppose that

$$\underline{e}(b_i) \leq \underline{e}(a_i) \quad , i = 1,2. \text{ If}$$

$$\underline{e}(a_1 + a_2) = \underline{e}(b_1 + b_2),$$

then $$\underline{e}(a_i) = \underline{e}(b_i),\ i = 1,2.$$

Proof: The statement (i) is a direct consequence of the definition of $\chi(b)$, $b \in M(D)$.

Suppose that $a_i,b_i \in M(D)$ and $\underline{e}(b_i) \leq \underline{e}(a_i)$, $i = 1,2$. Let $\underline{e}(a_1) \neq \underline{e}(b_1)$ or $\underline{e}(a_2) \neq \underline{e}(b_2)$. We can take that $\underline{e}(a_1) \neq \underline{e}(b_1)$. Now $\underline{e}(b_1) \lneq e(a_1)$.

Set

$$\underline{e}(a_1) = (e_1,\ldots,e_n),\underline{e}(a_2) = (e_{n+1},\ldots,e_{m+n}),$$
$$\underline{e}(b_1) = (f_1,\ldots,f_n),\underline{e}(b_2) = (f_{n+1},\ldots,f_{m+n}).$$

Choose a permutation σ of $\{1,\ldots,n+m\}$ so that

$$f_{\sigma^{-1}(1)} \geq f_{\sigma^{-1}(2)} \geq \ldots \geq f_{\sigma^{-1}(n+m)}.$$

Now

$$\underline{e}(b_1 + b_2) = (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n+m)}).$$

We can choose σ such that σ satisfies

$$1 \leq \sigma^{-1}(i) < \sigma^{-1}(j) \leq n \quad \Rightarrow \quad i < j,$$

$$m + 1 \leq \sigma^{-1}(i) < \sigma^{-1}(j) \leq n + m \Rightarrow i < j.$$

Now $\underline{e}(b_1) \lneqq \underline{e}(a_1)$ implies that

$$\underline{e}(b_1 + b_2) = (f_{\sigma^{-1}(1)}, \ldots, f_{\sigma^{-1}(n+m)}) \lneqq$$

$$\lneqq (e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n+m)}).$$

Since $(e_{\sigma^{-1}(1)}, \ldots, e_{\sigma^{-1}(n+m)}) \leq \underline{e}(a_1 + a_2)$, we have

$$\underline{e}(b_1 + b_2) \lneqq \underline{e}(a_1 + a_2).$$

This proves (ii) of the lemma.

5.5. <u>Lemma</u>: Fix $a \in M(D)$.

(i) There exist $m_b^a \in \mathbb{Z}_+$ such that

$$\lambda(a) = \sum_{b \in M(D)} m_b^a L(b).$$

If $m_b^a \neq 0$ then $\chi(a) = \chi(b)$ and $\underline{e}(b) \leq \underline{e}(a)$.

(ii) We have $m_a^a = 1$. Thus

$$\lambda(a) = L(a) + \sum_{\substack{\chi(b) = \chi(a) \\ \underline{e}(b) < \underline{e}(a)}} m_b^a L(b).$$

(iii) There exist $m(a,b) \in \mathbf{Z}$ so that

$$L(a) = \sum_{b \in M(D)} m(a,b)\lambda(b).$$

If $m(a,b) \neq 0$ then $\chi(a) = \chi(b)$ and $\underline{e}(b) \leq \underline{c}(a)$

(iv) We have $m(a,a) = 1$. Thus

$$L(a) \neq \lambda(a) + \sum_{\substack{\chi(b) = \chi(a) \\ \underline{e}(b) < \underline{e}(a)}} m(a,b)\ \lambda(b).$$

(v) Let $c \in M(D)$, $c \neq a$. Suppose that c satisfies the following condition

(C_1) The representation $L(c)$ is contained in $\lambda(a)$ and for each $d \in M(D)$ so that $\chi(d) = \chi(a)$ and $\underline{e}(d) < \underline{e}(a)$, we have

$$e(c) \not< e(d).$$

Then $m(a,c) \neq 0$.

(vi) Let $c \in M(D)$, $c \neq a$. Suppose that c satisfies the following condition:

(C_2) The representation $L(c)$ is contained in $\lambda(a)$ and for each $d \in M(D), d \neq a$, such that $\lambda(a)$ contains $L(d)$ we have

$$e(c) \not< e(d).$$

Then $m(a,c) \neq 0$.

<u>Proof</u>: The fact that $L(a)$ has multiplicity one in $\lambda(a)$ ([28]) and Lemma 5.3 implies (i) and (ii). We prove (iii), (iv) and (vi) by induction in the following set. Let $x \in M(\mathbb{C})$. Then we consider the set

$$X = \{a \in M(D);\ \chi(a) = x\} \quad .$$

Suppose that $a \in X$ and that there is no $b \in X$ such that $\underline{e}(b) < \underline{e}(a)$. Then (ii) implies (iii) and (iv) for such a. Now (vi) is obviously satisfied because there is no c which satisfies (C_2).

Let $a \in X$ be arbitrary. Suppose that (iii), (iv) and (vi) holds for $b \in X$ with $\underline{e}(b) < \underline{e}(a)$. Then (ii) implies

$$L(a) = \lambda(a) - \sum_{\substack{\chi(b) = \chi(a) \\ \underline{e}(b) < \underline{e}(a)}} m_b^a L(b) =$$

$$= \lambda(a) - \sum_{\substack{\chi(b) = \chi(a) \\ \underline{e}(b) < \underline{e}(a)}} m_b^a (\lambda(b) + \sum_{\substack{\chi(c) = \chi(b) \\ \underline{e}(c) < \underline{e}(b)}} m(b,c)\lambda(c))$$

After a gathering of the terms in the above formula for $L(a)$, we obtain (iii) and (iv). Suppose that $c \in M(D)$ satisfies (C_2) (clearly $c \in X$) and $m(a,c) = 0$. Therefore, there exist $b \in M(D)$ so that

$$m_b^a \neq 0$$

and

$$\underline{e}(c) < \underline{e}(b)$$

This contradicts with (C_2).

If $c \in M(D)$ satisfies (C_1), then c satisfies (C_2). Thus (v) holds.

Now we shall prove (U4).

5.6. <u>Proposition:</u> If $a,b \in M(D)$, then $L(a) \times L(b)$ containes $L(a+b)$, and multiplicity is one.

<u>Proof</u>: Similary as in the proof of Proposition 3.5. one obtains

$$L(a) \times L(b) = (\lambda(a) + \sum_{\substack{\chi(c) = \chi(a) \\ \underline{e}(c) < \underline{e}(a)}} m(a,c)\lambda(c)) \times$$

$$\times (\lambda(b) + \sum_{\substack{\chi(d) = \chi(b) \\ \underline{e}(d) < \underline{e}(b)}} m(b,d)\lambda(d)) =$$

$$= \lambda(a+b) + \sum_{\substack{\chi(d) = \chi(b) \\ \underline{e}(d) < \underline{e}(b)}} m(b,d)\lambda(a+d) +$$

$$+ \sum_{\substack{\chi(c) = \chi(a) \\ \underline{e}(c) < \underline{e}(a)}} m(a,c)\lambda(c+b) +$$

$$+ \sum_{\substack{\chi(c)=\chi(a)\\ \underline{e}(c)<\underline{e}(a)}} \sum_{\substack{\chi(d)=\chi(b)\\ \underline{e}(d)<\underline{e}(b)}} m(a,c)m(b,d)\lambda(c+d) =$$

$$= L(a+b) + \sum_{\substack{\chi(u)=\chi(a+b)\\ \underline{e}(u)<\underline{e}(a+b)}} m_u^{a+b} L(u) +$$

$$+ \sum_{\substack{\chi(d)=\chi(b)\\ \underline{e}(d)<\underline{e}(b)}} \sum_{\substack{\chi(u)=\chi(a+d)\\ \underline{e}(u)\le\underline{e}(a+d)}} m(b,d)m_u^{a+d} L(u) +$$

$$+ \sum_{\substack{\chi(c)=\chi(a)\\ \underline{e}(c)<\underline{e}(a)}} \sum_{\substack{\chi(u)=\chi(c+b)\\ \underline{e}(u)\le\underline{e}(c+d)}} m(a,c) L(u) +$$

$$+ \sum_{\substack{\chi(c)=\chi(a)\\ \underline{e}(c)<\underline{e}(a)}} \sum_{\substack{\chi(d)=\chi(b)\\ \underline{e}(d)<\underline{e}(b)}} \sum_{\substack{\chi(u)=\chi(c+d)\\ \underline{e}(u)\le\underline{e}(c+d)}} m(a,c)m(b,d)m_u^{c+d}L(u).$$

By (ii) of Lemma 5.4. we see that

$$L(a) \times L(b) = L(a+b) + \sum_{\substack{\pi\in Irr\\ \pi\neq L(a+b)}} m_\pi \pi \quad .$$

This is just the statement of the proposition.

Now we shall see that (U1) holds for $GL(n,\mathbb{R})$. Essentially B. Speh proved it in [20]

5.7. Proposition: Representations

$u(\delta,n)$

are unitarizable when $\delta \in D^u$ and $n \in \mathbb{N}$.

<u>Proof</u>: If $\mathrm{gr}\delta = 1$, then $u(\delta,n)$ is a unitary character of G_n obtained by composing δ with the determinant homomorphism. Thus $u(\delta,n)$ is unitary. It remains to consider the case $\mathrm{gr}\,\delta = 2$. Then $\delta = \delta(\beta,m)$ with $m \in \mathbb{N}, \beta \in \mathbb{C}$ so that $\mathrm{Re}\,\beta = 0$. If $\beta = 0$ then Theorem 3.5.3. of [20] implies unitarizability of $u(\delta(0,m),n)$. From $\delta(\beta,m) = \nu^{\beta}\delta(0,m)$ we obtain

$$u(\delta(\beta,m),n) = \nu^{\beta}u(\delta(0,m),n).$$

Since ν^{β} is a unitary character of G_{2n}, $u(\delta(\beta,m),n)$ is unitary.

One can also consult [10] for a proof of the above proposition.

The following proposition is related to (U2).

5.8. <u>Proposition</u>: Suppose that (U0) holds. Then (U2) holds, i.e.

$$\pi(u(\delta,n),\alpha)$$

are irreducible unitarizable for $\delta \in D^u, 0 < \alpha < 1/2$, $n \in \mathbb{N}$.

<u>Proof:</u> Note that several authors have proved that $\pi(u(\delta,n),\alpha)$ is irreducible unitarizable, if $\mathrm{gr}\,\delta = 1$.

First we shall prove that $\pi(u(\delta,n),\alpha)$ are irreducible. For this we do not need (U0).

Let $a = (\delta_1, \ldots, \delta_u)$, $b = (\delta_{u+1}, \ldots, \delta_{u+v}) \in M(D)$. Suppose that $\delta_i \times \delta_j \in \mathrm{Irr}$ for all $1 \leq i \leq u$, $u+1 \leq j \leq u+v$. Then

$$L(a) \times L(b) \in \mathrm{Irr}.$$

Such result was proved in [31] by A.V. Zelevinsky in the case of $GL(n)$ over non-archimedean fields. His proof applies after neccessary modifications also to the archimedean case. Let $gr(a+b) = n$. Suppose that σ is a permutation of $\{1, \ldots, u+v\}$ which satisfy the following assumptions:

$$1 \leq \sigma(i) < \sigma(j) \leq u \implies i < j,$$

$$u+1 \leq \sigma(i) < \sigma(j) \leq u+v \implies i < j.$$

Let π_σ be the parabolicaly induced representation from a suitable parabolic subgroup by

$$\delta_{\sigma(1)} \otimes \delta_{\sigma(2)} \otimes \cdots \otimes \delta_{\sigma(u+v)} \quad .$$

Induction by stages implies that all π_σ are isomorphic. Note that the representation τ parabolically induced by $L(a) \otimes L(b)$ is a quotient of π_{id}. The above consideration implies that τ has a unique irreducible quotient, and this quotient is isomorphic to $L(a+b)$. Repeating the above considerations with $\tilde{a}$ and $\tilde{b}$, one obtains that τ has a unique irreducible subrepresentation isomorphic to $L(a+b)$. The multiplicity one of $L(a+b)$ in $\lambda(a+b)$ implies that τ is irreducible.

Now we shall present a sufficient condition for $\delta_1, \delta_2 \in D^u$ that $\delta_1 \times \delta_2 \in \mathrm{Irr}$. Let $\delta_i = \gamma(x_i, y_i)$, $i = 1,2$. Suppose that

$$x_1 - x_2 \notin \mathbb{Z}, \text{ or}$$
$$x_1 - y_2 \notin \mathbb{Z}, \text{ or}$$
$$y_1 - x_2 \notin \mathbb{Z}, \text{ or}$$
$$y_1 - y_2 \notin \mathbb{Z}.$$

Then $\delta_1 \times \delta_2 \in \mathrm{Irr}$.

Note that the above four conditions are equivalent since

$$x_1 - y_1 \in \mathbb{Z} \quad \text{and} \quad x_2 - y_2 \in \mathbb{Z}.$$

Now we shall see that $\delta_1 \times \delta_2 \in \mathrm{Irr}$. Suppose that $\delta_1 \times \delta_2 \notin \mathrm{Irr}$. Set $a_0 = (\delta_1, \delta_2)$. Since $\delta_1 \times \delta_2$ is not irreducible, there exist $a \in M(D)$ so that

$$\chi(a) = \chi(a_0) \quad \text{and} \quad \underline{e}(a) < \underline{e}(a_0).$$

The condition $\chi(a) = \chi(a_0)$ implies that a is one of the following multisets

$$(\gamma(x_1, y_1),\ \gamma(x_2, y_2)),$$
$$(\gamma_{\varepsilon_1}(x_1), \gamma_{\varepsilon_2}(y_1), \gamma(x_2, y_2)),$$
$$(\gamma_{\varepsilon_1}(x_1), \gamma_{\varepsilon_2}(x_2), \gamma_{\varepsilon_3}(x_2), \gamma_{\varepsilon_u}(y_2)),$$

with $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_u \in \mathbb{Z}/2\mathbb{Z}$. Direct checking implies $\underline{e}(a) \geq \underline{e}(a_0)$

in all four cases. This is a contradiction.

An immediate consequence of the preceding two remarks on irreducibility is the fact that

$$\pi(u(\delta,n),\alpha) \in \mathrm{Irr}$$

for $0 < \alpha < 1/2$.

Now $\pi(u(\delta,n),0) \in \mathrm{Irr}^u$ by (UO) and thus

$$\pi(u(\delta,n),\alpha) \in \mathrm{Irr} \quad \text{for} \quad 0 \leqq \alpha < 1/2.$$

Well-known analytic properties of intertwining operators implies unitarizability of $\pi(u(\delta,n),\alpha)$, $0 < \alpha < 1/2$. For the formal proof one can apply B. Speh criterion in §3 of [19] for existance of complementary series, and the first part of our proof.

6. Real general linear group II

In this section we assume that $F = \mathbb{R}$. Here we prove that $u(\delta,n), \delta \in D$, are prime.

6.1. Lemma: Let $\delta \in \tilde{G}_1$ and $n \in \mathbb{N}$. The element $u(\delta,n)$ is a prime element of the factorial ring R.

Proof: Note that it is enough to consider the case of $n \geq 2$ by Corollary 1.4.

Let $\delta = \gamma_\varepsilon(x)$. Then

$$a(\delta,n) = (\gamma_\varepsilon(x + \frac{n-1}{2}), \gamma_\varepsilon(x + \frac{n-1}{2} - 1), \ldots, \gamma_\varepsilon(x, -\frac{n-1}{2}).$$

We consider $u(\delta,n)$ as a polynomial in indetermines D. Since $\chi(a(\delta,n))$ consists of n different elements we see that the degree of $u(\delta,n)$ in any indetermine is 0 or 1.

Set
$$X_i = \gamma_\varepsilon(x + \frac{n-1}{2} - i + 1),\ 1 \leq i \leq n;$$

$$X_{i,j} = \gamma(x + \frac{n-1}{2} - i + 1,\ x + \frac{n-1}{2} - j + 1),\ 1 \leq i < j \leq n;$$

$$a_0 = (X_1, \ldots, X_m);$$

$$a_{i,j} = (X_1, \ldots, X_{i-1}, X_{i,j}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n),$$
$$1 \leq i < j \leq n.$$

Fix $1 \leq i < n$. Now we shall show that $a_{i,i+1}$ satisfies condition (C_1) of Lemma 5.5 with respect to a_0. It is clear that $\lambda(a_0)$ contains $L(a_{i,i+1})$. Let $d \in M(D)$ so that

$\chi(d) = \chi(a_0)$ and $\underline{e}(d) < \underline{e}(a_0)$. One can see directly that there exist $1 \leq j < k \leq n$ so that

$$\underline{e}(d) \leq \underline{e}(a_{j,k}) \ .$$

Thus for the proof of the condition (C_1) for $a_{i,i+1}$, it is enough to see that

$$\underline{e}(a_{i,i+1}) \not< \underline{e}(a_{j,k})$$

for all $1 \leq j < k \leq n$. Suppose that such j,k exist, i.e.

$$\underline{e}(a_{i,i+1}) < \underline{e}(a_{j,k}) .$$

Note that $\underline{e}(a_0) = (\frac{n-1}{2}, \frac{n-1}{2} - 1, \dots, -\frac{n-1}{2}) + (\mathrm{Re} x)\ (1,1,\dots,1)$,

$$\underline{e}(a_{i,i+1}) = (\frac{n-1}{2}, \dots, \frac{n-1}{2} - i + 2, \frac{n-1}{2} - i + \frac{1}{2}, \frac{n-1}{2} - i + \frac{1}{2}, \frac{n-1}{2} - i - 1, \dots, -\frac{n-1}{2}) + (\mathrm{Re}\ x)(1,1,\dots,1).$$

Now one obtains directly that $\underline{e}(a_{i,i+1}) < \underline{e}(a_{j,k})$ implies $i \leq j$ and $k \leq i+1$. Thus $i = j$, $i+1 = k$. This is a contradiction.

Let $u(\delta,n) = P \times Q$ be a non-trivial decomposition. Corollary 1.6. implies that P and Q are homogenious elements of R. By (iv) of Lemma 5.5 graduations of P and Q are greater then zero. Set

$$P = \sum_{a \in M(D)} m(P,a)\lambda(a)$$

$$Q = \sum_{a \in M(D)} m(Q,a)\lambda(a).$$

We have

$$u(\delta,n) = \lambda(a_0) + \sum_{i=1}^{n-1} m(a_0,a_{i,i+1})\lambda(a_{i,i+1}) +$$

$$+ \sum_{\substack{a\in M(D) \\ a \neq a_0,\ a \neq a_{i,i+1}}} m(a_0,a)\lambda(a) = P \times Q.$$

where $m(a_0,a_{i,i+1}) \neq 0$ by (v) of Lemma 5.5. There exist

$a^v, b^v \in M(D)$ so that

$a^v + b^v = a_0,$

$m(P,a^v) \neq 0,$

$m(Q,b^v) \neq 0.$

Find a partition

$$\{1,\ldots,n\} = \{\sigma(1),\ldots,\sigma(u)\} \cup \{\tau(1),\ldots,\tau(v)\}$$

so that $a^v = (X_{\sigma(1)},\ldots,X_{\sigma(u)})$

$b^v = (X_{\tau(1)},\ldots,X_{\tau(v)}).$

Here $u+v = n$ and $u \geq 1$, $v \geq 1$. We can find $1 \leq i \leq u$, $1 \leq j \leq v$ so that

$$\{r,r+1\} = \{\sigma(i),\tau(j)\}$$

for some $1 \leq r \leq n$. Without lost of generality we suppose that $i = u$, $j = v$. Since $m(a_0,a_{r,r+1}) \neq 0$, there exist $c^v, d^v \in M(D)$ so that

$c^v + d^v = a_{r,r+1}$

$m(P,c^v) \neq 0$

$m(Q,d^v) \neq 0,$

$\mathrm{gr}\ c^v = u$

$\mathrm{gr}\ d^v = v.$

We can suppose that $X_{r,r+1}$ is in the support of c^V. Now one obtains directly that there exist

$$X \in \{X_1,\dots,X_{r-1},X_{r+2},\dots,X_n\}$$

so that X is in the support of d^V and in the support of a^V. This means that the degrees of P and of Q in the indetermine X are greater than zero. Thus, the degree of $u(\delta,n)$ in X is greater than or equal to 2. This is a contradiction which proves our lemma.

6.2. Lemma: Let $x,y \in \mathbb{C}$. Suppose that $x - y = k \in \mathbb{N}$.

(i) If $k \geq 3$, then

$$\gamma(x,y) \times \gamma(x+1,y+1) =$$

$$= L(\gamma(x,y),\gamma(x+1,y+1)) + m(\gamma(x,y+1) \times \gamma(x+1,y))$$

for some $m \in \mathbb{N}$.

(ii) If, $k = 1$, then

$$\gamma(x,y) \times \gamma(x+1,y+1) =$$

$$L(\gamma(x,y),\gamma(x+1),y+1)) + m(\gamma_0(x) \times \gamma_1(y+1) \times \gamma(x+1,y))$$

for some $m \in \mathbb{N}$. Note that $x = y+1$.

(iii) If $k = 2$, then there exist $m_0,m_1,m_2 \in \mathbb{Z}_+$ such that

$$\gamma(x,y) \times \gamma(x+1,y+1) = L(\gamma(x,y),\gamma(x+1,y+1)) +$$

$$+ m_2\ \gamma(x,y+1) \times \gamma(x+1,y)) + m_0 L((\gamma_0(x),\gamma_0(y+1),\gamma(y,x+1))) +$$

$$+ m_1 L(\gamma_1(x),\gamma_1(y+1),\gamma(y,x+1))$$

with

$$m_1 + m_2 + m_3 \geq 1.$$

The above lemma will be proved together with the following:

6.3. Lemma: Let $x, y \in \mathbb{C}$ and $r \in \mathbb{N}$. Suppose that $x - y = k \in \mathbb{N}$. Set

$$a_0 = (\gamma(x,y), \gamma(x+r, y+r)).$$

Let $a \in M(D)$ so that $\chi(a) = \chi(a_0)$ and $\underline{e}(a) < \underline{e}(a_0)$.

(i) We have $\underline{e}(a_0) = (r,r,0,0) + \mathrm{Re}\,(\frac{x+y}{2})(1,1,1,1)$ and $\underline{e}(a_0) > \underline{e}(a) \geq (r/2,r/2,r/2,r/2) + \mathrm{Re}\,(\frac{x+y}{2})\,(1,1,1,1)$.

(ii) If $k > 2r$ or $r = k$, then

$$\underline{e}(a) = (r/2,r/2,r/2,r/2) + \mathrm{Re}\,(\tfrac{x+y}{2})(1,1,1,1).$$

(iii) If $2r \geq k > r$, then $\underline{e}(a)$ equals to one of the following terms

$$(r/2,r/2,r/2,r/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1,1,1),$$
$$(k/2,r/2,r/2,r-k/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1,1,1).$$

(iv) If $r > k$, then $\underline{e}(a)$ equals to one of the following terms

$$(r/2,r/2,r/2,r/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1,1,1),$$

$$(r-k/2,r/2,r/2,k/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1,1,1),$$

$$(r/2+k/2,r/2+k/2,r/2-k/2,r/2-k/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\,(1,1,1,1).$$

Proof of Lemmas 6.2. and 6.3.: Let $x,y \in \mathbb{C}$ such that $x-y = k \in \mathbb{N}$. Set $a_0 = (\gamma(x,y),\gamma(x+r,y+r))$. Let $a \in M(D)$ so that $\chi(a) = \chi(a_0)$ and $\underline{e}(a) < \underline{e}(a_0)$. Note that if $\lambda(a_0)$ contains $L(a)$ and $a \neq a_0$, then two above conditions on a are fulfilled.

The condition $\chi(a) = \chi(a_0)$ implies that a is of one of the following forms

$$a_1 = (\gamma(x+r,y),\gamma(y+r,x))$$
$$a_2 = (\gamma(x+r,y),\gamma_{\varepsilon_1}(y+r),\gamma_{\varepsilon_2}(x))$$
$$a_3 = (\gamma_{\varepsilon_1}(x+r),\gamma_{\varepsilon_2}(y),\gamma(y+r,x))$$
$$a_4 = (\gamma_{\varepsilon_1}(x+r),\gamma_{\varepsilon_2}(y),\gamma_{\varepsilon_3}(y+r),\gamma_{\varepsilon_4}(x))$$
$$a_5 = (\gamma(x+r,x),\gamma(y+r,y))$$
$$a_6 = (\gamma_{\varepsilon_1}(x+r),\gamma_{\varepsilon_2}(x),\gamma(y+r,y))$$
$$a_7 = (\gamma(x+r,x),\gamma_{\varepsilon_1}(y+r),\gamma_{\varepsilon_2}(y))$$

for some $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 \in \mathbb{Z}/2\mathbb{Z}$. We have directly

$$\mathrm{Re}\ x = \mathrm{Re}\ (\tfrac{x+y}{2} + \tfrac{x-y}{2}) = (k/2) + \mathrm{Re}\ \tfrac{x+y}{2},$$

$$\mathrm{Re}\ y = -(k/2) + \mathrm{Re}\ \tfrac{x+y}{2},$$

$$\underline{e}(a_0) = (r,r,0,0) + \mathrm{Re}\ (\tfrac{x+y}{2})\ (1,1,1,1).$$

Note that $\mathrm{Tr}\ \underline{e}(a) = 2(r + \mathrm{Re}(x+y))$. This implies directly that

$$\underline{e}(a) \geq (r/2,r/2,r/2,r/2) + \mathrm{Re}\ \tfrac{x+y}{2}\ (1,1,1,1).$$

Thus (i) of Lemma 6.3. holds.

Since $e(\gamma_\varepsilon(x+r)) = (k/2) + r + \mathrm{Re}\,\frac{x+y}{2}$, the condition $\underline{e}(a) < \underline{e}(a_0)$ implies that a can not be equal to a_3, a_4, a_6. Since $e(\gamma_\varepsilon(y)) = -(k/2) + \mathrm{Re}\,\frac{x+y}{2}$, the condition $\underline{e}(a) < \underline{e}(a_0)$ implies $a \neq a_7$.

Thus a is equal to a_1, a_2 or a_5.

Suppose that $r > k$.

$$\underline{e}(a_1) = (r/2, r/2, r/2, r/2) + \mathrm{Re}\,(\tfrac{x+y}{2})\ (1,1,1,1)$$
$$\underline{e}(a_2) = (r-(k/2), r/2, r/2, k/2) + \mathrm{Re}(\tfrac{x+y}{2})\ (1,1,1,1)$$
$$\underline{e}(a_5) = ((r/2)+(k/2), (r/2)-(k/2), (r/2)-(k/2), (r/2)-(k/2)) + \mathrm{Re}\,(\tfrac{x+y}{2})\ (1,1,1,1).$$

This proves (iv) of Lemma 6.3.

Suppose that $k > r$. Then

$$\underline{e}(a_5) = ((r/2)+(k/2), (r/2)+(k/2), (r/2)-(k/2), (r/2)-(k/2)) + \mathrm{Re}\,(\tfrac{x-y}{2})\ (1,1,1,1).$$

Since $(r/2)-(k/2) < 0$ it can not be $\underline{e}(a_0) > \underline{e}(a_5)$. Thus if $k > r$ then $a = a_1$ or $a = a_2$. This implies (iii) of Lemma 6.3.

Let $k > 2r$. Then $k > r$ and thus $a = a_1$ or $a = a_2$. We have

$$\underline{e}(a_2) = (k/2, r/2, r/2, r-k/2) * \mathrm{Re}(\tfrac{x+y}{2})\ (1,1,1,1).$$

Since $r - r/2 < 0$, it can not be $\underline{e}(a_2) < \underline{e}(a_0)$. Thus $a = a_1$.

Therefore, we proved (ii) if $k > 2r$.

Let $k = r$. We showed that $a = a_1$ or $a = a_2$ or $a = a_5$. Now

$$\underline{e}(a_5) = (r,r,0,0) + \mathrm{Re}\left(\frac{x+y}{2}\right) \quad (1,1,1,1),$$

and thus $\underline{e}(a_5) = \underline{e}(a_0)$. Therefore $a = a_5$. Thus $a = a_1$ or $a = a_2$. Since $k = r$ we have $y + r = x$. Thus a_1 is not defined in this situation. This implies $a = a_2$. Now the rest of (ii) is obvious.

We have proved Lemma 6.3. In the rest we shall suppose $r = 1$. Now we shall prove Lemma 6.2.

By §2 of [19], $\gamma(x,y) \times \gamma(x+1,y+1)$ is reducible (see also §5 of [19]). We know that $L(a_0)$ is a composition factor of $\lambda(a_0)$ with multiplicity one. Thus there exist $a \in M(D)$ so that $L(a)$ is contained in $\lambda(a_0)$ and $a \neq a_0$. This implies that $\chi(a) = \chi(a_0)$ and $\underline{e}(a) < \underline{e}(a_0)$.

Let $k \geq 3$. Then $k > 2r$. Now the first part of the proof implies

$$a = a_1 = (\gamma(x+1,y), \gamma(y+1,x)).$$

This proves (i) of Lemma 6.2.

Suppose that $k = 1$. Then $k = r$ and the first part of the proof implies

$$a = a_2 = (\gamma(x+1,y), \gamma_{\varepsilon_1}(y+1), \gamma_{\varepsilon_2}(x)) =$$
$$= (\gamma(x+1,y), \gamma_{\varepsilon_1}(x), \gamma_{\varepsilon_2}(x))$$

since $y + 1 = x$. We need to determine which ε_1 and ε_2 are possible. Note that $\gamma(x,y)$ is a composition factor of $\gamma_0(x) \times \gamma_0(y)$ and $\gamma(x+1,y+1)$ is a composition factor of $\gamma_0(x+1) \times \gamma_0(y+1)$. Therefore $L(a)$ is a composition factor of

$$\gamma_0(x) \times \gamma_0(y) \times \gamma_0(x+1) \times \gamma_0(y+1).$$

Let I_4 be identity of G_4. Then $-I_4$ acts in the above representation trivialy, and thus $-I_4$ acts in $L(a)$ trivialy. Now $\gamma(y,x+1)$ is a composition factor of $\gamma_0(x+1) \times \gamma_1(y)$. Thus $L(a_2)$ is contained in

$$\gamma_0(x+1) \times \gamma_1(y) \times \gamma_{\varepsilon_1}(x) \times \gamma_{\varepsilon_2}(y+1).$$

Here $-I_4$ act as multiplication by $(-1)^{1+\varepsilon_1+\varepsilon_2}$. Thus $-I_4$ acts in $L(a_2)$ by $(-1)^{1+\varepsilon_1+\varepsilon_2}$. Since $L(a) = L(a_2)$ we have $\varepsilon_1+\varepsilon_2 = 1$ in $\mathbb{Z}/2\mathbb{Z}$. Thus

$$a = (\gamma_1(x), \gamma_0(x), \gamma(y,x+1)) \quad .$$

This proves (ii) of Lemma 6.2 since

$$e(\gamma_1(x)) = e(\gamma_0(x)) = e(\gamma(y,x+1))$$

We obtain (iii) of Lemma 6.2. in the same way as (i) and (ii)..

Now we can prove (U3).

6.4. <u>Proposition</u>: Let $\delta \in D$ and $n \in \mathbb{N}$. The representation $u(\delta,n)$ is a prime element of the factorial ring R.

<u>Proof</u>: By Lemma 6.1. we can suppose that $\delta \notin \tilde{G}_1$. Thus $\operatorname{gr}\delta = 2$ and we can choose $x,y \in \mathbb{C}$ so that $x - y = k \in \mathbb{N}$ and $\delta = \gamma(x,y)$.

Without lost of generality we can suppose

$$\operatorname{Re}(x+y) = 0$$

(if $\alpha \in \mathbb{R}$, then $L(a) \to \nu^{\alpha} L(a)$ lifts to an multiplicative automorphism of R). We need the above assumption only for simplifying notation. By Corollary 1.4. we can suppose that $n \geq 2$.

Set

$$a_0 = a(\gamma(x,y),n),$$

$$X_i = \gamma(x+\frac{n-1}{2} - i + 1,\ y + \frac{n-1}{2} - i + 1),$$

$i = 1,\dots,n$. Clearly

$$a_0 = (X_1,\dots,X_n).$$

Suppose that $u(\gamma(x,y),n) = L(a_0)$ is not prime. Let

$$L(a_0) = P \times Q$$

be a non-trivial decomposition (i.e. P and Q are not invertible in R). By Corollary 1.4. we know that P and Q are homogeneous in R. Let

$$P = \sum_{a \in M(D)} m(P,a)\lambda(a),$$

$$Q = \sum_{a \in M(D)} m(Q,a)\lambda(a).$$

By (iv) of Lemma 5.5 we have

$$L(a_0) = \lambda(a_0) + \sum_{\substack{\chi(a)=\chi(a_0) \\ \underline{e}(a)<\underline{e}(a_0)}} m(a_0,a)\lambda(a) =$$

$$= X_1 \times X_2 \times \ldots \times X_n + \sum_{\substack{\chi(a)=\chi(a_0) \\ \underline{e}(a)\ \ \underline{e}(a_0)}} m(a_0,a)\lambda(a).$$

The above formula implies gr $P \geq 1$, gr $Q \geq 1$. The definition of multiplication in R implies that there exist $a^V, b^V \in M(D)$ so that

$$a^V + b^V = a_0,$$
$$m(P,a^V) \neq 0,$$
$$m(Q,b^V) \neq 0.$$

Since gr $P > 0$ and gr $Q > 0$ we have $a^V \neq \emptyset$ and $b^V \neq \emptyset$. Take a partition

$$\{\sigma(1),\ldots,\sigma(u)\} \cup \{\tau(1),\ldots,\tau(v)\} = \{1,2,\ldots,n\}$$

where $u \geq 1, v \geq 1$, $u + v = n$, so that

$$a^V = (X_{\sigma(1)},\ldots,X_{\sigma(u)}),$$
$$b^V = (X_{\tau(1)},\ldots,X_{\tau(v)}).$$

Now gr $P = 2u$ and gr $Q = 2v$.

We can find $1 \leq t \leq n - 1$ and $1 \leq i \leq u$, $1 \leq j \leq v$ so that $\{t,t+1\} = \{\sigma(i),\tau(j)\}$. After a prenumeration we can take that $i = u$ and $j = v$, i.e. $\{t,t+1\} = \{\sigma(u),\tau(v)\}$.

We compute directly

$$\underline{e}(a_0) = (\frac{n-1}{2}, \frac{n-1}{2}, \frac{n-1}{2} - 1, \ldots, -\frac{n-1}{2}, -\frac{n-1}{2}).$$

Suppose that a_1 satisfies the condition (C_2) of Lemma 5.5. with respect to (X_t, X_{t+1}). Now the end of §3 of [21] (more precisely Corollary 3.15., Lemma 3.8 and Theorem 3.7. of [21]), Lemma 5.3. and Lemma 6.3. imply that

$$a_{t,t+1} = a_1 + (X_1, \ldots, X_{t-1}, X_{t+2}, \ldots, X_n)$$

satisfies the condition (C_2) of Lemma 5.5 with respect to a_0. By (vi) of the same lemma we have

$$m(a_0, a_{r,r+1}) \neq 0.$$

Now we shall choose a_1 as above. First we have

$$(X_t, X_{t+1}) = (\gamma(x + \frac{n-1}{2} - t + 1, y + \frac{n-1}{2} - t + 1), \gamma(x + \frac{n-1}{2} - t, y + \frac{n-1}{2} - t).$$

Suppose that $x - y = k \geq 3$. Then there is only one possible a_1 by Lemmmas 6.2. and 6.3.,

$$a_1 = (\gamma(x + \frac{n-1}{2} - t + 1, y + \frac{n-1}{2} - t), \gamma(x + \frac{n-1}{2} - t, y + \frac{n-2}{2} - t + 1)).$$

Set

$$Y_t = \gamma(x + \frac{n-1}{2} - t + 1, y + \frac{n-1}{2} - t).$$

$$Y_{t+1} = \gamma(x + \frac{n-1}{2} - t, y + \frac{n-1}{2} - t + 1).$$

Let $k = 2$. Then (iii) of Lemma 6.3. implies that there exist $m_0, m_1, m_2 \in \mathbb{Z}_+$ such that

$$\gamma(x+\frac{n-1}{2} - t+1, y+\frac{n-1}{2} - t+1) \times \gamma(x+\frac{n-1}{2} - t, y+\frac{n-1}{2} - t) =$$

$$= L((X_t, X_{t+1})) +$$

$$+ m_2 (\gamma(x+\frac{n-1}{2} - t + 1, y+\frac{n-1}{2} - t) \times \gamma(x+\frac{n-1}{2} - t, y+\frac{n-1}{2} - t+1)) +$$
$$+ m_0 L((\gamma(x+\frac{n-1}{2} - t+1, y+\frac{n-1}{2} - t), \gamma_0(x+\frac{n-1}{2} - t), \gamma_0(y+\frac{n-1}{2} - t+1)) +$$
$$+ m_1 L((\gamma(x+\frac{n-1}{2} - t+1, y+\frac{n-1}{2} - t), \gamma_1(x+\frac{n-1}{2} - t), \gamma_1(y+\frac{n-1}{2} - t+1)).$$

Suppose that there exist $\varepsilon \in \{0,1\}$ so that $m_\varepsilon \neq 0$. This case will be called non-standard. Set

$$Y_t = \gamma(x+\frac{n-1}{2} - t+1, y+\frac{n-1}{t} - t),$$

$$Y_{t+1}^{(x)} = \gamma_\varepsilon(x+\frac{n-1}{2} - t),$$

$$Y_{t+1}^{(y)} = \gamma_\varepsilon(y+\frac{n-1}{2} - t + 1),$$

$$Y_{t+1} = (Y_{t+1}^{(x)}, Y_{t+1}^{(y)}).$$

Now we take

$$a_1 = (Y_t, Y_{t+1}^{(x)}, Y_{t+1}^{(y)}).$$

If $m_0 = m_1 = 0$, then we define Y_t, Y_{t+1} and a_1 in the same manner as in the case of $k \geq 3$. This will be called the standard case.

Let $k = 1$. Put

$$Y_t = \gamma(x+\frac{n-1}{2} - t + 1, y+\frac{n-1}{2} - t),$$

$$Y_{t+1}^{(0)} = \gamma_0(x+\frac{n-1}{2} - t),$$

$$Y_{t+1}^{(1)} = \gamma_1(x+\frac{n-1}{2}-t),$$

$$Y_{t+1} = (Y_{t+1}^{(0)}, Y_{t+1}^{(1)}).$$

We take $a_1 = (Y_t, Y_{t+1}^{(0)}, Y_{t+1}^{(1)})$.

We define $a_{t,t+1}$ as before:

$$a_{t,t+1} = (X_1,\ldots,X_{t-1},X_{t+2},\ldots,X_n) + a_1.$$

As we have noticed, $m(a_0,a_{t,t+1}) \neq 0$. This implies that there exist $c^V, d^V \in M(D)$ so that

$$c^V + d^V = a_{t,t+1}$$
$$m(P,c^V) \neq 0$$
$$m(Q,d^V) \neq 0.$$

Clearly $gr\ c^V = 2u$ and $gr\ d^V = 2v$. Without lost of generality we can suppose that Y_t is in the support of c^V.

Let $k = 1$. Since the graduations of c^V and d^V are even, $Y_{t+1}^{(0)}$ is in the support of c^V if and only if $Y_{t+1}^{(1)}$ is also in the support of c^V. If $k = 2$ and we are in the non-standard case, then the same observation holds for $Y_{t+1}^{(x)}$ and $Y_{t+1}^{(y)}$. Therefore, it makes sence to say that Y_{t+1} is in support of c^V or d^V, even if $k = 1$ or $k = 2$ and we are in the non-standard case.

We consider now two possible cases. Let Y_{t+1} be in the support of c^V. We can decompose

$$\{1,2,\ldots,t-1,t+2,\ldots,n\} =$$

$$= \{\pi(1),\ldots,\pi(u-2)\} \cup \{\rho(1),\ldots,\rho(v)\}$$

so that

$$c^V = (X_{\pi(1)},\ldots,X_{\pi(u-2)},Y_t) + Y_{t+1} ,$$

$$d^V = (X_{\rho(1)},\ldots,X_{\rho(v)})$$

(we identify an element with the multiset of cordinality one, in a natural way). Simple consideration implies that

$$\{\rho(1),\ldots,\rho(v)\} \cap \{\sigma(1),\ldots,\sigma(u)\} \neq \emptyset .$$

Set $T = \{\sigma(1),\ldots,\sigma(u)\} \cup \{\rho(1),\ldots,\rho(v)\}$. The above relation implies

$$T \neq \{1,\ldots,n\} .$$

We shall denote by $\deg_T S$ the total degree of $S \in R$ in the indetermines $\{X_i, i \in T\}$. We know that

$$\deg_T P \geq u,$$

$$\deg_T Q \geq v.$$

Thus $\deg_T L(a_0) \geq u + v = n$. Considering the graduation one obtains

$$\deg_T L(a_0) = n.$$

Thus there exist $b_0 \in M(D)$ so that

$$\deg_T \lambda(b_0) = n,$$

$$m(a_0,b_0) \neq 0,$$

$$gr(b_0) = 2n,$$

$$\chi(b_0) = \chi(a_0),$$

$$\underline{e}(b_0) < \underline{e}(a_0).$$

Let $\lambda(b_0) = X_1^{\alpha_1} \times \ldots \times X_n^{\alpha_n}$, where $\alpha_i \in \mathbb{Z}_+$ for $i = 1,\ldots,n$ and $\alpha_1+\ldots+\alpha_n = n$. Since $T \neq \{1,\ldots,n\}$, there exist some i so that $\alpha_1 \neq 1$, i.e. there exist some $j \in T$ so that $\alpha_j \geq 2$.

We have

$$\lambda(a_0) = (x+\frac{n-1}{2}, x+\frac{n-1}{2}-1, x+\frac{n-1}{2}-2,\ldots,x-\frac{n-1}{2},$$
$$y+\frac{n-1}{2}, y+\frac{n-1}{2}-1,\ldots,y-\frac{n-1}{2}).$$

Let

$$i_0 = \min\{i;\alpha_i \neq 1\}.$$

Such i_0 exists since all α_i are not equal to 1. Now one sees directly that

$$x - \frac{n-1}{2} - i + 1$$

can not have the same multiplicity in $\chi(a_0)$ and $\chi(b_0)$. This is a contradiction.

It remains to consider the case when Y_{t+1} is in the support of d^v. Choose a partition

$$\{1,\ldots,t-1,t+2,n\} = \{\pi(1),\ldots,\pi(u-1)\} \cup \{\rho(1),\ldots,\rho(v-1)\}$$

so that

$$c^V = (X_{\pi(1)},\ldots,X_{\pi(u-1)},Y_r),$$
$$d^V = (X_{\rho(1)},\ldots,X_{\rho(v-1)}) + Y_{r+1}.$$

Let $T = \{\pi(1),\ldots,\pi(u-1),\tau(1),\ldots,\tau(v)\}$. The total degree of $f \in R$ in indetermines

$$\{Y_r\} \cup \{X_i, i \in T\}$$

is denoted by $\deg^*_T$. Note that

$$T \neq \{1,\ldots,n\}.$$

As before, we have

$$\deg^*_T L(a_0) = n.$$

Thus we can find $b_0 \in M(D)$ so that

$$\deg^*_T \lambda(b^0) = n,$$
$$\chi(b_0) = \chi(a_0) \quad .$$

Let $\lambda(b_0) = X_1^{\alpha_1} \times \ldots \times X_n^{\alpha_n} \times Y_r^{\alpha}$ where $\alpha_i \in \mathbb{Z}_+$, $i = 1,\ldots,n$, and

$$\alpha_1 + \ldots + \alpha_n + \alpha = n.$$

Previous considerations implies that there exist i so that $\alpha_i \neq 1$.

If $\alpha = 0$, then the preceding case implies that

$\chi(b_0) \neq \chi(a_0)$. Thus $\alpha \geq 1$. Considering $\chi(a_0)$ one obtains directly that $\alpha \leq 2$, since multiplicities in $\chi(a_0)$ are at most two.

Note that

$$2\,\mathrm{Tr}\,X_i \in (n-1) + 2\mathbb{Z}\,, \quad i = 1,\dots, n\,,$$

$$2\,\mathrm{Tr}\,Y_t \in n + 2\mathbb{Z}\,.$$

Therefore

$$2\,\mathrm{Tr}\,b_0 \in (n-1)(n-\alpha) + \alpha n + 2\mathbb{Z} = \alpha + 2\mathbb{Z}$$

and $$2\,\mathrm{Tr}\,a_0 \in n(n-1) + 2\mathbb{Z} = 2\mathbb{Z}\,.$$

Since $\mathrm{Tr}\,b_0 = \mathrm{Tr}\,a_0$ we have $\alpha \in 2\mathbb{Z}$.

Thus we proved that $\alpha = 2$.

We consider the case $\alpha = 2$. Recall that

$$Y_t = \gamma\left(x + \frac{n-1}{2} - t + 1,\ y + \frac{n-1}{2} - t\right)$$

$$X_i = \gamma\left(x + \frac{n-1}{2} - i + 1,\ y + \frac{n-1}{2} - i + 1\right),$$

$i = 1,\dots,n$. Direct consideration of support implies that $n \neq 2$. Put

$$i_0 = \min\{i;\ \alpha_i \neq 1\}\,.$$

One obtains that $i_0 \geq t$. Direct consequence of this is the fact that multiplicity of $x + \frac{n-1}{2} - t + 1$ in $\chi(b_0)$ is greater than or equal to three. This is impossible.

The last contradiction finishes our proof of the proposition.

Proposition 2.1., results of the fifth and the sixth section, together with the end of the third section implies that Theorem A holds if $F = \mathbb{R}$. Thus, Theorem A. holds in the general case.

7. A conjecture on $GL(n, \mathbb{R})$

In this section we assume $F = \mathbb{R}$.

7.1 Conjecture: Let $n,k \in \mathbb{N}$, $\varepsilon \in \mathbb{Z}/2\,\mathbb{Z}$ and $\beta \in \mathbb{C}$ so that $\operatorname{Re} \beta = 0$.

(i) If $k \geq 2$, then we have in R

$$\nu^{1/2}\, u\Big(\delta(\beta,k),n\Big) \times \nu^{-1/2}\, u\Big(\delta(\beta,k),n\Big) = $$
$$= u\Big(\delta(\beta,k),n+1\Big) \times u\Big(\delta(\beta,k),n-1\Big) + $$
$$+ u\Big(\delta(\beta,k+1),n\Big) \times u\Big(\delta(\beta,k-1),n\Big) .$$

(ii) For $k = 1$ we have in R

$$\nu^{1/2}\, u\Big(\delta(\beta,1),n\Big) \times \nu^{-1/2}\, u\Big(\delta(\beta,1),n\Big) = $$
$$= u\Big(\delta(\beta,1),n+1\Big) \times u\Big(\delta(\beta,1),n-1\Big) + $$
$$+ u\Big(\delta(\beta,2),n\Big) \times \Big[u\Big(\gamma_0(\beta),n\Big) \times u\Big(\gamma_1(\beta),n\Big)\Big]$$

(iii) We have in R

$$\nu^{1/2}\, u\Big(\gamma_\varepsilon(\beta),n\Big) \times \nu^{-1/2}\, u\Big(\gamma_\varepsilon(\beta),n\Big) = $$
$$= u\Big(\gamma_\varepsilon(\beta),n+1\Big) \times u\Big(\gamma_\varepsilon(\beta),n-1\Big) + u\Big(\delta(\beta,1),n\Big) .$$

REFERENCES

[1] Bargmann V., Irreducible unitary representations of the Lorentz group, Ann. of Math. 48 (1947), 568-640.

[2] Bernstein J.N., P-invariant distributions on GL(N) and the classification of unitary representations of GL(N) (non-archimedean case), in Lie Group Representations II, Proceedings, University of Maryland 1982-83, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1984, 50-102.

[3] Bernstein J.N., Deligne P., Kazhdan D., Vigneras M.F., Représentations des groupes réductifs sur un corps local, Herman, Paris, 1984.

[4] Borel A., Wallach N., Continuous cohomology, discrete subgroups, and representations of reductive groups, Ann. of Math. Studies 94, Princeton University Press, Princeton, 1980.

[5] Duflo M., Représentations unitaires irréducibles des groupes simples complexes de rang deux, Bull. Soc. Math. France 107 (1979), 55-96.

[6] Gelfand I.M., Neumark M.A., Unitäre Darstellungen der Klassischen Gruppen, Akademie Verlag, Berlin, 1957 (German translation).

[7] Gelfand I.M., Naimark M.A., Unitary representations of the Lorentz group, Inverstiya Akad. Nauk SSSR., Ser. Mat. 11 (1947), 411-504 (Russian).

[8] Godement R., Notes on Jacquet-Langlands' theory, Institute for Advanced Study, Princeton, 1970.

[9] Jacquet H., Generic representations, in Non-Commutative Harmonic Analysis, Lecture Notes in Math. 587, Springer-Verlag, 1977, 91-101.

[10] Jacquet H., On the residual spectrum of GL(n), in Lie Group Representations II, Proceedings, University of Maryland 1982-1983, Lecture Notes in Math. 1041, Springer-Verlag, Berlin, 1983, 185-208.

[11] Jacquet H., Principal L-functions of the linear group, in Proc. Sympos. Pure Math. vol. XXXIII, part 2, Amer. Math. Soc., Providence, R.I. 1979, 63-86.

[12] Kirillov A.A., Infinite dimensional representations of the general linear group, Dokl. Akad. Nauk SSSR 114 (1962), 37-39 (=Soviet Math. Dokl. 3(1962),652-655).

[13] Knapp A.W., Zuckerman G.J., Classification theorems for representations of semisimple Lie groups, in Non-Commutative Harmonic Analysis, Lecture Notes in Math. 587, Springer-Verlag, Berlin, 1977, 138-159.

[14] Langlands R.P., Automorphic representations, Shimura varieties, and motives. Ein märchen, in Proc. Sympos. Pure Math. vol. XXXIII, part 2, Amer. Math. Soc., Providence, R.I., 1979, 205-246.

[15] Langlands R.P., On the classification of irreducible representations of real algebraic groups, Preprint, Institute for Advanced Study, Princeton, 1974.

[16] Miličić D., On C*-algebras with bounded trace, Glasnik Mat. 8(28) (1973), 7-22.

[17] Olshansky G.I., Intertwining operators and complementary series in the class of representations of the general group of matrices over a locally compact division algebra, induced from parabolic subgroups, Mat. Sb. vol. 93, no 2 (1974), 218-253.

[18] Schmid W. On the characters of the discrete sereies (the Hermitian symmetric case), Invent. Math. 30 (1975), 47-144.

[19] Speh B., The unitary dual of $G\ell(3,\mathbb{R})$ and $GL(4,\mathbb{R})$ Math. Ann. 258 (1981), 113-133.

[20] Speh B., Unitary representations of $G\ell(n,\mathbb{R})$ with non-trivial $(\mathfrak{q},K)$-cohomology, Invent. Math. 71 (1983), 443-465.

[21] Speh B., Vogan D., Reducibility of generalized principal series representations, Acta Math. 145 (1980), 227-299.

[22] Stein E.M., Analysis in matrix spaces and some new representations of $SL(N,\mathbb{C})$, Ann. of Math. 86 (1967), 461-490.

[23] Tadić M., On the classification of irreducible unitary representations of $GL(n)$ and the conjectures of Bernstein and Zelevinsky (non-archimedean case), to appear in Ann. Scient. Éc. Norm. Sup.

[24] Tadić M., Proof of a conjecture of Bernstein, to appear in Math. Ann.

[25] Tadić M., Unitary dual of p-adic GL(n). Proof of Bernstein conjectures, to appear in Bull. Amer. Math. Soc.

[26] Tadić M., Spherical unitary dual of general linear group over local non-archimedean field, preprint at Max-Planck-Institut für Mathematik, Bonn, 1984.

[27] Tadić M., Solution of the unitarizability problem for general linear group (non-archimedean case), preprint at Max-Planck-Institut für Mathematik, Bonn, 1985.

[28] Vogan D., The algebraic structure of the representation of semisimple Lie group I, Ann. of Math. 109 (1979), 1-60.

[29] Vogan D., Representations or real reducitive groups, Birkhäuser, Boston, 1981.

[30] Vogan D., Understanding the unitary dual, in Lie Group Representations I, Proceedings, University of Maryland 1982-83, Lecture Notes in Math. 1024, Springer-Verlag, Berlin, 1983, 264-286.

[31] Zelevinsky A.V., Induced representations of reductive p-adic groups II, Ann. Scient. Ecole Norm. Sup. 13 (1980), 165-210.